Solutions to the Equations of Motion of Classical Relativistic Strings

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October 2005

In this report some solutions to the equations of motion of classical relativistic strings are presented. Rotating solutions of open and closed strings are found, including intermediate forms between open strings that are folded k and $3k$ times.

1 Introduction

In classical relativistic string theory the Howe and Tucker form of the action can be reduced to

$$S = -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu,$$

where T is the string tension, in combination with constraints

$$g_{\alpha\beta} - \frac{1}{2}(h \cdot g)h_{\alpha\beta} = 0.$$

This action leads to separate solutions for open and closed strings.

1.1 Open strings

For open strings we have the Neumann boundary conditions:

$$\partial_\sigma x^\mu(\tau, 0) = \partial_\sigma x^\mu(\tau, \pi) = 0.$$

The equations of motion for open strings is given by

$$x^\mu(\tau, \sigma) = q^\mu + \frac{1}{\pi T}P^\mu \tau + i\ell \sum_{n \neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \tag{1}$$

such that $\alpha_n^\mu = (\alpha_{-n}^\mu)^*$, where * denotes the complex conjugate.

The constraints can be written as

$$\begin{array}{l} \frac{1}{2}\dot{x}^2 + \frac{1}{2}(x')^2 = 0, \\ \dot{x} \cdot x' = 0, \end{array} \tag{2}$$

or equivalently, as shown in [1],

$$L_n \equiv \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0, \text{ for all } n, \tag{3}$$

where

$$\alpha_0^\mu = \frac{1}{\pi T \ell} P^\mu.$$

1.2 Closed Strings

For closed strings we have the boundary conditions

$$x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + \pi).$$

The equations of motion are

$$x^\mu(\tau, \sigma) = q^\mu + \frac{1}{\pi T} P^\mu \tau + \frac{i}{2} \ell \sum_{n \neq 0} \frac{1}{n} e^{-2in\tau} (\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}). \tag{4}$$

The constraints are equivalent to

$$\begin{array}{ll} L_n \equiv \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0, & \alpha_0^\mu = \frac{1}{2\pi \ell T} P^\mu, \\ \tilde{L}_n \equiv \frac{1}{2} \sum_m \tilde{\alpha}_m^\mu \tilde{\alpha}_{n-m,\mu} = 0, & \tilde{\alpha}_0^\mu = \alpha_0^\mu. \end{array} \tag{5}$$

2 The constraints

A simple solution of the equations of motion is given by

$$\left. \begin{array}{rcl} x^0 & = & \ell \tau \\ x^1 & = & \ell \cos\sigma \cos\tau \\ x^2 & = & \ell \cos\sigma \sin\tau \end{array} \right\}. \tag{6}$$

This solution corresponds to a rotating open string of wich the endpoints move with the speed of light. To verify that this solution indeed obeys the constraints we can use the equations (2) or we can calculate the coefficients α_n^μ. The only nonzero coefficients α_m^μ for this case are:

$\alpha_0^0 = 1$	$\alpha_1^1 = -\frac{i}{2}$	$\alpha_1^2 = \frac{1}{2}$
	$\alpha_{-1}^1 = \frac{i}{2}$	$\alpha_{-1}^2 = \frac{1}{2}$

2.1 General properties of the constraints

Before calculating the L_n it is convenient to make some general statements to see which of them actually have to be calculated. The first one indicates that it is only necessary to consider the L_n with positive n.

Theorem 2.1 *$L_n = 0$ if and only if $L_{-n} = 0$.*

Proof We use the fact that in the equations of motion (1) $\alpha_n^\mu = (\alpha_{-n}^\mu)^*$ where the * denotes the complex conjugate.

$$\begin{aligned}
L_n &= \sum_m \alpha_m^\mu \alpha_{n-m,\mu} \\
&= \sum_{m>0} \alpha_m^\mu \alpha_{n-m,\mu} + \sum_{m<0} \alpha_m^\mu \alpha_{n-m,\mu} + \alpha_0^\mu \alpha_{n,\mu} \\
&= \sum_{m>0} (\alpha_{-m}^\mu)^* (\alpha_{-n+m,\mu})^* + \sum_{m<0} (\alpha_{-m}^\mu)^* (\alpha_{-n+m,\mu})^* + (\alpha_0^\mu)^* (\alpha_{-n,\mu})^* \\
&= \sum_{m<0} (\alpha_{-m}^\mu)^* (\alpha_{-n-m,\mu})^* + \sum_{m>0} (\alpha_{-m}^\mu)^* (\alpha_{-n-m,\mu})^* + (\alpha_0^\mu)^* (\alpha_{-n,\mu})^* \\
&= \sum_m (\alpha_m^\mu)^* (\alpha_{-n-m,\mu})^* = \left(\sum_m \alpha_m^\mu \alpha_{-n-m,\mu} \right)^* = (L_{-n})^*
\end{aligned}$$

If the complex conjugate of L_{-n} equals zero, then L_{-n} is zero as well, and also the converse is true. $\blacksquare$

Another useful theorem is the next one. It states that we do not have to evaluate the conditions L_n for any $n > 2n_{\max}$, where $n_{\max}$ is the highest n of all non-zero α_n^μ, since they all vanish.

