Double Field Theory

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Bachelor thesis

University of Groningen

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July 12, 2012

Азбука, к мудрости ступенька

You have to learn to walk before you can run

Abstract

In this thesis, a theory called double field theory will be discussed. First, an introduction will be given into string theory. After this, compactifications of dimensions will be discussed and with it the notion of winding. This winding gives rise to T-duality which is an important symmetry of double field theory. We will discuss the weak constraint which says all fields and gauge parameters must be annihilated by the operator $\partial_i \tilde{\partial}^i$. An action to all orders in the fields will be constructed and it will turn out that a constraint stronger than the weak constraint is needed to ensure gauge invariance of the action and to keep T-duality as a symmetry. A possible stronger constraint is the strong constraint but it will be shown that upon satisfying this strong constraint it is always possible to perform an O(D,D) transformation after which our double field theory has become a single field theory. Finally, it is discussed what conditions are exactly needed to prove gauge invariance and to keep all the symmetries. These conditions are compared with the strong constraint and the possibility of an intermediate constraint will be discussed.

Contents

1 Introduction

This thesis will discuss a theory called double field theory. It is a theory that describes a massless subsector of closed string theory. Why was double field theory developed and why is it called 'double' field theory? At the moment string theory is the best candidate for a quantum gravity theory. It is, however, very hard to do calculations in string field theory. Double field theory contains some 'stringy' aspects and it is relatively simple to do calculations in double field theory. In this way we get a theory that describes strings and we can actually calculate things. We will give a short explanation why this is the case, but first we give a short introduction into string theory.

Around 1915 Einstein presented the Einstein field equations. These well known equations specify how the geometry of space and time is influenced by whatever matter is present, and form the core of Einstein's general theory of relativity. It is a theory of gravity but it does not contain quantum mechanics. This means that if we go to very small distances where quantum mechanical effects begin to play a role, Einstein's theory does not work any more. If we try to quantize general relativity, the theory becomes non-renormalizable. Whenever we want to calculate something, like a probability, we get infinite as an answer. This of course does not make any sense. String theory finds a way to avoid these infinities by using one dimensional vibrating strings instead of point particles. It is for now the most successful quantum gravity theory that we have. Before 1995 there were five known consistent nontrivial string theories, called super string theories. Although they are all fundamentally based on one-dimensional vibrating strings, in detail they look very different. So different that people did not think they might be related to each other. But after 1995 it was discovered that they actually *are* related by dualities. In this thesis a particular duality called T-duality will be discussed.

Why did people not see before that the different super string theories are related to each other? It is because an inconvenient notation was used to write down the theories. A notation in which the symmetries of the theories were not manifest. Doing calculations is a lot easier if there are more symmetries in a theory. Imagine doing calculations in quantum field theory without using notation where Lorentz invariance is manifest, i.e., without using x^μ but rather writing out all components. Calculations will become a mess. Finding symmetries to write formulae in a nice and simple way is what every theoretical physicist is dreaming of. In double field theory we double the coordinates to get new symmetries and we can write the theory in an O(D,D) covariant form. The calculations will be much simpler than in string field theory. Since double field theory does contain real 'stringy' aspects, it combines the easy calculation of a field theory, with aspects of string theory.

In what sense is double field theory 'stringy'? If we do double field theory, we work in D spacetime dimensions. But d of them will be compactified to form a d-dimensional torus and the other $n = D - d$ will be just ordinary Minkowski-space dimensions. We thus work in a $\mathbb{R}^{n-1,1} \times \mathrm{T}^d$ spacetime. As soon as we compactify a dimension, the notion of winding will appear. A string can wind around the compact dimension one or several times. This is something a particle cannot do, it is a real stringy feature. The winding w will turn out to have dimensions of momentum. Just like the momentum p is canonical to the coordinate x, we find that the winding w will we canonical to a coordinate $\tilde{x}$, which we will call the *dual* coordinate.

