Inflation by a massive scalar field

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Abstract

In this thesis inflation by a massive scalar field is investigated. Because scalar fields play an important role in high energy physics it is chosen to investigate inflation by a scalar field. As an introduction the Big Bang model is introduced, the problems of this model are discussed and the solution of these problems is worked out. The solution of the problems is to introduce a short period in which inflationary expansion occurs. Inflationary expansion occurs by introducing a cosmological constant but the problem is that in this case inflationary expansion does not come to an end. Therefore massive scalar field inflation is introduced. The dynamics of massive scalar field inflation is obtained both by solving the equations analytically in the so called slow roll approximation as by solving the equations numerically. By numerically solving the equations it is found that the slow roll approximation is a good approximation. By looking at density perturbations the mass of the scalar field is calculated as $m \approx 10^{-6}$. Furthermore the tilt of the spectrum of density perturbations is calculated as $n(60) \approx 0.97$. The calculated value of the tilt agrees with the value of the tilt derived from observations. It is found that the duration of inflation by a massive scalar field is typically $10^{-36}s \leq t_{infl} \leq 10^{-31}s$ and that the total number of e-foldings is typically $60 \leq N_t \leq 10^{13}$. Because this inflation model is not ruled out by observations and because the total amount of e-foldings that can be reached is much larger than the minimum amount of e-foldings necessary to solve the problems of the Big Bang model it is concluded that inflation by a massive scalar field is very plausible. The significance of the findings is that in order to describe nature in the very early stages of the universe one must take into account that it is plausible that a massive scalar field was present at this early epoch.

Contents

1 Introduction

The Big Bang model is the general accepted model to describe the evolution of our universe. However there are problems associated with the Big Bang model [1, Liddle][2, Perkins][3, Watson]. Among these problems are the horizon problem and the flatness problem. To solve these problems before the 'Big Bang' the universe had to go throughout a stage of accelerated expansion called inflation. In high energy physics scalar fields play an important role. Therefore a lot of inflation models are based on scalar fields [2, Perkins][3, Watson][4, Linde][5, Kinney][6, Riotto][7, Langlois]. In the last two decades different models have been suggested. In this thesis inflation in general and inflation by a massive scalar field in particular is investigated. Inflation by a scalar field not only is able to solve the problems associated with the Big Bang model but also generates density perturbations which can explain the large scale structure of the universe. It is even proposed that inflation by a scalar field can produce gravitational waves. Before discussing massive scalar field inflation first the standard Big Bang model, its problems and inflation in general is investigated in the first part of this thesis. Then in the second part of this thesis massive scalar field inflation is investigated. Massive scalar field inflation belongs to a more general inflation model called Chaotic Inflation invented by Linde around 1983 [4, Linde].

2 The standard Big Bang Model and Inflation

2.1 Introduction

In cosmology the universe is assumed isotropic and homogeneous on large scales. This is confirmed by observations. An important example of an observation is the fact that the Cosmic Microwave Background Radiation has the same temperature of about $2.73K$ in all directions. Temperature fluctuations are observed to be $\frac{\Delta T}{T} \approx 10^{-5}$. Furthermore it is assumed that the universe is homogeneous expanding. This follows from Hubble's observation that the speed v at which an object in the universe is flowing away is given by $v = H_0 d$, in which H_0 is the current Hubble parameter and d is the distance to the object [1, Liddle][2, Perkins][3, Watson].
In the case of a flat universe it is possible to define the following coordinate system,

$$d(x,t) = a(t)r(x) \tag{1}$$

where $d(x,t)$ is the physical distance, $a(t)$ is the scale parameter and $r(x)$ is the comoving distance [1, Liddle][2, Perkins][3, Watson]. The scale parameter which is only a function of time describes how large the universe is at a certain time. The meaning of a flat universe is explained in the next subsection. If the scale factor doubles in a certain amount of time by a factor of two this means that the radius of the universe has become twice as large as it was before. The comoving distance is not a function of time and only describes the spatial distance in a frame which moves with the expansion.

2.2 Equations describing the universe

In order to quantify the evolution of the universe it is necessary to consider some equations. The first equation is the well known Friedmann equation [1, Liddle]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \tag{2}$$

In this equation H is the Hubble parameter, a the scale parameter, ρ the density of the universe and k describes the curvature of space. A positive value of k means a spherical geometry, $k = 0$ means a flat geometry and a negative value of k means a hyperbolic geometry. In general a spherical geometry means the universe is closed and will collapse in the future. In general a hyperbolic geometry means the universe is open and will expand forever. In general a flat geometry means the universe is flat and will expand forever but now the expansion rate goes to zero when time approaches infinity. The curvature term in this equation is often rewritten in the following way

$$\frac{k}{a^2} \Rightarrow \frac{\pm 1}{a'^2} \qquad a' = \frac{a}{\sqrt{|k|}} \tag{3}$$

To make use of the Friedmann equation it is necessary to have an equation for the evolution of the density. This is the second equation called the fluid equation [1, Liddle]

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{4}$$

To use the equations it is only necessary to specify the equation of state which gives the pressure in terms of the density. The equation of state is simply written as

$$p = w\rho \tag{5}$$

Let's now try to obtain an equation for the acceleration of the scale factor. It will appear that such an equation will be useful when discussing inflation. In getting an expression for $\ddot{a}$ it might be useful to differentiate the Friendmann equation with respect to time because $\dot{a}$ appears in that equation

$$\frac{d}{dt}\frac{\dot{a}^2}{a^2} = \frac{1}{a^2}2\dot{a}\ddot{a} + \dot{a}^2\frac{-2}{a^3}\dot{a} = \frac{8\pi G}{3}\dot{\rho} + \frac{2\dot{a}}{a}\frac{k}{a^2} \tag{6}$$

Using equation (4) this is rewritten as

$$2\frac{\dot{a}}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right] = (-4\pi G)2\frac{\dot{a}}{a}(\rho + p) + \frac{2\dot{a}}{a}\frac{k}{a^2} \tag{7}$$

Cancelling the common factor and putting equation (2) into this equation it follows that

$$\ddot{a} = -4\pi G a\left(\frac{\rho}{3} + p\right) \tag{8}$$

Substituting the equation of state (5) into this equation it follows that

$$\ddot{a} = -4\pi G a\left(w + \frac{1}{3}\right)\rho \tag{9}$$

This equation is called the acceleration equation, in the literature this equation is often written in a somewhat different way. It is important to note the following
If $w > -\frac{1}{3}$ then $\ddot{a} < 0$ $\Rightarrow$ NO INFLATION
If $w < -\frac{1}{3}$ then $\ddot{a} > 0$ $\Rightarrow$ INFLATION

2.3 Cosmological models with flat geometry

Because observations imply that our universe is flat [1, Liddle][2, Perkins][3, Watson] and the equations are easily solved if $k = 0$ let's consider such models of the universe. Putting the equation of state (5) into the fluid equation it follows that

$$\dot{\rho} + 3(w + 1)\frac{\dot{a}}{a}\rho = 0 \tag{10}$$

This can be rewritten as

$$\frac{d}{dt}\left(\rho a^{3(w+1)}\right) = 0 \tag{11}$$

An expression for the density follows immediately

$$\rho = \frac{C}{a^{3(w+1)}} \tag{12}$$

in which the constant C is time independent. The Friedmann equation (2) becomes

$$\dot{a}^2 = \frac{8\pi GC}{3a^{3w+1}} \tag{13}$$

Trying $a(t) = Dt^m$ in which D is time independent this can be written as

$$m^2 D^2 t^{2m-2} = \frac{8\pi GC}{3D^{3w+1}} t^{-m(3w+1)} \tag{14}$$

If $w \neq -1$ then $a(t) = Dt^m$ is indeed a solution with $m = \frac{2}{3(w+1)}$ and
$D = (6\pi GC(w+1)^2)^{\frac{1}{3(w+1)}}$. The full solutions for $a(t)$ and $\rho(t)$ are then

$$a(t) = (6\pi GC(w+1)^2)^{\frac{1}{3(w+1)}}\, t^{\frac{2}{3(w+1)}} \tag{15}$$

$$\rho(t) = \frac{1}{6\pi G(w+1)^2\, t^2} \tag{16}$$

Now let's look at the evolution of the scale factor and the evolution of the density for a matter dominated universe and a radiation dominated universe.

Matter dominated universe

Because matter exerts no pressure [2, Perkins] it follows that $w = 0$. The evolution of the scale factor and the evolution of the density are the following

$$a \propto t^{\frac{2}{3}} \tag{17}$$

$$\rho_m \propto \frac{1}{a^3} \propto \frac{1}{t^2} \tag{18}$$

Furthermore $\ddot{a} < 0$ as follows from the acceleration equation (9).

Radiation dominated universe

Radiation exerts pressure given by $p = \frac{\rho}{3}$ [2, Perkins]. It follows that $w = \frac{1}{3}$. The evolution of the scale factor and the evolution of the density are the following

$$a \propto t^{\frac{1}{2}} \tag{19}$$

$$\rho_r \propto \frac{1}{a^4} \propto \frac{1}{t^2} \tag{20}$$

Furthermore also for a radiation dominated universe $\ddot{a} < 0$.

