University of Groningen

Bachelor Thesis

The Riemann Zeta Function and the connection to Hamiltonians in Physics

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1. INTRODUCTION

In November 1859, Bernhard Riemann published his one and only paper named "Über die Anzahl der Primzahlen unter einer gegebenen Größe", translated this means "On the Number of Primes Less Than a Given Magnitude". As the title suggests Riemann was looking for an estimate of the prime numbers. In his search he used the function $\zeta(s) = \sum_{n=0}^{\infty} n^{-s}$, which is only defined for s with $\mathrm{Re}(s) > 1$. But Riemann used a tool from complex analysis, called meromorphic extension, to extend the function to the entire complex plane. This function now is called the Riemann zeta function.

In his paper Riemann only mentioned briefly that he suspects all the non-real zeroes of his function to lie on the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1/2\}$. This now is better known as the Riemann hypothesis. Still this hypothesis is neither proved nor disproved. But, since the proof or disproof of this hypothesis was not Riemann's purpose of his paper — he only wanted an estimate for the prime numbers —, he proceeded without mentioning the hypothesis again.

Up until now, a lot of people have been searching for a proof using different techniques, as well were David Hilbert and George Pólya. In the early days of quantum mechanics they suggested a physical way to verify the Riemann hypothesis.

On 3rd January 1982, Pólya sent a letter to Andrew Odlyzko. In his letter he wrote the following.

See http://www.dtc.umn.edu/~odlyzko/polya/ for the scanned letter.

> *Dear Mr Odlyzko,*
>
> *Many thanks for your letter of December 8. I can only tell you what happened to me.*
>
> *I spent two years in Göttingen ending around the begin of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true?" This would be the case, I answered, if the non-trivial zeroes of the ξ-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.*
>
> *I never published this remark, but somehow it became known and it is still remembered.*
>
> *With best regards.*
>
> *Your sincerely,*
>
> *George Pólya*

The ξ-function in this letter is the Riemann xi function and is an adapted form of the Riemann zeta function, as we will see in this thesis.

In this thesis we will give an introduction to the mathematical properties of the Riemann zeta function. We will state the Riemann hypothesis and the Hilbert-Pólya conjecture, which implies the Riemann hypothesis. We will look at three Hamiltonians that could provide a possible solution for the Hilbert-Pólya conjecture. We will derive some properties of the Hamiltonian that can solve the Hilbert-Pólya conjecture, by comparing the counting function for the number of non-trivial zeroes of the Riemann zeta function with the counting function for the number of energy eigenstates of physical systems.

2. PRELIMINARIES

2.1 Sets

For clarity, we specify some sets and their notation that will appear in this thesis.

Notation 2.1 (Natural numbers). The set of natural numbers is denoted as

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

This notation agrees with ISO 80000-2 clause 6.1.

Notation 2.2 (Integers). The set of integers is denoted as

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

This notation agrees with ISO 80000-2 clause 6.2.

Notation 2.3 (Prime numbers). The set of prime numbers is denoted as

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}.$$

This notation agrees with ISO 80000-2 clause 6.6.

2.2 Notations

Notation 2.4 (Big-O). The notation $f(x) = \mathrm{O}(g(x))$, $f(x)$ is big-O of $g(x)$, means

$$\left|\frac{f(x)}{g(x)}\right|$$

is bounded from above in the limit implied by the context.

This notation agrees with ISO 80000-2 clause 11.9.

Notation 2.5 (Little-o). The notation $f(x) = \mathrm{o}(g(x))$, $f(x)$ is little-o of $g(x)$, means

$$\frac{f(x)}{g(x)} \to 0.$$

in the limit implied by the context.

This notation agrees with ISO 80000-2 clause 11.10.

The symbol "=" in the notations is used for historical reasons and does not have the meaning of equality, because transitivity does not apply.

2.3 Complex exponential and logarithmic functions

This notation agrees with ISO 80000-2 clause 12.5, which also states: "$\log(x)$ shall not be used in place of $\ln(x)$."

Definition 2.1 (Natural logarithm). The *natural logarithmic function*, often abbreviated as *natural logarithm*, $\ln\colon \mathbb{R}_{>0} \to \mathbb{R}$, is the function defined by

$$\ln(x) := \int_1^x \frac{1}{t}\,\mathrm{d}t.$$

Definition 2.2 (Argument). The *argument* of a complex number s is denoted as $\arg(s)$ and is defined to be the set

$$\arg(s) := \tan^{-1}\left(\frac{\operatorname{Im}(s)}{\operatorname{Re}(s)}\right)$$

which contains infinitely many values.

Definition 2.3 (Natural logarithm (continued)). If $s \neq 0$ is a complex number, then we define $\ln(s)$ to be the set

$$\ln(s) := \int_1^{|s|} \frac{1}{t}\,\mathrm{d}t + \mathrm{i}\arg(s),$$

such that it is equal to Definition 2.1 when s is restricted to $\mathbb{R}_{>0}$.

Definition 2.4. If s is a complex number, then the complex-valued function $\exp\colon \mathbb{C} \to \mathbb{C}$ is implicitly defined by

$$\ln(\exp(s)) = s.$$

One can also write e^s instead of $\exp(s)$.

Definition 2.5. If α is a complex constant and $s \neq 0$ a complex number, then s^α is defined by

$$s^\alpha := \mathrm{e}^{\alpha \ln(s)}.$$

Definition 2.6. If $f(x) = g(x) + \mathrm{i}h(x)$ is a complex-valued function of a real variable x and the real and imaginary parts $g(x)$ and $h(x)$ are differentiable functions of x, then the derivative of $f(x)$ with respect to x is defined to be

$$f'(x) := g'(x) + \mathrm{i}h'(x).$$

2.4 Complex analysis

Definition 2.7 (Holomorphic). A complex-valued function f is said to be *holomorphic* on an open set O if at every point s_0 of O, the limit

$$\lim_{\Delta s \to 0} \frac{f(s_0 + \Delta s) - f(s_0)}{\Delta s}$$

exists. This limit is called the *derivative* of f at s_0.

The word holomorphic derives from the Greek words ολος (holos), meaning "whole", and μορφη (morphe), meaning "form" or "appearance".

Another frequently used word for holomorphic is *analytic* and sometimes it is called *complex differentiable*.

Theorem 2.1 (Cauchy-Riemann condition). *Let $f(s)$ be a complex-valued function defined in some open set O. If the first partial derivatives of $f(s)$ with respect to* $\mathrm{Re}(s)$ *and* $\mathrm{Im}(s)$ *exist, are continuous and satisfy the* Cauchy-Riemann condition

$$\frac{\partial f(s)}{\partial \mathrm{Re}(s)} = \frac{1}{\mathrm{i}} \frac{\partial f(s)}{\partial \mathrm{Im}(s)}$$

at all points of O, then f is holomorphic in O.

Proof. See Saff and Snider (2003) for a proof. □

Definition 2.8 (Meromorphic). A complex-valued function f is said to be *meromorphic* in a *domain* D (an open and connected set) if at every point of D it is either holomorphic or has a pole (a zero of the denominator).

The word meromorphic derives from the Greek words μερος (meros), meaning "part", and μορφη (morphe), meaning "form" or "appearance".

Theorem 2.2. *If f is holomorphic and non-zero at each point of a simple closed positively oriented contour C and is meromorphic inside C, then*

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(s)}{f(s)} \, \mathrm{d}s = N - P,$$

where N and P are, respectively, the number of zeroes and the number of poles, included multiplicity, of f inside C.

Proof. See Saff and Snider (2003) for a proof. □

3. MATHEMATICS

3.1 The Riemann zeta function

It was Euler who studied the properties of the series $\sum_{n=1}^{\infty} 1/n^s$ for integer values of s. Later, Dirichlet looked at this series for real s greater than one. But Riemann went even further than that, he allowed s to attain complex values.

Definition 3.1 (Riemann zeta function). The *Riemann zeta function*, $\zeta\colon \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\} \to \mathbb{C}$, is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Riemann zeta function is a special type of the Dirichlet series. The Dirichlet series attached to a complex valued function χ defined on the natural numbers is given by

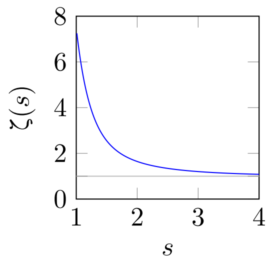
$$D(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

so $\zeta(s) = D(1, s)$. Most results in this thesis can be generalized to other Dirichlet series.

This series converges when $\mathrm{Re}(s) > 1$, because it converges absolutely. To show this, we will use the widely used notation $\sigma := \mathrm{Re}(s)$ and $t := \mathrm{Im}(s)$, so $s = \sigma + it$. With this notation we have $|n^{-s}| = \left|n^{-\sigma - \mathrm{i}t}\right| = |n^{-\sigma}|\left|n^{-\mathrm{i}t}\right| = |n^{-\sigma}|\left|\mathrm{e}^{-\mathrm{i}t\ln n}\right| = |n^{-\sigma}| = n^{-\sigma}$. And we know that the series $\sum_{n=1}^{\infty} n^{-\sigma}$ converges when $\sigma > 1$.

The Riemann zeta function is now only defined for input values with real part greater than one. But Riemann meromorphically extended this function to the entire complex plane. This means he found a meromorphic function in $\mathbb{C}$ such that this function is equal to $\zeta(s)$ when s is restricted to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$. This meromorphic function turns out to be a holomorphic function, except at the point $s = 1$, there is has a simple pole, that is, a pole of order one. Moreover, this meromorphic extension is unique, that means if $f(s)$ and $g(s)$ are meromorphic functions in $\mathbb{C}$ such that they are equal to $\zeta(s)$ when s is restricted to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$, then they are equal everywhere in the complex plane. This is a property of all meromorphic functions and it is called the identity theorem. Instead of looking at two functions, we could also look at their difference.

The Riemann zeta function for $s > 1$.



Theorem 3.1 (Identity theorem). *Let f be a holomorphic function on a domain D. If there exists a point s_0 in D such that $f^{(n)}(s_0) = 0$ for all $n \in \mathbb{N}$, then $f(s) = 0$ for all s in D.*

Proof. Let $A = \left\{s \in D \mid f^{(n)}(s) = 0 \ \forall n \in \mathbb{N}\right\}$, which is non-empty by assumption. We will show that A and $D \setminus A$ are open sets, which means $A = D$ because $A \neq \emptyset$. And then we are done, because $A = D$ implies $f^{(n)}(s) = 0$ for all $n \in \mathbb{N}$ and all $s \in D$, and which implies $f(s) = 0$ for all $s \in D$.

(Proof that A is open) Let $a \in A$ be arbitrary. Since D is open, there exists a radius $R > 0$ such that $B(a, R) \subseteq D$ and

$$f(s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(s - a)^n$$

for $\|s-a\| < R$. Since $a \in A$, each $f^{(n)}(a) = 0$, so f is identically zero on $B(a,R)$. This means that $B(a,R) \subseteq A$. Because $a \in A$ was arbitrary is holds for all points in A and since the union of open sets is again open, A is an open set.

(Proof that $D \setminus A$ is open) Let $b \in D \setminus A$ be arbitrary. Then there exists an $n_0 \in \mathbb{N}$ with $f^{(n_0)}(b) \neq 0$. Since $f^{(n_0)}$ is a continuous function, there exists an $r > 0$ such that $f^{(n_0)}(s) \neq 0$ on $B(b,r)$. This means $B(b,r) \subseteq D \setminus A$. Because $b \in D \setminus A$ was arbitrary is holds for all points in $D \setminus A$ and since the union of open sets is again open, $D \setminus A$ is an open set.

So $D = A = \{s \in D \mid f^{(n)}(s_0) = 0 \; \forall n \in \mathbb{N}\} \subseteq \{s \in D \mid f(s_0) = 0\}$. This proves the theorem. □

This extended function is called the *Riemann zeta function*. In the next section we will show how the function in Definition 3.1 can be meromorphically extended to $\mathbb{C}$ with its single pole at $s = 1$.

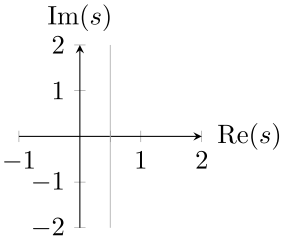
The Riemann hypothesis states that all non-real zeroes of the Riemann zeta function lie on the the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1/2\}$. The non-real zeroes are called *non-trivial zeroes* because the real zeroes are easy to calculate.

Conjecture 3.1 (Riemann hypothesis). All non-trivial zeroes of $\zeta(s)$ have the form

$$\rho = \frac{1}{2} + \mathrm{i}t,$$

where t is a real number.

The location of the non-trivial zeroes in the complex plane if the Riemann hypothesis is true is represented by the gray line.



3.2 The zeroes of the Riemann zeta function

The goal of this section is to get some insight in the locations of the zeroes of the Riemann zeta function. It turns out that all the non-trivial zeroes are located in the vertical strip $\{s \in \mathbb{C} \mid 0 \leqslant \mathrm{Re}(s) \leqslant 1\}$. We are going to prove this in parts. First we will show that there are no zeroes in the plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.

Theorem 3.2 (Euler product formula). *The Riemann zeta function is equal to*

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}},$$

for $\mathrm{Re}(s) > 1$.