We now let all our fields depend on both the normal coordinates x, and the dual coordinates $\tilde{x}$. We have therefore doubled the coordinates, but in principle we should only do this for the d compact directions since only these have winding (you cannot wind around a non-compact dimension) and therefore only these will have a dual coordinate. However, it is convenient to also double the non-compact coordinates which we will call x_μ. In this way we can write the theory in an O(D,D) covariant form which is what we wanted. We can always restrict our fields in such a way that they will not depend on these doubled noncompact coordinates.

The question is whether these extra degrees of freedom are redundant or not. It turns out that in order to make sure the theory is consistent, we have to impose a constraint on the fields and gauge parameters. One possible constraint we can use that leads to a correct theory is called the *strong constraint*. However, this strong constraint is actually so strong that it implies that the extra degrees of freedom are in fact redundant and we can always write our theory in such a way that the fields are independent of the dual coordinates $\tilde{x}$. We will discuss the possibility to create a different constraint which is less strong than the strong constraint, but will nevertheless still lead to a consistent theory.

2 Classical closed string theory

In this section we will first discuss the relativistic point particle since it will prove to be helpful. The construction of the relativistic string action will be analogous to the construction of the action for the relativistic point particle. We will derive the equations of motion for a relativistic string, choose our world-sheet parameterization and finally obtain a mode expansion of the string coordinates.

2.1 The relativistic point particle

Here we will construct the relativistic action S of a free point particle. How do we do this? Let us first look at the action S_{nr} of a nonrelativistic free particle. It is given by the time integral of the kinetic energy:

$$S_{nr} = \int dt\, \frac{1}{2} m v^2. \tag{2.1}$$

The equation of motion which follows by Hamilton's principle is

$$\frac{d\vec{v}}{dt} = 0. \tag{2.2}$$

Why is this action nonrelativistic? A simple answer is that the action allows the particle to move with *any* constant velocity, even one that exceeds the velocity of light. Therefore S_{nr} cannot describe a relativistic point particle. Also, the action is not invariant under a Galilean boost $\vec{v} \to \vec{v} + \vec{v_0}$ with constant $\vec{v_0}$. This means that the action is not Lorentz invariant. Normally we require the action to be a Lorentz scalar to make sure that the equations of motion are Lorentz invariant. In this case however, we see that the equation of motion *is* Lorentz invariant even though the action is not. This might leave you wondering whether Lorentz invariance is too strong a constraint on the action. But for what will come, we will always demand the action to be a Lorentz scalar.

Lorentz invariance imposes strong constraints on the possible forms of the action. The correct action turns out to be

$$S = -mc \int_{s_i}^{s_f} ds = -mc^2 \int_{t_i}^{t_f} dt \sqrt{1 - \frac{v^2}{c^2}}. \tag{2.3}$$

Here the constant m is needed to make the action dimensionless. It is no surprise that the proper time $\tau = t\sqrt{1 - \frac{v^2}{c^2}}$ enters the actions, since all Lorentz observers agree on the amount of time that elapses on a clock carried by the moving particle. Note that the action makes no sense if $\mathrm{v} > \mathrm{c}$ since it ceases to be real: the constraint of maximal velocity is implemented.

When a particle moves, it traces out a line in spacetime, called the *world-line* of the particle. The equations of motion are obtained from (2.3) through the principle of stationary action. We know that the action is proportional to the proper time which, if multiplied by c, gives the 'proper length' ds. This means that the particle moves in such a way as to minimize this invariant length ds between its starting and ending point. We will see this is analogous to the case of a relativistic string, only in that case the string will trace out a *world-sheet* instead of a world-line. And the string will move in such a way as to minimize the Lorentz invariant *area* of this worldsheet.