Our universe

In the standard Big Bang model the universe is assumed radiation dominated between $t = t_P$ and $t = t_{dec}$ where $t_P \approx 5 \cdot 10^{-44} s$ is the Planck time and t_{dec} is the time of the decoupling of matter and radiation. Before decoupling radiation is assumed to be the dominant form of energy and after decoupling matter is assumed to be the dominant form of energy. Matter is the dominant form of energy between $t = t_{dec}$ and $t = t_{pr}$ where t_{pr} is the present age of the universe [1, Liddle][2, Perkins][3, Watson]. Before $t = t_P$ a classical description of the universe is not possible [4, Linde].
The age of the universe is determined to be [8, Komatsu et al.]

$$t_{pr} \approx 13.7 \cdot 10^9 yr \approx 4 \cdot 10^{17} s \tag{21}$$

Given the age of the universe it is possible to determine the time of decoupling. According to equation (17) $a \propto t^{\frac{2}{3}}$ and the fact that $aT = constant$ in a matter dominated universe, because the number of relativistic degrees of freedom is conserved in a matter dominated universe [6, Riotto], it follows that

$$t_{dec} = t_{pr} \left(\frac{a(t_{dec})}{a(t_{pr})} \right)^{\frac{3}{2}} = t_{pr} \left(\frac{T_{pr}}{T_{dec}} \right)^{\frac{3}{2}} \tag{22}$$

The temperature at the time of decoupling has been calculated to be about $0.3eV$ [5, Kinney]. The temperature at the present time is the temperature of the Cosmic Microwave Background Radiation and is about $2 \cdot 10^{-4} eV$. The time of decoupling is then given by

$$t_{dec} \approx 10^{13} s \tag{23}$$

2.4 Problems of the Big Bang model

The Big Bang model has a number of problems which must be solved. As remarked before among the important problems are the horizon problem and the flatness problem. [1, Liddle][2, Perkins][3, Watson]

2.4.1 The flatness problem

First rewrite the Friedmann equation (2) in the form

$$\rho = \frac{3H^2}{8\pi G} + \frac{3k}{8\pi G a^2} \tag{24}$$

Now consider the density needed to make the universe flat which is called the critical density. By setting $k = 0$ in the Friedmann equation the critical density is obtained

$$\rho_c = \frac{3H^2}{8\pi G} \tag{25}$$

Consider the difference between the actual density and the critical density divided by the critical density

$$\frac{\Delta\rho}{\rho_c} = \frac{\rho - \rho_c}{\rho_c} = \frac{\frac{3k}{8\pi G a^2}}{\frac{3H^2}{8\pi G}} = \frac{k}{H^2 a^2} = \frac{k}{\dot{a}^2} \tag{26}$$

Using equation (17) it follows that for a matter dominated universe

$$\frac{\Delta\rho}{\rho_c} \propto t^{\frac{2}{3}} \tag{27}$$

and using equation (19) it follows that for a radiation dominated universe

$$\frac{\Delta\rho}{\rho_c} \propto t \tag{28}$$

Using these two relations it follows that

$$\frac{\Delta\rho}{\rho_c}(t_P) = \left(\frac{t_P}{t_{dec}}\right)\left(\frac{t_{dec}}{t_{pr}}\right)^{\frac{2}{3}}\frac{\Delta\rho}{\rho_c}(t_{pr}) \tag{29}$$

As remarked before observations imply that our universe is almost flat at the present time

$$\frac{\Delta\rho}{\rho_c}(t_{pr}) = O(1) \tag{30}$$

From this it follows that

$$\frac{\Delta\rho}{\rho_c}(t_P) \approx \left(\frac{5\cdot 10^{-44}}{10^{13}}\right)\left(\frac{10^{13}}{4\cdot 10^{17}}\right)^{\frac{2}{3}} O(1) = O(10^{-60}) \tag{31}$$

This means that at the Big Bang the universe had to be extremely close to flat. The standard Big Bang scenario provides no explanation for the fact why the universe was very flat from the beginning. This problem is called the flatness problem. However one must note that this calculation is based on the solutions for the scale factor in the case that $k = 0$. If this is not the case then the result can be different.

2.4.2 The horizon problem

Consider a light ray which leaves a certain point at $t = t_1$ and reaches another point at $t = t_2$. Between t and $t + dt$ where $t_1 < t < t_2$ the light ray travels a distance dt. At time $t = t_3$ where $t_3 \geq t_2$ this distance has increased due to the expansion of the universe. The distance the light has traveled between t and $t + dt$ at time t_3 is given by

$$d_{t,t+dt}(t_3) = dt\,\frac{a(t_3)}{a(t)} \tag{32}$$

The distance $d_{t_1,t_2}(t_3)$ the light has traveled between $t = t_1$ and $t = t_2$ at time t_3 is then given by

$$d_{t_1,t_2}(t_3) = \int_{t_1}^{t_2}\frac{a(t_3)}{a(t)}\,dt = a(t_3)\int_{t_1}^{t_2}\frac{dt}{a(t)} \tag{33}$$

Consider the distance light could have traveled between the Big Bang and the time of decoupling at the present time. By using equation (33), equation (19) and equation (17)

$$d_{t_P,t_{dec}}(t_{pr}) = a(t_{pr})\int_0^{t_{dec}}\frac{dt}{a(t_{dec})\left[\frac{t}{t_{dec}}\right]^{\frac{1}{2}}} = \frac{a(t_{pr})}{a(t_{dec})}2t_{dec} = 2[t_{dec}]^{\frac{1}{3}}[t_{pr}]^{\frac{2}{3}} \tag{34}$$

For simplicity the lower limit of the integral is set equal to zero which increases the integral a little bit, but this is negligible which can be seen when comparing the time of

decoupling (23) with the Plank time.
The calculated distance is half the distance between two points in the universe which could be in thermal equilibrium in the standard Big Bang scenario.
Also consider the distance light could have traveled between the time of decoupling and the present time at the present time

$$d_{t_{dec},t_{pr}}(t_{pr}) = a(t_{pr}) \int_{t_{dec}}^{t_{pr}} \frac{dt}{a(t_{pr}) \left[\frac{t}{t_{pr}}\right]^{\frac{2}{3}}} = 3[t_{pr}]^{\frac{2}{3}} \left([t_{pr}]^{\frac{1}{3}} - [t_{dec}]^{\frac{1}{3}}\right) \tag{35}$$

In the small angle approximation an expression for the angle $\theta(t_{pr})$ that spans that part of the sky which could have reached thermal equilibrium is given by

$$\frac{1}{\theta}(t_{pr}) = \frac{d_{t_{dec},t_{pr}}(t_{pr})}{2 \cdot d_{t_P,t_{dec}}(t_{pr})} = \frac{3t_{pr} - 3[t_{dec}]^{\frac{1}{3}}[t_{pr}]^{\frac{2}{3}}}{2 \cdot 2[t_{dec}]^{\frac{1}{3}}[t_{pr}]^{\frac{2}{3}}} = \frac{3}{4}\left[\left(\frac{t_{pr}}{t_{dec}}\right)^{\frac{1}{3}} - 1\right] \tag{36}$$

$$\theta(t_{pr}) = \frac{4}{3\left[\left(\frac{t_{pr}}{t_{dec}}\right)^{\frac{1}{3}} - 1\right]} \tag{37}$$

Putting the age of the universe (21) and the time of decoupling (23) in equation (37) it follows that

$$\theta(t_{pr}) \approx 0.04\, rad \approx 2^{\circ} \tag{38}$$

Thus it was allowed to use the small angle approximation.
If the standard Big Bang scenario is correct it is expected that the Cosmic Microwave Background Radiation is anisotropic when looking at angular scales larger than $\approx 2^{\circ}$. But as remarked before when looking at the measured Cosmic Microwave Background Radiation one concludes that the Cosmic Microwave Background Radiation is isotropic. This problem is called the horizon problem.

2.5 Inflation by a cosmological constant

Let's consider a more exotic forms of energy in the universe by adding a constant vacuum energy density ρ_Λ to the total energy density [1, Liddle][2, Perkins][3, Watson]. One can think of this energy representing the so called zero-point energy of the universe. It has been tried to relate this energy to the zero-point energy predicted by quantum field theory [5, Kinney]. However quantum field theory predicts this energy 120 orders of magnitude larger than the critical energy density. This is an important problem in quantum field theory. Not worrying anymore were this vacuum energy comes from let's assume that this vacuum energy density is present. Often this vacuum energy density is represented by a cosmological constant Λ which is related to the vacuum energy density in the following way

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \tag{39}$$

Using this definition the Friendmann equation (2) is now given by

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{40}$$

Because the cosmological constant is constant as its name implies it follows from the fluid equation (4) that $p = -\rho$ or in term of the equation of state (5) that $w = -1$. According to the acceleration equation (9) it follows that $\ddot{a} > 0$. This means that a vacuum energy density causes inflationary expansion if it is the dominant form of energy. According to equations (18) and (20) the matter and radiation density terms are soon redshifted away as also the curvature term does. After that has take place the Friedmann equation can be simplified to

$$H^2 = \frac{\Lambda}{3} \tag{41}$$

Because $\dot{a} = \sqrt{\frac{\Lambda}{3}}a$ the time evolution of the scale factor is given by

$$a(t) = a(t_0) \exp\left[\sqrt{\frac{\Lambda}{3}} \cdot (t - t_0)\right] = a(t_0)e^{H\cdot(t-t_0)} \tag{42}$$

This means that the universe expands exponential if a cosmological constant is the dominant form of energy.
One can define the number of e-foldings $N(t)$ during inflationary expansion [3, Watson]

$$a(t) \equiv a(t_0)e^{N(t)} \tag{43}$$

In the literature one finds different start points as also is the case in [3, Watson]. One e-folding means that the scale factor has grown by a factor of e. By definition the number of e-foldings is only defined during inflationary expansion. The number of e-foldings $N(t)$ as a function of the scale factor is then written as

$$N(t) = \ln\frac{a(t)}{a(t_0)} \tag{44}$$

Putting equation (42) into this equation gives

$$N(t) = H \cdot (t - t_0) \tag{45}$$

Assuming that all the vacuum energy decays away at $t = t_{end}$ the total number of e-foldings reached during inflationary expansion is given by

$$N_t = H \cdot (t_{end} - t_0) \tag{46}$$

It is common to choose $t_0 = t_P$ where t_P is the Plank time because before $t = t_P$ a classical description of the universe is not possible as remarked before.