Theorem 2.2 *For any set of non-zero coefficients α_n^μ let $\alpha_{n_{\max}}^\mu$ be the coefficient with highest n. Then $L_k = 0$ if $k > 2n_{\max}$.*

Proof It is convenient to split the sum in a part with $m < 0$ and a part with $m \geq 0$, so

$$L_k = \sum_m \alpha_m^\mu \alpha_{k-m,\mu} = \sum_{m \geq 0} \alpha_m^\mu \alpha_{k-m,\mu} + \sum_{m<0} \alpha_m^\mu \alpha_{k-m,\mu}.$$

First consider the part where m runs negative:

$$\sum_{m<0} \alpha_m^\mu \alpha_{k-m,\mu} = \alpha_{-1}^\mu \alpha_{k+1,\mu} + \alpha_{-2}^\mu \alpha_{k+2,\mu} + \cdots$$

Since $k > n_{\max}$ all factors are zero and so this part does not contribute to the sum.

Next $m \geq 0$.

$$\sum_{m\geq 0} \alpha^\mu_m \alpha_{k-m,\mu} = \alpha^\mu_0 \alpha_{k,\mu} + \alpha^\mu_1 \alpha^\mu_{k-1,\mu} + \cdots + \alpha^\mu_{n_{\max}} \alpha_{k-n_{\max},\mu} + \cdots$$

The contravariant components, starting from $m = 0$, can be nonzero up to $m = n_{\max}$. In that region the index of the covariant components runs from k down to $k - n_{\max}$. If $k > 2n_{\max}$ it follows that $k - n_{\max} > n_{\max}$, so all covariant components that are multiplied by possible nonzero contravariant components indeed vanish. $\blacksquare$

Furthermore there is a condition for the coefficients α^μ_n with highest n.

Theorem 2.3 *Let $n_{\max}$ be the index of the non-zero coefficient α^μ_n with the highest value of n. Then*

$$\alpha^\mu_{n_{\max}} \alpha_{n_{\max},\mu} = 0.$$

Proof Consider $L_{2n_{\max}}$ and again split it in two parts.

$$L_{2n_{\max}} = \sum_m \alpha^\mu_m \alpha_{2n_{\max}-m,\mu} = \sum_{m\geq 0} \alpha^\mu_m \alpha_{2n_{\max}-m,\mu} + \sum_{m<0} \alpha^\mu_m \alpha_{2n_{\max}-m,\mu}.$$

In the sum where $m \geq 0$ we have

$$\sum_{m\geq 0} \alpha^\mu_m \alpha_{2n_{\max}-m,\mu} = \alpha^\mu_0 \alpha_{2n_{\max},\mu} + \alpha^\mu_1 \alpha_{2n_{\max}-1,\mu} + \cdots + \alpha^\mu_{n_{\max}-1} \alpha_{n_{\max}+1,\mu} + \alpha^\mu_{n_{\max}} \alpha_{n_{\max},\mu} + \cdots$$

where the only nonzero term is given by $\alpha^\mu_{n_{\max}} \alpha_{n_{\max},\mu}$. In the part where m runs negative we have

$$\sum_{m<0} \alpha^\mu_m \alpha_{2n_{\max}-m,\mu} = \alpha^\mu_{-1} \alpha_{2n_{\max}+1,\mu} + \alpha^\mu_{-2} \alpha_{2n_{\max}+2,\mu} + \cdots = 0$$

So the condition $L_{2n_{\max}} = 0$ is equivalent to

$$\alpha^\mu_{n_{\max}} \alpha_{n_{\max},\mu} = 0. \tag{7}$$

$\blacksquare$

Finally the number of L_n that have to be calculated can be decreased by the following theorem:

Theorem 2.4 *In the constraints L_n the term $\alpha^\mu_k \alpha_{l,\mu}$ is contained only in L_{l+k}.*

Proof The sum can be written as

$$L_n = \alpha^\mu_{-m} \alpha_{n+m,\mu} + \cdots + \alpha^\mu_{n-k} \alpha_{k,\mu} + \cdots + \alpha^\mu_k \alpha_{n-k,\mu} + \cdots + \alpha^\mu_m \alpha_{n-m,\mu}$$

so in the whole sum α^μ_k appears twice (If $n > 2k$ the two terms in the middle should be interchanged). In both cases it is multiplied by α^μ_{n-k}. We have $\alpha^\mu_l = \alpha^\mu_{n-k}$ only when $n = l + k$ $\blacksquare$

From this theorem it is possible to consider only the terms that can contribute to the L_n. For example, if the coefficients α_n^μ exist for only the spatial coordinates of $n = \pm 1$, the only way they can contribute to the L_n is when $\alpha_1^\mu \alpha_{1,\mu}$ or $\alpha_{-1}^\mu \alpha_{1,\mu}$ appear in the sum. This happens only in L_0 and L_2.

The condition L_0 can be seen as a normalization of the components. If $\alpha_0^0 = 1$ it can be shown that

$$\sum_{n \neq 0} |\alpha_n^\mu|^2 = \frac{1}{2} \tag{8}$$

2.2 Calculating the constraints

For calculating the constraints on the coefficients of the equations (6) we only need to calculate L_0 and L_2 as can be seen from the above theorems.

$$\begin{aligned} L_0 &= \sum_m \alpha_m^\mu \alpha_{-m,\mu} = \alpha_0^\mu \alpha_{0,\mu} + \alpha_1^\mu \alpha_{-1,\mu} + \alpha_{-1}^\mu \alpha_{1,\mu} \\ &= -1 - \frac{i}{2}\frac{i}{2} + \frac{1}{2}\frac{1}{2} - \frac{i}{2}\frac{i}{2} + \frac{1}{2}\frac{1}{2} = 0 \\ L_2 &= \alpha_1^\mu \alpha_{1,\mu} = (-\frac{i}{2})^2 + (\frac{1}{2})^2 = 0 \end{aligned}$$

So by the constraints it can be seen that solution (6) is indeed valid. The same result is obtained when using the constraints given by (2).