2.1.1 Reparametrization invariance

Here we will discuss a very important concept: reparametrization invariance. To evaluate the integral in (2.3), it is useful to parameterize the particle world-line. Reparametrization invariance of the action means that the value of the action is independent of the parametrization we choose. To parameterize a world line, we only need one parameter, for example τ. As τ ranges in the interval $[\tau_i, \tau_f]$ it describes the motion of the particle. The coordinates x^μ are now functions of τ. Normally, time is used as a parameter. But here even the time coordinate x^0 is parameterized since we want to treat space and time on equal footing.

We rewrite the action (2.3) by using $ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$:

$$S = -mc \int_{\tau_i}^{\tau_f} d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}, \tag{2.4}$$

where have used $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. It can easily be shown that S is reparametrization invariant by changing the parameter τ to τ' and using the chain rule.

The notion of reparametrization invariance is important since it allows us to choose one that simplifies our equations of motion. We will use this to our advantage when we calculate the equations of motion of the relativistic closed string.

2.2 The relativistic closed string

In this section we will first derive the action for a relativistic string. We will do this in a way analogous to what we have done in the point particle case. Finally, we will give the equations of motion and simplify them by choosing a particular parametrization for our world-sheet, where we make use of reparametrization invariance. We will see that the equations of motion are wave equations.

2.2.1 The Nambu-Goto string action

Just as a particle traces out a line in spacetime, a string traces out a surface. This two-dimensional surface is called the *world-sheet*. An open string will trace out a strip and a closed string will trace out a tube. We will only be interested in closed strings since only these will give rise to winding, which will be explained in section 4.

Just as a world-line can be described by one parameter, a world-sheet requires *two* parameters. We will call these parameters σ and τ. It will turn out that τ is related to time on the strings and σ is related to positions along the strings. We use them to construct the Lorentz invariant area

$$A = \int d\sigma d\tau \sqrt{(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma})^2 - (\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau})(\frac{\partial X^\nu}{\partial \sigma}\frac{\partial X_\nu}{\partial \sigma})}. \tag{2.5}$$

Here $X^\mu(\tau,\sigma) = (X^0(\tau,\sigma), X^1(\tau,\sigma), \ldots, X^d(\tau,\sigma))$ are the string coordinates in (d+1)-dimensional spacetime.

Let us imagine the world-sheet of a closed string string, a tube. Since we take τ to be related to time on the string, planes of constant τ that intersect the world-sheet will define what we call our string at that specific time. So by fixing τ we now where our string is at that moment, we know which string we are talking about. Then by specifying σ, we now where on the string we are. Finally, X^μ tells us where that specific point on the string is in our (d+1)-dimensional spacetime. For bosonic string theory, which we are discussing, the number of spacetime dimensions is 26, the proof of which can be read in [3].

We now use the invariant area (2.5) to construct the action. This action is called the *Nambu-Goto string action*

$$S_{NG} = -T \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} d\tau d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}. \tag{2.6}$$

Here the constant T plays the same role as m did for the point particle case, it makes the action dimensionless. Since it multiplies a surface area, it has dimensions $[\text{length}]^{-2}$. We have chosen our σ parameter to lie in the interval $[0, \sigma_1]$ where for closed strings we will set σ_1 equal to 2π. We have also introduced the notation $\dot{X} \equiv \frac{\partial X^\mu}{\partial \tau}$, $X^{\mu\prime} \equiv \frac{\partial X^\mu}{\partial \sigma}$. The speed of light c was set equal to one in (2.6). From now on we will work in natural units.

2.2.2 The equations of motion

We can obtain the equations of motion for the relativistic string setting the variation of the action (2.6) equal to zero. This leads to

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0, \tag{2.7}$$

where we have used $\delta X^\mu(\tau_i, \sigma) = \delta X^\mu(\tau_f, \sigma) = 0$ (which means that we fix the time values at the beginning and ending of the string's trajectory) and for closed strings $\delta X^\mu(\tau, \sigma = 0) = \delta X^\mu(\tau, \sigma = 2\pi)$ (This periodicity condition says that if we pick a point on the string and move around the string once, we get back at the same point where we started). We have also introduced $\mathcal{P}^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$ and $\mathcal{P}^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}}$.