2.6 Solving the problems

2.6.1 The horizon problem

Because during inflationary expansion the scale factor increases exponentially, it can be guessed that inflationary expansion may be able to solve the horizon problem when looking at equation (33). Because at time $t_2 = t_{end}$ the scale factor is much larger then at time $t_1 = t_0$ the contribution to the distance is very large at times just after t_0. Now a quantitative analysis to solve the horizon problem is given.

In order to solve the horizon problem light must have been traveled between two points on opposite parts of the sky at the present time. This means that

$$d_{t_P,t_{dec}}(t_{pr}) \geq 2d_{t_{dec},t_{pr}}(t_{pr}) = 6[t_{pr}]^{\frac{2}{3}}\left([t_{pr}]^{\frac{1}{3}} - [t_{dec}]^{\frac{1}{3}}\right) \tag{47}$$

Instead of assuming that the universe is radiation dominated between $t = t_P$ and $t = t_{dec}$ it is now assumed that the universe is dominated by vacuum energy between $t = t_P$ and $t = t_{end}$ and that the universe is radiation dominated between $t = t_{end}$ and $t = t_{dec}$. At $t = t_{end}$ the Big Bang occurs and the universe reheats to a temperature $T = T_R$. In this case $d_{t_P,t_{dec}}(t_{pr})$ is given by

$$t_P, t_{dec}(t_{pr}) = a(t_{pr})\left[\int_{t_P}^{t_{end}} \frac{dt}{a(t_{end})\left[\frac{e^{Ht}}{e^{Ht_{end}}}\right]} + \int_{t_{end}}^{t_{dec}} \frac{dt}{a(t_{dec})\left[\frac{t}{t_{dec}}\right]^{\frac{1}{2}}}\right]$$

$$= a(t_{pr})\left[\frac{e^{Ht_{end}}}{a(t_{end})}\frac{-1}{H}\left(e^{-Ht_{end}} - e^{-Ht_P}\right) + \frac{2[t_{dec}]^{\frac{1}{2}}}{a(t_{dec})}\left([t_{dec}]^{\frac{1}{2}} - [t_{end}]^{\frac{1}{2}}\right)\right] \tag{48}$$

Before proceeding with this calculation let's first determine a realistic value for t_{end}. According to equation (19) $a(t) \propto t^{\frac{1}{2}}$ and assuming that aT = constant when $t > t_{end}$ by assuming the number of relativistic degrees of freedom is conserved (which in practice is not the case in a radiation dominated universe, however even in the case of phase transitions it is a good approximation [6, Riotto]) it follows that

$$t_{end} = t_{rec}\left(\frac{T_{dec}}{T_R}\right)^2 \tag{49}$$

in which $T_R \equiv T_{end}$ is the reheating temperature after inflation. The reheating temperature is taken here as $T_R \approx 10^{15} GeV$. In [4, Linde] this temperature is taken as an upper limit. Using $T_{dec} \approx 0.3\, eV$ a realistic approximation for t_{end} is given by

$$t_{end} \approx 10^{-36} s \tag{50}$$

A smaller reheating temperature T_R will give a larger value of t_{end}. Neglecting $[t_{end}]^{\frac{1}{2}}$ with respect to $[t_{dec}]^{\frac{1}{2}}$ in the second term and rearranging terms equation (48) is rewritten as

$$d_{t_P,t_{dec}}(t_{pr}) = \frac{a(t_{pr})}{a(t_{dec})}\left[\frac{a(t_{dec})}{a(t_{end})}\frac{e^{H[t_{end}-t_P]} - 1}{H} + 2t_{dec}\right] \tag{51}$$

Using equation (47) it follows that

$$\frac{e^{H[t_{end}-t_P]} - 1}{H} \geq \frac{a(t_{end})}{a(t_{dec})}\left[6\frac{a(t_{dec})}{a(t_{pr})}\left(t_{pr} - [t_{pr}]^{\frac{2}{3}}[t_{dec}]^{\frac{1}{3}}\right) - 2t_{dec}\right]$$

$$= \left(\frac{t_{end}}{t_{dec}}\right)^{\frac{1}{2}}\left[6\left(\frac{t_{dec}}{t_{pr}}\right)^{\frac{2}{3}}\left(t_{pr} - [t_{pr}]^{\frac{2}{3}}[t_{dec}]^{\frac{1}{3}}\right) - 2t_{dec}\right]$$

$$= \sqrt{t_{end}}\left[6[t_{dec}]^{\frac{1}{6}}[t_{pr}]^{\frac{1}{3}} - 8[t_{dec}]^{\frac{1}{2}}\right] \approx 6 \cdot 10^8 \sqrt{t_{end}} \tag{52}$$

if $t_{end} \ll t_{dec}$ or in terms of reheating temperature $T_R \gg T_{dec}$ which in practice is satisfied.

Using equation (46), equation (49) and equation (50), equation (52) is rewritten as

$$\frac{e^{N_t} - 1}{N_t} \geq \frac{6 \cdot 10^8}{\sqrt{t_{end}}} = 6 \cdot 10^{26}\frac{T_R}{10^{15} GeV} \tag{53}$$

Because the right hand side of this equation is very large the number of e-foldings N_t to solve the horizon problem is given by

$$N_t - \ln N_t \geq \ln\left[6 \cdot 10^{26} \frac{T_R}{10^{15} GeV}\right] \tag{54}$$

Taking $T_R = 10^{15} GeV$ corresponding to $t_{end} \approx 10^{-36} s$

$$N_t(T_R = 10^{15} GeV) \geq 66 \tag{55}$$

This expression is not very sensitive to the reheating temperature T_R. If one for example takes $T_R = 10^{12} GeV$ corresponding to $t_{end} \approx 10^{-30} s$ one finds

$$N_t(T_R = 10^{12} GeV) \geq 59 \tag{56}$$

One must remember that these calculations are based on the assumption that the number of relativistic degrees of freedom is conserved throughout the radiation dominated era.
Let's now consider the relation between the vacuum energy density and the reheating temperature. According to equation (39) and equation (41)

$$\rho_v = \frac{3H^2}{8\pi G} \tag{57}$$

and according to equation (46) and equation (49)

$$H^2 \approx \frac{N_t^2}{t_{end}^2} = \frac{N_t^2}{t_{dec}^2}\left(\frac{T_R}{T_{dec}}\right)^4 \tag{58}$$

When also using equation (54) the following constraint on the vacuum energy density is found

$$\rho_v \geq \frac{3}{8\pi G t_{dec}^2}[N_t(T_R)]^2 \left(\frac{T_R}{T_{dec}}\right)^4 \tag{59}$$

The horizon problem is solved if the vacuum energy density ρ_v is large enough. For example if $T_R = 10^{15} GeV$ the vacuum energy density must satisfy

$$\rho_v(T_R = 10^{15} GeV) \geq \frac{3}{8\pi \cdot 6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}(10^{13} s)^2} 66^2 \left(\frac{10^{15} GeV}{0.3 eV}\right)^4 \approx 10^{85} \frac{kg}{m^3} \tag{60}$$

This is a very large density, about 70 orders of magnitude larger than nuclear energy densities. However the minimum value of the vacuum energy density ρ_v depends strongly on the reheating temperature T_R, approximately $(\rho_v)_{min} \propto T_R^4$.

2.6.2 The flatness problem

Using equation (42) and equation (45) in a vacuum energy dominated universe

$$\dot{a(t)} = a(t_P) H e^{H \cdot (t - t_P)} \propto e^{N(t)} \tag{61}$$

Using equation (26) it follows that for a vacuum energy dominated universe

$$\frac{\Delta \rho}{\rho_c} \propto e^{-2N(t)} \tag{62}$$

If a stage of inflationary expansion precedes the Big Bang equation (29) reads

$$\frac{\Delta\rho}{\rho_c}(t_P) = \frac{e^{+2N(t_{end})}}{e^{+2N(t_P)}}\left(\frac{t_{end}}{t_{dec}}\right)\left(\frac{t_{dec}}{t_{pr}}\right)^{\frac{2}{3}}\frac{\Delta\rho}{\rho_c}(t_{pr})$$
$$= e^{+2N_t}\left(\frac{t_{end}}{t_{dec}}\right)\left(\frac{t_{dec}}{t_{pr}}\right)^{\frac{2}{3}}\frac{\Delta\rho}{\rho_c}(t_{pr}) \tag{63}$$

This can be rewritten as

$$N_t = \frac{1}{2}\ln\left[\frac{t_{dec}}{t_{end}}\left(\frac{t_{pr}}{t_{dec}}\right)^{\frac{2}{3}}\left(\frac{\frac{\Delta\rho}{\rho_c}(t_P)}{\frac{\Delta\rho}{\rho_c}(t_{pr})}\right)\right] \tag{64}$$

Using equation (49) and setting $\frac{\Delta\rho}{\rho_c}(t_P) \geq O(1)$

$$N_t \geq \frac{1}{2}\ln\left[\left(\frac{t_{pr}}{t_{dec}}\right)^{\frac{2}{3}}\left(\frac{T_R}{T_{dec}}\right)^2\right] = \frac{1}{3}\ln\left[\frac{t_{pr}}{t_{dec}}\right] + \ln\left[\frac{T_R}{T_{dec}}\right] \tag{65}$$

If for example $T_R = 10^{15}GeV$ then the flatness problem is solved if

$$N_t \geq \frac{1}{3}\ln\left[\frac{4\cdot 10^{17}}{10^{13}}\right] + \ln\left[\frac{10^{15}GeV}{0.3eV}\right] = 60 \tag{66}$$

2.6.3 Problem of inflation by a cosmological constant

In the last subsections it appeared that a constant vacuum energy density i.e. a cosmological constant can solve important problems of the standard Big Bang scenario. However in the section on inflation it was assumed that all the vacuum energy decays away at $t = t_{end}$. Because there is no reason to expect why this happens this is a problem. It would be nice to have a model in which the decay of the vacuum energy can be explained. It will appear in the following part of this thesis that inflation by a (massive) scalar field gives the solution of this problem.

