Born-Infeld charged black holes

Bachelorthesis

C.G.F. van der Kwaak
(student number 1348124)

Supervisor: Prof. Dr. M. de Roo

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Rijks*universiteit* Groningen
Faculty of Mathematics and Natural Sciences
Department of Theoretical Physics

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Chapter 1

Introduction

In 1872 James Clerk Maxwell was the first to unite electricity and magnetism into one theory, which was eventually named after him. About forty years later around 1916 Albert Einstein finally managed to come up with his theory of general relativity, on which he had worked for almost ten years. In this new theory the gravitational fields were described by the energy-momentum tensor T_{ab}. Because of the equivalence of energy and mass all forms of energy act as sources for the gravitational fields. This implies that Maxwell's linear electromagnetic theory also causes spacetime to curve when the Maxwell energy-momentum tensor is substituted in the full field equations of general relativity.

When looking for a static, asymptotically flat, spherically symmetric solution of these Einstein-Maxwell equations we obtain the so-called Reissner-Nordstrøm solution. In the early twentieth century however, some physicists were not satisfied with Maxwell's theory. When solving the Maxwell's equations for a distribution of point charges, they came up with a field that was divergent at the location of the point charges.

To solve this problem they started looking at nonlinear electrodynamics, in order to remove the infinities at the locations of the point charges. The first attempt was made by the famous physicist Max Born, and his student Leopold Infeld. They introduced the so-called Born-Infeld Lagrangian from which nonlinear field equations can be derived. Contrary to the Maxwell theory, these new equations were not divergent at the location of the point charges anymore.

The main concern of this paper will be to research the consequences for the energy- momentum tensor, the field equations of general relativity, and the charged black holes, when using the Born-Infeld Lagrangian instead of the Maxwell Lagrangian. The new energy-momentum tensor and the new field equations will be referred to as the Einstein-Born-Infeld energy-momentum tensor, respectively the Einstein-Born-Infeld field equations. The question is whether this new choice for the Lagrangian leads to new predictions when compared with Reissner-Nordstrøm theory. Eventually a comparison between Reissner-Nordstrøm theory and Einstein-Born-Infeld theory will be made.

In order to investigate this, the reader will first be introduced to Reissner-Nordstrøm theory in chapter 2. After this, the reader will be introduced to the Born-Infeld Lagrangian, and a generalization of the nonlinear electromagnetic field tensor in chapter 3, from which nonlinear electromagnetic field equations and an expression for the nonlinear Born-Infeld electric field will be derived. Next the Einstein-Born-Infeld energy-momentum tensor in chapter 4 will be derived. The Einstein-Born-Infeld field equations will be derived in chapter 5. The solutions for the Einstein-Born-Infeld field equations will be discussed in chapter 6. In chapter 7 the results and their importance will be discussed, together with some suggestions for follow-up research.

Some concepts that will be used in this thesis are supposed to be known to the reader. These concepts have probably been introduced to the reader in a course on general relativity and on classical electrodynamics. Other concepts that will be used in this thesis are the Lagrangian and the Lagrangian density. One can write the action S as an integral of the Lagrangian over time $S = \int \Lambda dt = \int L d^4x$. The Lagrangian Λ is in general no Lorentz- invariant quantity. The Lagrangian density, on the other hand, has to be Lorentz-invariant, in order to guarantee Lorentz-invariant action. Since in this thesis only the Lagrangian density will be used, the quantity L will be called the Lagrangian throughout this thesis. The units which will be used coincide with the units used in [2], from which the author used the Born-Infeld lagrangian. These are units in which $c = \bar{h} = \epsilon_0 = \mu_0 = G = 1$. This implies that the parameter b, which is contained in the Born-Infeld lagrangian, has the dimension of length squared. This implies that the electric and magnetic charges are dimensionless. Since $\epsilon_0 = 1$, ϵ will be used as the symbol for the electric charge.

Chapter 2

Reissner-Nordstrøm charged black holes

In this chapter the Reissner-Nordstrøm solution for charged black holes caused by a charged mass point will be investigated. Although the derivation can be found in [1] it will be repeated in this section in order to facilitate the comparison in chapter 7. As mentioned in the introduction, the Reissner-Nordstrøm solution is the static, asymptotically flat, spherically symmetric solution of the Einstein-Maxwell field equations. The Einstein-Maxwell equations are:

$$G_{ab} = 8\pi T_{ab} \tag{2.1}$$

where T_{ab} is the Maxwell energy-momentum tensor. Since the energy-momentum tensor is tracefree, the Ricci scalar vanishes and (2.1) becomes

$$R_{ab} = 8\pi T_{ab} \tag{2.2}$$

The Maxwell tensor F_{ab} must also satisfy Maxwell's equations in source-free regions

$$\nabla_b F^{ab} = 0 \tag{2.3}$$

$$\partial_{[a} F_{bc]} = 0 \tag{2.4}$$

The assumption of spherical symmetry implies the conversion from Cartesian coordinates to the spherical coordinates (t, r, θ, ϕ). The line element then becomes:

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5}$$

where λ and ν are functions of t and r. Since the solution is static by assumption, λ and ν can only be functions of r:

$$\nu = \nu(r), \ \ \lambda = \lambda(r) \tag{2.6}$$

In the first paragraph of the chapter it was assumed that the field is caused by a charged particle, which, by assumption, is at the origin of coordinates. This implies that the line element and the Maxwell tensor become singular there. The charged particle will also cause a purely radial field. Therefore the Maxwell tensor must take on the form:

$$F_{ab} = E(r) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

By plugging the assumptions (2.5) - (2.7) into (2.3) and (2.4) it is found that (2.4) is automatically satisfied, and that (2.3) reduces to one equation:

$$(e^{-\frac{1}{2}(\nu+\lambda)} r^2 E(r))' = 0 \tag{2.8}$$

where the prime denotes differentiation with respect to r. After integrating this it is found that:

$$E = e^{\frac{1}{2}(\nu+\lambda)} \epsilon / r^2 \tag{2.9}$$

where ϵ is a constant of integration. The assumption that the solution is asymptotically flat requires:

$$\nu, \lambda \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty \tag{2.10}$$

This leads to $E \sim \epsilon/r^2$ asymptotically. Since this is the same as the classical result for the electric field of a point particle of charge ϵ situated at the origin of the coordinate system, it is clear that ϵ must be interpreted as the charge of the particle. The general equation for the Maxwell energy-momentum tensor (for derivation, see [1] page 163) is given by:

$$T_{ab} = \frac{1}{4\pi}(-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd}) \tag{2.11}$$

By using (2.5) to (2.9) and (2.11) the Maxwell energy-momentum tensor T_{ab} is obtained for Reissner-Nordstrøm. After substitution of the Maxwell energy-momentum tensor into (2.2), the following equations are obtained from the 00 and 11 equations:

$$\lambda' + \nu' = 0 \tag{2.12}$$

By assumption (2.10), this results in $\lambda = -\nu$. The 22 equation is the one remaining equation and after rewriting the following equation is obtained:

$$(re^{\nu})' = 1 - \epsilon^2/r^2 \tag{2.13}$$

After integration this results in:

$$e^{\nu} = 1 - 2m/r + \epsilon^2/r^2 \tag{2.14}$$

where m is a constant of integration. This solution is known as the Reissner-Nordstrøm solution.

$$ds^2 = (1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2})dt^2 - (1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2})^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.15}$$

When $\epsilon = 0$, the Reissner-Nordstrøm line element reduces to the Schwarzschild line element. Therefore the integration constant m is identified as the geometric mass.