2.2.3 choosing a σ parametrization

If one would try to obtain the equations of motion by expanding (2.7), one would soon realize this gives very complicated equations. To simplify these equations we will use the fact that the area (2.5) is reparametrization invariant. We can choose a τ and σ parametrization that suits us best. First we concentrate on the σ parametrization.

Imagine the two-dimensional world-sheet of the string parameterized by σ and τ. Lines of constant τ and σ can be drawn on the world-sheet to form a grid. Now $\dot{X} = \frac{\partial X^\mu}{\partial \tau}$ and $X^{\mu\prime} = \frac{\partial X^\mu}{\partial \sigma}$ are tangent vectors along lines of constant τ and σ respectively. We can impose the following two constraints:

$$\dot{X}^\mu X'_\mu = 0, \quad \dot{X}^2 + X'^2 = 0. \tag{2.8}$$

The first constraint tells us that the lines of constant τ are orthogonal to the lines of constant σ. The second constraint[1] specifies the length of the tangent vectors. Once we have specified our τ parameterization (which we will do in a minute), our σ parameterization will be completely fixed by the two constraints since we now have specified both a length and a direction. The two constraints can be conveniently packaged together as

$$(\dot{X}^2 \pm X')^2 = 0. \tag{2.9}$$

2.2.4 light-cone coordinates and a τ parameterization

In what will come, we will work in light-cone coordinates. The reason for this is that the quantization of the relativistic string will turn out to be easier in these coordinates. The two light-cone coordinates X^+ and X^- are defined[2]

[1]Note that in this constraint, since τ is related to time on the strings, $\dot{X}^2$ is the length of a timelike vector, and is therefore negative.

[2] We note that X^+ and X^- both have equal right to be called a time coordinate since they depend on the string time coordinate X^0 in the same way. However, neither is a time coordinate in the standard sense of time. Light-cone time is not the same as ordinary time! The most familiar property of time is that it goes forward for any physical motion of a particle. Imagine a spacetime diagram with X^0 and X^1 as orthogonal axes. The

as follows

$$X^+ \equiv \frac{1}{\sqrt{2}}(X^0 + X^1) \tag{2.10}$$

$$X^- \equiv \frac{1}{\sqrt{2}}(X^0 - X^1), \tag{2.11}$$

all the other coordinates $(X^2,\ldots,X^d)$ are left the same, they are called the transverse light-cone coordinates X^I. To make it extra clear, we have

$$X^+,\ X^-,\ \underbrace{X^2, X^3, \ldots, X^{26}}_{X^I}, \tag{2.12}$$

where the total number of spacetime dimensions is 26 because we are talking about closed bosonic string theory.

In light-cone coordinates the invariant interval ds^2 takes the form

$$-ds^2 = -2dX^+dX^- + (dX^2)^2 + \cdots + (dX^d)^2. \tag{2.13}$$

Note that, if we are given ds^2, solving for dX^- or for dX^+ does not require taking a square root. This is an important feature of light-cone coordinates and it is the reason why they are easy to work with.

We fix our τ parameterization by choosing

$$X^+(\tau, \sigma) = \alpha' p^+ \tau. \tag{2.14}$$

Here α' is defined as $\frac{1}{2\pi T}$ (where T is the same constant as in (2.6)) and is called the *slope parameter*. Also p^+ is the momentum of the string in the + direction. The choice (2.14) is called the *light-cone gauge*.

2.2.5 the wave equation

Already by choosing our σ-parameterization, the expressions for $\mathcal{P}^\tau_\mu$ and $\mathcal{P}^\sigma_\mu$ have simplified. We can calculate them using (2.6) and (2.8). This gives

$$\begin{aligned} \mathcal{P}^\tau_\mu &= \frac{1}{2\pi\alpha'}\dot{X}_\mu, \\ \mathcal{P}^\sigma_\mu &= -\frac{1}{2\pi\alpha'}X'_\mu, \end{aligned} \tag{2.15}$$

which we can use together with (2.7) to get the simplified equations of motion:

$$\ddot{X}^\mu - X''^\mu = 0. \tag{2.16}$$

We recognize this as a wave equation.