Proof. By Definition 3.1, the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

for $\mathrm{Re}(s) > 1$. Now, we repeat the following kind of operations

$$\begin{aligned}
\frac{1}{2^s}\zeta(s) &= \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \cdots \\
\left(1 - \frac{1}{2^s}\right)\zeta(s) &= 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \cdots \\
\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) &= \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \cdots \\
\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) &= 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \cdots
\end{aligned}$$

to obtain

$$\cdots\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1$$

as a consequence of the fundamental theorem of arithmetic, the unique factorization of integers into prime numbers. Rewriting this equation gives

$$\zeta(s)=\frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\cdots}$$

or equivalently

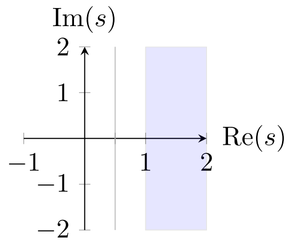
$$\zeta(s)=\prod_{p\in\mathbb{P}}\frac{1}{1-p^{-s}}$$

for $\operatorname{Re}(s)>1$. This proves the theorem. □

This equation is the famous relation between the Riemann zeta function and the prime numbers.

Proposition 3.1. *The Riemann zeta function has no zeroes in the complex plane where* $\operatorname{Re}(s)>1$. *In other words,* $\zeta(s)\neq 0$ *for* $\operatorname{Re}(s)>1$.

The blue shaded area is the area without zeroes.



Proof. Again, we will use the notation $\sigma := \operatorname{Re}(s)$ and $t := \operatorname{Im}(s)$, so $s=\sigma+it$. Now we have $|p^{-s}|=p^{-\sigma}$. By Theorem 3.2 we have

$$\begin{aligned}|\zeta(s)|&=\prod_{p\in\mathbb{P}}\frac{1}{|1-p^{-s}|}\geqslant\prod_{p\in\mathbb{P}}\frac{1}{1+|p^{-s}|}=\prod_{p\in\mathbb{P}}\frac{1}{1+p^{-\sigma}}=\exp\left(-\sum_{p\in\mathbb{P}}\ln\left(1+p^{-\sigma}\right)\right)\\&=\exp\left(\sum_{p\in\mathbb{P}}\left(-p^{-\sigma}+\frac{1}{2}p^{-2\sigma}-\frac{1}{3}p^{-3\sigma}+\mathrm{O}\left(p^{-4\sigma}\right)\right)\right)>\exp\left(-\sum_{p\in\mathbb{P}}p^{-\sigma}\right)>0,\end{aligned}$$

because $\sum_{p\in\mathbb{P}}p^{-\sigma}$ converges for $\sigma>1$. In the fifth step, we used the Taylor expansion of $\ln(1+x)$ around the point $x=0$:

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{\ln^{(n)}(1)}{n!}x^n=x-\frac{1}{2}x^2+\frac{1}{3}x^3+\mathrm{O}\left(x^4\right),$$

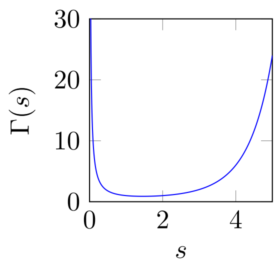
which converges when $|x|<1$. Now, $|\zeta(s)|>0$ or equivalently $\zeta(s)\neq 0$, for $\operatorname{Re}(s)>1$. This proves the proposition. □

To meromorphically extend the Riemann zeta function to the complex plane, we will use the so called *Gamma function*.

Definition 3.2 (Gamma function). The *Gamma function*, $\Gamma\colon \{s\in\mathbb{C}\mid \operatorname{Re}(s)>0\}\to\mathbb{C}$, is defined by

$$\Gamma(s):=\int_0^\infty x^{s-1}\mathrm{e}^{-x}\,\mathrm{d}x.$$

The Gamma function for $s>1$.



This notation is introduced by Legendre. Riemann used another notation, he used the notation introduced by Gauß:

$$\Pi(s):=\int_0^\infty x^s\mathrm{e}^{-x}\,\mathrm{d}x,$$

for $\mathrm{Re}(s) > -1$. Although the definition of $\Pi(s)$ looks more natural, today almost everyone uses the definition of $\Gamma(s)$ instead of $\Pi(s)$. Therefore, we will use it here too.

Let's proof that the Gamma function converges. We show that it even converges absolutely. Setting $\sigma := \mathrm{Re}(s)$, the absolute value of the integrand equals $|x^{s-1}\mathrm{e}^{-x}| = x^{\sigma-1}\mathrm{e}^{-x}$. Splitting the integral at the point $s = 1$, gives

$$\begin{aligned}\int_0^\infty \left|x^s \mathrm{e}^{-x}\right| \mathrm{d}x &= \int_0^\infty x^{\sigma-1}\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^1 x^{\sigma-1}\mathrm{e}^{-x}\,\mathrm{d}x + \int_1^\infty x^{\sigma-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &< \int_0^1 x^{\sigma-1}\,\mathrm{d}x + \int_1^\infty x^{\sigma-1}\mathrm{e}^{-x}\,\mathrm{d}x.\end{aligned}$$

The first integral converges if $\sigma > 0$, and the second integral converges because its a product of a polynomial and e^{-x}.

Proposition 3.2. *The Gamma function defines a holomorphic function if* $\mathrm{Re}(s) > 0$.

Proof. We use the Cauchy-Riemann condition, Theorem 2.1, to proof this. We have by definition of the Gamma function, Definition 3.2

$$\begin{aligned}\frac{\partial\,\Gamma(\sigma+\mathrm{i}t)}{\partial\sigma} &= \frac{\partial}{\partial\sigma}\lim_{N\to\infty}\int_0^N x^{\sigma+\mathrm{i}t-1}\mathrm{e}^{-x}\,\mathrm{d}x = \lim_{N\to\infty}\frac{\partial}{\partial\sigma}\int_0^N x^{\sigma+\mathrm{i}t-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \lim_{N\to\infty}\int_0^N \frac{\partial x^{\sigma+\mathrm{i}t-1}\mathrm{e}^{-x}}{\partial\sigma}\,\mathrm{d}x = \lim_{N\to\infty}\int_0^N x^{\sigma+\mathrm{i}t-1}\mathrm{e}^{-x}\ln(x)\,\mathrm{d}x \\ &= \lim_{N\to\infty}\int_0^N \frac{\partial x^{\sigma+\mathrm{i}t-1}\mathrm{e}^{-x}}{\partial(\mathrm{i}t)}\,\mathrm{d}x = \frac{1}{\mathrm{i}}\lim_{N\to\infty}\int_0^N \frac{\partial x^{\sigma+\mathrm{i}t-1}\mathrm{e}^{-x}}{\partial t}\,\mathrm{d}x \\ &= \frac{1}{\mathrm{i}}\lim_{N\to\infty}\frac{\partial}{\partial t}\int_0^N x^{\sigma+\mathrm{i}t-1}\mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{\mathrm{i}}\frac{\partial}{\partial t}\lim_{N\to\infty}\int_0^N x^{\sigma+\mathrm{i}t-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \frac{1}{\mathrm{i}}\frac{\partial\,\Gamma(\sigma+\mathrm{i}t)}{\partial t},\end{aligned}$$

where in steps three and seven the differentiations and integrations are interchanged by the Leibniz rule. In steps two and eight the derivatives and limits could be interchanged because of pointwise convergence of the integrals and uniform convergence of their derivatives. □

The Gamma function satisfies a special property called the *Reduction Formula*, sometimes also called the *functional equation of the factorial function*.

The Gamma function is a generalization of the factorial. If $n \in \mathbb{N}$ then

$$\begin{aligned} n! = \Gamma(n+1) &= n\,\Gamma(n) \\ &= n(n-1)\,\Gamma(n-1) \\ &= \cdots \\ &= n(n-1)\cdots 1\,\Gamma(1) \\ &= \prod_{i=1}^n i,\end{aligned}$$

since by definition

$$\begin{aligned}\Gamma(1) &= \int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x \\ &= -\mathrm{e}^{-x}\big|_{x=0}^\infty = 1.\end{aligned}$$

Theorem 3.3 (Reduction Formula). *The Gamma function satisfies the* Reduction Formula

$$\Gamma(s+1) = s\,\Gamma(s)$$

for $\mathrm{Re}(s) > 0$.

Proof. Using integration by parts, we can see that

$$\Gamma(s) = \int_0^\infty x^{s-1}\mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{s}\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x^s = \frac{1}{s}x^s\mathrm{e}^{-x}\Big|_{x=0}^\infty + \frac{1}{s}\int_0^\infty x^s\mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{s}\,\Gamma(s+1)$$

for $\mathrm{Re}(s) > 0$, since $x^s|_{x=0}$ is only well-defined for $\mathrm{Re}(s) > 0$. This implies the Reduction Formula. □

Yet, the Gamma function is only defined for $\mathrm{Re}(s) > 0$, but we need to extend it to the entire complex plane in order to use it for the extension of the Riemann zeta function. Looking at the Reduction Formula, we see that we can calculate $\Gamma(s+1)$ using $\Gamma(s)$. But we also could use this Reduction Formula to go backwards. Therefore, for $-1 < \mathrm{Re}(s) \leqslant 0$ we can define $\Gamma(s)$ by

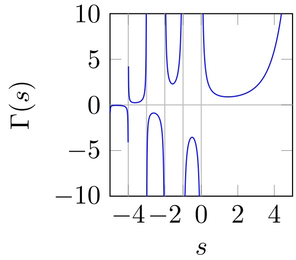
$$\Gamma(s) = \frac{\Gamma(s+1)}{s}.$$

This is allowed, since $\Gamma(s+1)$ is perfectly defined for $-1 < \mathrm{Re}(s) \leqslant 0$. From this definition we see that $\Gamma(s)$ has a pole at $s = 0$. Continuing in this way, we can define the Gamma function in the left side of the complex plane as follows.

Definition 3.3 (Gamma function (continued)). The *Gamma function*, $\Gamma\colon \mathbb{C} \to \mathbb{C}$, is defined for $-(n+1) < \mathrm{Re}(s) \leqslant -n$, where $n \in \mathbb{N}$, by

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{\prod_{m=0}^{n}(s+m)}.$$

The extended Gamma function for s real.



Therefore, the Gamma function can be meromorphically extended to $\mathbb{C}$, where it has all its poles at $s \in \mathbb{Z}_{\leqslant 0}$, those poles are simple poles. It turns out that the extended Gamma function is never zero.

Theorem 3.4. *The Riemann zeta function in Definition 3.1 admits a meromorphic extension to $\mathbb{C}$ which satisfies the equation*

$$\xi(s) = \xi(1-s) \tag{3.1}$$

where $\xi(s)$ is the so called Riemann xi function *defined by*

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{3.2}$$

Proof. Substituting $x = n^2\pi y$ into the Gamma function gives

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right) &= \int_0^\infty x^{\frac{s}{2}-1}\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty (n^2\pi y)^{\frac{s}{2}-1}\mathrm{e}^{-n^2\pi y}n^2\pi\,\mathrm{d}y \\ &= n^s\pi^{\frac{s}{2}}\int_0^\infty y^{\frac{s}{2}-1}\mathrm{e}^{-n^2\pi y}\,\mathrm{d}y\end{aligned}$$

for $\mathrm{Re}(s) > 0$. Substitution of this result into the definition of $\xi(s)$ gives for $\mathrm{Re}(s) > 1$

$$\begin{aligned}\frac{2}{s(s-1)}\xi(s) &= \pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^{\infty}\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \sum_{n=1}^{\infty}\int_0^\infty y^{\frac{s}{2}-1}\mathrm{e}^{-n^2\pi y}\,\mathrm{d}y \\ &= \int_0^\infty y^{\frac{s}{2}-1}\left(\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi y}\right)\mathrm{d}y = \int_0^\infty y^{\frac{s}{2}-1}\omega(y)\,\mathrm{d}y,\end{aligned}$$

where $\omega(y) := \sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi y}$ and the sum and integral are exchanged for $\mathrm{Re}(s) > 1$ because of absolute convergence. The Poisson summation formula implies the following functional equation

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2\pi/y} = \sqrt{y}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2\pi y},$$

see for example Edwards (1974), hence $2\omega(1/y)+1=\sqrt{y}(2\omega(y)+1)$, or equivalently

$$\omega\left(\frac{1}{y}\right)=\sqrt{y}\omega(y)+\frac{\sqrt{y}}{2}-\frac{1}{2}.$$

The equation for $\xi(s)$ yields, by splitting the integral at $y=1$ and making a change of variables $y\mapsto 1/y$ in the first integral,

$$\begin{aligned}
\frac{2}{s(s-1)}\,\xi(s) &= \int_0^1 y^{\frac{s}{2}-1}\omega(y)\,\mathrm{d}y+\int_1^\infty y^{\frac{s}{2}-1}\omega(y)\,\mathrm{d}y\\
&= \int_1^\infty y^{-\frac{s}{2}-1}\omega\left(\frac{1}{y}\right)\mathrm{d}y+\int_1^\infty y^{\frac{s}{2}-1}\omega(y)\,\mathrm{d}y\\
&= \int_1^\infty y^{-\frac{s}{2}-1}\left(\sqrt{y}\omega(y)+\frac{\sqrt{y}}{2}-\frac{1}{2}\right)\mathrm{d}y+\int_1^\infty y^{\frac{s}{2}-1}\omega(y)\,\mathrm{d}y\\
&= \int_1^\infty y^{-\frac{s}{2}-1}\left(\frac{\sqrt{y}}{2}-\frac{1}{2}\right)\mathrm{d}y+\int_1^\infty\left(y^{-\frac{s}{2}-\frac{1}{2}}+y^{\frac{s}{2}-1}\right)\omega(y)\,\mathrm{d}y,\\
\xi(s) &= \frac{1}{2}+\frac{s(s-1)}{2}\int_1^\infty\left(y^{-\frac{s}{2}-\frac{1}{2}}+y^{\frac{s}{2}-1}\right)\omega(y)\,\mathrm{d}y \qquad (3.3)
\end{aligned}$$

for $\operatorname{Re}(s)>1$. The right hand side is properly defined on $\mathbb{C}\setminus\{1\}$, because $\omega(y)=\mathrm{O}(\mathrm{e}^{-\pi y})$ as $y\to\infty$. Furthermore, this equation is invariant under the change of variables $s\mapsto 1-s$, therefore $\xi(s)=\xi(1-s)$. □

So, we have, combining (3.1) with (3.2), the following relation for the Riemann zeta function

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \qquad (3.4)$$

and (3.3) in combination with (3.2) gives

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left(\frac{1}{s(s-1)}+\int_1^\infty\left(x^{-\frac{s}{2}-\frac{1}{2}}+x^{\frac{s}{2}-1}\right)\omega(x)\,\mathrm{d}x\right). \qquad (3.5)$$

One can see from this equation that $\zeta(s)$ is properly defined on $\mathbb{C}\setminus\{0,1\}$. Let's take a closer look at the points $s=0$ and $s=1$.