It is also worth considering the coefficients:

$$g_{00} = -(g_{11})^{-1} = 1 - 2m/r + \epsilon^2/r^2 = Q/r^2 \tag{2.16}$$

where

$$Q = r^2 - 2mr + \epsilon^2 \tag{2.17}$$

The discriminant of the quadratic Q is:

$$\Delta = m^2 - \epsilon^2 \tag{2.18}$$

and, if this is negative, i.e. $\epsilon^2 > m^2$, the quadratic has no real roots and is positive for all values of r. This implies that the line element (2.15) is non-singular for all values of r except at the origin $r = 0$, which is obvious since this is the place where the point charge producing the field is located. This is confirmed by calculating the Riemann invariant$R^{abcd}R_{abcd}$. This means there is a naked singularity, instead of the usual case where the singularity is hidden behind the horizon. It is also worth exploring what happens when $\epsilon^2 \leq m^2$. In this case the metric possesses singularities at $r = r_+$ and $r = r_-$, because Q vanishes in these points. $r_\pm$ is given by:

$$r_\pm = m \pm (m^2 - \epsilon^2)^{\frac{1}{2}} \tag{2.19}$$

In figure 2.1 a plot of g_{00} is drawn for $\epsilon^2 < m^2$ and compared with the Schwarzschild coefficient ${}_sg_{00} = 1 - 2m/r$.
The line element is regular in the regions:
I. $r_+ < r < \infty$
II. $r_- < r < r_+$
III. $0 < r < r_-$

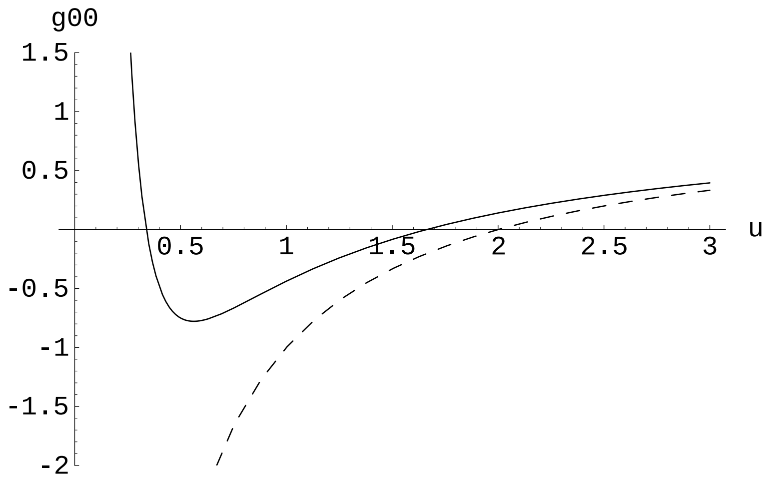

Figure 2.1: *Graphs of g_{00} versus $u = r/m$ for Reissner-Nordstrøm and Schwarzschild (dashed line) solutions. The zero that is left corresponds to u_-, and the zero that is right corresponds to u_+ in the Reissner-Nordstrøm case. In the Schwarzschild case the zero is given by $u = 2$.*

If $\epsilon^2 = m^2$, then only regions I and III exist. The regions are separated by the null hypersurfaces $r = r_+$ and $r = r_-$. The situation at $r = r_+$ is rather similar to the Schwarzschild case at $r = 2m$. The coordinates t and r are timelike and spacelike, respectively, in the regions I and III, but change their character in region II. Thus, regions I and III are static, but region II is not. Because of the totally different orientations of the lightcones on either side of the null hypersurfaces $r = r_\pm$ these coordinates suggest that the regions I, II and III are totally disconnected, just as in the case of the Schwarzschild solution.

Chapter 3

Nonlinear generalization of the electromagnetic field tensor

In this chapter the Born-Infeld lagrangian will be investigated, and the way in which the electromagnetic field tensor and the electric field will be different from the results that are obtained from the Maxwell Lagrangian. Just as for the Reissner- Nordstrøm solution, a static, asymptotically flat, spherically symmetric solution of the Einstein-Born-Infeld field equations is assumed. This implies that the Maxwell tensor (2.7) can be used in the Born-Infeld Lagrangian. In the limit of $b \rightarrow 0$ the Maxwell Lagrangian, the Maxwell tensor, and the electric field (2.9) should be obtained from the Born-Infeld Lagrangian, the Born-Infeld tensor, and the Born-Infeld electric field. First the Born-Infeld Lagrangian is obtained from [2]:

$$L = \frac{1}{b^2}\{\sqrt{-g} - \sqrt{-det(g_{ab} + bF_{ab})}\} \tag{3.1}$$

which can be written as:

$$L = \frac{1}{b^2}\{(-g)^{\frac{1}{2}} - (-a)^{\frac{1}{2}}\} \tag{3.2}$$

With $g = det(g_{ab})$ and $a = det(a_{ab})$. Then a_{ab} is defined as follows:

$$a_{ab} = g_{ab} + bF_{ab} \tag{3.3}$$

Just as in chapter 2, the assumption of spherical symmetry implies the conversion from Cartesian coordinates to the spherical coordinates (t, r, θ, ϕ). Again, the line element becomes:

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.4}$$

where λ and ν are functions of t and r. Since the solution is static by assumption, λ and ν can only be functions of r:

$$\nu = \nu(r), \;\; \lambda = \lambda(r) \tag{3.5}$$

From the line element (3.4) the determinant of the metric can be calculated:

$$g = -e^{\nu+\lambda} r^4 \sin^2\theta \tag{3.6}$$

In the first paragraph of the chapter it was shown that the Maxwell tensor from chpater 2 is still valid, because of the assumption of spherical symmetry. So the Maxwell tensor is again given by (2.7):

$$F_{ab} = E(r) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

and F^{ab} is equal to:

$$F^{ab} = E(r)e^{-\nu-\lambda} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.8}$$

Now it is possible to calculate a_{ab}:

$$a_{ab} = g_{ab} + bF_{ab} = \begin{pmatrix} e^{\nu} & -bE(r) & 0 & 0 \\ bE(r) & -e^{\lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{3.9}$$

And therefore the determinant of a_{ab} is equal to:

$$a = (-e^{\nu+\lambda} + b^2E^2(r))r^4\sin^2\theta \tag{3.10}$$

After substitution into (3.2) the following Born-Infeld Lagrangian is obtained:

$$L = \frac{1}{b^2}\{e^{\frac{1}{2}(\nu+\lambda)}r^2\sin\theta - (e^{\nu+\lambda} - b^2E^2(r))^{\frac{1}{2}}r^2\sin\theta\} \tag{3.11}$$