3 More general rotating solutions

The solutions to the equations of motions (6) can be made more general. If we consider a system with 2 spatial dimensions and 1 time dimension such that there are only five non-zero complex coefficients the system is almost the same as the above example. The time is given by $x^0 = \ell\tau$ so $\alpha_0^0 = 1$. It is convenient to use the polar form for the complex coefficients. The non-zero coefficients are given below.

$\alpha_0^0 = 1$	$\alpha_k^1 = ae^{i\phi}$ $(\alpha_k^1)^* = ae^{-i\phi}$	$\alpha_k^2 = a_2 e^{i\phi_2}$ $(\alpha_k^2)^* = a_2 e^{-i\phi_2}$

Since the only non-zero values of α_n are for $n = \pm k$, by theorem (2.4) we can see that $\alpha_{\pm k}^\mu \alpha_{\pm k,\mu}$ only appears in $L_{\pm 2k}$ and $\alpha_k^\mu \alpha_{-k,\mu}$ only in L_0. By theorem (2.1) we only have to make sure that L_{2k} and L_0 vanish.

From theorem (2.3) it follows that $\alpha_k^\mu \alpha_{k,\mu} = 0$, so $\alpha_k^2 = \pm i \alpha_k^1$, and this satisfies the constraint L_{2k}. If we take $\alpha_k^1 = ae^{i\phi}$ we have, using that $i = e^{i\pi/2}$, the following set of α_k^μ:

$\alpha_0^0 = 1$	$\alpha_k^1 = ae^{i\phi}$ $\alpha_{-k}^1 = ae^{-i\phi}$	$\alpha_k^2 = ae^{i(\phi \pm \frac{\pi}{2})}$ $\alpha_{-k}^2 = ae^{-i(\phi \pm \frac{\pi}{2})}$

where a is a positive real number.

For L_0 to vanish we need

$$\begin{aligned} L_0 &= \sum_m \alpha_m^\mu \alpha_{-m,\mu} \\ &= \alpha_0^\mu \alpha_{0,\mu} + \alpha_k^\mu \alpha_{-k,\mu} + \alpha_{-k}^\mu \alpha_{k,\mu} \\ &= -1 + 4a^2 = 0 \end{aligned}$$

With $a = \frac{1}{2}$ this leads to the following solution:

$$\left.\begin{aligned} x^0 &= \ell\tau \\ x^1 &= \tfrac{\ell}{k}(\sin k\tau \cos\phi - \cos k\tau \sin\phi)\cos k\sigma \\ x^2 &= \pm\tfrac{\ell}{k}(-\sin k\tau \sin\phi - \cos k\tau \cos\phi)\cos k\sigma \end{aligned}\right\}. \tag{9}$$

This solution corresponds to a rotating string. It is an open string for $k = 1, 3, 5\ldots$ For $k > 1$ the string is folded k times on itself. At $\tau = 0$, the angle the string makes with the x^2-axis is given by ϕ. If the sign of the x^2-part of solution (9) is positive the string rotates counter-clockwise.

When $\phi = 0$ the equations of motion are given by

$$\left.\begin{aligned} x^0 &= \ell\tau \\ x^1 &= \tfrac{\ell}{k}\sin k\tau \cos k\sigma \\ x^2 &= \mp\tfrac{\ell}{k}\cos k\tau \cos k\sigma \end{aligned}\right\}. \tag{10}$$

4 More complicated solutions of open strings

A more complicated solution of an open string can be obtained when we have a solution like (9) with an additional function of order $l > k$. First we consider the case in which l and k are chosen arbitrarily.

4.1 Arbitrary higher order terms

For arbitrary l the nonzero coefficients α_n^μ are as follows:

$\alpha_0^0 = 1$	$\alpha_l^1 = ae^{i\phi}$ $\alpha_k^1 = be^{i\theta}$ $\alpha_{-k}^1 = be^{-i\theta}$ $\alpha_{-l}^1 = ae^{-i\phi}$	$\alpha_l^2 = a_2e^{i\phi_2}$ $\alpha_k^2 = b_2e^{i\theta_2}$ $\alpha_{-k}^2 = b_2e^{-i\theta_2}$ $\alpha_{-l}^2 = a_2e^{-i\phi_2}$	$\alpha_k^3 = b_3e^{i\theta_3}$ $\alpha_{-k}^3 = b_3e^{i\theta_3}$

The solution is completely general if the $\alpha_{\pm l}^3$ is chosen to be zero. Such a term inevitably leads to a rotating solution, since it is build up of sine and cosine functions and we can alway choose a coordinate system such that this motion is restricted to the x_1, x_2-plane. The coefficients of highest order are easiest fixed by theorem (2.3) if they are restricted to two spatial dimensions. We use theorem (2.4) to see which of the constraints L_n have to be calculated. The only way that non-zero multiplications of the coefficients α_n^μ occur is in $\alpha_l^\mu \alpha_{l,\mu}$, $\alpha_l^\mu \alpha_{k,\mu}$, $\alpha_l^\mu \alpha_{-k,\mu}$, $\alpha_l^\mu \alpha_{-l,\mu}$, $\alpha_k^\mu \alpha_{k,\mu}$, $\alpha_k^\mu \alpha_{-k,\mu}$, and they emerge in the constraints $L_{2l}, L_{l+k}, L_{l-k}, L_0, L_{2k}, L_0$, respectively.

First we calculate L_{2l}. It follows from theorem (2.3) that

$$\begin{aligned} a_2e^{i\phi_2} &= \pm i\alpha_l^1 \\ &= ae^{i(\phi\pm\pi/2)} \end{aligned} \tag{11}$$

so $a_2 = a$ and $\phi_2 = \phi \pm \pi/2$.