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light-cone axes $X^\pm$ now have a slope of 45°. If we imagine a light ray traveling in the negative X^1 direction, X^+ remains zero, i.e., light-cone time will freeze! Nevertheless, we will take X^+ as the light-cone time coordinate and think of X^- as a spatial coordinate. We will also choose our τ parameter to be proportional to X^+.

2.2.6 mode expansion and commutation relations

We saw in section 2.2.5 that the equations of motion take the form of a wave equation. The general solution of this wave equation is

$$X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma), \tag{2.17}$$

where X_L^μ (the L stand for left) is a left-moving wave and X_R^μ (the R stands for right) is a right-moving wave. We can thus imagine the closed string as a closed piece of rope on which waves can move around. The left-moving waves are independent of the right-moving waves and together they form standing waves on the string. To visualize this, see Figure 1. The solution in (2.17) is, however, not the only solution. Another solution to the wave equation is given by

$$\tilde{X}^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) - X_R^\mu(\tau-\sigma). \tag{2.18}$$

The coordinates in (2.17) are the string coordinates and the coordinates in (2.18) are called the *dual string coordinates.* These dual coordinates will play an important role when we compactify some of the dimensions. When we include them, they give rise to a symmetry called T-duality. This will be discussed in section 4.

As said before, for closed strings we choose our σ parameter to lie in the interval $[0,2\pi]$.[3] If we pick a point on a string and let σ run 2π further, we are back at exactly the same point (if we take τ to be constant). This means $\sigma = 0$ and $\sigma = 2\pi$ represent the *same* point on the closed string.

The parameter space (τ,σ) for closed strings is a cylinder, so, to describe closed strings properly we compactify the world-sheet coordinate σ:

$$\sigma \sim \sigma + 2\pi. \tag{2.19}$$

We demand that the string coordinate X^μ assumes the same value at any two points that are identified with each other in parameter space:

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi), \quad \forall\, \tau,\sigma. \tag{2.20}$$

We emphasize that the condition (2.20) only applies in a space in which every closed string can be continuously shrunk into a point. This means that the space in which the string propagates has to be a simply connected space, for example Minkowski space. If a spatial direction is curled up into a circle, closed string can wrap around this circle and cannot be continuously shrunk into a point any more. In this case (2.20) must be modified. We will do this

[3] We can in fact use any interval of the form $[\sigma_0, \sigma_0 + 2\pi]$ to describe the closed strings. The choice $\sigma_0 = 0$ is just one of the possible choices.

in section 4 .

We will construct the mode expansion of the string coordinates and their dual coordinates. First two new variables are introduced:

$$u \equiv \tau + \sigma, \tag{2.21}$$

$$v \equiv \tau - \sigma. \tag{2.22}$$

In terms of these new variables, equation (2.17) becomes

$$X^\mu = X^\mu_L(u) + X^\mu_R(v). \tag{2.23}$$

When we let $\sigma \to \sigma + 2\pi$, u increases by 2π and v decreases by 2π. Equation (2.20) can therefore be written as

$$X^\mu_L(u) + X^\mu_R(v) = X^\mu_L(u+2\pi) + X^\mu_R(v-2\pi), \tag{2.24}$$

or, equivalently,

$$X^\mu_L(u+2\pi) - X^\mu_L(u) = X^\mu_R(v) - X^\mu_R(v-2\pi). \tag{2.25}$$

Note that for the dual coordinates, we get an expression similar to (2.25) but with an extra minus sign on the right hand side.