3 Inflation by a massive scalar field

3.1 Introduction

As mentioned before a lot of inflation models are based on scalar fields because scalar fields play an important role in high energy physics. Here emphasis is put on inflation by a massive scalar field which belong to a more general model of inflation called Chaotic Inflation due to Andrei Linde [4, Linde]. In the Chaotic Inflation model it is assumed that the universe is initially filled with a scalar field $\phi(\vec{x}, t)$ with mass m and potential energy density $V(\phi)$. The potential energy density has a minimum at $\phi = 0$. This model is called Chaotic Inflation because the scalar field is assumed not to be homogeneous in space. After the Planck time $t = t_P$ after the birth of the universe a classical description of the universe becomes possible. Before this time physical laws are not known. To describe nature before this time general relativity has to be combined with quantum mechanics. Until now it is not known how to do this. At the Planck time in some parts of the universe the scalar field is large and in other parts of the universe the scalar field is low. It will appear that in those parts of the universe where the scalar field is large at the Planck time inflationary expansion can take place. In other places where the scalar field is low at the Planck time inflationary expansion will not occur. In the following a sufficiently small piece of the universe is considered in which the scalar field is homogeneous in space. One can always divide an inhomogeneous part of the early universe into approximately homogeneous parts.

3.2 The equation of motion

In classical mechanics we can define for each system of particles a Lagrangian L. In the case of scalar fields it is not possible to use the Lagrangian to do calculations. Instead of using the Lagrangian the Langrangian density $\mathcal{L}$ can be used which is related to the Lagrangian L by [3, Watson]

$$L = \int \mathcal{L}\, d^3x \tag{67}$$

In the case of a scalar field the Lagrangian density $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{68}$$

It is useful to consider the stress-energy tensor for scalar fields. This tensor is given by [3, Watson]

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \tag{69}$$

When considering a sufficient small piece of the universe where the scalar field is homogeneous we have $\partial^1 = \partial^2 = \partial^3 = 0$. The Lagrangian density is then given by

$$\mathcal{L} = T(\phi) - V(\phi) = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{70}$$

The universe is considered to be a perfect fluid. The stress-energy tensor for a perfect fluid is given by [3, Watson]

$$T^{\mu\nu} = diag(\rho, p, p, p) \tag{71}$$

In a sufficient small part of the universe the stress-energy tensor is given by

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right] \tag{72}$$

The density of the scalar field is given by

$$\rho = T^{00} = \partial^0\phi\partial^0\phi - g^{00}\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right] = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{73}$$

The pressure of the scalar field is given by

$$p = T^{11} = T^{22} = T^{33} = \partial^1\phi\partial^1\phi - g^{11}\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right] = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{74}$$

Because the universe is a perfect fluid the fluid equations holds (4) which is given here again

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{75}$$

Putting the results for the pressure and the density into the fluid equation gives

$$\dot{\phi}\ddot{\phi} + \dot{\phi}\frac{d}{dt}V(\phi) + 3H\dot{\phi}^2 = 0 \tag{76}$$

When dividing by $\dot{\phi}$ the equation of motion follows

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{77}$$

It is important to note that the equation of motion remains the same when adding a constant term to the potential energy density.

3.2.1 Massive scalar field

If $V(\phi) \propto \phi^2$ then the equation of motion describes a damped harmonic oscillator with friction term $3H\dot{\phi}$. A massive scalar field satisfies this condition because a massive scalar field has the following potential energy density [4, Linde]

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{78}$$

The equation of motion for a massive scalar field is given by

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \tag{79}$$

In the following sections the scalar field is considered to be a massive scalar field.

3.3 Equations describing scalar field inflation

To describe the dynamics of scalar field inflation the Friedmann equation and the equation of motion must be solved simultaneously. Putting equation (73) into equation (2) and defining $G = {M_p}^{-2} = 1$ the following two equations must be solved simultaneously

$$H^2 = \frac{8\pi}{3}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2\right] - \frac{k}{a^2} \tag{80}$$

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \tag{81}$$

3.3.1 Constraints on the scalar field for inflationary expansion to occur

According to equation (9) inflationary expansion occurs as long as $p < -\frac{1}{3}\rho$. Equation (5) reads using equations (74) and (73)

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2}{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2} = \frac{T - V}{T + V} < -\frac{1}{3} \tag{82}$$

This condition is satisfied if and only if

$$V > 2T \tag{83}$$

or in terms of the scalar field

$$\phi > \frac{\sqrt{2}}{m}\dot{\phi} \tag{84}$$

This means that inflation does only occur if the scalar field is large enough or in terms of kinetic and potential energy densities that the potential energy density is more than two times larger than the kinetic energy density in the Friedmann equation.

3.4 The slow roll approximation

These equations cannot be solved algebraically for a massive scalar field. It is necessary to make some approximations. From equation (83) it appeared that inflation only occurs if the potential energy density is more than two times larger than the kinetic energy density. Let's now assume that the potential energy density is much larger than the kinetic energy density. This can be done because in the Chaotic Inflation scenario there had to be some domains in the early universe where the kinetic energy density was much smaller than the potential energy density.

$$\frac{1}{2}\dot{\phi}^2 \ll \frac{1}{2}m^2\phi^2 \tag{85}$$

In this case equation (82) reads using equations (74) and (73)

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2}{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2} \approx -1 \tag{86}$$

According to equation (9) this means that inflationary expansion occurs. Furthermore in a section on inflationary expansion it was shown that if $w = -1$ the scale factor expands exponentially. The conclusion is that if there is a domain in which the kinetic energy density is small compared to the potential energy density almost exponential expansion will occur. Because in this case it is expected that the curvature density is redshifted away in a short time a further approximation is to neglect the curvature term. A final approximation is to neglect the 'acceleration' of the field. Using these approximations the full equations are simplified to

$$H^2 = \frac{4\pi}{3}m^2\phi^2 \tag{87}$$

$$3H\dot{\phi} + m^2\phi = 0 \tag{88}$$

These two equations are called the slow roll equations and approximating the full equations by the slow roll equations is called the slow roll approximation [4, Linde]. The slow roll equations will be solved exactly and the full equations will be solved numerically using Mathematica. The numerical results will be compared with the analytical results.

3.5 Solving the slow roll equations

The slow roll equations can be solved exactly. First an expression for the time evolution for the scalar field is derived. After having also derived the value of the scalar field at which inflation ends the duration of inflation follows. Thereafter an expression for the time evolution of the scale factor is derived. From this expression an expression for the number of e-folding follows.

3.5.1 Time evolution of the scalar field

Putting equation (88) into equation (87) gives

$$H^2\left(1 - \frac{12\pi\dot{\phi}^2}{m^2}\right) = 0 \tag{89}$$

The 'velocity' of the scalar field follows immediately

$$\dot{\phi} = -\frac{m}{\sqrt{12\pi}} \tag{90}$$

The minus sign appears when looking at equation (88). It follows that $\dot{\phi} < 0$. The time evolution of the field is given by

$$\phi(t) = \phi(t_0) - \frac{m}{\sqrt{12\pi}}(t - t_0) \tag{91}$$

In the slow roll approximation the scalar field decreases linear as a function of time. This linear relationship is expected to become exact if both the field is large and the curvature density can be neglected. Thus when the curvature density has already been redshifted away.

3.5.2 The end of inflation

In the last section it appeared that inflationary expansion occurs only if the scalar field is large enough. Referring to equation (91) at a certain moment the scalar field reaches a value at which inflationary expansion terminates. In the following the final value ϕ_f for the scalar field is calculated at which inflation terminates in the slow roll approximation.
In the case of inflation $\ddot{a} > 0$. In relating the definition of inflation to the Hubble parameter let's differentiate the Hubble parameter with respect to time

$$\frac{dH}{dt} = \frac{d}{dt}\frac{\dot{a}}{a} = \frac{\ddot{a}}{a} - \frac{\dot{a}}{a}^2 \tag{92}$$

Making use of the definition of inflation it follows that

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \tag{93}$$

The Hubble parameter is given by equation (87). Because also the time derivate of the Hubble parameter appears in the above equation let's differentiate equation (87) with respect to time

$$\dot{H} = \frac{1}{2H}\frac{8\pi}{3}m^2\phi\dot{\phi} = \frac{4\pi m^2\phi^2}{3}\left(\frac{\dot{\phi}}{\phi}\right)\frac{1}{H} \tag{94}$$

Using equations (87) and (88) this can be rewritten as

$$\dot{H} = H^2\left(\frac{-m^2}{3H}\right)\frac{1}{H} = \frac{-m^2}{3} \tag{95}$$

Equation (93) describing the condition for inflation can now be written as

$$\frac{4\pi}{3}m^2\phi^2 > \frac{m^2}{3} \tag{96}$$

Inflation in the slow roll approximation proceeds until the scalar field has reached the final value

$$\phi_f = \frac{1}{\sqrt{4\pi}} \tag{97}$$

Now let's look at the kinetic and potential energy densities of the scalar field when it has reached the final value. Looking at equation (90) it appears that the kinetic energy density is constant and given by

$$T = \frac{1}{2}\dot{\phi}^2 = \frac{m^2}{24\pi} \tag{98}$$

Using equation (97) it follows that the potential energy density at $\phi = \phi_f$ is given by

$$V(\phi_f) = \frac{1}{2}m^2\phi_f^2 = \frac{m^2}{8\pi} \tag{99}$$

It follows that $V(\phi_f) = 3T$. Comparing this result with result (83) which states that inflationary expansion continues as long as $V(\phi) \geq 2T$ one concludes that the end of inflationary expansion is well approximated if the slow roll equations are used. The equation of state (82) reads at $\phi = \phi_f$

$$w(\phi_f) = \frac{T - V(\phi_f)}{T + V(\phi_f)} = -\frac{1}{2} \tag{100}$$

Thus in the slow roll approximation inflation terminates in time.