Next we consider L_{l+k}:

$$\begin{aligned} L_{l+k} &= \alpha_l^\mu \alpha_{k,\mu} \\ &= abe^{i\phi+\theta} + ab_2e^{i(\phi+\theta_2\pm\pi/2)} = 0 \end{aligned}$$

It follows that $b_2 = b$ and $\theta_2 = \theta \pm \frac{\pi}{2}$, where the sign is the same as in equations (11). So we have

$$\alpha_k^1 = be^{i\theta} \text{ and } \alpha_k^2 = be^{i(\theta\pm\pi/2)}, \tag{12}$$

This means that if condition L_{2k} is to be satisfied, α_k^3 must vanish. Furthermore, if we consider condition L_{k-l} we see that

$$\begin{aligned} L_{l-k} &= \alpha_l^\mu \alpha_{-k,\mu} + \alpha_{-k}^\mu \alpha_{l,\mu} \\ &= 2(\alpha_l^1 \alpha_{-k,1} + \alpha_l^2 \alpha_{-k,2}) \\ &= 2(abe^{i\phi}e^{i\theta} + abe^{i(\phi\pm\pi/2)}e^{-i(\theta\pm\pi/2)}) \\ &= 4abe^{i(\phi-\theta)} \end{aligned}$$

wich can be zero only if

$$a = 0 \text{ or } b = 0.$$

This implies that either the coefficients α_k^μ vanish, or the coefficients α_l^μ. This shows that an arbitrary combination of l and k is not possible.

4.2 Reducing the number of constraints

Above we saw that the constraints for a system with coefficients of order l and k are $L_{2l}, L_{2k}, L_{l+k}, L_{l-k}$ and L_0. If we choose l such that two of the constraints become the same, we can effectively remove one of them. By choosing $l = 3k$, the constraint L_{l-k} becomes identical to L_{2k}. There is no other way in which two of the constraints can be combined. The constraint L_{6k} still leads to equation (11), and L_{4k} leads to (12), where $3k$ must be substituted for l.

Now we have the following set of coefficients:

$\alpha_0^0 = 1$	$\alpha_{3k}^1 = ae^{i\phi}$ $\alpha_k^1 = be^{i\theta}$ $\alpha_{-k}^1 = be^{-i\theta}$ $\alpha_{-3k}^1 = ae^{-i\phi}$	$\alpha_{3k}^2 = \pm iae^{i\phi}$ $\alpha_k^2 = \pm ibe^{i\theta}$ $\alpha_{-k}^2 = \mp ibe^{-i\theta}$ $\alpha_{-3k}^2 = \mp iae^{-i\phi}$	$\alpha_k^3 = b_3 e^{i\theta_3}$ $\alpha_{-k}^3 = b_3 e^{-i\theta}$

The value of α_{3k} can be found using L_{2k}.

$$\begin{aligned} L_{2k} &= \sum_m \alpha_m^\mu \alpha_{2k-m,\mu} \\ &= \alpha_k^\mu \alpha_{k,\mu} + \alpha_{3k}^\mu \alpha_{-k,\mu} + \alpha_{-k}^\mu \alpha_{3k,\mu} \\ &= (\alpha_k^3)^2 + 2(abe^{-i\theta}e^{i\phi} + abe^{i(\phi \pm \frac{\pi}{2})}e^{-i(\theta \pm \frac{\pi}{2})}) \\ &= (\alpha_k^3)^2 + 4abe^{i(\phi-\theta)} \end{aligned}$$

It follows that

$$\alpha_k^3 = \pm 2i\sqrt{ab}e^{i(\phi-\theta)/2} = 2\sqrt{ab}e^{i(\phi-\theta\pm\pi)/2}.$$

Finally, using L_0 , we can calculate that

$$L_0 = -1 + 2(2b^2 + 4ab + 2a^2) = -1 + 4(a+b)^2,$$

so $(b+a)^2 = \frac{1}{4}$, where both a and b are positive real numbers. We have $b = \frac{1}{2} - a$ where $0 \leq a, b \leq \frac{1}{2}$.

4.3 The equations of motion

Now we can write down the equations of motion. Putting together all values of the α_n^μ, remembering that the sign in front of α_{3k}^2 should be the same as the one in front of α_k^2, we get the following sytem of coefficients.

$\alpha_0^0 = 1$	$\alpha_{3k}^1 = ae^{i\phi}$ $\alpha_k^1 = be^{i\theta}$ $\alpha_{-k}^1 = be^{-i\theta}$ $\alpha_{-3k}^1 = ae^{-i\phi}$	$\alpha_{3k}^2 = \pm iae^{i\phi}$ $\alpha_k^2 = \pm ibe^{i\theta}$ $\alpha_{-k}^2 = \mp ibe^{-i\theta}$ $\alpha_{-3k}^2 = \mp ae^{-i\phi}$	$\alpha_k^3 = \pm 2i\sqrt{ab}e^{i(\phi-\theta)/2}$ $\alpha_{-k}^3 = \mp 2i\sqrt{ab}e^{-i(\phi-\theta)/2}$

and putting the coefficients in equation (1) we obtain:

$$\left.\begin{array}{rcl} x^0 &=& \ell\tau \\ x^1 &=& \frac{\ell}{k}\Big(2b(\cos\theta\sin k\tau - \sin\theta\cos k\tau)\cos k\sigma + \frac{2}{3}a(\cos\phi\sin 3k\tau - \sin\phi\cos 3k\tau)\cos 3k\sigma\Big) \\ x^2 &=& \mp\frac{\ell}{k}\Big(2b(\cos\theta\cos k\tau + \sin\theta\sin k\tau)\cos k\sigma + \frac{2}{3}a(\cos\phi\cos 3k\tau - \sin\phi\sin 3k\tau)\cos 3k\sigma\Big) \\ x^3 &=& \mp\frac{\ell}{k}4\sqrt{ab}(\,\cos\frac{\phi-\theta}{2}\cos k\tau + \sin\frac{\phi-\theta}{2}\sin k\tau)\cos k\sigma \end{array}\right\}. \tag{13}$$

The values of ϕ and θ determine the phase of the different parts of the motion. The motion itself corresponds to a rotating string with a twist, wich is rotating as well. The motion when $k = 1$ shown in figure (1) below.