Since u and v are independent variables, if we would take the derivative with respect to u, the right hand side must vanish. Therefore we have $X'^\mu_L(u) = X'^\mu_L(u+2\pi)$ where a prime denotes a derivative with respect to the argument. We can also take the derivative with respect to v and thus we find that both $X'^\mu_L(u)$ and $X'^\mu_R(v)$ are periodic functions with a period of 2π. Note that this argument also holds for the dual coordinates. We can therefore write the mode expansions

$$X'^\mu_L(u) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \bar{\alpha}^\mu_n e^{-inu},$$

$$X'^\mu_R(v) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha^\mu_n e^{-inv}. \tag{2.26}$$

The α^μ_n and $\bar{\alpha}^\mu_n$ are just constants. If we demand X^μ to be real, we get $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$ and $\bar{\alpha}^\mu_{-n} = (\bar{\alpha}^\mu_n)^*$.

We can now integrate these expressions to obtain formulae for $X^\mu_L(u)$ and $X^\mu_R(v)$

$$X^\mu_L(u) = \frac{1}{2}x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}^\mu_0 u + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\bar{\alpha}^\mu_n}{n} e^{-inu},$$

$$X^\mu_R(v) = \frac{1}{2}x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}}\alpha^\mu_0 v + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha^\mu_n}{n} e^{-inv}, \tag{2.27}$$

Figure 1: *Vibrating string* [7]

where the coordinate zero modes $x_0^{L\mu}$ and $x_0^{R\mu}$ are constants of integration. Only the sum of these zero modes will play a role in the string coordinates.

By applying (2.25) we see that

$$2\pi\sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^\mu = 2\pi\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu, \tag{2.28}$$

and therefore

$$\bar{\alpha}_0^\mu = \alpha_0^\mu. \tag{2.29}$$

This is an important result and it tells us that closed string theory has only *one* momentum operator (it is shown below that the momentum p^μ is proportional to α_0^μ). We emphasize that this only holds in a simply connected space, since this result was derived from (2.20) which is invalid when we use compact dimensions.

We will write down the full solution for our string coordinates and dual coordinates by substituting (2.27) in (2.17) and (2.18), and using (2.29):

$$\begin{aligned} X^\mu(\tau,\sigma) &= \frac{1}{2}(x_0^{L\mu} + x_0^{R\mu}) + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\bar{\alpha}_n^\mu e^{-in\sigma} + \alpha_n^\mu e^{in\sigma}), \\ \tilde{X}^\mu(\tau,\sigma) &= \frac{1}{2}(x_0^{L\mu} - x_0^{R\mu}) + \sqrt{2\alpha'}\alpha_0^\mu\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\bar{\alpha}_n^\mu e^{-in\sigma} - \alpha_n^\mu e^{in\sigma}). \end{aligned} \tag{2.30}$$

The expansions have a nice interpretation. For the normal coordinate the first term is a constant, it gives the 'starting coordinates' of the string. It tells us where the string is at τ =0. The last term describes the excitations on the string. It says that our string can wiggle and can have all sorts of waves propagating on it. To make clear how this looks like, see Figure 1. Finally, the string coordinates have a term proportional to τ. Since τ represents a time, this term can be seen as a momentum; it is the momentum of the string as a whole. Indeed, we can calculate the momentum by integrating the first equation in (2.15) over σ, since $\mathcal{P}^{\tau\mu}(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$ is a momentum density. This gives us

$$p^\mu = \int_0^{2\pi} \mathcal{P}^{\tau\mu}(\tau,\sigma) d\sigma = \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu, \tag{2.31}$$

and thus

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu. \tag{2.32}$$

We see that the dual coordinates do not satisfy constraint (2.20) and therefore they are not physical. As noted before, this periodicity condition will change for compactified coordinates. This is why we *will* use the dual coordinates when our space is noncompact, as will be discussed in section 4.