First, we take a look at $s=0$. We take the limit of (3.5) as $s\to 0$. Because $1/\Gamma(s)=s+\mathrm{o}(s)$ as $s\to 0$, it holds that

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)}\simeq\frac{s}{2},$$

as $s\to 0$. Substituting this into (3.5) yields the following limit

$$\begin{aligned}
\lim_{s\to 0}\zeta(s) &= \lim_{s\to 0}\frac{\pi^{\frac{s}{2}}s}{2}\left(\frac{1}{s(s-1)}+\int_1^\infty\left(x^{-\frac{s}{2}-\frac{1}{2}}+x^{\frac{s}{2}-1}\right)\omega(x)\,\mathrm{d}x\right)\\
&= \frac{\pi^{\frac{s}{2}}}{2}\left(\frac{1}{s-1}+s\int_1^\infty\left(x^{-\frac{s}{2}-\frac{1}{2}}+x^{\frac{s}{2}-1}\right)\omega(x)\,\mathrm{d}x\right)\Bigg|_{s=0}\\
&= \frac{1}{2}(-1+0)=-\frac{1}{2}.
\end{aligned}$$

And now we take a look at $s=1$. Evaluating (3.4) at this point we obtain the following relation

$$\zeta(1)=\frac{\pi^{\frac{1}{2}}\,\Gamma(0)\,\zeta(0)}{\Gamma(1/2)}.$$

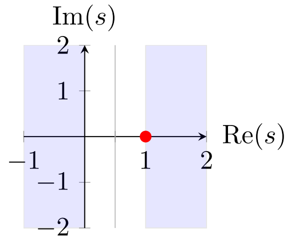
Because $\Gamma(0)$ is a simple pole and all other terms at the right hand side have a finite value, $\zeta(1)$ has to be a simple pole too.

We now have a meromorphical extension of the Riemann zeta function in $\mathbb{C}$, since we can set $\zeta(0) = -1/2$. The only pole of the Riemann zeta function, which is a simple pole, lies at $s = 1$.

We have already proved that there are no zeroes in the plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$. Now we will show what the locations of the zeroes in the plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$ are.

Proposition 3.3. *The only zeroes of the Riemann zeta function in the complex plane where* $\mathrm{Re}(s) < 0$ *are located at all even negative integers. Otherwise stated,* $\zeta(s) \neq 0$ *for* $\mathrm{Re}(s) < 0$, *except in the points where* $s \in 2\mathbb{Z}_{<0}$.

The blue shaded area is the area without non-trivial zeroes.



The symbol "•" represents the simple pole.

Proof. Let us look at $\mathrm{Re}(s) < 0$. All the factors on the right hand side of (3.4) are then non-zero, but the factor $\Gamma(s/2)$ on the left hand side has poles at all $s \in 2\mathbb{Z}_{\leqslant 0}$. This means that, since $\pi^{-s/2}$ is never zero, $\zeta(s)$ has to be zero. So, for $\mathrm{Re}(s) < 0$, $\zeta(s) = 0$ if and only if $s \in 2\mathbb{Z}_{<0}$. Those zeroes are called the *trivial zeroes*. This proves the proposition. □

So, we have already scored out a lot of possible positions of zeroes in the complex plane. But what about the strip $\{s \in \mathbb{C} \mid 0 \leqslant \mathrm{Re}(s) \leqslant 1\}$, can we say something about that?

First, we show that the Riemann zeta function has no zeroes in the interval $0 < s < 1$.

Lemma 3.1. *For* $\mathrm{Re}(s) > 0$ *the Riemann zeta function is equal to*

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,\mathrm{d}x. \tag{3.6}$$

The notation $\lfloor x \rfloor$ means floor x, the least integer greater then or equal to the real number x (ISO 8000-2 clause 9.17).

Proof. By Definition 3.1 we have for $\mathrm{Re}(s) > 1$

Observe that $n = \lfloor x \rfloor$ if $x \in (n, n+1)$, where $n \in \mathbb{N}$.

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty n\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) = \sum_{n=1}^\infty n \left.\frac{-1}{x^s}\right|_{x=n}^{n+1} = s\sum_{n=1}^\infty n\int_n^{n+1}\frac{1}{x^{s+1}}\,\mathrm{d}x\\ &= s\int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}}\,\mathrm{d}x = s\left(\int_1^\infty \frac{1}{x^s}\,\mathrm{d}x - \int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,\mathrm{d}x\right)\\ &= \frac{s}{s-1} - s\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,\mathrm{d}x.\end{aligned}$$

The second step is maybe a little clearer when observing the first few terms:

$$\begin{aligned}\sum_{n=1}^\infty \frac{1}{n^s} &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\\ &= \frac{1}{1^s} + \left(-\frac{1}{2^s} + \frac{2}{2^s}\right) + \left(-\frac{2}{3^s} + \frac{3}{3^s}\right) + \cdots\\ &= 1\left(\frac{1}{1^s} - \frac{1}{2^s}\right) + 2\left(\frac{1}{2^s} - \frac{1}{3^s}\right) + 3\left(\frac{1}{3^s} - \frac{1}{4^s}\right) + \cdots\\ &= \sum_{n=1}^\infty n\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right).\end{aligned}$$

But we see that the right hand side of the first equation is also valid for $\operatorname{Re}(s) > 0$. This proves the lemma. □

From (3.6) we can see that for $0 < s < 1$

$$\left|\zeta(s) - \frac{s}{s-1}\right| = \left|s\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,\mathrm{d}x\right| < \left|s\int_1^\infty \frac{1}{x^{s+1}}\,\mathrm{d}x\right| = \left|\frac{1}{x^s}\Big|_{x=1}^\infty\right| = 1,$$

or equivalently

$$-1 < \zeta(s) - \frac{s}{s-1} < 1.$$

Therefore,

$$\zeta(s) < 1 + \frac{s}{s-1} = \frac{2s-1}{s-1}.$$

And for $1/2 < s < 1$, we have $(2s-1)/(s-1) < 0$, so $\zeta(s) \neq 0$ in the interval $1/2 < s < 1$. From (3.4) we see that the zeroes of $\zeta(s)$ are symmetrically located around the line $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = 1/2\}$ in the region $0 < \operatorname{Re}(s) < 1$, since the factors in front of the Riemann zeta functions cannot be zero in this region. Thus, $\zeta(s)$ has no zero in the interval $0 < s < 1$.

3.3 Properties of the Riemann xi function

The Riemann xi function has some special properties and is often more useful than the Riemann zeta function.

According to ISO 80000-2 clause 14.6 the notations $\overline{s}$ and $s^\star$ both mean the complex conjugate of s, where $\overline{s}$ is meanly used in mathematics and $s^\star$ is meanly used in physics and engineering.

Lemma 3.2. *A holomorphic function f in $\mathbb{C}$ satisfies $f(\overline{s}) = \overline{f(s)}$ for all $s \in \mathbb{C}$ if and only if f is real-valued when restricted to the real values.*

Proof. A holomorphic function can always be expanded as a power series around zero, that is for all $s \in \mathbb{C}$ it holds that

$$f(s) = \sum_{n=0}^\infty a_n s^n,$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} f(\overline{s}) = \overline{f(s)}\ \forall s \in \mathbb{C} \Leftrightarrow \sum_{n=0}^\infty a_n \overline{s}^n = \sum_{n=0}^\infty \overline{a_n s^n}\ \forall s \in \mathbb{C} \Leftrightarrow a_n = \overline{a_n}\ \forall n \in \mathbb{N} \\ \Leftrightarrow a_n \in \mathbb{R}\ \forall n \in \mathbb{N} \Leftrightarrow \sum_{n=0}^\infty a_n s^n \in \mathbb{R}\ \forall s \in \mathbb{R} \Leftrightarrow f(s) \in \mathbb{R}\ \forall s \in \mathbb{R}, \end{aligned}$$

which proves the lemma. □

A list of useful properties.

- We already saw that the Riemann xi function is holomorphic in $\mathbb{C} \setminus \{1\}$. But from (3.2) and the fact that $\zeta(1)$ is a simple pole, we can see that that the Riemann xi function is holomorphic in $\mathbb{C}$, since the simple zero from $s-1$ cancels the simple pole from $\zeta(s)$.
- From (3.1) we can see that the Riemann xi function is symmetric in the complex plane around the line $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = 1/2\}$.

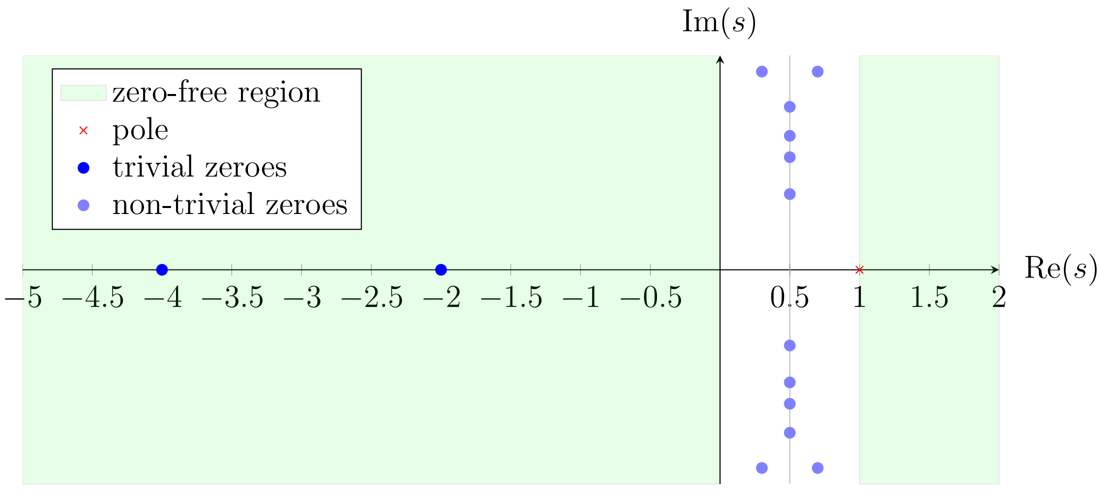


Fig. 3.1: The location of zeroes of the Riemann zeta function.

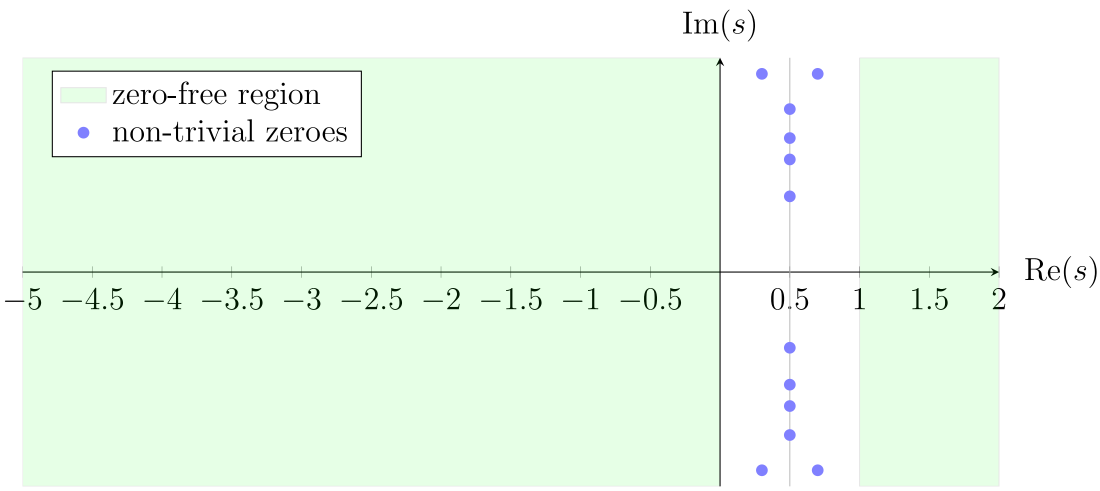
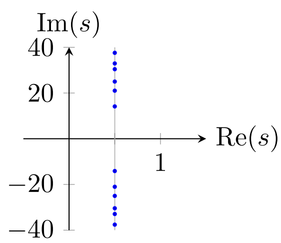


Fig. 3.2: The location of zeroes of the Riemann xi function.