Which can be rewritten as:

$$L = \frac{1}{b^2}\{e^{\frac{1}{2}(\nu+\lambda)}r^2\sin\theta - e^{\frac{1}{2}(\nu+\lambda)}(1 - b^2E^2(r)e^{-\nu-\lambda})^{\frac{1}{2}}r^2\sin\theta\} \tag{3.12}$$

In the limit of $b \to 0$ the Maxwell Lagrangian must be recovered from the Born-Infeld Lagrangian. The Maxwell Lagrangian is given by [1], page 163:

$$L_E = \frac{(-g)^{\frac{1}{2}}}{8\pi} g^{ac}g^{bd}F_{ab}F_{cd} \tag{3.13}$$

After substitution of the metric from (3.4), and the equations (3.6), and (3.7), this becomes:

$$L = \frac{-1}{4\pi}E^2(r)e^{-\frac{1}{2}(\nu+\lambda)}r^2\sin\theta \tag{3.14}$$

When the limit of $b \to 0$ is taken for equation (3.12), the following expression is obtained for the Born-Infeld Lagrangian:

$$L = \frac{1}{2}E^2(r)e^{-\frac{1}{2}(\nu+\lambda)}r^2\sin\theta \tag{3.15}$$

So it is clear that the Born-Infeld Lagrangian has to be multiplied with a factor $-1/2\pi$, in order to obtain the Maxwell Lagrangian from the Born-Infeld Lagrangian in the limit of $b \to 0$. The Born-Infeld Lagrangian then becomes:

$$L = \frac{-1}{2\pi b^2}\{(-g)^{\frac{1}{2}} - (-a)^{\frac{1}{2}}\} \tag{3.16}$$

Now a derivation of the Maxwell equation (2.3) is given. After this it will be clear how to derive the Born-Infeld tensor, the nonlinear equivalent of the Maxwell tensor, since it is derived in an analoguous way. The Maxwell Lagrangian is given in (3.13), where now the original defition of F_{ab} will be used:

$$F_{ab} = \phi_{a,b} - \phi_{b,a} \tag{3.17}$$

Variation with respect to ϕ_a and $\phi_{a,b}$ gives:

$$\frac{\delta L_E}{\delta \phi_a} = 0 \tag{3.18}$$

$$\frac{\partial L_E}{\partial \phi_{a,b}} = \frac{(-g)^{\frac{1}{2}}}{2\pi} F^{ab} \tag{3.19}$$

The Euler-Lagrange equations are given by:

$$\frac{\delta L}{\delta \phi_a} - \partial_b(\frac{\delta L}{\delta \phi_{a,b}}) = 0 \tag{3.20}$$

The equations of motion can now be found by substitution into the Euler-Lagrange equations:

$$\partial_b(\frac{(-g)^{\frac{1}{2}}}{2\pi} F^{ab}) = 0 \tag{3.21}$$

In general relativity this becomes:

$$\nabla_b(\frac{(-g)^{\frac{1}{2}}}{2\pi} F^{ab}) = 0 \tag{3.22}$$

However, since

$$\nabla_b((-g)^{\frac{1}{2}}) = 0 \tag{3.23}$$

the equation of motion becomes equal to the Maxwell equation (2.3):

$$\nabla_b F^{ab} = 0 \tag{3.24}$$

Because of equation (3.17) the tensor a_{ab} can also be written as:

$$a_{ab} = g_{ab} + b(\phi_{a,b} - \phi_{b,a}) \tag{3.25}$$

variation with respect to $\phi_{a,b}$ gives:

$$\frac{\delta L}{\delta \phi_{a,b}} = \frac{\delta L}{\delta a}\frac{\delta a}{\delta a_{cd}}\frac{\delta a_{cd}}{\delta \phi_{a,b}} = \frac{1}{4\pi b}(-a)^{\frac{1}{2}} p^{dc}(\delta^a_c \delta^b_d - \delta^a_d \delta^b_c) = \frac{1}{4\pi b}(-a)^{\frac{1}{2}}(p^{ba} - p^{ab}) = K^{ab} \tag{3.26}$$

Where p^{ba} is the inverse of a_{ab} and is given by:

$$p^{ba} = \begin{pmatrix} \frac{e^{\lambda}}{e^{\nu+\lambda}-b^2E^2(r)} & \frac{-bE(r)}{e^{\nu+\lambda}-b^2E^2(r)} & 0 & 0 \\ \frac{bE(r)}{e^{\nu+\lambda}-b^2E^2(r)} & \frac{-e^{\nu}}{e^{\nu+\lambda}-b^2E^2(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{3.27}$$

By substitution of K^{ab} into the Euler-Lagrange equations (3.20), the equation of motion belonging to the Born-Infeld lagrangian is obtained:

$$\partial_b K^{ab} = 0 \tag{3.28}$$

So in general relativity the following equation is obtained:

$$\nabla_b K^{ab} = 0 \tag{3.29}$$

Equation (3.23) is still valid:

$$\nabla_b((-g)^{\frac{1}{2}}) = 0 \tag{3.30}$$

Therefore K^{ab} is written as $\frac{(-g)^{\frac{1}{2}}}{2\pi} H^{ab}$ with

$$H^{ab} = \frac{1}{2b} \frac{(-a)^{\frac{1}{2}}}{(-g)^{\frac{1}{2}}} (p^{ba} - p^{ab}) \tag{3.31}$$

This way, the nonlinear equivalent of (2.3), which will be referred to as the Born-Infeld equation, becomes:

$$\nabla_b H^{ab} = 0 \tag{3.32}$$

Where H^{ab}, which will be referred to as the Born-Infeld tensor, is given by:

$$H^{ab} = \frac{E(r)e^{-\frac{1}{2}(\nu+\lambda)}}{(e^{\nu+\lambda} - b^2E^2(r))^{\frac{1}{2}}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.33}$$

It is obvious that F^{ab} and the Maxwell equation (2.3) are recovered from H^{ab}, respectively the Born-Infeld equation (3.32), as $b \to 0$.