3.5.3 The duration of inflation

It is straightforward to calculate the duration of inflation in the slow roll approximation. Putting the final value of the scalar field (97) into the equation for the time evolution of the scalar field (91) it follows that

$$\frac{1}{\sqrt{4\pi}} = \phi_f = \phi(t_f) = \phi(t_0) - \frac{m}{\sqrt{12\pi}}(t_f - t_0) \tag{101}$$

The duration of inflation in the slow roll approximation is then given by

$$t_{infl} = t_f - t_0 = \frac{\sqrt{12\pi}}{m}\left(\phi(t_0) - \frac{1}{\sqrt{4\pi}}\right) \tag{102}$$

3.5.4 Time evolution of the scale factor

To solve for $a(t)$ equation (88) is rewritten as

$$\frac{\dot{a}}{a} = \frac{-m^2\phi}{3\dot{\phi}} \tag{103}$$

Putting equation (91) into this equation gives

$$\frac{\dot{a}(t)}{a(t)} = \frac{-m^2\left[\phi(t_0) - \frac{m}{\sqrt{12\pi}}(t - t_0)\right]}{-3\frac{m}{\sqrt{12\pi}}} = \sqrt{\frac{4\pi}{3}}m\phi(t_0) - \frac{m^2}{3}(t - t_0) = B + C(t - t_0) \tag{104}$$

$$B = \sqrt{\frac{4\pi}{3}}m\phi(t_0) \qquad C = \frac{-m^2}{3} \tag{105}$$

Using $t' = t - t_0$ the equation to be solved becomes

$$\frac{da(t')}{dt'} = a(t')[B + Ct'] \tag{106}$$

Trying $a(t') = D \exp g(t')$ gives

$$\frac{da(t')}{dt'} = a(t')\frac{dg(t')}{dt'} = a(t')\left[B + Ct'\right] \tag{107}$$

The function $g(t')$ is then given by

$$g(t') = A + Bt' + \frac{C}{2}(t')^2 \tag{108}$$

This gives the solution

$$a(t) = e^{A \ln D} \exp\left[\sqrt{\frac{4\pi}{3}} m\phi(t_0)(t - t_0) - \frac{m^2}{6}(t - t_0)^2\right] \tag{109}$$

Setting $t = t_0$ gives $e^{A \ln D} = a(t_0)$. The solution becomes

$$a(t) = a(t_0) \exp\left[\sqrt{\frac{4\pi}{3}} m\phi(t_0)(t - t_0) - \frac{m^2}{6}(t - t_0)^2\right] \tag{110}$$

Of one writes the scale factor as

$$a(t) = a(t_0) \exp\left[\sqrt{\frac{4\pi}{3}} m\phi(t_0)(t - t_0)\left[1 - \sqrt{\frac{1}{48\pi}} \frac{m}{\phi(t_0)}(t - t_0)\right]\right] \tag{111}$$

one can easy see that the scale factor in the initial stage of inflation is approximately given by

$$\lim_{t \to t_0} a(t) = a(t_0) \exp\left[\sqrt{\frac{4\pi}{3}} m\phi(t_0)(t - t_0)\right] \tag{112}$$

This confirms that inflation is approximately exponential in the initial stage of inflation. One can see from equation (111) that the larger the value of the initial scalar field $\phi_(t_0)$ the longer it takes for the approximation (112) to become invalid.

Time evolution of the Hubble parameter

Let's express the Hubble parameter is expressed as a function of time instead of a function of the scalar field. Looking at equation (104) one can see that the result has already been obtained

$$H(t) = \sqrt{\frac{4\pi}{3}} m\phi(t_0) - \frac{1}{3}m^2(t - t_0) \tag{113}$$

From this one can see that the Hubble parameter linearly decreases as a function of time

$$\dot{H} = -\frac{1}{3}m^2 \tag{114}$$

In the initial stage of inflation the Hubble parameter is given by

$$\lim_{t \to t_0} H(t) = \sqrt{\frac{4\pi}{3}} m\phi(t_0) \tag{115}$$

Comparing this result with equation (112) it appears that

$$\lim_{t \to t_0} a(t) = a(t_0) \exp\left[H(t_0) \cdot (t - t_0)\right] \tag{116}$$

This is equal to the time evolution of the scale parameter if a vacuum energy with energy density ρ_v as described in the section on inflationary expansion in the first part of this thesis is the dominant form of energy.

3.5.5 The number of e-foldings

Using equation (44) for the number of e-foldings $N(t)$

$$N(t) = \ln \frac{a(t)}{a(t_0)} \tag{117}$$

The number of e-foldings can be calculated by putting equation (110) into this equation. It gives

$$N(t) = \sqrt{\frac{4\pi}{3}} m(t - t_0) \left[\phi(t_0) - \frac{m}{\sqrt{48\pi}} (t - t_0) \right] \tag{118}$$

Using equation (102) the term between the brackets can be rewritten as

$$[...] = \phi(t_0) - \frac{m}{\sqrt{48\pi}} \frac{\sqrt{12\pi}}{m} \left[\phi(t_0) - \phi(t)\right] = \frac{1}{2} \left[\phi(t_0) + \phi(t)\right] \tag{119}$$

Using equation (102) again the equation for the number of e-folding can be simplified to

$$N(t) = \sqrt{\frac{4\pi}{3}} m \frac{\sqrt{12\pi}}{m} \left[\phi(t_0) - \phi(t)\right] \frac{1}{2} \left[\phi(t_0) + \phi(t)\right] = 2\pi \left[\phi^2(t_0) - \phi^2(t)\right] \tag{120}$$

As a consistency check for previous calculations the number of e-foldings N is also calculated in another way. Let's differentiate equation (44) with respect to the field ϕ .

$$\frac{dN}{d\phi} = \left(\frac{dN}{dt}\right) \frac{dt}{d\phi} = \frac{H}{\dot{\phi}} \tag{121}$$

Using equation (87) and equation (88) the number of e-foldings N can be written as

$$N(t) = \int_{\phi(t_0)}^{\phi(t)} \frac{H}{\dot{\phi}} \, d\phi = \int_{\phi(t)}^{\phi(t_0)} \frac{3H^2}{V'(\phi)} \, d\phi = 8\pi \int_{\phi(t)}^{\phi(t_0)} \frac{V(\phi)}{V'(\phi)} \, d\phi \tag{122}$$

In the case of a massive scalar field the number of e-foldings is given by

$$N(t) = 8\pi \int_{\phi(t)}^{\phi(t_0)} \frac{m^2\phi^2}{2m^2\phi} \, d\phi = 4\pi \int_{\phi(t)}^{\phi(t_0)} \phi \, d\phi = 2\pi \left[\phi^2(t_0) - \phi^2(t)\right] \tag{123}$$

This is exactly the same result as equation (120). It is important to note that the number of e-foldings in independent of the mass of the scalar field. Using equation (97) the total number of e-foldings $N_t = N(t_f)$ reached during slow roll inflation is given by

$$N_t = 2\pi\phi^2(t_0) - \frac{1}{2} \tag{124}$$

One can write $N_t = N'(t) + N(t)$ such that $N'(t) = N_t - N(t)$ is the number of e-foldings which will be reached after when the field has value $\phi(t)$. It is given by

$$N'(t) = \left(2\pi\phi^2(t_0) - \frac{1}{2}\right) - \left(2\pi\phi^2(t_0) - 2\pi\phi^2(t)\right) = 2\pi\phi^2(t) - \frac{1}{2} \tag{125}$$

it will appear useful in the next subsection if this is rewritten as

$$\phi^2(t) = \frac{2N'(t) + 1}{4\pi} \tag{126}$$

3.6 Quantum fluctuations, density perturbations and the large scale structure of the universe

As remarked before scalar field inflation is not only useful in solving problems like the horizon problem and the flatness problem but also gives an explanation for the observed large scale structure of the universe.

3.6.1 Quantum fluctuations of a scalar field

In the case of scalar field inflation it is believed that a homogeneous scalar field $\phi(\vec{x})$ is not constant in time but also has quantum fluctuations [3, Watson][4, Linde][5, Kinney][6, Riotto][7, Langlois]. A quantum fluctuation has amplitude $\delta\phi(\vec{x},t)$ and wavelength $\lambda(t)$. Using equation (110) in an universe expanding due to a massive scalar field the time evolution of the wavelength $\lambda(t)$ is given by

$$\lambda(t) = \lambda(t_0)\exp\left[\sqrt{\frac{4\pi}{3}}m\phi(t_0)(t-t_0) - \frac{m^2}{6}(t-t_0)^2\right] \tag{127}$$