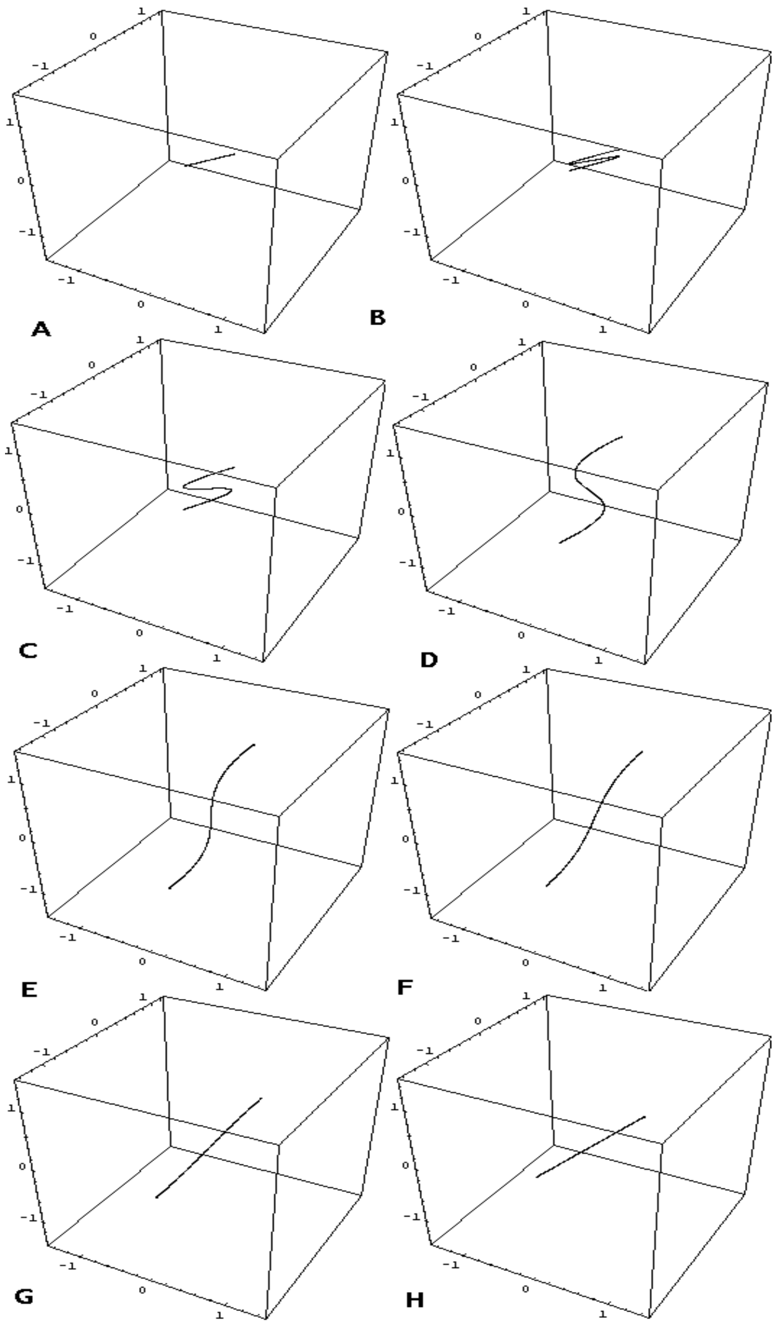

Figure 1: The string as given in equations (13) is drawn here for different values of a and b, but all with the same value of τ. In (**A**) the motion is that of a rotating open string folded three times on itself. In this case $a = \frac{1}{2}$, and $b = 0$. In (**B**) we have $a = 0.499$ and $b = 0.001$. In the following pictures a is made smaller and smaller, and the string is stretching until is becomes a rotating open string in (**H**).

5 Closed Strings

For closed strings there are two sets of coefficients, a_n^μ and $\tilde{a}_n^\mu$ (see equation [4]). They are independent, but $\alpha_0^0 = \tilde{\alpha}_0^0$. Since $\alpha_0^\mu = \frac{1}{2\pi\ell T}P^\mu$, we have $x^0 = \ell\tau$ if $\alpha_0^0 = \tilde{\alpha}_0^0 = \frac{1}{2}$. First the case with the following coefficients is considered:

$\alpha_0^0 = \frac{1}{2}$	$\alpha_k^1 = \frac{1}{4}e^{i\theta}$ $\alpha_{-k}^1 = \frac{1}{4}e^{-i\theta}$	$\alpha_k^2 = \pm i\frac{1}{4}e^{i\theta}$ $\alpha_{-k}^2 = \mp i\frac{1}{4}e^{-i\theta}$

and for $\tilde{a}_n^\mu$:

$\alpha_0^0 = \frac{1}{2}$	$\alpha_m^1 = \frac{1}{4}e^{i\tilde{\theta}}$ $\alpha_{-m}^1 = \frac{1}{4}e^{-i\tilde{\theta}}$	$\alpha_m^2 = \pm i\frac{1}{4}e^{i\tilde{\theta}}$ $\alpha_{-m}^2 = \mp i\frac{1}{4}e^{-i\tilde{\theta}}$