From (3.32), the following equations are obtained:

$$\partial_1 H^{01} + (\frac{1}{2}\nu' + \frac{1}{2}\lambda' + 2r^{-1})H^{01} = 0 \tag{3.34}$$

$$\partial_0 H^{10} = 0 \tag{3.35}$$

But since the solution is by assumption static, the second equation is automatically satisfied. Substituting the expression for H^{ab} into (3.32), the following equation is obtained:

$$\frac{1}{2\pi} \frac{e^{-\frac{1}{2}(\nu+\lambda)}}{(e^{\nu+\lambda} - b^2E^2(r))^{\frac{3}{2}}} (-\frac{1}{2}(\nu'+\lambda')E(r)e^{\nu+\lambda} + E'(r)e^{\nu+\lambda} + 2r^{-1}E(r)e^{\nu+\lambda} - 2b^2r^{-1}E^3(r)) = 0 \tag{3.36}$$

This can be rewritten in the following way:

$$(r^2 E(r) e^{-\frac{1}{2}(\nu+\lambda)})' = 2b^2 r e^{-\frac{3}{2}(\nu+\lambda)} E^3(r) \tag{3.37}$$

Again it is obvious that as $b \to 0$, equation (2.8) is retrieved. If $F = r^2 E(r) e^{-\frac{1}{2}(\nu+\lambda)}$ equation (3.37) becomes:

$$\frac{dF}{dr} = \frac{2b^2 F^3}{r^5} \tag{3.38}$$

After integration, the following equation is obtained:

$$\frac{1}{2} F^{-2} = 2b^2 (-\frac{1}{4} r^{-4}) + \alpha \tag{3.39}$$

And after substituting back $F = r^2 E(r) e^{-\frac{1}{2}(\nu+\lambda)}$ into equation (3.39), an expression for the electric field is obtained:

$$E^2(r) = \frac{e^{\nu+\lambda}}{b^2 - 2\alpha r^4} \tag{3.40}$$

So the electric field, which will be referred to as the Born-Infeld electric field, is given by:

$$E(r) = \pm \frac{e^{\frac{1}{2}(\nu+\lambda)}}{\sqrt{b^2 - 2\alpha r^4}} \tag{3.41}$$

It is possible to determine the constant α, since in the limit of $b \to 0$, equation (3.41) should become equation (2.9). Therefore the positive root of (3.41) is taken. This gives for $E(r)$:

$$E(r) \approx \frac{e^{\frac{1}{2}(\nu+\lambda)}}{\sqrt{-2\alpha r^4}} = \frac{1}{\sqrt{-2\alpha}} \frac{e^{\frac{1}{2}(\nu+\lambda)}}{r^2} \tag{3.42}$$

Therefore:

$$\epsilon = \frac{1}{\sqrt{-2\alpha}} \quad \Rightarrow \quad \alpha = \frac{-1}{2\epsilon^2} \tag{3.43}$$

So the Born-Infeld electric field becomes:

$$E(r) = \frac{1}{\sqrt{1 + \frac{b^2 \epsilon^2}{r^4}}} \frac{e^{\frac{1}{2}(\nu+\lambda)} \epsilon}{r^2} \tag{3.44}$$

Again the electric field (2.9) is recovered from the Born-Infeld electric field as $b \to 0$. It is obvious that the negative root of equation (3.41) can be obtained from equation (3.44) by substituting a neagative value for the charge ϵ, just as in the case of the classical electric field.

Chapter 4

Einstein-Born-Infeld energy-momentum tensor

After the derivation of the Born-Infeld tensor and the Born-Infeld electric field, the Einstein-Born-Infeld energy-momentum tensor (EBI energy-momentum tensor) will be derived in this chapter. The condition that in the limit of $b \to 0$ the Einstein-Born-Infeld theory has to go over into the Reissner-Nordstrøm theory is still valid. Therefore this condition should also be valid for the EBI energy-momentum tensor. This means that the Einstein-Maxwell energy-momentum tensor will have to be recovered from the EBI energy-momentum tensor in the limit of $b \to 0$. In general the energy-momentum tensor can be derived from the following equation, see [1] page 151:

$$\frac{\delta L}{\delta g^{ab}} = -(-g)^{\frac{1}{2}} T_{ab} \tag{4.1}$$

which is valid for any arbitrary Lagrangian L. Now, for the derivation of the EBI energy-momentum tensor, the Born-Infeld Lagrangian (3.16) from chapter 3 is considered:

$$L = \frac{-1}{2\pi b^2}\{(-g)^{\frac{1}{2}} - (-a)^{\frac{1}{2}}\} \tag{4.2}$$

In order to find the EBI energy-momentum tensor it is obvious from (4.1) that the Born-Infeld Lagrangian has to be varied with respect to the contravariant metric g^{ab}:

$$\frac{\delta L}{\delta g^{ab}} = \frac{\delta}{\delta g^{ab}}(\frac{-1}{2\pi b^2}\{(-g)^{\frac{1}{2}} - (-a)^{\frac{1}{2}}\}) = \frac{-1}{2\pi b^2}(\frac{\delta(-g)^{\frac{1}{2}}}{\delta g^{ab}} - \frac{\delta(-a)^{\frac{1}{2}}}{\delta g^{ab}}) \tag{4.3}$$

Differentiation of $(-g)^{\frac{1}{2}}$ and $(-a)^{\frac{1}{2}}$ with respect to g^{ab} gives:

$$\frac{\delta(-g)^{\frac{1}{2}}}{\delta g^{ab}} = \frac{1}{2}(-g)^{\frac{1}{2}} g^{cd}(-g_{c\{a}g_{b\}d}) = -\frac{1}{2}(-g)^{\frac{1}{2}} g_{ab} \tag{4.4}$$

$$\frac{\delta(-a)^{\frac{1}{2}}}{\delta g^{ab}} = \frac{\delta(-a)^{\frac{1}{2}}}{\delta a}\frac{\delta a}{\delta a_{cd}}\frac{\delta a_{cd}}{\delta g^{ab}} = \frac{1}{2}(-a)^{\frac{1}{2}} p^{dc}(-g_{c\{a}g_{b\}d}) = \frac{1}{4}(-a)^{\frac{1}{2}}(-p_{ba} - p_{ab}) \tag{4.5}$$

where p_{ba} is the covariant of p^{ba} from chapter (3), equation (3.27), and is given by:

$$p_{ba} = \begin{pmatrix} \frac{e^{\nu}}{1-b^2E^2(r)e^{-\nu-\lambda}} & \frac{bE(r)}{1-b^2E^2(r)e^{-\nu-\lambda}} & 0 & 0 \\ \frac{-bE(r)}{1-b^2E^2(r)e^{-\nu-\lambda}} & \frac{-e^{\lambda}}{1-b^2E^2(r)e^{-\nu-\lambda}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{4.6}$$

And so the variation of L with respect to the contravariant metric is given by:

$$\frac{\delta L}{\delta g^{ab}} = \frac{-(-g)^{\frac{1}{2}}}{8\pi b^2}(-2g_{ab} + \frac{(-a)^{\frac{1}{2}}}{(-g)^{\frac{1}{2}}}(p_{ba} + p_{ab})) \tag{4.7}$$

Now the EBI energy-momentum tensor can be obtained from equation (4.1), and it is obvious from equation (4.7) that it is given by:

$$T_{ab} = \frac{1}{8\pi b^2}(-2g_{ab} + \frac{(-a)^{\frac{1}{2}}}{(-g)^{\frac{1}{2}}}(p_{ba} + p_{ab})) \tag{4.8}$$