The amplitude $\delta\phi(\vec{x},t)$ of a quantum fluctuation is 'frozen in' if the end parts of its wavelength are not longer in causal contact [6, Riotto], thus if the end points are receding away from each other faster than the speed of light. Thus if

$$\frac{d\lambda(t)}{dt} = \left[\sqrt{\frac{4\pi}{3}}m\phi(t_0) - \frac{1}{3}m^2(t-t_0)\right]\lambda(t) = H(t)\lambda(t) \geq 1 \tag{128}$$

where equation (113) has been substituted. Thus the amplitude is 'frozen in' if

$$\lambda(t) \geq \frac{1}{H(t)} \tag{129}$$

The wavelength a quantum fluctuation had when its amplitude became 'frozen in' is called λ_F. A relation between λ_F and the value the Hubble parameter H had when a quantum fluctuation became 'frozen in' is then given by

$$\lambda_F = \frac{1}{H} \tag{130}$$

By substituting equation (87) into this equation gives a relation between λ_F and the value the scalar field ϕ had when a quantum fluctuation became 'frozen in'

$$\lambda_F = \frac{1}{H} = \frac{\sqrt{\frac{3}{4\pi}}}{m\phi} \tag{131}$$

3.6.2 Density inhomogeneities

In order to relate λ_F to the amplitude of density inhomogeneities a relation is needed between ϕ and the amplitude of density inhomogeneities. This is a very difficult to do. Different people have derived the wanted equation. In [4, Linde] a general result is given as

$$\delta_H = \frac{\delta\rho}{\rho} = C\frac{H^2}{2\pi\dot{\phi}} \tag{132}$$

in which $C \approx 1$. Let's substitute equation (87) and equation (90) into this equation. It gives

$$\delta_H = C\frac{\frac{4\pi}{3}m^2\phi^2}{-2\pi\frac{m}{\sqrt{12\pi}}} = -4C\sqrt{\frac{\pi}{3}}m\phi^2$$
$$= -4C\sqrt{\frac{\pi}{3}}m\frac{\frac{3}{4\pi}}{m^2\lambda_F^2} = -C\sqrt{\frac{3}{\pi}}\frac{1}{m\lambda_F^2} \tag{133}$$

Because quantum fluctuations are produced continuously during scalar field inflation, at each time or at each value of the scalar field there are perturbations whose wavelength becomes equal to λ_F. Amplitudes of quantum perturbations are 'frozen in' continuously during scalar field inflation. This causes that a spectrum of density perturbations is created.
Using equation (126) equation (133) is rewritten as

$$\delta_H = \frac{C}{\sqrt{3\pi}}m\left[2N' + 1\right] \tag{134}$$

The minus sign is neglected because the sign is not important. By rewriting this equation an equation for the mass of the scalar field is obtained

$$m = \frac{\sqrt{3\pi}}{C}\frac{\delta_H(N')}{2N' + 1} \tag{135}$$

3.6.3 Assigning a numerical value to the mass

When defining N_{min} as the minimum number of e-foldings to solve the horizon problem this means that $\delta_H(N_{min})$ denotes the amplitude of density perturbations at the time of the decoupling of matter and radiation. Observations of the Cosmic Microwave Background Radiation are used to assign $\delta_H(N_{min})$ a value. The following value for $\delta_H(N_{min})$ is derived from WMAP data combined with distance measurements from Type Ia supernovae (SN) and Baryon Acoustic Oscillations (BAO) in the distribution of galaxies [8, Komatsu et al.]:

$$\delta_H(N_{min}) = \sqrt{2.445 \cdot 10^{-9}} \approx 5 \cdot 10^{-5} \tag{136}$$

In has been shown before that the minimum number of e-foldings to solve the horizon (and the flatness problem) is $N_{min} \approx 60$. By using this approximation and the observed value of $\delta_H(N_{min})$ the mass of the scalar field can be approximated by using equation (135)

$$m = \sqrt{3\pi}\frac{5 \cdot 10^{-5}}{2 \cdot 60 + 1} \approx 10^{-6} \tag{137}$$

The mass is not very sensitive to the exact value of N_{min} as can be seen by looking at equation (135).

3.6.4 The tilt of the spectrum of density perturbations

An important observational parameter is the tilt $n - 1$ of the spectrum of density perturbations defined by [7, Langlois]

$$n - 1 \equiv \frac{d\ln(\delta_H^2)}{d\ln(aH)} = \frac{2}{\delta_H}\frac{d\delta_H}{d\ln(aH)} \tag{138}$$

Substituting of equation (133) into this equation gives

$$n-1=\frac{2}{\delta_H}\frac{d\left(-4C\sqrt{\frac{\pi}{3}}m\phi^2\right)}{d\ln(aH)}=\frac{2}{\delta_H}\frac{2\delta_H}{\phi}\frac{d\phi}{d\ln(aH)}=\frac{4\dot{\phi}}{\phi}\left[\frac{dt}{d\ln(aH)}\right]$$
$$=-\frac{2}{\sqrt{3\pi}}\frac{m}{\phi}\left[\frac{d\ln(aH)}{dt}\right]^{-1} \qquad (139)$$

Equation (90) is used to substitute for the velocity of the scalar field. Let's rewrite the part between brackets

$$\frac{d\ln(aH)}{dt}=\frac{1}{H}\frac{dH}{dt}+\frac{d\ln a}{dt} \qquad (140)$$

Using equation (110), equation (113) and equation (114) this is rewritten as

$$\frac{d\ln(aH)}{dt}=-\frac{m^2}{3H}+\frac{d\ln a(t_0)\exp\left[\sqrt{\frac{4\pi}{3}}m\phi(t_0)(t-t_0)-\frac{m^2}{6}(t-t_0)^2\right]}{dt}$$
$$=-\frac{m^2}{3H}+H=H\left(1-\frac{m^2}{3H}\right)=\sqrt{\frac{4\pi}{3}}m\phi\left(1-\frac{1}{4\pi\phi^2}\right) \qquad (141)$$

where equation (87) is used to substitute for the Hubble parameter.
The tilt $n-1$ is then given by

$$n-1=-\frac{2}{\sqrt{3\pi}}\frac{m}{\phi}\left[\sqrt{\frac{3}{4\pi}}\frac{1}{m\phi}\frac{1}{1-\frac{1}{4\pi\phi^2}}\right]=\frac{-4}{4\pi\phi^2-1} \qquad (142)$$

By substituting equation (126) into this equation the tilt is obtained as a function of N'

$$[n-1](N')=-\frac{2}{N'} \qquad (143)$$

Let's compute the tilt $n-1$ and the associated value of n in the case $N'=N_{min}=60$.

$$[n-1](60)=-\frac{2}{60}\approx-0.033$$
$$n(60)=1-\frac{2}{60}\approx 0.97 \qquad (144)$$

3.6.5 Observations of the tilt compared with theory

The following values for $n(N_{min})$ and $\sigma_{n(N_{min})}$ are derived from WMAP data combined with distance measurements from Type Ia supernovae (SN) and Baryon Acoustic Oscillations (BAO) in the distribution of galaxies [8, Komatsu et al.]:

$$n(N_{min})=0.960\pm0.013 \qquad (145)$$

When comparing theory with observations it can be concluded that the theoretical result is confirmed by observations if $N_{min}=60$. More general, it is found, using

$$\frac{2}{N_{min}(min)}=1-0.947=0.053$$
$$\frac{2}{N_{min}(max)}=1-0.973=0.027$$

that

$$38 \leq N_{min} \leq 74 \tag{146}$$

for the theoretical result to be in the interval $0.947 \leq n(N_{min}) \leq 0.973$. It is very important to note that this value says nothing about the total number of e-foldings N_t reached during inflation. The total number of e-foldings is allowed to be many orders of magnitude larger than the values mentioned above. What this values are meaning is that to compare theory with observations one must have a good idea of what the value for N_{min} must be because this value is put into equation (135) and equation (143) which are giving theoretical results for the mass and the tilt.
Referring to the first part of this thesis the reheating temperature as a function of the minimum number of e-foldings N_{min} needed to solve the horizon problem, using equation (53), is written as

$$T_R = \frac{10^{15}GeV}{6 \cdot 10^{26}} \frac{e^{N_{min}}}{N_{min}} \tag{147}$$

in order for $38 \leq N_{min} \leq 74$ the reheating temperature must be in the interval

$$10^3 GeV \leq N_{min} \leq 10^{18} GeV \tag{148}$$

Again it must be noted that this result is based on the fact that the number of relativistic degrees of freedom is assumed constant during the radiation era.

3.7 Numerically solving the equations

Now let's look how good to slow roll approximation is. For this purpose the full equations describing scalar field inflation are solved numerically with Mathematica. The source text listings of the Mathematica program written are given in appendix A.
The full equations describing massive scalar field inflation are given here again

$$H^2 = \frac{8\pi}{3}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2\right] - \frac{k}{a^2} \tag{149}$$

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \tag{150}$$

Within the slow roll approximation the curvature energy density, the kinetic energy density and the acceleration of the scalar field were neglected. Curvature energy density can be important at the initial stages of inflation. During inflation it is redshifted away. The kinetic energy density term and the acceleration of the scalar field term become important at the end of the inflationary stage.
In order to get numerical results it is necessary to assign a value to the mass of the scalar field. Referring to the result (137) the mass is taken to be

$$m = 10^{-6}$$

3.7.1 Initial conditions

The initial value of the scalar field must be set. Because it has shown before that the minimum number of e-foldings $N_{min} \approx 60$ let's take for the initial value of the scalar field the value of the field which in the slow roll approximation gives that amount of e-foldings. Thus

$$N_t(SR) = 60$$

Also if initially there is a nonzero curvature energy density present the ratio of initial curvature density divided by the initial potential energy density must be set. This ratio is given by

$$\frac{K(t_P)}{V(t_P)} = \frac{\frac{3|k|}{8\pi a^2(t_P)}}{\frac{1}{2}m^2\phi^2(t_P)} = \frac{3}{4\pi m^2\phi^2(t_P)}\frac{|k|}{a^2(t_P)} \tag{151}$$

By the rescaling $k \Rightarrow \pm 1$ for $k \neq 0$ the initial value $a(t_P)$ determines this ratio.