- By (3.4), the trivial zeroes of $\zeta(s)$ are cancelled by $\Gamma(s/2)$. Therefore, the zeroes of the Riemann xi function are the non-trivial zeroes of the Riemann zeta function.

- From (3.6), it is clear that the restriction of the Riemann xi function to the real numbers is real-valued, and therefore satisfies $\xi(\overline{s}) = \overline{\xi(s)}$. As result of this, the zeroes of the Riemann xi function, and by (3.2) also of the Riemann zeta function, are located symmetrically around the line $\{s \in \mathbb{C} \mid \mathrm{Im}(s) = 0\}$, the real axis.



The first six non-trivial zeroes of the Riemann xi and zeta function.

3.4 The counting function

We need a function which counts the number of zeroes on the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1/2\}$ of the Riemann zeta function between 0 and a given height it.

Definition 3.4 (Counting function). We define the *counting function*, $N\colon \{T \in \mathbb{R} \mid T > 0\} \to \mathbb{Q}$, by

$$N(T) := \sum_{n=1}^{\infty} \theta(t - t_n),$$

where $\theta\colon \mathbb{R} \to \mathbb{Q}$ is the *Heaviside step function*, also knows as the *unit step function*, given by

$$\theta(x) := \begin{cases} 0 & \text{if } x < 0; \\ \frac{1}{2} & \text{if } x = 0; \\ 1 & \text{if } x > 0, \end{cases}$$

and t_n is the nth zero of the Riemann zeta function above the real line.

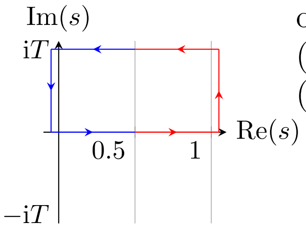
We can use Theorem 2.2 to calculate the number of zeroes in the region $R = \{s \in \mathbb{C} \mid -\epsilon \leqslant \mathrm{Re}(s) \leqslant 1 + \epsilon, 0 \leqslant \mathrm{Im}(s) \leqslant T\}$. To calculate this number, we will use the Riemann xi function instead of the Riemann zeta function, for the following reasons. The Riemann xi function is

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}}\,\Gamma\!\left(\frac{s}{2}\right)\zeta(s).$$

The term $s - 1$ is there to remove the simple pole of $\zeta(s)$ at $s = 1$, the term s makes $\xi(s)$ symmetric and the factor $1/2$ is there for historical reasons. This makes the Riemann xi function a holomorphic function in the entire complex plane. Furthermore, it has the same number of zeroes in R as the Riemann zeta function when $0 \leqslant \epsilon < 1$. So, the number of zeroes of the Riemann zeta function in R is given by

$$N(T) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial R} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s,$$

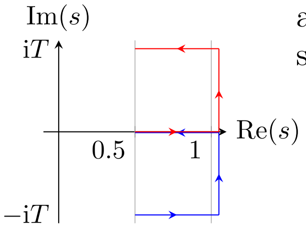
where ∂R is the boundary of R oriented in the usual counterclockwise direction. We can split the contour ∂R up into two curves γ_1 and γ_2. The curve γ_1 consists of three straight line segments between the points $(0.5, \mathrm{i}T)$, $(-\epsilon, \mathrm{i}T)$, $(-\epsilon, 0)$, and $(0.5, 0)$. The curve γ_2 consists of three straight line segments between the points $(0.5, 0)$, $(1+\epsilon, 0)$, $(1+\epsilon, \mathrm{i}T)$, and $(0.5, \mathrm{i}T)$. And now we have



The curves γ_1 (blue) and γ_2 (red).

$$\begin{aligned} N(T) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_2} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s \\ &= -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} \frac{\xi'(1-s)}{\xi(1-s)}\,\mathrm{d}s + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_2} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_2} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s, \end{aligned}$$

where in the second step we used (3.1) and in the third step we made a change of variables $1 - s \mapsto s$. When we changed the variables, the curve γ_1 changed to another curve, which we call γ_3. So the curve γ_3 consists of three straight line segments between the points $(0.5, -\mathrm{i}T)$, $(1+\epsilon, -\mathrm{i}T)$, $(1+\epsilon, 0)$, and $(0.5, 0)$.



The curves γ_3 (blue) and γ_2 (red).

By the chain rule we have

$$\frac{\mathrm{d}\ln(\xi(s))}{\mathrm{d}s} = \frac{\xi'(s)}{\xi(s)}.$$

By using the fundamental theorem of calculus (six times) and letting the argument of the logarithm lie in the interval $(-\pi, \pi]$, we can evaluate the integral as follows

$$\begin{aligned}
N(T) &= \frac{1}{2\pi i}\ln(\xi(s))\bigg|_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} = \frac{1}{2\pi i}\ln|\xi(s)|\bigg|_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} + \frac{1}{2\pi}\arg(\xi(s))\bigg|_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \\
&= \frac{1}{2\pi}\arg\left(\xi\left(\frac{1}{2}+iT\right)\right) - \frac{1}{2\pi}\arg\left(\xi\left(\frac{1}{2}-iT\right)\right) = \frac{1}{\pi}\arg\left(\xi\left(\frac{1}{2}+iT\right)\right) \\
&= \frac{1}{\pi}\arg\left(\left(\frac{1}{2}+iT\right)\left(-\frac{1}{2}+iT\right)\right) + \frac{1}{\pi}\arg\left(\pi^{-\frac{1}{4}-\frac{iT}{2}}\right) + \frac{1}{\pi}\arg\left(\Gamma\left(\frac{1}{4}+\frac{iT}{2}\right)\right) \\
&\quad + \frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+iT\right)\right) \\
&= 1 - \frac{T}{2\pi}\ln(\pi) + \frac{1}{\pi}\arg\left(\Gamma\left(\frac{1}{4}+\frac{iT}{2}\right)\right) + \frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+iT\right)\right).
\end{aligned}$$

To computation $\arg(\Gamma(1/4 + iT/2))$, we use the expansion of $\ln(\Gamma(s))$

$$\ln(\Gamma(s)) = \frac{1}{2}\ln(2\pi) + \left(s - \frac{1}{2}\right)\ln(s) - s + \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \cdots$$

which is often called *Stirling's series* and can be derived from the asymptotic series of $\Gamma(s)$. The computation is as follows

$$\begin{aligned}
\arg\left(\Gamma\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)\right) &= \operatorname{Im}\left(\ln\left(\Gamma\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)\right)\right)\\
&= \operatorname{Im}\left(\left(-\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)\ln\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)-\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)+\frac{1}{12\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)}\right.\\
&\quad\left.-\frac{1}{360\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)^3}+\cdots\right)\\
&= \frac{T}{2}\operatorname{Re}\left(\ln\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)\right)-\frac{1}{4}\operatorname{Im}\left(\ln\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)\right)-\frac{T}{2}\\
&\quad+\frac{-\frac{T}{2}}{12\left(\frac{1}{16}+\frac{T^2}{4}\right)}-\frac{\operatorname{Im}\left(\left(\frac{1}{4}-\frac{\mathrm{i}T}{2}\right)^3\right)}{360\left(\frac{1}{16}+\frac{T^2}{4}\right)^3}+\cdots\\
&= \frac{T}{2}\ln\sqrt{\left(\frac{T}{2}\right)^2\left(1+\frac{1}{4T^2}\right)}-\frac{1}{4}\left(\frac{\pi}{2}-\arctan\left(\frac{\frac{1}{4}}{\frac{T}{2}}\right)\right)-\frac{T}{2}\\
&\quad-\frac{1}{6T\left(1+\frac{1}{4T^2}\right)}-\frac{\frac{T^3}{8}+3\left(-\frac{T}{2}\right)\left(-\frac{1}{4}\right)^2}{360\left(\frac{T^2}{4}\right)^2\left(1+\frac{1}{4T^2}\right)^3}+\cdots\\
&= \frac{T}{2}\ln\left(\frac{T}{2}\right)+\frac{T}{4}\ln\left(1+\frac{1}{4T^2}\right)-\frac{\pi}{8}+\frac{1}{4}\arctan\left(\frac{1}{2T}\right)-\frac{T}{2}\\
&\quad-\frac{1}{6T}\left(1+\frac{1}{4T^2}\right)^{-1}-\frac{1}{45T^3}\left(1+\frac{1}{4T^2}\right)^{-3}\\
&\quad+\frac{1}{60T^5}\left(1+\frac{1}{4T^2}\right)^{-3}+\cdots\\
&= \frac{T}{2}\ln\left(\frac{T}{2}\right)+\frac{T}{4}\left(\frac{1}{4T^2}-\frac{1}{2}\left(\frac{1}{4T^2}\right)^2+\cdots\right)-\frac{\pi}{8}\\
&\quad+\frac{1}{4}\left(\frac{1}{2T}-\frac{1}{3}\left(\frac{1}{2T}\right)^3+\cdots\right)-\frac{T}{2}-\frac{1}{6T}\left(1-\frac{1}{4T^2}+\cdots\right)\\
&\quad-\frac{1}{45T^3}\left(1-\frac{3}{4T^2}+\cdots\right)+\frac{1}{60T^5}\left(1+\frac{1}{4T^2}\right)^{-3}+\cdots
\end{aligned}$$

This gives finally

$$\arg\left(\Gamma\left(\frac{1}{4}+\frac{\mathrm{i}T}{2}\right)\right)=\frac{T}{2}\ln\left(\frac{T}{2}\right)-\frac{T}{2}-\frac{\pi}{8}+\mathrm{O}\left(\frac{1}{T}\right).$$

When we substitute this into our obtained formula for the counting function, we get

$$N(T)=\frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right)-\frac{T}{2\pi}+\frac{7}{8}+\mathrm{O}\left(\frac{1}{T}\right)+\frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+\mathrm{i}T\right)\right).$$

Often, this counting function function is separated into a smooth part and a fluctuating part

$$N(T)=\langle N(T)\rangle+N_{\mathrm{fl}}(T), \tag{3.7}$$

where

$$\langle N(T)\rangle:=\frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right)-\frac{T}{2\pi}+\frac{7}{8}+\mathrm{O}\left(\frac{1}{T}\right), \tag{3.8}$$

is the smooth part and

$$N_{\mathrm{fl}}(T) := \frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+\mathrm{i}T\right)\right) \tag{3.9}$$

is the fluctuating part.

To rewrite (3.9) into a different form, we use the Euler product formula, (3.2), disregarding the fact that this does not converges at these values, and the Taylor expansion of the function $\ln(1+s)$ around the point $s=0$, which is given by

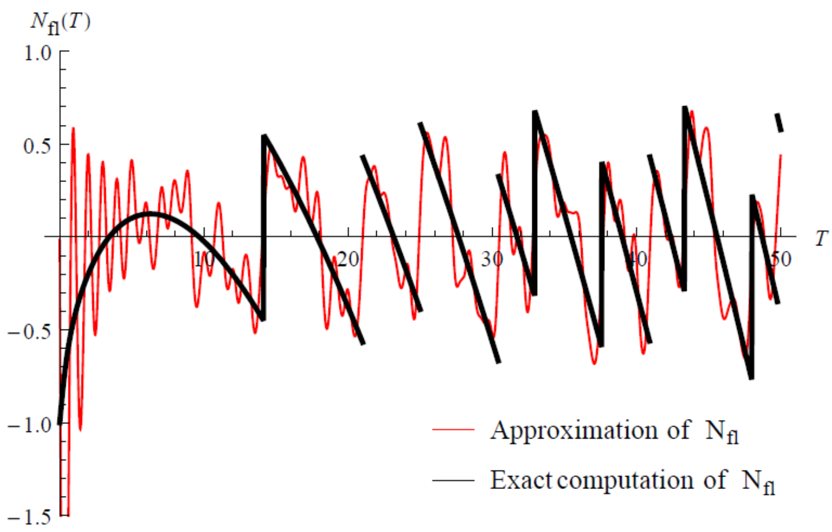
$$\ln(1+s) = \sum_{m=0}^{\infty}\frac{\ln^{(m)}(1)}{m!}s^m = s - \frac{1}{2}s^2 + \frac{1}{3}s^3 - \cdots = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}s^m.$$