By looking at equation (4.8), it is clear that there are only diagonal components, and they are given by:

$$T_{00} = -\frac{1}{4\pi}\frac{e^{\nu}}{b^2}(1 - \frac{1}{(1-b^2E^2(r)e^{-\nu-\lambda})^{\frac{1}{2}}}) \tag{4.9}$$

$$T_{11} = \frac{1}{4\pi}\frac{e^{\lambda}}{b^2}(1 - \frac{1}{(1-b^2E^2(r)e^{-\nu-\lambda})^{\frac{1}{2}}}) \tag{4.10}$$

$$T_{22} = \frac{1}{4\pi}\frac{r^2}{b^2}(1 - (1-b^2E^2(r)e^{-\nu-\lambda})^{\frac{1}{2}}) \tag{4.11}$$

$$T_{33} = \frac{1}{4\pi}\frac{r^2\sin^2\theta}{b^2}(1 - (1-b^2E^2(r)e^{-\nu-\lambda})^{\frac{1}{2}}) \tag{4.12}$$

At the beginning of the chapter the condition was formulated that in the limit of $b \rightarrow 0$ the Maxwell energy-momentum tensor had to be recovered from the EBI energy-momentum tensor. Since the Maxwell energy-momentum tensor is given by

$$T_{ab} = \frac{1}{4\pi}(-g^{cd}F_{ac}F_{bd} + \frac{1}{4}g_{ab}F_{cd}F^{cd}), \tag{4.13}$$

it can be seen that as $b \rightarrow 0$ the Maxwell energy-momentum tensor is indeed recovered from the Einstein-Born-Infeld energy-momentum tensor.

Chapter 5

Einstein-Born-Infeld field equations

Now that the Born-Infeld electric field and the Einstein-Born-Infeld energy-momentum tensor have been derived, the next step is to substitute these expressions into the general field equations of relativity. This way, expressions for the Einstein-Born- Infeld line element for a charged particle at the origin of the coordinate system can be found. The general field equations are given by equation (2.1):

$$G_{ab} = 8\pi T_{ab} \tag{5.1}$$

Because a static, asymptotically flat, spherically symmetric solution of the Einstein- Born-Infeld field equations is sought, the metric can be found from the same line element as in equation (2.5):

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.2}$$

The assumptions made in chapter 2 are still valid. Therefore the assumption that the solution is asymptotically flat requires:

$$\nu, \lambda \to 0 \quad \text{as} \quad r \to \infty \tag{5.3}$$

After substitution of the Einstein-Born-Infeld energy-momentum tensor into (2.1), the following equation is again obtained from the 00 and 11 equations:

$$\lambda' + \nu' = 0 \tag{5.4}$$

By assumption (5.3), this results in $\lambda = -\nu$. The 22 equation is the one remaining equation and after rewriting the following equation is obtained:

$$(r^2 e^{\nu} \nu')' = \frac{4r^2}{b^2}(1 - (1 - b^2 E^2(r) e^{-\nu-\lambda})^{\frac{1}{2}}) \tag{5.5}$$

By substituting the equation for the nonlinear Born-Infeld electric field (3.44), the following expression is obtained:

$$(r^2 e^{\nu} \nu')' = \frac{4r^2}{b^2}(1 - (1 - \frac{\frac{b^2\epsilon^2}{r^4}}{1 + \frac{b^2\epsilon^2}{r^4}})^{\frac{1}{2}}) = \frac{4r^2}{b^2}(1 - \frac{1}{\sqrt{1 + \frac{b^2\epsilon^2}{r^4}}}) \tag{5.6}$$

The solution for this differential equation is given in [3] and [4], and is repeated below with one difference: in article [3] and [4] $1/b$ is used, instead of b. This has to do with the definition of the Born-Infeld Lagrangian.

$$e^{\nu} = 1 - \frac{2m}{r} + \frac{2}{3}\frac{r^2}{b^2}(1 - \sqrt{1 + \frac{b^2\epsilon^2}{r^4}}) + \frac{4\epsilon^2}{3r}g(r) \tag{5.7}$$

where

$$g'(r) = \frac{dg}{dr} = -(r^4 + b^2\epsilon^2)^{-1/2} \tag{5.8}$$

The first condition that must be valid for this solution is that it corresponds to the Reissner-Nordstrøm solution in the limit of large distances, $r \to \infty$. The second condition is that in the limit of $b \to 0$, the Reissner-Nordstrøm solution must also be obtained, just as throughtout this entire paper the linear theory has to be recovered from the nonlinear theory in the limit of $b \to 0$. The limits of integration of the function $g(r)$ are undetermined. The first solution corresponds to

$$g(r) = \int_r^{\infty} \frac{ds}{\sqrt{s^4 + b^2\epsilon^2}} = \frac{1}{2\sqrt{b\epsilon}}F(\arccos\{\frac{r^2 - b\epsilon}{r^2 + b\epsilon}\}, \frac{1}{\sqrt{2}}) \tag{5.9}$$

where F is the Legendre's elliptic function of the first kind. The second solution corresponds to

$$g(r) = -\int_0^{r} \frac{ds}{\sqrt{s^4 + b^2\epsilon^2}} = -\frac{1}{2\sqrt{b\epsilon}}F(\arccos\{\frac{b\epsilon - r^2}{b\epsilon + r^2}\}, \frac{1}{\sqrt{2}}) \tag{5.10}$$

The difference between the two solutions is in the behavior at the origin. The solution for e^{ν}, with $g(r)$ given in equation (5.9), diverges at $r \to 0$ (even when $m = 0$), and therefore corresponds to the black hole solution. This solution is called the GSP solution. The other solution of e^{ν}, with $g(r)$ given in equation (5.10), is a solution that is finite at the origin for $m = 0$, and is called the Demianski solution. The integrals of equations (5.9) and (5.10) are related by

$$\int_r^{\infty} \frac{ds}{\sqrt{s^4 + b^2\epsilon^2}} = -\int_0^{r} \frac{ds}{\sqrt{s^4 + b^2\epsilon^2}} + const \tag{5.11}$$

The constant is fixed by the condition that the Reissner-Nordstrøm solution has to be recovered from the Einstein-Born-Infeld solution, in the limit of $b \to 0$. In the articles [3] and [4] the constant is then found to be equal to $const = \frac{1}{\sqrt{b\epsilon}}\mathrm{K}(\frac{1}{2})$.

$$\int_r^{\infty} \frac{ds}{\sqrt{s^4 + b^2\epsilon^2}} + \int_0^{r} \frac{ds}{\sqrt{s^4 + b^2\epsilon^2}} = \frac{1}{\sqrt{b\epsilon}}\mathrm{K}(\frac{1}{2}) \tag{5.12}$$

where $\mathrm{K}(\frac{1}{2})$ is the complete elliptic integral of the first kind.

Chapter 6

Einstein-Born-Infeld charged black holes

In the previous chapter a solution was found for the metric function e^{ν}. This solution contained a function $g(r)$. The function $g(r)$ depended on the integration limits, chosen after integration of $g'(r)$. Because the Demianksi solution (5.10) is not divergent at $r = 0$ (for $m = 0$), the GSP solution (5.9), which is divergent at $r = 0$, is the solution which has to be considered for the charged black hole case. Different values of the parameters ϵ, m and the Born-Infeld parameter b will have different consequences for the metric function e^{ν}. The aim of this chapter will be to investigate these consequences. These consequences can be found in the articles [3] and [4] and will be partially repeated in this chapter, in order to facilitate the comparison between the Reissner-Nordstrøm theory and the Einstein-Born-Infeld theory in the next chapter.