The factor $\frac{1}{4}$ in front of the coefficients comes from condition L_0 with $\alpha_0^0 = \frac{1}{2}$. This results in the following equations of motion:

$$\left.\begin{array}{rcl} x^0 &=& \ell\tau \\ x^1 &=& \frac{\ell}{4}\Big(\frac{1}{k}\sin\left(2k\tau - \theta - 2k\sigma\right) + \frac{1}{m}\sin\left(2m\tau - \tilde{\theta} + 2m\sigma\right)\Big) \\ x^2 &=& \mp\frac{\ell}{4}\Big(\frac{1}{k}\cos\left(2k\tau - \theta - 2k\sigma\right) \pm \frac{1}{m}\cos\left(2m\tau - \tilde{\theta} + 2m\sigma\right)\Big) \end{array}\right\}. \tag{14}$$

When $k = m$ and when we have a $(+)$ in the middle of x^2 the motion corresponds to a closed string collapsed to a line that is rotating. If the middle sign is a $(-)$ the motion corresponds to a circle of which the radius is periodically changing between 0 and $\frac{\ell}{2}$. The sign in front of x^2 determines the direction of the rotation. The angles θ and $\tilde{\theta}$ determine the phase of the motion. A few examples of the motion when $k \neq m$ are shown in figure (2).

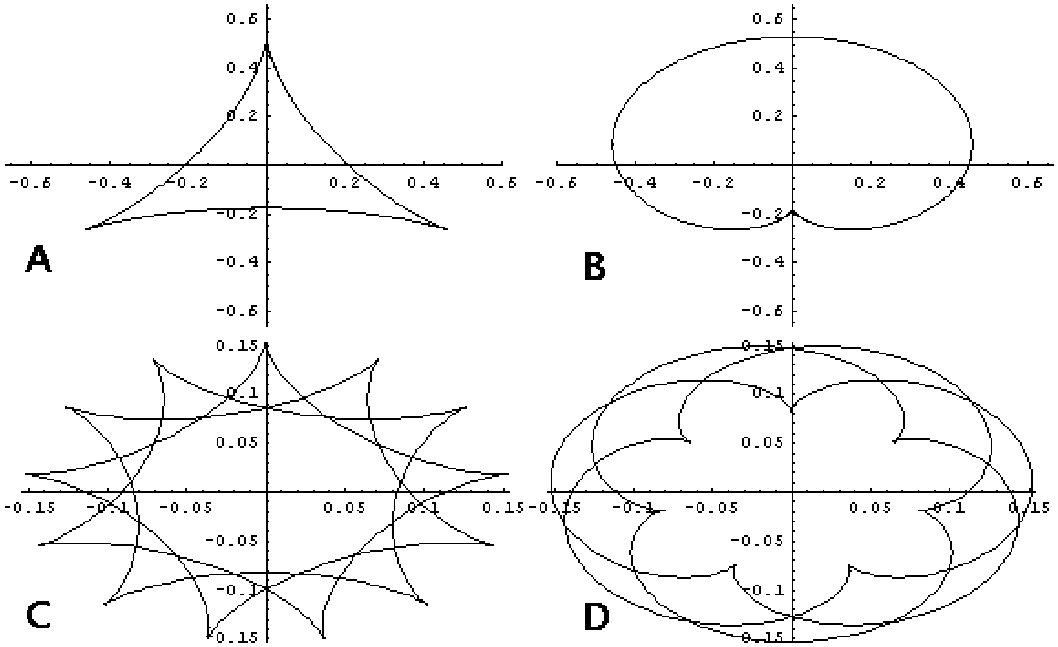

Figure 2: Some examples for different values of m, k and the $\pm$-sign between the two terms of x^2. In **A** and **B** we have $m = 1$, $k = 2$ with a plus and a minus sign respectively. The same applies for **C** and **D** where $m = 10$, $k = 3$. The strings are all rotating with time.

More general are the equations of motion following from the following system of coefficients, obtained in the same way as in the open string case:

$\alpha_0^0 = \frac{1}{2}$	$\alpha_{3k}^1 = be^{i\beta}$ $\alpha_k^1 = ae^{i\alpha}$ $\alpha_{-k}^1 = ae^{-i\alpha}$ $\alpha_{-3k}^1 = be^{-i\beta}$	$\alpha_{3k}^2 = \pm ibe^{i\beta}$ $\alpha_k^2 = \pm iae^{i\alpha}$ $\alpha_{-k}^2 = \mp iae^{-i\alpha}$ $\alpha_{-3k}^2 = \mp be^{-i\beta}$	$\alpha_k^3 = \pm 2i\sqrt{ab}e^{i(\beta-\alpha)/2}$ $\alpha_{-k}^3 = \mp 2i\sqrt{ab}e^{-i(\beta-\alpha)/2}$

The same applies to the coefficients $\tilde{\alpha}_n^\mu$:

	$\tilde{\alpha}^1_{3k} = \tilde{b}e^{i\tilde{\beta}}$	$\tilde{\alpha}^2_{3k} = \pm i\tilde{b}e^{i\tilde{\beta}}$	
	$\tilde{\alpha}^1_{k} = \tilde{a}e^{i\tilde{\alpha}}$	$\tilde{\alpha}^2_{k} = \pm i\tilde{a}e^{i\tilde{\alpha}}$	$\tilde{\alpha}^3_{k} = \pm 2i\sqrt{\tilde{a}\tilde{b}}e^{i(\tilde{\beta}-\tilde{\alpha})/2}$
$\tilde{\alpha}^0_0 = \frac{1}{2}$	$\tilde{\alpha}^1_{-k} = \tilde{a}e^{-i\tilde{\alpha}}$	$\tilde{\alpha}^2_{-k} = \mp i\tilde{a}e^{-i\tilde{\alpha}}$	$\tilde{\alpha}^3_{-k} = \mp 2i\sqrt{\tilde{a}\tilde{b}}e^{-i(\tilde{\beta}-\tilde{\alpha})/2}$
	$\tilde{\alpha}^1_{-3k} = \tilde{b}e^{-i\tilde{\beta}}$	$\tilde{\alpha}^2_{-3k} = \mp\tilde{b}e^{-i\tilde{\beta}}$	

By condition L_0 we must have $a + b = \tilde{a} + \tilde{b} = \frac{1}{4}$. This leads to the following solutions for the equations of motion:

$$\left.\begin{aligned}
x^0 &= \ell\tau \\
x^1 &= \ell\Big(\tfrac{1}{k}\{\tfrac{b}{3}\sin\left(6k\tau - \beta - 6k\sigma\right) + a\sin\left(2k\tau - \alpha - 2k\sigma\right)\} \\
&\quad +\tfrac{1}{m}\{\tfrac{\tilde{b}}{3}\sin\left(6m\tau - \tilde{\beta} + 6m\sigma\right) + \tilde{\alpha}\sin\left(2m\tau - \tilde{\alpha} + 2m\sigma\right)\}\Big) \\
x^2 &= \mp\ell\Big(\tfrac{1}{k}\{\tfrac{b}{3}\cos\left(6k\tau - \beta - 6k\sigma\right) + a\cos\left(2k\tau - \alpha - 2k\sigma\right)\} \\
&\quad \mp\tfrac{1}{m}\{\tfrac{\tilde{b}}{3}\cos\left(6m\tau - \tilde{\beta} + 6m\sigma\right) + \tilde{\alpha}\cos\left(2m\tau - \tilde{\alpha} + 2m\sigma\right)\}\Big) \\
x^3 &= \mp\ell\Big(\tfrac{2\sqrt{ab}}{k}\cos\left(2k\tau - \tfrac{\beta-\alpha}{2} - 2k\sigma\right) \\
&\quad \pm\tfrac{2\sqrt{\tilde{a}\tilde{b}}}{m}\cos\left(2m\tau - \tfrac{\tilde{\beta}-\tilde{\alpha}}{2} + 2m\sigma\right)\Big)
\end{aligned}\right\} . \qquad (15)$$

An example of these solutions is given below.

6 Summary

Solutions to the equations of classical relativistic strings can be expressed in term of the coefficients α^μ_n, and they are subject to the constraints L_n. It is shown that some of the constraints can be combined, and more terms can be added to the equations of motion.