This series converges whenever $|s|<1$. Now, the fluctuating part becomes

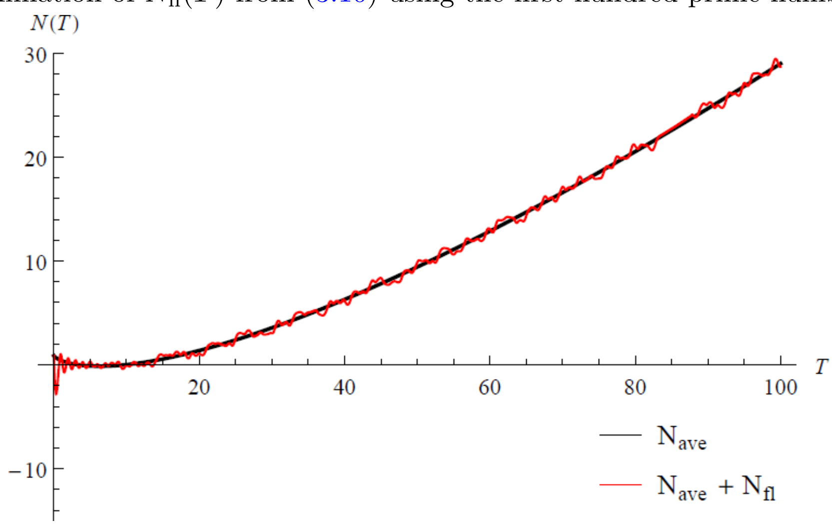
$$\begin{aligned}
N_{\mathrm{fl}}(T) &= \frac{1}{\pi}\arg\left(\prod_{p\in\mathbb{P}}\frac{1}{1-p^{-\frac{1}{2}-\mathrm{i}T}}\right) = -\frac{1}{\pi}\operatorname{Im}\sum_{p\in\mathbb{P}}\ln\left(1-p^{-\frac{1}{2}-\mathrm{i}T}\right)\\
&= -\frac{1}{\pi}\sum_{p\in\mathbb{P}}\operatorname{Im}\left(\ln\left(1-\mathrm{e}^{\left(-\frac{1}{2}-\mathrm{i}T\right)\ln(p)}\right)\right)\\
&= -\frac{1}{\pi}\sum_{p\in\mathbb{P}}\operatorname{Im}\left(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left(-\mathrm{e}^{\left(-\frac{1}{2}-\mathrm{i}T\right)\ln(p)}\right)^m\right)\\
&= -\frac{1}{\pi}\sum_{p\in\mathbb{P}}\sum_{m=1}^{\infty}\frac{(-1)^{2m+1}\mathrm{e}^{-\frac{1}{2}m\ln(p)}}{m}\operatorname{Im}\left(-\mathrm{e}^{-\mathrm{i}mT\ln(p)}\right)\\
&= -\frac{1}{\pi}\sum_{p\in\mathbb{P}}\sum_{m=1}^{\infty}\frac{-\mathrm{e}^{-\frac{1}{2}m\ln(p)}}{m}\sin(-mT\ln(p))
\end{aligned}$$

to conclude with

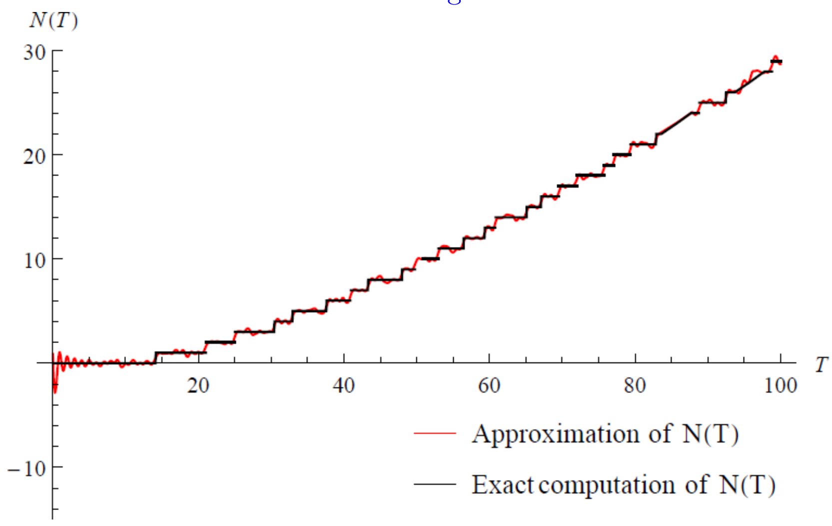
$$N_{\mathrm{fl}}(T) = -\frac{1}{\pi}\sum_{p\in\mathbb{P}}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-\frac{1}{2}m\ln(p)}}{m}\sin(mT\ln(p)). \tag{3.10}$$



(a) The black line is the exact computation of $N_{fl}(T)$ from (3.9). The red line is the approximation of $N_{fl}(T)$ from (3.10) using the first hundred prime numbers.



(b) The black line is the approximation of $\langle N(T)\rangle$ from (3.8). The red line is the sum of this black line and the red line from Figure 3.3a.



(c) The black line is the exact computation of $N(T)$. The red line is the same as the red line from Figure 3.3b.

Fig. 3.3: The counting function, $N(T)$, of the number of zeroes of the Riemann zeta function.

4. PHYSICS

4.1 Classical mechanics

4.1.1 Lagrange's equation of motion

Definition 4.1 (Lagrangian). If the kinetic energy of a particle is denoted by T and the potential energy of that particle by U, then the *Lagrangian* is defined by

$$L(x, \dot{x}) := T(\dot{x}) - U(x),$$

where x is the position of the particle and $\dot{x}$ is the derivative of the position of the particle with respect to time, called the velocity.

The time is denoted by t. Both position and velocity are time-dependent, that is, $x = x(t)$ and $\dot{x} = \dot{x}(t)$. The Lagrangian can be generalized to a more particle system by summing over all particles. *Hamilton's principle* states that the path a particle follows from $x(t_1)$ to $x(t_2)$ is that one that minimizes the time integral of the Lagrangian:

$$\delta \int_{t_1}^{t_2} L(x, \dot{x}) \, \mathrm{d}t = 0.$$

Note that this path starts at $x(t_1)$ and ends at $x(t_2)$. Actually this equation corresponds to an extremum, but luckily in most cases this is equal to a minimum. One may interchange integration and differentiation by the Leibniz rule. So

$$\begin{aligned} 0 = \delta \int_{t_1}^{t_2} L(x, \dot{x}) \, \mathrm{d}t = \int_{t_1}^{t_2} \delta L(x, \dot{x}) \, \mathrm{d}t = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) \mathrm{d}t \\ = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} \delta x - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} \delta x \right) \mathrm{d}t + \frac{\partial L}{\partial \dot{x}} \delta x \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} \right) \delta x \, \mathrm{d}t, \end{aligned} \tag{4.1}$$

where in the fourth step we used partial integration. The integrated term vanishes because the begin and endpoints of all possible paths are equal by definition, and thus the differences between the paths are zero: $\delta x(t_1) = 0$ and $\delta x(t_2) = 0$. Because the integral in (4.1) must vanish for all possible paths, the integrand itself must vanish:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} = 0. \tag{4.2}$$

This equation is knowns as *Euler's equation* or, when applied to mechanical systems, as the *Euler-Lagrange equation*.

One can show that Lagrange's equation of motion (4.2) is equivalent to Newton's equation of motion

$$-\frac{\partial U}{\partial x} =: F = m\ddot{x}.$$

4.1.2 Hamilton's equations of motion

When the Lagrangian is denoted by L, the *momentum* is defined by

$$p := \frac{\partial L}{\partial \dot{x}}$$

where x is the position of the particle and $\dot{x}$ is the derivative of the position of the particle with respect to time, the velocity. The time is denoted by t. Both position and velocity are time-dependent, that is, $x = x(t)$ and $\dot{x} = \dot{x}(t)$. The momentum can be generalized to a more particle system by summing over all particles. We can use the definition of momentum to express $\dot{x}$ in terms of x and p, that is, $\dot{x} = \dot{x}(x, p)$. Using (4.2), we can see that the time derivative of p is given by

$$\dot{p} = \frac{\partial L}{\partial x}.$$

Definition 4.2 (Hamiltonian). The *Hamiltonian* is defined by

$$H(x, p) := p\dot{x} - L(x, \dot{x}),$$

which we observe as a function of only x and p, by using $\dot{x} = \dot{x}(x, p)$.

Taking the total differential of the left hand side of the definition of the Hamiltonian, gives

$$\mathrm{d}H = \frac{\partial H}{\partial x}\,\mathrm{d}x + \frac{\partial H}{\partial p}\,\mathrm{d}p. \tag{4.3}$$

The total differential of the right hand side is

$$\begin{aligned} \mathrm{d}H = \mathrm{d}(p\dot{x}) - \mathrm{d}L(x, \dot{x}) &= \dot{x}\,\mathrm{d}p + p\,\mathrm{d}\dot{x} - \frac{\partial L}{\partial x}\,\mathrm{d}x - \frac{\partial L}{\partial \dot{x}}\,\mathrm{d}\dot{x} \\ &= \dot{x}\,\mathrm{d}p + p\,\mathrm{d}\dot{x} - \dot{p}\,\mathrm{d}x - p\,\mathrm{d}\dot{x} = -\dot{p}\,\mathrm{d}x + \dot{x}\,\mathrm{d}p. \end{aligned} \tag{4.4}$$

After identifying the coefficients of $\mathrm{d}x$ and $\mathrm{d}p$ in (4.3) and (4.4), we find

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}, \\ \dot{p} = -\dfrac{\partial H}{\partial x}. \end{cases} \tag{4.5}$$

Those equations are called *Hamilton's equations of motion*. By construction, they are equivalent to Lagrange's equation of motion.

4.2 Quantum mechanics

If $\Psi = \Psi(x,t)$ is the *wave function* of a particle, then the *Schrödinger equation* reads

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U\Psi,$$

where x is the position and m the mass of the particle, t is the time and U the potential energy.

For a particle in the state $\Psi = \Psi(x,t)$, the expectation value of the observable for position x is

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} \Psi^\star x\Psi\,\mathrm{d}x.$$

The expectation value for momentum p is

$$\begin{aligned}
\langle p\rangle &= \left\langle m\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle = m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = m\int_{-\infty}^{\infty} x\frac{\partial}{\partial t}|\Psi|^2\,\mathrm{d}x = m\int_{-\infty}^{\infty} x\frac{\partial}{\partial t}\Psi^\star\Psi\,\mathrm{d}x\\
&= m\int_{-\infty}^{\infty} x\left(\frac{\partial\Psi^\star}{\partial t}\Psi + \Psi^\star\frac{\partial\Psi}{\partial t}\right)\mathrm{d}x = \frac{\mathrm{i}\hbar}{2}\int_{-\infty}^{\infty} x\left(-\frac{\partial^2\Psi^\star}{\partial x^2}\Psi + \Psi^\star\frac{\partial^2\Psi}{\partial x^2}\right)\mathrm{d}x\\
&= \frac{\mathrm{i}\hbar}{2}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left(-\frac{\partial\Psi^\star}{\partial x}\Psi + \Psi^\star\frac{\partial\Psi}{\partial x}\right)\mathrm{d}x = \frac{\mathrm{i}\hbar}{2}\int_{-\infty}^{\infty}\left(\frac{\partial\Psi^\star}{\partial x}\Psi - \Psi^\star\frac{\partial\Psi}{\partial x}\right)\mathrm{d}x\\
&= \frac{\hbar}{\mathrm{i}}\int_{-\infty}^{\infty}\Psi^\star\frac{\partial\Psi}{\partial x}\,\mathrm{d}x = \int_{-\infty}^{\infty}\Psi^\star\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}\Psi\,\mathrm{d}x,
\end{aligned}$$

where in the eighth and ninth steps we used integration by parts, the integrated term vanishes because the wave function Ψ goes to zero at $\pm\infty$. We say that the *operator* x represents position, and we denote this operator by

$$\widehat{x} := x,$$

and the operator $(\hbar/\mathrm{i})(\partial/\partial x)$ represents momentum, and this operator is denoted by

$$\widehat{p} := \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}.$$

The *expectation value* of an observable $H(x,p)$ is expressed as

$$\langle H(x,p)\rangle = \int_{-\infty}^{\infty}\psi^\star\widehat{H}(\widehat{x},\widehat{p})\psi\,\mathrm{d}x =: \left\langle\psi,\widehat{H}(\widehat{x},\widehat{p})\psi\right\rangle.$$

An operator $\widehat{Q}$, constructed from the observable $Q(x,p)$, is called *hermitian* if

$$\left\langle\psi,\widehat{Q}\psi\right\rangle = \left\langle\widehat{Q}\psi,\psi\right\rangle$$

for all $\psi(x)$.

4.3 The Hilbert-Pólya conjecture

From the letter in the Introduction, we know George Pólya suggested a physical approach to prove the Riemann hypothesis.

Conjecture 4.1 (Hilbert-Pólya conjecture). If

$$\rho_n = \frac{1}{2} + \mathrm{i}t_n$$

are the non-trivial zeroes of the Riemann zeta function, then the t_n's correspond to the eigenvalues of a hermitian operator.

Also David Hilbert is associated with this conjecture, although there is no evidence he conjectured it. When we formulate this into a conjecture, we get the following.