It can be shown that a singularity exists for the GSP solution at $r = 0$ of order at least $1/r^6$, due to the mass distribution. This also occurs in the case of Reissner-Nordstrøm and Schwarzschild (for the Schwarzschild case $R^{abcd}R_{abcd} = 48m^2/r^6$). Furthermore it is clear that the contribution due to the Born-Infeld electric field diverges at $r = 0$ and therefore, there is only one singularity at $r = 0$ in this spacetime.

The first thing to notice is that the zeros of the metric function e^{ν} indicate the existence of coordinate singularities which can be eliminated by a change of coordinates. The number of zeros and their positions depend on the parameters ϵ, m and the Born-Infeld parameter b. Second, the position of the horzion is identified in the Einstein-Born-Infeld theory, and is defined as the value r_h for which $g_{00}(r_h) = e^{\nu}(r_h) = 0$. The position of the horzion is also dependent on the parameters ϵ, m and b. The zeros of e^{ν} however, have to be identified numerically. It is possible to distinguish different cases in which e^{ν} has two, one or none zeros at all, depending on the relation ϵ/m and b. It can be seen that e^{ν} has no solution for $\epsilon/m > 1$ (hyperextreme case, $\epsilon > m$), except for very large values of b, see figure 6.1.

The behavior resembles the Schwarzschild behavior for $\epsilon/m < 0.4$, independently of the Born-Infeld parameter b. In this case the gravitational field due to the mass dominates the (linear or nonlinear) electromagnetic field, because of the small charge. This can clearly be seen in figure 6.2, where $\epsilon = m/10$.

There are also certain ranges of ϵ/m, in which the value of the Born-Infeld parameter determines the structure of e^{ν}. For $0.5 < \epsilon/m < 0.9$, the metric function e^{ν} resembles Schwarzschild if $b > 10m/7$. This corresponds to the weak electromagnetic field. In the case of $b < 10m/7$ e^{ν} has two zeros and e^{ν} grows to infinity $(+\infty)$ near $r = 0$. The larger the value of ϵ/m, the larger the value that is needed for b to defeat the gravitational attraction at $r = 0$. This last conclusion becomes especially clear from a comparison between figure 6.1 and figure 6.3. In figure 6.1 the b-values are given by $b = m$, $b = 10m$, and $b = 20m$, while they are given by $b = m/5$, $b = m/2$, and $b = 2m$ in figure 6.3. This means that as the mass becomes smaller with respect to the charge, a smaller nonlinear electromagnetic field (equation (3.44)) is needed to overwhelm gravitation. When $\epsilon/m > 0.5$ and $b < m/2$, e^{ν} goes to infinity $(+\infty)$ at $r \to 0$. This also implies that as b becomes larger (remaining smaller than $b = m/2$), the point where e^{ν} becomes zero is nearer the origin, which means that the horizon shrinks. This can be seen in 6.3.

The last case, which is an important one, is the case in which $\epsilon = m$. This case corresponds to the extreme black hole. In this case the function e^{ν} shows a very sensitive behavior with respect to the value b in the neighborhood of $b = 2m$; in the vicinity of this value three cases are possible: one horizon, two horizons or no horizon at all. For $b = m/0.5225$ e^{ν} possesses one horizon and $e^{\nu} \to \infty$ for $r \to 0$. For $b < m/0.5225$, e^{ν} has no zeros at all, and this corresponds to a naked singularity. The final case, in which $m/0.5225 < b < m/0.516$, two horizons are present. For $b \geq m/0.516$, gravitation dominates and $e^{\nu} \to -\infty$ as $r \to 0$, just as in the Schwarzschild case. The behavior for $\epsilon = m$ is shown in figure 6.4.

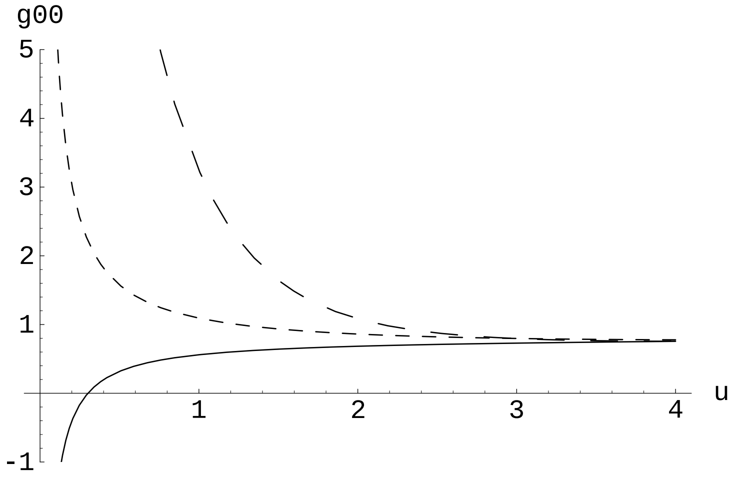

Figure 6.1: *The behavior of the function $g_{00} = e^{\nu}$ for the hyperextreme case $\epsilon = 2m$ for distinct values of b is shown. These values are given by (from the upper graph downward) by $b = m$, $b = 10m$, and $b = 20m$. The plot of e^{ν} is in terms of the variable $u = r/m$.*

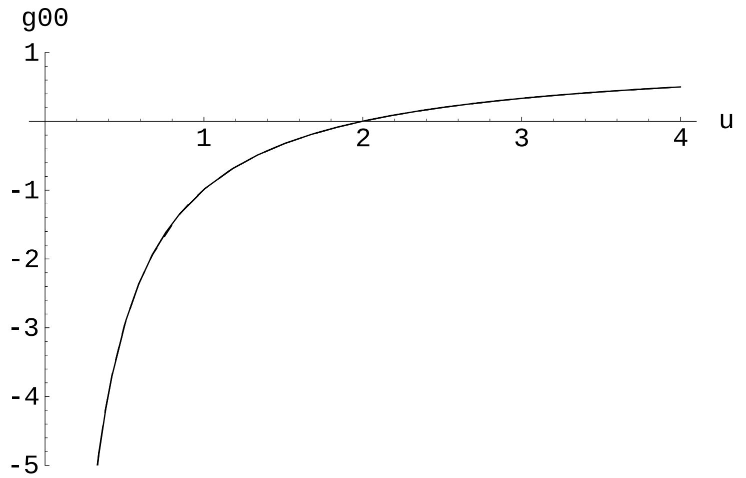

Figure 6.2: *The behavior of the function $g_{00} = e^{\nu}$ for the $\epsilon = m/10$ for distinct values of b is shown. These values are given by $b = 10m$, $b = m/10$, and $b = m/100$. It is clear that the solution resembles the Schwarzschild solution independent of the value of the Born-Infeld parameter b. The plot of e^{ν} is in terms of the variable $u = r/m$.*