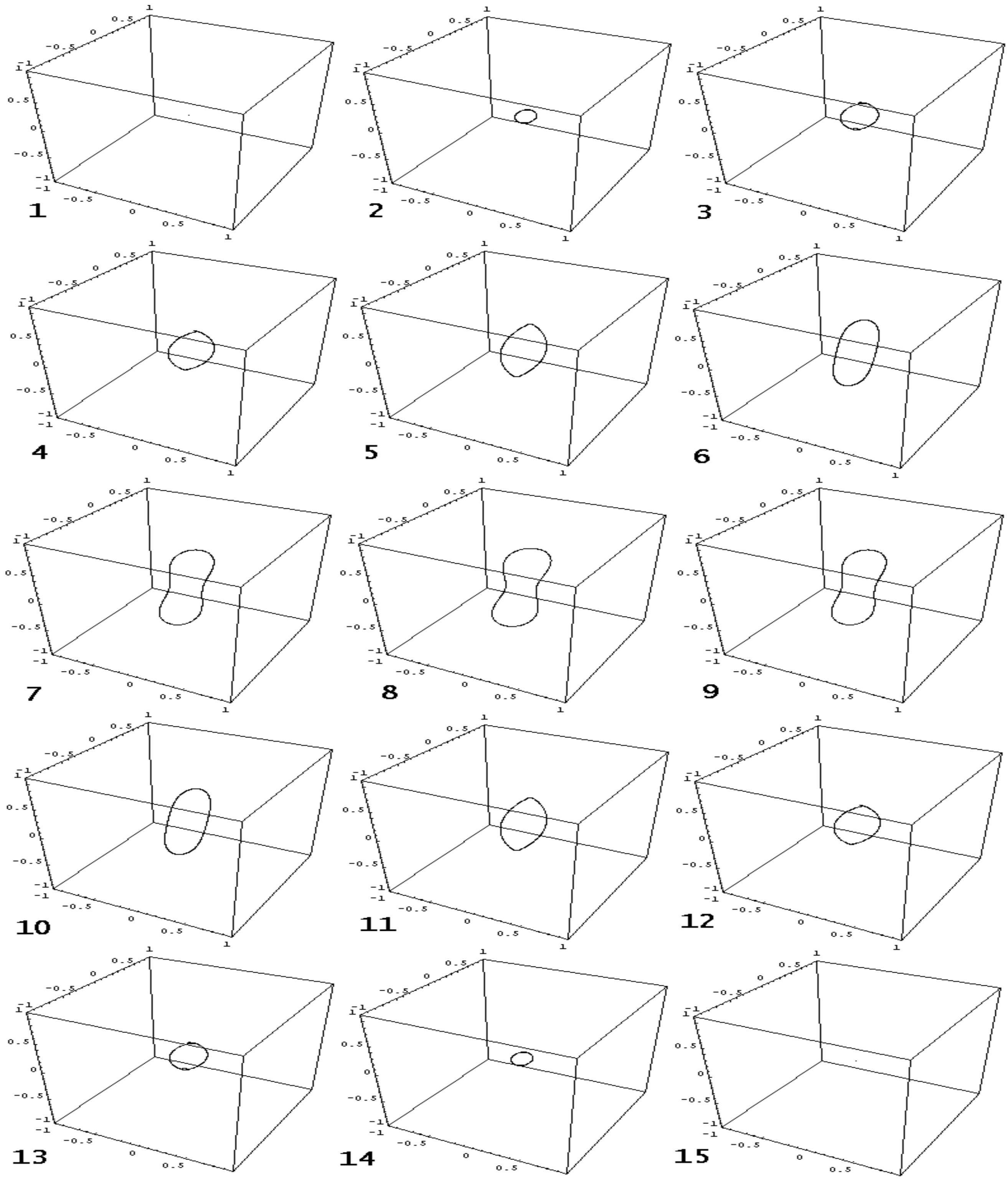

Figure 3: The movement of the string according to equations (15) with $k = m = 1$, $\alpha = \tilde{\alpha} = \beta = \tilde{\beta} = 0$ and $a = b = \tilde{a} = \tilde{b} = \frac{1}{8}$. The sign in the middle of x^2 is $(-)$, all other $(+)$. The pictures have different values of τ such that 1 period of its motion is covered.

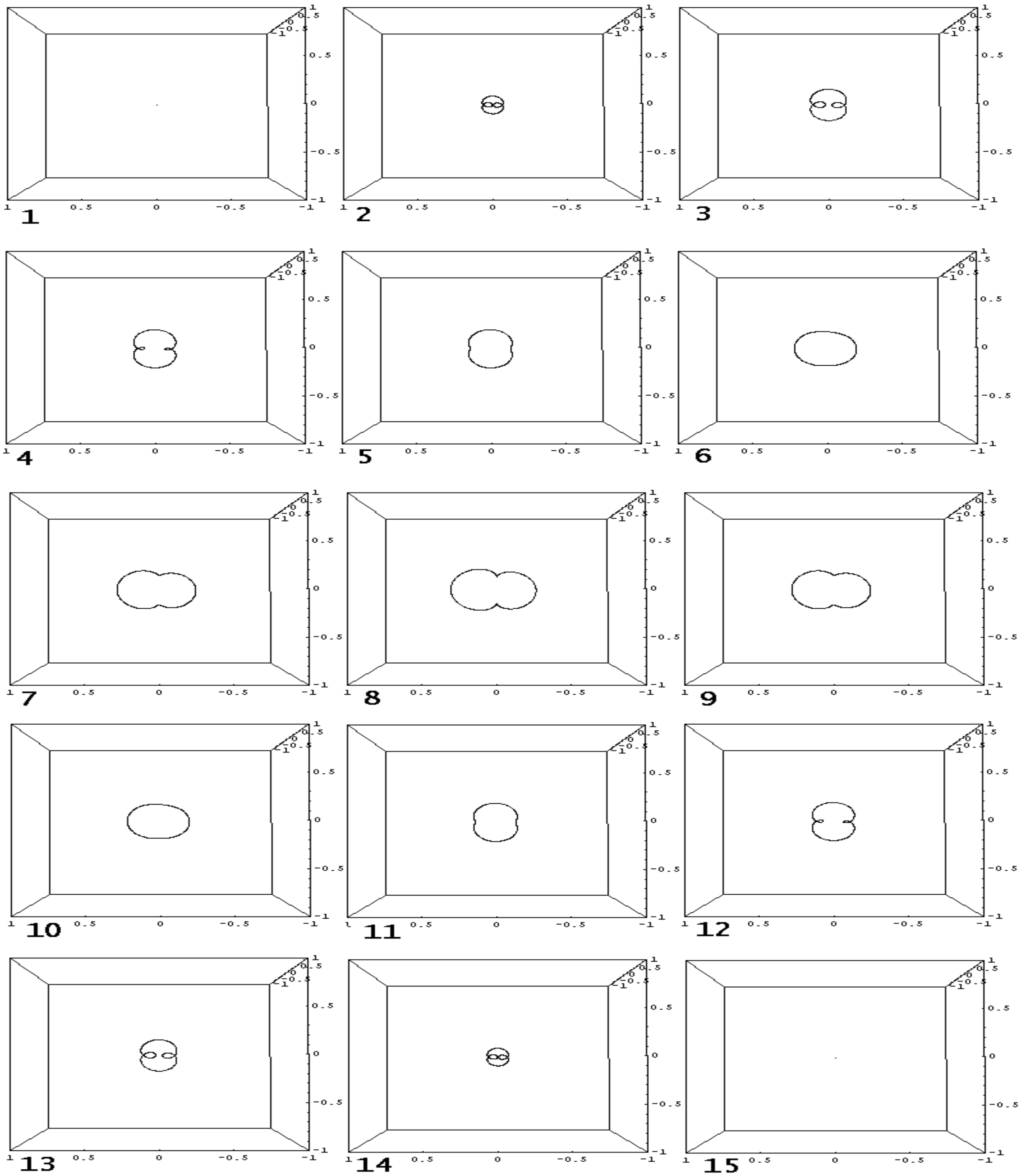

Figure 4: The same as in figure (3), but this time the box in which the string is drawn is viewed from above.

7 Acknowledgment

I would like to express my gratitude to Dr. M. de Roo for his support and advice.

References

[1] M. de Roo, *Basic String Theory*, April 2004.