Theorem 4.1. *The Hilbert-Pólya conjecture, Conjecture 4.1, implies the Riemann hypothesis, Conjecture 3.1.*

Proof. Note that if $\rho_n = 1/2 + \mathrm{i}t_n$ are the non-trivial zeroes of the Riemann zeta function, then the t_n's do not necessarily have to be real. Suppose $t_n = a_n + \mathrm{i}b_n$, then

$$\rho_n = \frac{1}{2} + \mathrm{i}t_n = \frac{1}{2} + \mathrm{i}(a_n + \mathrm{i}b_n) = \frac{1}{2} - b_n + \mathrm{i}a_n.$$

So ρ_n can be indeed any non-trivial zero. Now suppose the Hilbert-Pólya conjecture is true. That is, the t_n's correspond to the eigenvalues of a hermitian operator, say $\widehat{H}$. That means that if a t_n is a eigenvalues corresponding to the eigenfunction ψ_n of such an operator then

According to ISO 80000-2 clause 14.6 the notations $\overline{s}$ and $s^\star$ both mean the complex conjugate of s, where $\overline{s}$ is meanly used in mathematics and $s^\star$ is meanly used in physics and engineering.

$$\begin{aligned} t_n\langle\psi_n,\psi_n\rangle &= \langle t_n\psi_n,\psi_n\rangle = \left\langle \widehat{H}\psi_n,\psi_n\right\rangle = \left\langle \psi_n,\widehat{H}\psi_n\right\rangle \\ &= \left\langle \widehat{H}\psi_n,\psi_n\right\rangle^\star = \langle t_n\psi_n,\psi_n\rangle^\star = t_n^\star\langle\psi_n,\psi_n\rangle^\star = t_n^\star\langle\psi_n,\psi_n\rangle, \end{aligned}$$

so $t_n = t_n^\star$, therefore t_n has to be real. That means the non-trivial zeroes have the form $\rho = 1/2 + \mathrm{i}t$, where t is real. And that is exactly what the Riemann hypothesis states. □

4.4 Equivalences between the fluctuating parts of the counting functions

Our goal is to find a hermitian operator. This hermitian operator can be obtained by quantizing a Hamiltonian of a dynamical system which *spectrum*, that is, the set of eigenvalues, matches the non-trivial zeroes of the Riemann zeta function. A necessary condition for this is that the counting function of the number of states of an "energy" between 0 and $E \geqslant 0$, denoted as $\mathcal{N}(E)$, equals the counting function of the number of non-trivial zeroes with imaginary part between 0 and $T \geqslant 0$, which we denoted by $N(T)$, see (3.7).

In mathematical physics the counting function $\mathcal{N}(E)$ of the number of states with an energy between 0 and E always contains a fluctuating part. That means $\mathcal{N}(E)$ can be written as

$$\mathcal{N}(E) = \langle\mathcal{N}(E)\rangle + \mathcal{N}_{\mathrm{fl}}(E).$$

Tab. 4.1: Correspondences between the fluctuating parts of the "Riemann" (3.10) and "Quantum" (4.7) counting functions.

	Riemann (N_{fl})	Quantum ($\mathcal{N}_{\text{fl}}$)
Dimensionless actions	$mT\ln(p)$	$mS_p(E)/\hbar$
Periods	$m\ln(p)$	mT_p
Stabilities	$\ln(p)/2$	$\lambda_p T_p/2$
Asymptotics	$T\to\infty$	$\hbar\to 0$

The fluctuating part is given by the *Gutzwiller formula*, Berry and Keating (1999),

$$\mathcal{N}_{\text{fl}}(E) \simeq \frac{1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{\sin(mS_p(E)/\hbar - \pi m\mu_p/2)}{m\sqrt{\left|\det\left(\boldsymbol{M}_p^m - \boldsymbol{I}\right)\right|}}, \tag{4.6}$$

where the index p labels the primitive periodic orbits (the smallest periodic orbits of a given set of initial conditions) and the index m labels their repetitions. The term $S_p(E)$ is the action of primitive periodic orbit p, and given by

$$S_p(E) = \oint_p \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{p}$ is the momentum vector and $\boldsymbol{x}$ is the position vector. The period of orbit p is

$$T_p = \frac{\partial S_p}{\partial E}.$$

The matrix $\boldsymbol{M}_p$ denotes the monodromy matrix and the symbol μ_p is the Maslov phase. For large T_p it holds that

$$\det\left(\boldsymbol{M}_p^m - \boldsymbol{I}\right) \simeq \mathrm{e}^{m\lambda_p T_p},$$

where λ_p is the Liapunov instability exponent of the orbit p. So, substituting this back into the Gutzwiller formula (4.6), gives

$$\begin{aligned}\mathcal{N}_{\text{fl}}(E) &\simeq \frac{1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{\sin(mS_p(E)/\hbar - \pi m\mu_p/2)}{m\mathrm{e}^{\frac{1}{2}m\lambda_p T_p}}\\ &= \frac{1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-\frac{1}{2}m\lambda_p T_p}}{m}\sin\left(\frac{mS_p(E)}{\hbar} - \frac{1}{2}\pi m\mu_p\right). \end{aligned}\tag{4.7}$$

Now we can equate (3.10) with (4.7) and observe four equivalences, which are represented in Table 4.1. From the periods and the stabilities we can see that $\lambda_p = 1$.

Berry and Keating (1999) summarized the known properties of the proposed Hamiltonian.

1. The Hamiltonian H has a classical counterpart called the *Riemann dynamics*, corresponding to a hamiltonian flow, or a symplectic transformation, in a phase space.
   This is based on the resemblance between (3.10) and (4.7). In (4.7) we sum over classical periodic orbits.

2. The Riemann dynamics is chaotic, that is, unstable and bounded.
   This is due to the fact that all orbits p have the same instability exponent, the Liapunov instability exponent $\lambda_p = 1$.

3. The Riemann dynamics does not have time-reversal symmetry.
   Because the prefactor in (3.10) is $1/\pi$ instead of $2/\pi$.

4. The Riemann dynamics is homogeneously unstable.
   This is also due to the fact that all orbits p have the same Liaponov instability exponent $\lambda_p = 1$.

5. The classical periodic orbits of the Riemann dynamics have periods that are independent of "energy" E, and given by multiples of logarithms of prime numbers.
   This is the observation that the period is given by $m \ln(p)$, where $m \in \mathbb{N}$ and $p \in \mathbb{P}$. And this is independent of T.

6. The Maslov phases associated with the orbits are all π.
   The negative sign in (3.10) can only be obtain when the Maslov phases $\pi m \mu_p/2$ in (4.7) are of the form $\pi + 2\pi k$, where $k \in \mathbb{Z}$. Restricted to the interval $[0, 2\pi)$ their value is π for all orbits.

7. The Riemann dynamics possesses complex periodic orbits whose periods are multiples of $\mathrm{i}\pi$.

8. For the Riemann operator, leading order semi-classical mechanics is exact: $\zeta(1/2 + \mathrm{i}T)$ is a product over classical periodic orbits, without corrections.

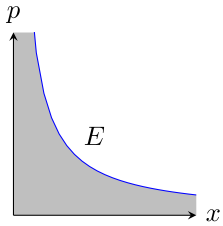
9. The Riemann dynamics is quasi-one dimensional.
   There are two indications of this. First, the number of zeroes less than T increases as $T \ln(T)$; for a N-dimensional scaling system, with energy parameter $\alpha(E)$ proportional to $1/\hbar$, the number of energy levels increases as $\alpha(E)^N$. Second, the presence of the factor $p^{-m/2}$ in (3.10), rather than the determinant in the more general Gutzwiller formula (4.6), suggests that there is a single expanding direction and no contracting direction.

10. The functional equation for $\zeta(s)$ resembles the corresponding relation — a consequence of hermiticity — for the quantum spectral determinant.

4.5 The Hamiltonian $H = xp$

4.5.1 The equations of motion

Let us consider the Hamiltonian $H = xp$, where x is the position of a particle and p is the conjugate momentum of that particle moving in one dimension. This Hamiltonian is also considered by Sierra and Townsend (2008). Classically, the trajectories of this particle are given by Hamilton's equations of motion (4.5):



The level curve $0 < E = xp$ in the first quadrant. The area under this curve is infinite.

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = x, \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -p, \end{cases}$$

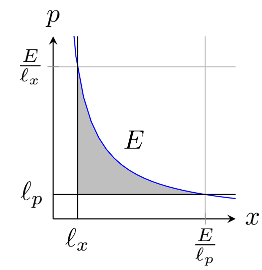
From the first equation of motion, $\dot{x} = x$, we can see that the orbit does not have time-reversal symmetry, as required by the third property of the ten properties in section 4.4. By separation of variables, it follows that

$$\begin{cases} \displaystyle\int_{x_0}^{x} \frac{1}{x'}\,\mathrm{d}x' = \int_{t_0}^{t} \mathrm{d}t', \\ \displaystyle\int_{p_0}^{p} \frac{1}{p'}\,\mathrm{d}p' = -\int_{t_0}^{t} \mathrm{d}t', \end{cases}$$

which has as solution

$$\begin{cases} x(t) = x_0 \mathrm{e}^{t-t_0}, \\ p(t) = p_0 \mathrm{e}^{-(t-t_0)}. \end{cases}$$

As required by the second property, this system is uniformly unstable, because all trajectories tend away from the origin if $E \neq 0$. But, these trajectories are unbounded, which is not allowed by the second property of the ten properties in section 4.4. Moreover, having unbounded trajectories means that we have a continuous spectrum. That is not what we want, we want a discrete spectrum such that we can compare it with the discrete non-trivial zeroes of the Riemann zeta function. But we know that we can get a discrete spectrum by going to the small scales, the quantum level. Therefore, we introduce a minimal position and a minimal momentum such that the system is bounded and gives a discrete spectrum. We impose the conditions $|x| \geqslant \ell_x$ and $|p| \geqslant \ell_p$, where ℓ_x is the minimal position and ℓ_p is the minimal momentum. The product of the minimal position and the minimal momentum is defined by $\ell_x \ell_p := 2\pi\hbar$, and is called the *Planck quantum*.



The truncated level curve $0 < E = xp$ in the first quadrant. The area under this curve is finite.

4.5.2 A corresponding hermitian operator

We want to check whether the eigenvalues of a hermitian operator for the Hamiltonian $H = xp$ with boundary conditions $|x| \geqslant \ell_x$ and $|p| \geqslant \ell_p$ correspond to the non-trivial zeroes of the Riemann zeta function. If we can prove this, we are done, because we have a proof of the Hilbert-Pólya conjecture and therefore a proof of the Riemann hypothesis.

Let us look at the operator

$$\widehat{H} = \widehat{x}\widehat{p} = x\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\hbar}{\mathrm{i}}x\frac{\mathrm{d}}{\mathrm{d}x}.$$

To use this operator for a proof of the Hilbert-Pólya conjecture, and therefore of the Riemann hypothesis, this operator has to be hermitian. Let us check this.

Remember that to be hermitian, $\widehat{H}$ must satisfy $\left\langle \psi, \widehat{H}\psi \right\rangle = \left\langle \widehat{H}\psi, \psi \right\rangle$. But it does not, because

$$\begin{aligned}
\left\langle \psi, \widehat{H}\psi \right\rangle &= \left\langle \psi, \frac{\hbar}{\mathrm{i}} x \frac{\mathrm{d}}{\mathrm{d}x}\psi \right\rangle = \int_{\ell_x}^{\infty} \psi^\star \frac{\hbar}{\mathrm{i}} x \frac{\mathrm{d}\psi}{\mathrm{d}x}\,\mathrm{d}x = \frac{\hbar}{\mathrm{i}} \int_{\ell_x}^{\infty} \psi^\star x \frac{\mathrm{d}\psi}{\mathrm{d}x}\,\mathrm{d}x \\
&= \frac{\hbar}{\mathrm{i}} \int_{\ell_x}^{\infty} x \left(\frac{\mathrm{d}\psi^\star\psi}{\mathrm{d}x} - \frac{\mathrm{d}\psi^\star}{\mathrm{d}x}\psi \right) \mathrm{d}x \\
&= \frac{\hbar}{\mathrm{i}} \int_{\ell_x}^{\infty} x \left(\frac{\mathrm{d}\psi^\star\psi}{\mathrm{d}x} \right) \mathrm{d}x - \frac{\hbar}{\mathrm{i}} \int_{\ell_x}^{\infty} x \frac{\mathrm{d}\psi^\star}{\mathrm{d}x}\psi\,\mathrm{d}x \neq -\frac{\hbar}{\mathrm{i}} \int_{\ell_x}^{\infty} x \frac{\mathrm{d}\psi^\star}{\mathrm{d}x}\psi\,\mathrm{d}x \\
&= \int_{\ell_x}^{\infty} \left(\frac{\hbar}{\mathrm{i}} x \frac{\mathrm{d}\psi}{\mathrm{d}x} \right)^\star \psi\,\mathrm{d}x = \left\langle \frac{\hbar}{\mathrm{i}} x \frac{\mathrm{d}}{\mathrm{d}x}\psi, \psi \right\rangle = \left\langle \widehat{H}\psi, \psi \right\rangle,
\end{aligned}$$

for general ψ.