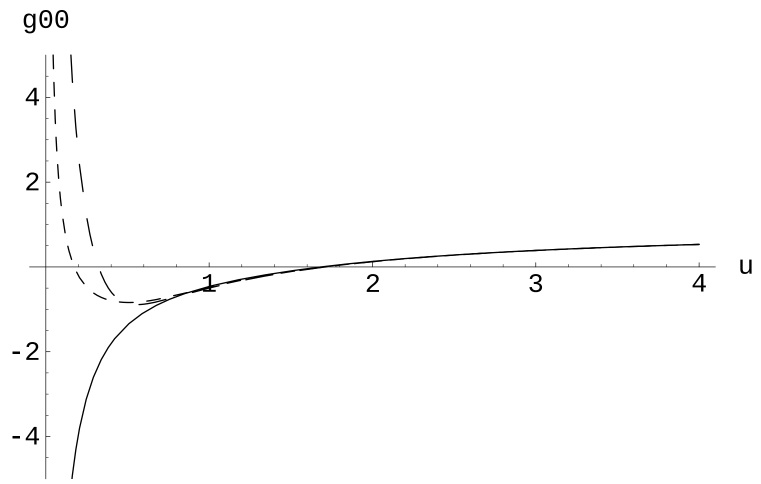

Figure 6.3: *The behavior of the function $g_{00} = e^{\nu}$ for the $\epsilon = 7m/10$ case for distinct values of b is shown. These values are given by (from the upper graph downward) by $b = m/5$, $b = m/2$, and $b = 2m$. The plot of e^{ν} is in terms of the variable $u = r/m$.*

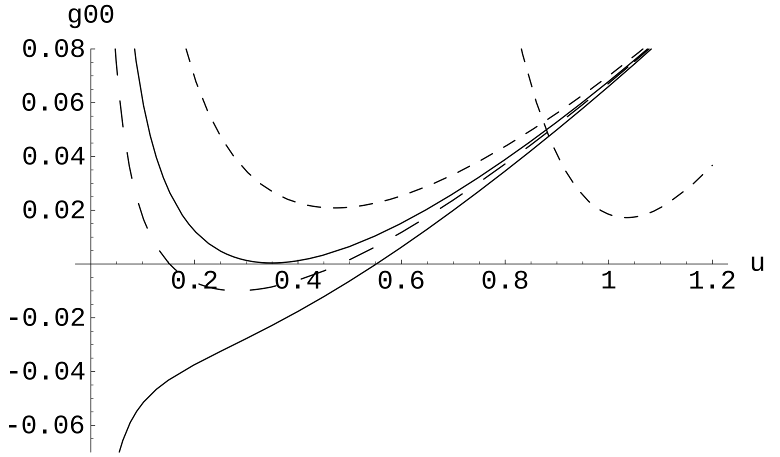

Figure 6.4: *The behavior of the function* $g_{00} = e^{\nu}$ *for the extreme case* $\epsilon = m$ *for distinct values of* b *is shown. These values are given by (from the upper graph downward) by* $b = m/0.53$, $b = m/0.5225$, $b = m/0.52$, and $b = m/0.516$. *The dashed graph on the right corresponds to* $b = m/10$. *The plot of* e^{ν} *is in terms of the variable* $u = r/m$.

Chapter 7

Discussion and Conclusions

The main concern of this research has been to investigate the conesquences for charged black holes when not the usual Maxwell Lagrangian was used, but the nonlinear Born-Infeld lagrangian. In order to obtain the answer to this question, first the Born-Infeld tensor, the Born-Infeld equation, and the Einstein-Born-Infeld energy-momentum tensor were calculated. After this stage the field equations were found, and the solution could be calculated with the help of articles [3] and [4]. The main goal of this paper, however, was to research the consequences of this new choice for the Lagrangian for charegd black holes.

The first thing to notice, and a condition we imposed on the Einstein-Born-Infeld theory (EBI theory), was that in the limit of $b \to 0$, the Reissner-Nordstrøm theory (RN theory)was recovered from the Einstein-Born-Infeld theory. This was shown to be the case, and so it can be concluded that EBI theory and RN theory are very much alike for small b. The same conclusion can be drawn for $r \to \infty$. This makes sense since the Born-Infeld tensor and energy-momentum tensor contain a $(1 - b^2E^2(r)e^{-\nu-\lambda})^{\pm\frac{1}{2}}$ term, which is after substitution of (3.44) equal to the $(1 + \frac{b^2\epsilon^2}{r^4})^{\mp\frac{1}{2}}$ term. This term is also present in the Born-Infeld electric field. The fraction is becoming small either when b becomes small, or when r grows large. In this last case the same expansion is made as when $b \to 0$, so that again the RN theory is recovered from the EBI theory.

The second thing to look at is the number of zeros in EBI theory. From the analysis of e^ν in the previous chapter it is obvious that just as in RN theory, it is possible to have either two horizons, one horzion, or no horizon at all, depending on the parameters involved in the theory. For RN theory, these parameters are the mass m and the charge ϵ. The first difference, however, is that in EBI theory a third parameter is introduced, the Born-Infeld parameter b. It can be seen that different values of the parameter b give totally different solutions for e^ν.

When looking at specific cases the following conclusions can be drawn from the figures 7.1 and 7.3. It is clear that as b becomes smaller, the RN solution is approached. Since the condition was imposed that in the limit of $b \to 0$ the RN solution had to be obtained from the EBI theory, this is a logical result. It can also be seen that with increasing b the zeros of the function e^ν are getting closer to $r = 0$, while still going to infinity $(+\infty)$. This implies that the horizon (or the bending point in case of no zero) is shrinking with decreasing nonlinear electromagnetic field. Eventually, however, b becomes so strong that

e^{ν} is flipped, and ends up going to minus infinity $(-\infty)$ for $r \to 0$. In this case, e^{ν} has one zero, and the solution looks a lot like the Schwarzschild case. However, the point where e^{ν} becomes zero, is now nearer the horizon than for the Schwarzschild case, i.e. the horizon shrinks.

For $\epsilon = m/10$ it can be concluded that the RN theory and the EBI theory coincide with the Schwarzschild theory. The reason is clear. The charge is too weak to with respect to the charge, and so gravitation wins. Again the RN theory is recovered from the EBI theory.

When the behavior of e^{ν} in RN theory is compared with the behavior in EBI theory in the case of $\epsilon = m$, it is obvious that there is one zero, and one horizon in RN theory. In EBI theory, however, e^{ν} can have three different solutions, depending on the parameter b. For $b \to 0$ the RN solution for e^{ν} is recovered from the EBI solution . For $b = m/0.5225$, e^{ν} contains one zero, and hence one horizon. This is the same solution as in RN theory, although the size of the horizon is smaller in EBI theory. But for $b < m/0.53$, contrary to the RN theory, e^{ν} doesn't contain a zero, and instead of one horizon, a naked singularity is present. For $m/0.5225 < b < m/0.516$ two horizons are present. e^{ν} is going to infinity $(+\infty)$ for these cases, and the closest zero to $r = 0$ (or the bending point, if no zero present), is coming nearer the origin. But again, for $b \geq m/0.516$ gravitation dominates and e^{ν} has one zero, is going to minus infinity $(-\infty)$, and has the same behavior as in the Schwarzschild black hole.