3.7.2 The different approaches

In solving the equations describing massive scalar field inflation four different cases are considered:

1. Numerically solving the slow roll equations (for comparison)
2. Solving the full equations for negative initial curvature $(k < 0)$ with $K(t_P) \approx 250 \cdot V(t_P)$
3. Solving the full equations for zero initial curvature $(k = 0)$
4. Solving the full equations for positive initial curvature $(k > 0)$ with $K(t_P) \approx 0.97 \cdot V(t_P)$

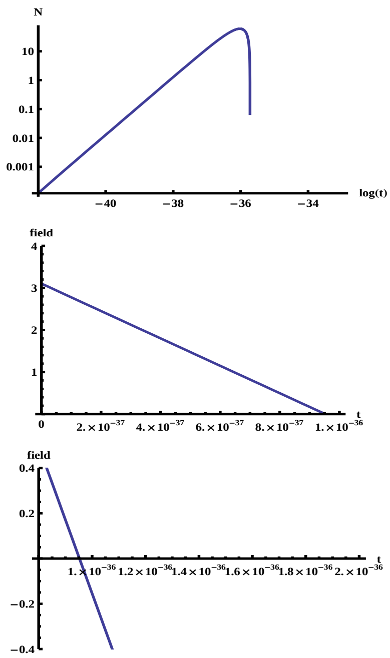
In the case of positive initial curvature the initial curvature density is of the same order of magnitude as the initial potential energy density. If the initial curvature density is chosen to be larger than the initial potential energy density the scale factor becomes an imaginary number which is physically not allowed. Thus the amount of initial curvature density chosen is almost equal to the upper limit.
In the case of negative initial curvature the initial curvature density is much larger than the initial potential energy density. Here one can take the amount of initial curvature density as large as possible.
In each case the following results are presented:

1. The duration of inflation t_{infl}. Equation (84) is used to calculate the end of inflation.
2. The total number of e-foldings N_t.
3. A graph in which $N = N_t - N'$ is plotted as a function of time.
4. Two graphs in which the scalar field ϕ is plotted as a function of time.

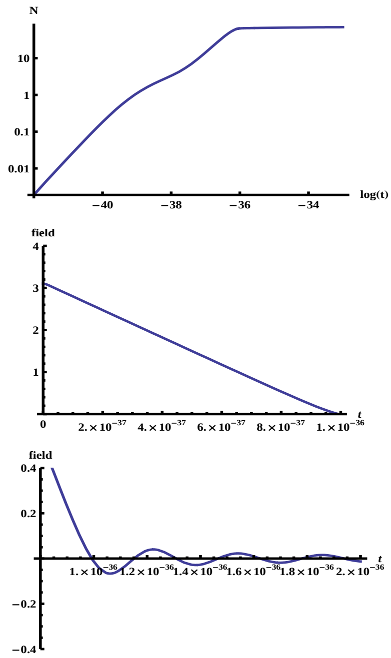
Also the duration of inflation in the slow roll approximation is calculated for comparison.

3.7.3 Results and discussion

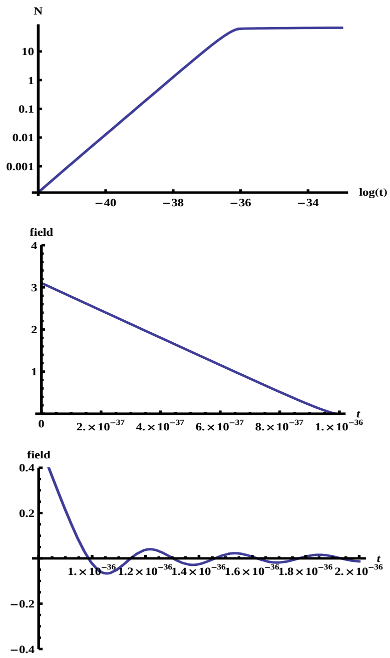
The results are presented in figure 1. In each case one can compare the calculated total number of e-foldings N_t with the total number of e-foldings $N_t(SR) = 60$ reached in the slow roll approximation. Also one can compare the duration of inflation t_{infl} with the duration of inflation calculated in the slow roll approximation. The duration of inflation in the slow roll approximation is calculated as $t_{infl}(SR) \approx 9 \cdot 10^{-37}s$. The results can be summarized as follows:



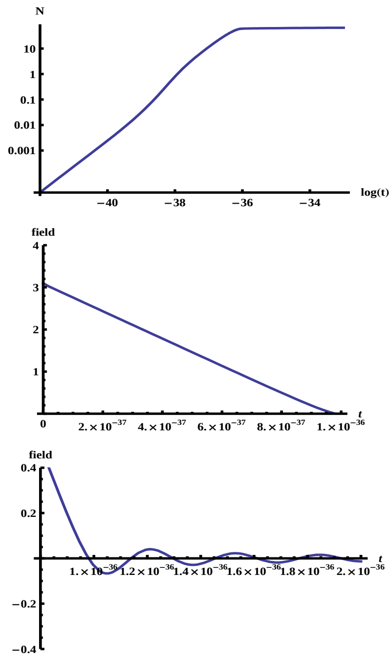
(a) Solutions of the Slow Roll equations. $t_{infl} \approx 9 \cdot 10^{-37}s$ and $N_t \approx 60$



(b) Solutions for negative initial curvature with $K(t_P) \approx 250 \cdot V(t_P)$. $t_{infl} \approx 9 \cdot 10^{-37}s$ and $N_t \approx 64$



(c) Solutions for zero initial curvature. $t_{infl} \approx 9 \cdot 10^{-37}s$ and $N_t \approx 61$



(d) Solutions for positive initial curvature with $K(t_P) \approx 0.97 \cdot V(t_P)$. $t_{infl} \approx 9 \cdot 10^{-37}s$ and $N_t \approx 60$

Figure 1: Numerical solutions of the equations with $m = 10^{-6}$ and $N_t(SR) = 60$

1. In each case the duration of inflation is $\approx 9 \cdot 10^{-37}s$ which is equal to the duration of inflation calculated analytically in the slow roll approximation.

2. In the case when the full equations are solved for zero initial curvature the total number of e-foldings N_t reached during inflation is ≈ 1 e-folding more than in the case when the slow roll equations are solved. The total amount of e-foldings N_t reached when solving the slow roll equations is approximately equal to the analytical calculated amount of $N_t(SR) = 60$.

3. In the case of positive initial curvature the total number of e-foldings N_t is ≈ 1 e-folding less than in the case of zero initial curvature and in the case of negative initial curvature the total number of e-foldings N_t is ≈ 3 e-foldings more than in the case of zero initial curvature. It must be noted that the ratio of initial curvature density divided by the initial potential energy was different for the two cases considered.

4. During inflation the plotted results for $N(t)$ and $\phi(t)$ based on solving the slow roll equations are approximately equal to the plotted results of $N(t)$ and $\phi(t)$ based on solving the full equations for zero initial curvature. As expected after the end of inflation the plotted results for $N(t)$ and $\phi(t)$ based on solving the Slow Roll equations become meaningless.

5. The slope related to the time evolution of $N(t)$ is approximately equal to 1 in the case of zero initial curvature which means that inflation is approximately exponential as expected from the slow roll solutions. After $t = t_{infl}$ the slope of $N(t)$ becomes approximately zero which confirms that inflation does not take place anymore. In the case of non negligible curvature energy density the deviation from exponential behaviour becomes larger but still is approximately exponential.

6. Independent of the initial curvature density during inflation the scalar field density $\phi(t)$ linear decreases as a function of time as expected from the Slow Roll solutions. After $t \approx t_{infl}$ the field begins to oscillate rapidly.

The conclusion is that approximating the full equations by the slow roll equations for $N_t(SR) = 60$ is a very good approximation even if the initial curvature density is two orders of magnitude larger then the initial potential energy density. As will be discussed in next section the total number of e-foldings N_t reached during inflation is expected to be (much) larger than $N_t = 60$. Because the slow roll approximation becomes even a better approximation if the total amount of e-foldings N_t reached during inflation becomes larger, the final conclusion is that it is always allowed to use the slow roll approximation.

3.8 Typical values for the total amount of e-foldings and the duration of massive scalar field inflation

Having confirmed that the slow roll approximation is a very good approximation let's look a possible values for the total amount of e-foldings N_t and the duration of inflation t_{infl}.
First let's give again the equation for the total number of e-foldings (124)

$$N_t = 2\pi\phi^2(t_P) - \frac{1}{2} \tag{152}$$

Also let's give again the equation for the duration of inflation (102)

$$t_{infl}(N_t) \approx \sqrt{3} \cdot 10^6 \left[\sqrt{2N_t + 1} - 1\right] \tag{153}$$

in which the equation above is used to substitute for $\phi(t_0)$ and equation (137) is used to substitute a typical value for the mass of the scalar field.
It has been shown before that the minimum value of the total number e-foldings to solve the horizon and flatness problem is $N_{min} \approx 60$. In this case the duration of inflation in seconds is

$$t_{infl}(N_t = 60) \approx \sqrt{3} \cdot 10^6 \left[\sqrt{120 + 1} - 1\right] \cdot 5 \cdot 10^{-44} s \approx 10^{-36} s \tag{154}$$

This result was also mentioned in the section before.
Now let's look at the maximum number of e-foldings that can be reached classically. In order for a classical description to be valid the potential energy density must satisfy [4, Linde]

$$V(\phi) = \frac{1}{2} m^2 \phi^2 \leq 1 \tag{155}$$

This means that the maximum value of the massive scalar field ϕ_{max} is given by

$$\phi_{max} = \frac{\sqrt{2}}{m} \approx \sqrt{2} \cdot 10^6 \tag{156}$$

The maximum value for the total number of e-foldings $N_{t,max}$ is then given by using equation (152)

$$N_{t,max} = 2\pi \cdot 2 \cdot 10^{12} - \frac{1}{2} \approx 10^{13} \tag{157}$$

In this case the duration of inflation is

$$t_{infl}(N_t = 10^{13}) \approx \sqrt{3} \cdot 10^6 \left[\sqrt{10^{13}} - 1\right] \cdot 5 \cdot 10^{-44} s \approx 10^{-31} s \tag{158}$$

Because any initial value of the field at the Planck time t_P occurs with the same probability the total number of e-foldings N_t can assume any value between the minimum and maximum value. Let's summarize the results

$$\begin{aligned} 60 \leq N_t \leq 10^{13} \\ 10^{-36} s \leq t_{infl} \leq 10^{-31} s \end{aligned} \tag{159}$$

Because some approximations are made one must not take these limits too strictly.
In the first part of this thesis it was shown that in order to solve the horizon problem $N_t \geq 66$ if $t_{end} = 10^{-36} s$ and $N_t \geq 59$ if $t_{end} = 10^{-30} s$ where $t_{end} \approx t_{infl}$.
One can conclude that scalar field inflation gives enough e-foldings within a time $10^{-31} s$ to solve the horizon problem and the flatness problem. However is it somewhat uncertain what the lower limit on N_t actually is. Note, assuming massive scalar field inflation is the correct inflationary scenario, that the probability that we live in a universe in which the total number of e-foldings reached during inflationary expansion was $N_t \approx 60$ is negligible because each value of the scalar field at the Planck time occurred with the same probability. So the probability is very high that we live in a universe in which the total number of e-foldings reached during inflationary expansion was $N_t \gg 60$ and in this case the horizon problem and the flatness problem are easily solved.