But we can make an operator which is hermitian and corresponds to the Hamiltonian $H = xp$. Just take the simple symmetric operator

$$\widehat{H} = \frac{1}{2}(\widehat{x}\widehat{p} + \widehat{p}\widehat{x}) = \frac{1}{2}\left(x \frac{\hbar}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\hbar}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x} x \right) = \frac{\hbar}{\mathrm{i}} \left(x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2} \right).$$

To find the eigenfunction corresponding to the eigenvalue E of this operator, we have to solve

$$\widehat{H}\psi = E\psi.$$

Substitution the operator into this equation gives

$$\begin{aligned}
\frac{\hbar}{\mathrm{i}} \left(x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2} \right) \psi &= E\psi \\
\frac{\hbar}{\mathrm{i}} x \frac{\mathrm{d}\psi}{\mathrm{d}x} &= \left(E - \frac{\hbar}{2\mathrm{i}} \right) \psi \\
\int_{\psi_0}^{\psi} \frac{1}{\psi'}\,\mathrm{d}\psi' &= \int_{x_0}^{x} \left(\frac{\mathrm{i}E}{\hbar} - \frac{1}{2} \right) \frac{1}{x'}\,\mathrm{d}x' \\
\ln\left| \frac{\psi}{\psi_0} \right| &= \left(\frac{\mathrm{i}E}{\hbar} - \frac{1}{2} \right) \ln\left| \frac{x}{x_0} \right| \\
\psi(x) &= \pm \frac{\psi_0}{(x/x_0)^{1/2 - \mathrm{i}E/\hbar}}.
\end{aligned}$$

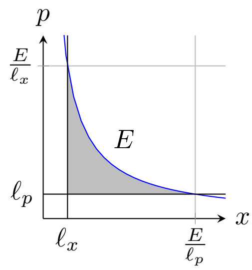
Because

$$\begin{aligned}
\left\langle \psi, \widehat{H}\psi \right\rangle &= \left\langle \psi, \frac{\hbar}{\mathrm{i}} \left(x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2} \right) \psi \right\rangle = \int_{\ell_x}^{\infty} \psi^\star \frac{\hbar}{\mathrm{i}} \left(x \frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{1}{2}\psi \right) \mathrm{d}x \\
&= \frac{\hbar}{\mathrm{i}} \int_{\ell_x}^{\infty} \left(x \left(\frac{\mathrm{d}\psi^\star\psi}{\mathrm{d}x} - \frac{\mathrm{d}\psi^\star}{\mathrm{d}x}\psi \right) + \frac{1}{2}\psi^\star\psi \right) \mathrm{d}x \\
&= \frac{\hbar}{\mathrm{i}} \int_{\ell_x}^{\infty} \left(-x \frac{\mathrm{d}\psi^\star}{\mathrm{d}x}\psi + \frac{\mathrm{d}x\psi^\star\psi}{\mathrm{d}x} - \frac{1}{2}\psi^\star\psi \right) \mathrm{d}x \\
&= \frac{\hbar}{\mathrm{i}} \int_{\ell_x}^{\infty} \left(-x \frac{\mathrm{d}\psi^\star}{\mathrm{d}x}\psi - \frac{1}{2}\psi^\star\psi \right) \mathrm{d}x = \int_{\ell_x}^{\infty} \left(\frac{\hbar}{\mathrm{i}} \left(x \frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{1}{2}\psi \right) \right)^\star \psi\,\mathrm{d}x \\
&= \left\langle \frac{\hbar}{\mathrm{i}} \left(x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2} \right) \psi, \psi \right\rangle = \left\langle \widehat{H}\psi, \psi \right\rangle,
\end{aligned}$$

for general ψ, this operator $\widehat{H}$ is hermitian.

4.5.3 The smooth part of the counting function

We already know that the number of states with an energy between 0 and E can be written as the sum of a smooth part and a fluctuating part, where the fluctuating part is the Gutzwiller formula, (4.6). At the semi-classical level, the smooth part of the number of states between an energy of 0 and E is given by *Weyl's law*, which states that the number of states equals the total volume of phase space divided by the volume of one state, that is the Planck quantum



The truncated level curve $0 < E = xp$ in the first quadrant. The area under this curve is finite.

$$\langle \mathcal{N}(E) \rangle = \frac{V(E)}{2\pi\hbar}.$$

In this case, the volume of phase space is

$$\begin{aligned} V(E) &= \int_{\ell_p}^{E/\ell_x} \int_{\ell_x}^{E/p} \mathrm{d}x\,\mathrm{d}p = \int_{\ell_p}^{E/\ell_x} \left(\frac{E}{p} - \ell_x \right) \mathrm{d}p = E \ln\left(\frac{E}{\ell_x \ell_p} \right) - \ell_x \left(\frac{E}{\ell_x} - \ell_p \right) \\ &= E \ln\left(\frac{E}{\ell_x \ell_p} \right) - E + \ell_x \ell_p = E \ln\left(\frac{E}{2\pi\hbar} \right) - E + 2\pi\hbar. \end{aligned}$$

Therefore, the semi-classical number of states below an energy of E is

$$\langle \mathcal{N}(E) \rangle = \frac{V(E)}{2\pi\hbar} = \frac{E}{2\pi\hbar} \ln\left(\frac{E}{2\pi\hbar} \right) - \frac{E}{2\pi\hbar} + 1.$$

This formula agrees, asymptotically, with the smooth part of the counting function for the non-trivial zeroes of the Riemann zeta function in the strip where the imaginary part lies between 0 and T, (3.8). One could obtain an extra term of $-1/8$ in the form of a Maslov phase by choosing certain boundary conditions for the eigenfunctions. In that case, even the constant term agrees:

$$\mathcal{N}(E) = \frac{E}{2\pi\hbar} \ln\left(\frac{E}{2\pi\hbar} \right) - \frac{E}{2\pi\hbar} + \frac{7}{8}.$$

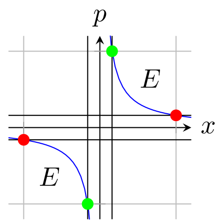
But these boundary conditions are not yet found.

4.5.4 Remarks

The Hamiltonian $H = xp$ has a lot of good properties, but there are still a few problems unsolved.

The truncations to bound the system came out of nowhere. There are no arguments to justify it. The truncations should arise by introducing boundary conditions for the hermitian operator. But no such conditions are found.

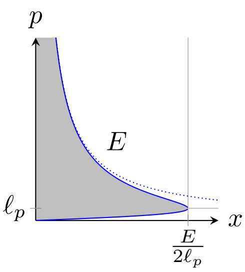
By the fifth property of the ten properties in section 4.4, the system needs to have periodic orbits related to prime numbers. But you cannot have periodic orbits if the trajectories are not closed, which is the case for the Hamiltonian $H = xp$. The trajectory starts at $(x, p) = (\ell_x, E/\ell_x)$ and ends at $(x, p) = (E/\ell_p, \ell_p)$. Berry and Keating (1999) suggest the identifications $x = -x$ and $p = -p$. This indeed closes the trajectories. In this case the trajectory starts at $(x, p) = (\ell_x, E/\ell_x)$ and moves in the first quadrant to $(x, p) = (E/\ell_p, \ell_p)$, then, by the identifications $x = -x$ and $p = -p$, the particle moves in the third quadrant from $(x, p) = (-E/\ell_p, -\ell_p)$ to $(x, p) = (-\ell_x, -E/\ell_x)$, and by the same identifications it ends at $(x, p) = (\ell_x, E/\ell_x)$. However, also for this procedure there are no arguments to justify it. Moreover, the relation to the prime numbers remains still unclear.



The truncated level curve $0 < E = xp$ with the identification $x = -x$ and $p = -p$. The two •'s are identified and the two •'s are identified.

4.6 The Hamiltonian $H = x\bigl(p + \ell_p^2/p\bigr)$

4.6.1 The equations of motion



The level curve $0 < E = x\bigl(p + \ell_p^2/p\bigr)$ in the first quadrant. The area under this curve is infinite.

Because the classical Hamiltonian $H = xp$ gives a continuous spectrum instead of a discrete spectrum, it fails to be a candidate for a Hamiltonian for the Riemann zeta function. A more promising result than the Hamiltonian $H = xp$ is the Hamiltonian

$$H = x\left(p + \frac{\ell_p^2}{p}\right).$$

This Hamiltonian is also considered by Sierra and Rodríguez-Laguna (2011). Let us look again at Hamilton's equations of motion:

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = x\left(1 - \dfrac{\ell_p^2}{p^2}\right), \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -p - \dfrac{\ell_p^2}{p}, \end{cases}$$

were t is the time. To solve those equations, we start with the second equation, because we need that solution for the first one. The second equation is a first-order non-linear ordinary differential equation. We can multiply both side by p to obtain

$$p\frac{\mathrm{d}p}{\mathrm{d}t} + p^2 = -\ell_p^2$$

and now we see that it can be written as a first-order linear differential equation

$$\frac{1}{2}\frac{\mathrm{d}(p^2)}{\mathrm{d}t} + \bigl(p^2\bigr) = -\ell_p^2. \tag{4.8}$$

To solve this differential equation, we need to know the following.

Theorem 4.2. *The solution of the initial value problem where the first-order linear differential equation is given by*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x), \tag{4.9}$$

where $y(x_0) = y_0$ is

$$y(x) = y_0 \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right) + \int_{x_0}^{x} q(x') \exp\left(-\int_{x'}^{x} p(x'')\,\mathrm{d}x''\right)\mathrm{d}x'.$$

Proof. The homogeneous differential equation corresponding to (4.9) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0, \tag{4.10}$$

and is obtained by taking $q(x) = 0$ in (4.9). It can be solved by separation of variables:

$$\int_{y_0}^{y} \frac{1}{y'}\,\mathrm{d}y' = -\int_{x_0}^{x} p(x')\,\mathrm{d}x'$$

$$y(x) = y_0 \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right). \tag{4.11}$$

To find the solution of (4.9), we use *the method of variation of parameters* due to Lagrange. We assume that the constant y_0 in (4.11) is now a function of x, that is

$$y(x) = y_0(x) \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right). \tag{4.12}$$

We now want to find $y_0(x)$ such that this equation becomes a solution of (4.9). Therefore, we substitute (4.12) into (4.9) to obtain

$$\begin{aligned} q(x) &= \frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y \\ &= \frac{\mathrm{d}y_0}{\mathrm{d}x} \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right) - y_0(x)p(x) \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right) \\ &\quad + y_0(x)p(x) \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right) \\ &= \frac{\mathrm{d}y_0}{\mathrm{d}x} \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right), \end{aligned}$$

or equivalently

$$\frac{\mathrm{d}y_0}{\mathrm{d}x} = q(x) \exp\left(\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right). \tag{4.13}$$

Integrating this equation gives the function

$$y_0(x) = \int_{x_0}^{x} q(x') \exp\left(\int_{x_0'}^{x'} p(x'')\,\mathrm{d}x''\right) \mathrm{d}x'.$$

Now, substituting this back into (4.12), we obtain the final solution

$$\begin{aligned} y(x) &= \int_{x_0}^{x} q(x') \exp\left(\int_{x_0'}^{x'} p(x'')\,\mathrm{d}x''\right) \mathrm{d}x' \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right) \\ &= \int_{x_0}^{x} q(x') \exp\left(\int_{x_0'}^{x'} p(x'')\,\mathrm{d}x'' - \int_{x_0}^{x} p(x'')\,\mathrm{d}x''\right) \mathrm{d}x' \\ &= y_0 \exp\left(-\int_{x_0}^{x} p(x')\,\mathrm{d}x'\right) + \int_{x_0}^{x} q(x') \exp\left(-\int_{x'}^{x} p(x'')\,\mathrm{d}x''\right) \mathrm{d}x'. \end{aligned}$$

□

Corollary 4.1. *The solution from Theorem 4.2 in the particular case when* $p(x) = p$ *and* $q(x) = q$ *simply reduces to*

$$y(x) = \left(y_0 - \frac{q}{p}\right) \mathrm{e}^{-p(x-x_0)} + \frac{q}{p}.$$

Proof. We substitute $p(x) = p$ and $q(x) = q$ into the solution of Theorem 4.2 and obtain

$$\begin{aligned} y(x) &= y_0 \exp\left(-\int_{x_0}^{x} p\,\mathrm{d}t\right) + \int_{x_0}^{x} q \exp\left(-\int_{t}^{x} p\,\mathrm{d}s\right) \mathrm{d}t \\ &= y_0 \mathrm{e}^{-p(x-x_0)} + \int_{x_0}^{x} q\mathrm{e}^{-p(x-t)}\,\mathrm{d}t = y_0 \mathrm{e}^{-p(x-x_0)} + \frac{q}{p}\left(1 - \mathrm{e}^{-p(x-x_0)}\right) \\ &= \left(y_0 - \frac{q}{p}\right) \mathrm{e}^{-p(x-x_0)} + \frac{q}{p}. \end{aligned}$$

This proves the corollary. □

So, using Corollary 4.1, the solution of (4.8) is

$$p(t) = \pm\sqrt{\left(p_0^2 + \ell_p^2\right)\mathrm{e}^{-2(t-t_0)} - \ell_p^2}.$$

We can get rid of the t_0 by rescaling the time, $t - t_0 \mapsto t$, such that we get the momentum as function of the time

$$p(t) = \pm\sqrt{\left(p_0^2 + \ell_p^2\right)\mathrm{e}^{-2t} - \ell_p^2}$$

The first equation of Hamilton's equations of motion is dependent of p, we also should remember that p is a function of the time t. So we get

$$\begin{aligned}\int_{x_0}^{x} \frac{1}{x'}\,\mathrm{d}x' &= \int_{t_0}^{t}\left(1 - \frac{\ell_p^2}{p(t')^2}\right)\mathrm{d}t' = \int_{t_0}^{t}\left(1 - \frac{\ell_p^2}{\left(p_0^2 + \ell_p^2\right)\mathrm{e}^{-2t'} - \ell_p^2}\right)\mathrm{d}t' \\ &= t - t_0 + \int_{t_0}^{t} \frac{-\ell_p^2\mathrm{e}^{2t'}}{p_0^2 + \ell_p^2 - \ell_p^2\mathrm{e}^{2t'}}\,\mathrm{d}t'.\end{aligned}$$

After solving the integrals, we get

$$\ln\left|\frac{x}{x_0}\right| = t - t_0 + \frac{1}{2}\ln\left|\frac{p_0^2 + \ell_p^2 - \ell_p^2\mathrm{e}^{2t}}{p_0^2 + \ell_p^2 - \ell_p^2\mathrm{e}^{2t_0}}\right|.$$