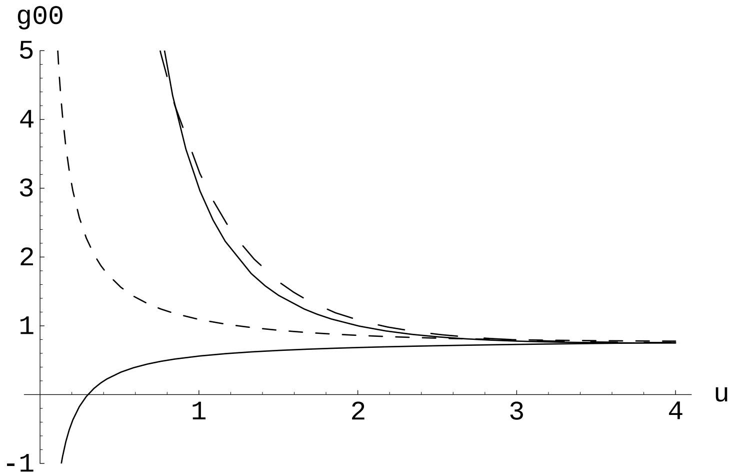

Figure 7.1: *e^{ν} versus $u = r/m$ is shown for $\epsilon = 2m$. Figure 6.1 is repeated, but the Reissner-Nordstrøm graph is included (drawn line).*

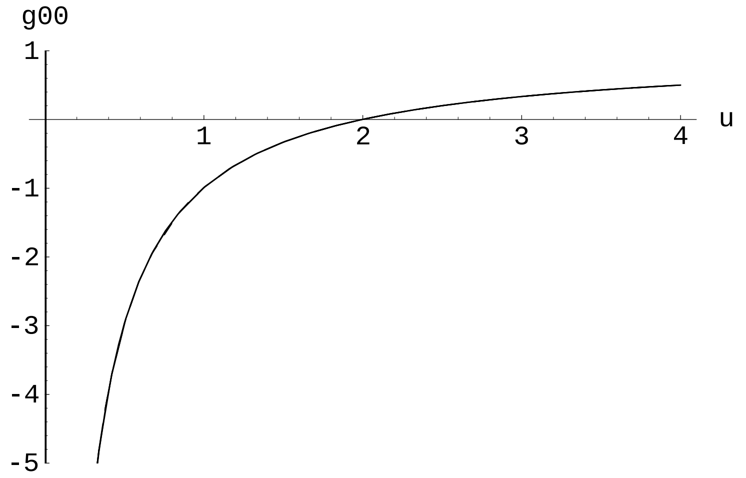

Figure 7.2: e^{ν} *versus* $u = r/m$ *is shown for* $\epsilon = m/10$. *Figure 6.2 is repeated, but the Reissner-Nordstrøm graph is included, but not visible, because it coincides, just as the EBI graphs, with the Schwarzschild graph.*

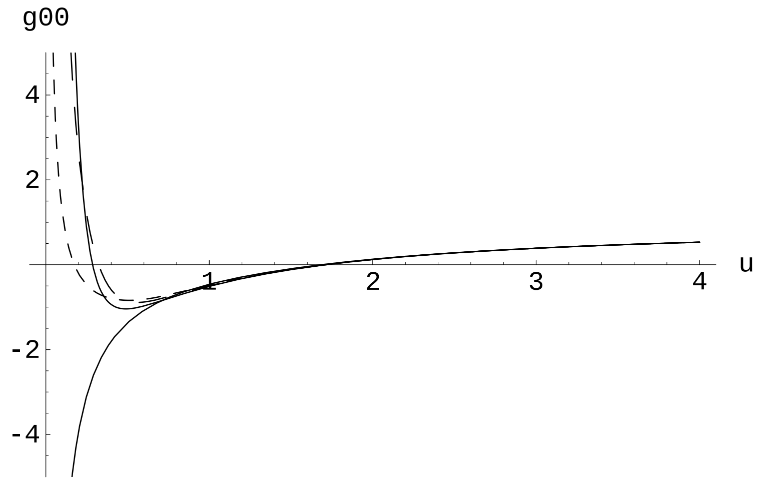

Figure 7.3: e^{ν} *versus* $u = r/m$ *is shown for* $\epsilon = 7m/10$. *Figure 6.3 is repeated, but the Reissner-Nordstrøm graph is included (drawn line).*

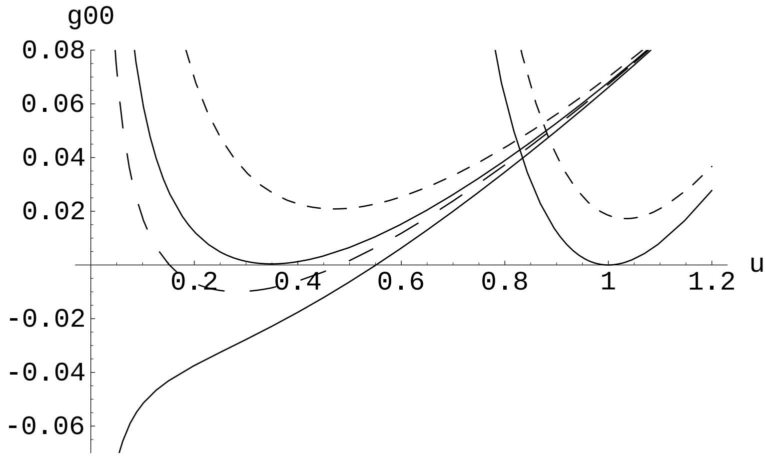

Figure 7.4: *Again e^ν versus $u = r/m$ is shown for $\epsilon = m$. Figure 6.4 is repeated, but the Reissner-Nordstrøm graph is included (drawn line).*

The general conclusion that can be drawn from the four figures, is that in the limit of $b \to 0$ the Reissner-Nordstrøm theory is recovered from the EBI theory. While the $\epsilon = m$ case is often invariant with respect to variations in a theory, the solution e^ν does depend on the Born-Infeld parameter b. It was also found that when b exceeds a certain value for a certain charge mass ratio, the function e^ν starts going to minus infinity ($-\infty$), instead of infinity ($+\infty$). This was not concluded in the articles [3] and [4]. During the research, the author wasn't able to find the reason for these results, and it might be interesting to perform a follow-up research on these questions.

Chapter 8

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Bibliography

[1] R. d'Inverno. *Introducing Einstein's Relativity*. Oxford University Press. 1992.

[2] H.J.L. van der Heiden. *Generalizations of Born-Infeld theory*. 2006

[3] N. Bretón, *Phys. Rev.* D **67,** 124004 (2003)

[4] N. Bretón, *Horzion structure of Born-Infeld black hole*, submitted to **World Scientific** on June 5, 2006