It is convenient to write down the τ and σ derivatives of X^μ. This will come in handy later. With the help of (2.17) we note that

$$\begin{aligned} \dot{X}^\mu &= X_L^{\mu'}(\tau+\sigma) + X_R^{\mu'}(\tau-\sigma), \\ X^{\mu'} &= X_L^{\mu'}(\tau+\sigma) - X_R^{\mu'}(\tau-\sigma). \end{aligned} \tag{2.33}$$

After adding and subtracting these equations we find

$$\begin{aligned} \dot{X}^\mu + X^{\mu'} &= 2X_L^{\mu'}(\tau+\sigma) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}, \\ \dot{X}^\mu - X^{\mu'} &= 2X_R^{\mu'}(\tau-\sigma) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in(\tau-\sigma)}. \end{aligned} \tag{2.34}$$

Note that the barred oscillators do not mix with the unbarred oscillators in these combinations of derivatives.

3 Quantization of closed string theory

We will now begin the quantization of closed string theory, derive an important condition and obtain a mass formula for closed string states. Finally, the closed string state space will be constructed. We will be particularly interested in the massless level. It will turn out that at this level, the spectrum of closed strings contains three fields: gravity fields $h_{\mu\nu}$, Kalb-Ramond fields $B_{\mu\nu}$ and dilaton fields ϕ. The fact that closed string theory gives rise to gravitons, is why it is called a quantum gravity theory.

3.1 The level matching condition

In this section we will derive the level matching condition. This condition will play an important role in double field theory where it is called the weak constraint, which will be discussed in section 5. Before deriving the level matching condition we will introduce commutation relations between the α_n^I modes.

3.1.1 Commutation relations

The wave equation continues to hold in the quantum theory, so we can still use the mode expansion. It will turn out that the classical modes α_n^I become quantum operators with nontrivial commutation relations. Will not derive the commutation relations here since this is not important for this thesis. We will only be interested in the results. The derivation is quite straightforward and leads to [3]:

$$\begin{aligned} \left[\bar{\alpha}_m^I, \bar{\alpha}_n^J\right] &= m\,\delta_{m+n,0}\,\eta^{IJ}, \\ \left[\alpha_m^I, \alpha_n^J\right] &= m\,\delta_{m+n,0}\,\eta^{IJ}. \end{aligned} \tag{3.1}$$

The left-moving and right-moving oscillators are independent, they do not see each other. We therefore have $[\alpha_m^I, \bar{\alpha}_n^J] = 0$. Note that α_n^+ and α_n^- do not appear in (3.1). This is because they do not give any new information. To obtain a full solution, α_n^+ and α_n^- are not needed. In fact, by comparing the light-cone gauge (2.14) with the mode expansion (2.30), we see that $\alpha_n^+ = 0$ for $n \neq 0$. Also, it can be shown that the minus oscillators α_n^- can be expressed in terms of the transverse oscillators.

To be able to construct the state space of closed strings, we need to introduce annihilation and creation operators. We will do this by first making the definition

$$\alpha_n^\mu = a_n^\mu\sqrt{n}, \quad \alpha_{-n}^\mu = a_n^{\mu *}\sqrt{n}, \quad n \geq 1. \tag{3.2}$$

Up until now, both α and a have been classical variables. They will now become operators. Classical variables that are complex conjugates of each other become operators that are Hermitian conjugates to each other in the quantum theory. We can therefore preserve the first of the above definitions, but the second must be changed. For our light-cone modes $\mu = I$ we take

$$\alpha_n^I = a_n^I\sqrt{n} \quad \text{and} \quad \alpha_{-n}^I = a_n^{I\dagger}\sqrt{n}, \quad n \geq 1. \tag{3.3}$$