Again, by rescaling the time, $t - t_0 \mapsto t$, which is equal to setting t_0 equal to 0, we obtain

$$\ln\left|\frac{x}{x_0}\right| = t + \frac{1}{2}\ln\left|\frac{p_0^2 + \ell_p^2 - \ell_p^2\mathrm{e}^{2t}}{p_0^2}\right|$$

So the position as function of the time is

$$\begin{aligned}x(t) &= x_0\mathrm{e}^{t + \frac{1}{2}\ln\left|\left(p_0^2 + \ell_p^2 - \ell_p^2\mathrm{e}^{2t}\right)/p_0^2\right|} = \frac{x_0}{|p_0|}\mathrm{e}^{t}\sqrt{p_0^2 + \ell_p^2 - \ell_p^2\mathrm{e}^{2t}} \\ &= \frac{x_0}{|p_0|}\mathrm{e}^{2t}\sqrt{\left(p_0^2 + \ell_p^2\right)\mathrm{e}^{-2t} - \ell_p^2}\end{aligned}$$

So, the equations of motion are given by

$$\begin{cases} x(t) = \dfrac{x_0}{|p_0|}\mathrm{e}^{2t}\sqrt{\left(p_0^2 + \ell_p^2\right)\mathrm{e}^{-2t} - \ell_p^2}, \\ p(t) = \pm\sqrt{\left(p_0^2 + \ell_p^2\right)\mathrm{e}^{-2t} - \ell_p^2}. \end{cases}$$

4.6.2 A corresponding hermitian operator

The simplest hermitian operator corresponding to the Hamiltonian $H = x\left(p + \ell_p^2/p\right)$ is

$$\widehat{H} = \sqrt{\widehat{x}}\left(\widehat{p} + \frac{\ell_p^2}{\widehat{p}}\right)\sqrt{\widehat{x}},$$

where the normal ordering prescription is used. Here $1/\widehat{p}$ is the Green's function satisfying

$$\widehat{p}\frac{1}{\widehat{p}} = \boldsymbol{I} = \frac{1}{\widehat{p}}\widehat{p}$$

with matrix elements

$$\left\langle x, \frac{1}{\widehat{p}}x' \right\rangle = \frac{\mathrm{i}}{\hbar}\theta(x - x'),$$

where $\theta(x - x')$ is the *Heaviside step function* and is equal to

$$\theta(x) = \begin{cases} 0 & \text{if } x < x'; \\ \frac{1}{2} & \text{if } x = x'; \\ 1 & \text{if } x > x', \end{cases}$$

The Heaviside step function is often denoted by $H(x)$, but in this case it would be confusing because the Hamiltonian is also denoted by H.

The operator $\widehat{H}$ acts on a wave function ψ as follows

$$\begin{aligned} \widehat{H}\psi(x) &= \left(\sqrt{\widehat{x}}\left(\widehat{p} + \frac{\ell_p^2}{\widehat{p}}\right)\sqrt{\widehat{x}}\right)\psi(x) = \left(\sqrt{\widehat{x}}\widehat{p}\sqrt{\widehat{x}}\right)\psi(x) + \left(\sqrt{\widehat{x}}\frac{\ell_p^2}{\widehat{p}}\sqrt{\widehat{x}}\right)\psi(x) \\ &= \sqrt{x}\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\big(\sqrt{x}\psi(x)\big) + \sqrt{x}\ell_p^2\int_{\ell_x}^{\infty}\frac{\mathrm{i}}{\hbar}\theta(x - x')\sqrt{x'}\psi(x')\,\mathrm{d}x' \\ &= \frac{\hbar}{\mathrm{i}}\sqrt{x}\left(\frac{1}{2\sqrt{x}}\psi(x) + \sqrt{x}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right) + \frac{\mathrm{i}\ell_p^2}{\hbar}\sqrt{x}\int_{\ell_x}^{\infty}\theta(x - x')\sqrt{x'}\psi(x')\,\mathrm{d}x' \\ &= \frac{\hbar}{2\mathrm{i}}\psi(x) + \frac{\hbar}{\mathrm{i}}x\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} + \frac{\mathrm{i}\ell_p^2}{\hbar}\sqrt{x}\int_{\ell_x}^{\infty}\theta(x - x')\sqrt{x'}\psi(x')\,\mathrm{d}x'. \end{aligned}$$

This operator is hermitian if both wave functions satisfy the non-local boundary condition

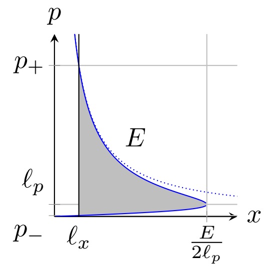
$$\hbar\sqrt{\ell_x}\mathrm{e}^{\mathrm{i}\vartheta}\psi(\ell_x) + \ell_p\int_{\ell_x}^{\infty}\sqrt{x}\psi(x)\,\mathrm{d}x = 0,$$

where $\vartheta \in [0, 2\pi)$. To derive, one has to assume that $\psi(x)$ decays asymptotically faster than $1/\sqrt{x}$, (Sierra and Rodríguez-Laguna, 2011).

4.6.3 The smooth part of the counting function

Because the area under the level curve $E = x\big(p + \ell_p^2/p\big)$ is infinite, we must truncate this curve too. But now we only make a truncation $|x| \geqslant \ell_x$.

To calculate the smooth part of the counting function, we can use Weyl's law again. But in this case, it is a more difficult, because our Hamiltonian is more involved. Let us look at the equation for a level curve



The truncated level curve $0 < E = x\big(p + \ell_p^2/p\big)$ in the first quadrant. The area under this curve is finite.

$$E = x\left(p + \frac{\ell_p^2}{p}\right).$$

We get the following quadratic equation for p by multiplying both sides by p

$$xp^2 - Ep + x\ell_p^2 = 0,$$

which gives the solutions of p in terms of x

$$p_\pm(x) = \frac{E}{2x} \pm \sqrt{\left(\frac{E}{2x}\right)^2 - \ell_p^2}.$$

At the endpoints, the values of p are

$$p_\pm := p_\pm(\ell_x) = \frac{E}{2\ell_x} \pm \sqrt{\left(\frac{E}{2\ell_x}\right)^2 - \ell_p^2}$$

In the next calculation, we have to make an approximation of the square root, and we will do that as follows. A Taylor expansion of $f(x) = (1 + x)^n$ at the point $x = 0$ gives

$$(1 + x)^n = \sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}x^m = 1 + nx + n(n-1)x^2 + n(n-1)(n-2)x^3 + \cdots$$

So when $x \to 0$ we have $(1+x)^n \simeq 1+nx$, and in our case $\sqrt{1+x} \simeq 1+x/2$. Now, the area under the level curve is equal to

$$\begin{aligned}
V(E) &= \int_{p_0}^{p_1}\int_{\ell_x}^{Ep/\left(p^2+\ell_p^2\right)} \mathrm{d}x\,\mathrm{d}p = \int_{p_0}^{p_1}\left(\frac{Ep}{p^2+\ell_p^2}-\ell_x\right)\mathrm{d}p \\
&= \frac{E}{2}\ln\left(\frac{p_1^2+\ell_p^2}{p_0^2+\ell_p^2}\right) - \ell_x(p_1-p_0) \\
&= \frac{E}{2}\ln\left(\frac{\left(\frac{E}{2\ell_x}\right)^2+\frac{E}{\ell_x}\sqrt{\left(\frac{E}{2\ell_x}\right)^2-\ell_p^2}+\left(\frac{E}{2\ell_x}\right)^2-\ell_p^2+\ell_p^2}{\left(\frac{E}{2\ell_x}\right)^2-\frac{E}{\ell_x}\sqrt{\left(\frac{E}{2\ell_x}\right)^2-\ell_p^2}+\left(\frac{E}{2\ell_x}\right)^2-\ell_p^2+\ell_p^2}\right) - 2\ell_x\sqrt{\left(\frac{E}{2\ell_x}\right)^2-\ell_p^2} \\
&= \frac{E}{2}\ln\left(\frac{2\left(\frac{E}{2\ell_x}\right)^2+\frac{E}{\ell_x}\sqrt{\left(\frac{E}{2\ell_x}\right)^2-\ell_p^2}}{2\left(\frac{E}{2\ell_x}\right)^2-\frac{E}{\ell_x}\sqrt{\left(\frac{E}{2\ell_x}\right)^2-\ell_p^2}}\right) - \sqrt{E^2-4\ell_x^2\ell_p^2} \\
&= \frac{E}{2}\ln\left(\frac{E+\sqrt{E^2-4\ell_x^2\ell_p^2}}{E-\sqrt{E^2-4\ell_x^2\ell_p^2}}\right) - \sqrt{E^2-4\ell_x^2\ell_p^2} \\
&= \frac{E}{2}\ln\left(\frac{E+E\sqrt{1-4\left(\frac{\ell_x\ell_p}{E}\right)^2}}{E-E\sqrt{1-4\left(\frac{\ell_x\ell_p}{E}\right)^2}}\right) - E\sqrt{1-4\left(\frac{\ell_x\ell_p}{E}\right)^2} \\
&\simeq \frac{E}{2}\ln\left(\frac{E+E\left(1-2\left(\frac{\ell_x\ell_p}{E}\right)^2\right)}{E-E\left(1-2\left(\frac{\ell_x\ell_p}{E}\right)^2\right)}\right) - E\left(1-2\left(\frac{\ell_x\ell_p}{E}\right)^2\right) \\
&= \frac{E}{2}\ln\left(\frac{E-\frac{\ell_x^2\ell_p^2}{E}}{\frac{\ell_x^2\ell_p^2}{E}}\right) - E + \frac{2\ell_x^2\ell_p^2}{E} = E\ln\sqrt{\frac{E^2}{\ell_x^2\ell_p^2}-1} - E + \frac{2\ell_x^2\ell_p^2}{E} \\
&\simeq E\ln\left(\frac{E}{\ell_x\ell_p}\right) - E = E\ln\left(\frac{E}{2\pi\hbar}\right) - E.
\end{aligned}$$

And by using Weyl's law, we get

$$\langle\mathcal{N}(E)\rangle = \frac{V(E)}{2\pi\hbar} \simeq \frac{E}{2\pi\hbar}\ln\left(\frac{E}{2\pi\hbar}\right) - \frac{E}{2\pi\hbar},$$

which agrees, asymptotically, with the smooth part of the counting function for the non-trivial zeroes of the Riemann zeta function in the strip where the imaginary part lies between 0 and T, (3.8).

4.6.4 Remarks

Sierra and Rodríguez-Laguna (2011) say this adapted Hamiltonian, $H = x\left(p+\ell_p^2/p\right)$, solves the truncation problem of the Hamiltonian $H = xp$, because at the point $(x,p) = \ell_x, p_-$ the particle could bounce off, meaning that its momentum p_- becomes p_+. This would be analogue to a change in the momentum, $p \mapsto -p$, of a particle hitting a wall. So, they made an identification between the points $(x,p) = \ell_x, p_-$ and $(x,p) = \ell_x, p_+$. But still there is no relation between the periodic orbits and the prime numbers.

5. CONCLUSIONS

We saw that the zeroes of the Riemann zeta function could be divided into the trivial ones and the non-trivial ones. The non-trivial zeroes lie in the strip $\{s \in \mathbb{C} \mid 0 \leqslant \mathrm{Re}(s) \leqslant 1\}$. The Riemann hypothesis says that they all lie one the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1/2\}$.

If we write the non-trial zeroes as $\rho_n = 1/2 + \mathrm{i}t_n$, then the Hilbert-Pólya conjecture states that the t_n's correspond to a hermitian operator. This conjecture immediately would imply the Riemann hypothesis. One could construct such hermitian operator by quantizing a Hamiltonian corresponding to a dynamical system.

Because the locations of the non-trivial zeroes are hard to find, we looked at the counting function, $N(T)$, for the number of non-trivial zeroes in the strip $\{s \in \mathbb{C} \mid 0 \leqslant \mathrm{Im}(s) \leqslant T\}$. This counting function can be written as the sum of a smooth part and a fluctuating part, $N(T) = \langle N(T) \rangle + N_{\mathrm{fl}}(T)$. Also, the number of states of a quantum physical system with energy between 0 and E can be written as a sum of a smooth part and a fluctuating part, $\mathcal{N}(E) = \langle \mathcal{N}(E) \rangle + \mathcal{N}_{\mathrm{fl}}(E)$, where $\langle \mathcal{N}(E) \rangle$ can be calculated by Weyl's law and $\mathcal{N}_{\mathrm{fl}}(E)$ is given by the Gutzwiller formula. By comparing the fluctuating parts of $N(T)$ and $\mathcal{N}(E)$, we can obtain a lot of information for the dynamical system to satisfy.

The Hamiltonian $H = xp$ could be a candidate for the Hilbert-Pólya conjecture, since it satisfies most of the conditions. The problem with this Hamiltonian is that there are no argument for the applied truncation and the prime numbers do not appear in the periodic orbits. The Hamiltonian $H = x\big(p + \ell_p^2/p\big)$ solves this problem for the truncation, but not the problem for the prime numbers.

So, there is still a lot of research that can be done. One could for example search for other Hamiltonians or modifications of those two Hamiltonians for a proof of the Hilbert-Pólya conjecture and hence for the Riemann hypothesis.

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