4 Conclusions

It was found that a period of inflationary expansion called inflation can solve the problems of the Big Bang model. Because in the case of inflation by a cosmological constant inflation does not end inflation by a massive scalar field was introduced. In this model the duration of inflation is finite and is typically $10^{-36}s \leq t_{infl} \leq 10^{-31}s$. Also this model typically produces a very large amount of e-foldings which easily solves the flatness and horizon problem. The total number of e-foldings is typically $60 \leq N_t \leq 10^{13}$. This model also gives an explanation for the observed large scale structure of the universe. The mass of the scalar field is calculated as $m \approx 10^{-6}$. The tilt of the spectrum of density perturbations is calculated as $n(60) \approx 0.97$ which agrees with the value of the tilt derived from observations. Because this inflation model is not ruled out by observations and because the total amount of e-foldings that can be reached is much larger than the minimum amount of e-foldings necessary to solve the problems of the Big Bang model the conclusion is that inflation by a massive scalar field is very plausible. In describing nature in the very early stages of the universe on must take into account that it is plausible that a massive scalar field was present at this early epoch.

A Source text listings of the Mathematica program

```
(* CHAOTIC INFLATION WITH MASSIVE SCALAR FIELD *)
(* Written by : DENNIS VISSER *)
Clear[m, a0, ffinal, efoldingssr, f0, derf, durationsr, solutions,
scalefactor, field, duration, efoldings, k,
b, c, d];
m = 10^(−6);
a0[1] = 3 ∗ 10^5; (*arbitrary*)
a0[2] = 10^4; (*determines curvature density for k = −1*)
a0[3] = 3 ∗ 10^5; (*arbitrary*)
a0[4] = 1.6 ∗ 10^5; (*determines curvature density for k = +1,
 must be larger than 1.57 ∗ 10^5( if N = 60) in order to obtain
 real solutions for the scale factor *)
ffinal = Sqrt[1/(4 ∗ Pi)];
efoldingssr = 60;
f0 = Sqrt[(2 ∗ efoldingssr + 1)/(4 ∗ Pi)]; derf = (−1) ∗ Sqrt[1/(12 ∗ Pi)] ∗ m;
V[f_] = 0.5 ∗ m^2 ∗ f[t]^2;
F[t_] = f0 − (m/Sqrt[12 ∗ Pi]) ∗ t;
durationsr = t/.N[Solve[ffinal == F[t/(5 ∗ 10^(−44))], t]][[1]];
solutions[1] =
NDSolve[
{(a'[t]/a[t])^2 == (8 ∗ Pi/3) ∗ V[f],
3 ∗ (a'[t]/a[t]) ∗ f'[t] + (D[V[f], f[t]]) == 0,
a[1] == a0[1], f[1] == f0},
{a, f},
{t, 1, 10^11}, MaxSteps → 1000000];
scalefactor[1] = a/.solutions[1][[1]];
field[1] = f/.solutions[1][[1]];
duration[1] = 5 ∗ 10^(−44) ∗ t/.FindRoot[Abs[field[1][t]] − (Sqrt[2]/m)∗
```

```
Abs[field[1]'[t]], {t, durationsr/(5 * 10^(-44))}];
efoldings[1] = Log[(a[t/.FindRoot[Abs[field[1][t]] - (Sqrt[2]/m)*
Abs[field[1]'[t]], {t, durationsr/(5 * 10^(-44))}]]/.solutions[1][[1]])/a0[1]];
b[1] = LogPlot[Log[(scalefactor[1][10^(t)/(5 * 10^(-44))])/a0[1]], {t, -42, -33},
Ticks -> {{-40, -38, -36, -34}, {0.001, 0.01, 0.1, 1, 10}},
AxesLabel -> {"log(t)", "N"}, AxesStyle -> Thick, PlotStyle -> Thick,
LabelStyle -> Bold];
c[1] = Plot[field[1][t/(5 * 10^(-44))], {t, 10^(-44), 10^(-36)},
PlotRange -> {0, 4}, AxesLabel -> {"t", "field"}, AxesStyle -> Thick,
PlotStyle -> Thick, LabelStyle -> Bold];
d[1] = Plot[field[1][t/(5 * 10^(-44))], {t, 8 * 10^(-37), 2 * 10^(-36)},
PlotRange -> {-0.4, 0.4}, AxesLabel -> {"t", "field"}, AxesStyle -> Thick,
PlotStyle -> Thick, LabelStyle -> Bold];
Print["Neglect the error messages, the solutions are not unique,
 the solutions from which the error messages appear are not used
because they have no physical meaning"];
k = -1;
For[i = 2, i <= 4, i++,
solutions[i] =
NDSolve[
{(a'[t]/a[t])^2 + k/a[t]^2 == (8 * Pi/3) * (V[f] + 0.5 * f'[t]^2),
f"[t] + 3 * (a'[t]/a[t]) * f'[t] + (D[V[f], f[t]]) == 0,
a[1] == a0[i], f[1] == f0, f'[1] == derf},
{a, f, f'},
{t, 1, 10^11}, MaxSteps -> 1000000];
scalefactor[i] = a/.solutions[i][[1]];
field[i] = f/.solutions[i][[1]];
duration[i] = 5 * 10^(-44) * t/.FindRoot[Abs[field[i][t]] - (Sqrt[2]/m)*
Abs[field[i]'[t]], {t, durationsr/(5 * 10^(-44))}];
efoldings[i] = Log[(a[t/.FindRoot[Abs[field[i][t]] - (Sqrt[2]/m)*
Abs[field[i]'[t]], {t, durationsr/(5 * 10^(-44))}]]/.solutions[i][[1]])/a0[i]];
b[i] = LogPlot[Log[(scalefactor[i][10^(t)/(5 * 10^(-44))])/a0[i]], {t, -42, -33},
Ticks -> {{-40, -38, -36, -34}, {0.001, 0.01, 0.1, 1, 10}},
AxesLabel -> {"log(t)", "N"}, AxesStyle -> Thick, PlotStyle -> Thick,
LabelStyle -> Bold];
c[i] = Plot[field[i][t/(5 * 10^(-44))], {t, 10^(-44), 10^(-36)},
PlotRange -> {0, 4}, AxesLabel -> {t, "field"}, AxesStyle -> Thick,
PlotStyle -> Thick, LabelStyle -> Bold];
d[i] = Plot[field[i][t/(5 * 10^(-44))], {t, 8 * 10^(-37), 2 * 10^(-36)},
PlotRange -> {-0.4, 0.4},
AxesLabel -> {t, "field"}, AxesStyle -> Thick,
PlotStyle -> Thick, LabelStyle -> Bold];
k++;
];
Print["************************************************************
*******************************************************************
**********"];
Print[""]; Print["CHAOTIC INFLATION WITH A
 MASSIVE SCALAR FIELD"];
Print["Mass of the scalar field: "
```

```
" m= ", N[m]];
Print["The total number of e-foldings in the Slow Roll approximation:"
" N= ", efoldingssr];
Print["-> The initial field in the SRA is then calculated as: "
" f0= ", N[f0]];
Print["-> The duration of inflation in the SRA is then calculated as: "
" T= ", durationsr, " s"];
Print[" "]; Print["Numerical solutions of the Slow Roll equations :"];
Print["The duration of inflation : T= ", duration[1], " s"];
Print["The total number of e-foldings : N= ", efoldings[1]];
Print[GraphicsArray[{b[1], c[1], d[1]}]];
Print["Numerical solutions of the exact equations :"]; Print[" "];
k = -1;
For[i = 2, i ≤ 4, i++,
Print["k= ", k];
Print["Curvature energy density / potential energy density: K / V = ",
N[(3 * Abs[k])/(4 * Pi * (a0[i])^2 * m^2 * (f0)^2)]];
Print["The duration of inflation : T= ", duration[i], " s"];
Print["The total number of e-foldings : N= ", efoldings[i]];
Print[GraphicsArray[{b[i], c[i], d[i]}]];
k++;
];
(* Exporting the plots to .pdf files *)
Export["sr.pdf", GraphicsGrid[{{b[1]}, {c[1]}, {d[1]}}]];
Export["k_neg.pdf", GraphicsGrid[{{b[2]}, {c[2]}, {d[2]}}]];
Export["k_0.pdf", GraphicsGrid[{{b[3]}, {c[3]}, {d[3]}}]];
Export["k_pos.pdf", GraphicsGrid[{{b[4]}, {c[4]}, {d[4]}}]];
```

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