Note that whereas the α_n^I modes are defined for all integers n, the a_n^I and $a_n^{I\dagger}$ operators are only defined for positive n. By using (3.1) it is straightforward to show that

$$\left[a_m^I, a_n^{J\dagger}\right] = \delta_{m,n}\eta^{IJ}, \tag{3.4}$$

and also that $\left[a^I_m, a^J_n\right] = \left[a^{I\dagger}_m, a^{J\dagger}_n\right] = 0$. This means that $(a^I_n, a^{I\dagger}_m)$ satisfy the commutation relations of the canonical annihilation and creation operators of a quantum simple harmonic oscillator. We can therefore interpret $a^{I\dagger}_n$ as a creation operator, and a^I_n as an annihilation operator. Oscillators corresponding to different mode numbers or different light-cone coordinates commute with each other. In terms of the α operators we have

$$\begin{aligned} &\alpha^I_n \quad \text{are annihilation operators,} \\ &\alpha^I_{-n} \quad \text{are creation operators } (n \geq 1). \end{aligned} \tag{3.5}$$

3.1.2 The level matching condition

We here derive an important condition. Once the space state is obtained, all states not satisfying this condition are considered to be non-physical. The condition will thus be of great importance when we investigate the allowed states and the fields which they represent. Because of its importance we will derive it explicitly.

We use the definition of the relativistic dot product in light-cone coordinates (see (2.13)) to write the constraint equations (2.9) as

$$-2(\dot{X}^+ \pm X^{+'})(\dot{X}^- \pm X^{-'}) + (\dot{X}^I \pm X^{I'})^2 = 0, \tag{3.6}$$

and rewrite this as

$$\dot{X}^- \pm X^{-'} = \frac{1}{2\alpha' p^+}(\dot{X}^I \pm X^{I'})^2. \tag{3.7}$$

Here we have used the light-cone gauge (2.14) (i.e. we used that $\dot{X}^+ = \alpha' p^+$) and we used $X^{+'} = 0$ since X^+ does not depend on σ.[4] Note, like we said before, that to solve for the derivatives of X^-, we did not have to take a square root. This is of course because of the off-diagonal term in the metric when using light-cone coordinates. The light-cone gauge was useful since it made X^+ into a constant.

We can use (2.34) to write

$$\begin{aligned} (\dot{X}^I + X^{I'})^2 &= 4\alpha' \sum_{n\in Z}\Big(\frac{1}{2}\sum_{p\in Z} \bar{\alpha}^I_p\, \bar{\alpha}^I_{n-p}\Big)e^{-in(\tau+\sigma)} \equiv 4\alpha' \sum_{n\in Z} \bar{L}^\perp_n e^{-in(\tau+\sigma)}, \\ (\dot{X}^I - X^{I'})^2 &= 4\alpha' \sum_{n\in Z}\Big(\frac{1}{2}\sum_{p\in Z} \alpha^I_p\, \alpha^I_{n-p}\Big)e^{-in(\tau-\sigma)} \equiv 4\alpha' \sum_{n\in Z} L^\perp_n e^{-in(\tau-\sigma)} \end{aligned} \tag{3.8}$$

[4] We have also assumed that $p^+ \neq 0$. It can happen that p^+ is equal to zero, but for this to occur the momentum p^1 must exactly cancel the energy p^0. This only happens if a massless particle travels exactly in the negative x^1 direction. This does not occur very often, so we will assume $p^+ \neq 0$.

Here we have defined two sets of transverse *Virasoro operators*:

$$\bar{L}_n^{\perp} = \frac{1}{2}\sum_{p\in Z} \bar{\alpha}_p^I \bar{\alpha}_{n-p}^I, \quad L_n^{\perp} = \frac{1}{2}\sum_{p\in Z} \alpha_p^I \alpha_{n-p}^I. \tag{3.9}$$

Equation (3.8) can be put into (3.7) to obtain

$$\dot{X}^- + X^{-\prime} = \frac{2}{p^+}\sum_{n\in Z} \bar{L}_n^{\perp} e^{-in(\tau+\sigma)}, \quad \dot{X}^- - X^{-\prime} = \frac{2}{p^+}\sum_{n\in Z} L_n^{\perp} e^{-in(\tau-\sigma)}. \tag{3.10}$$

There is another way in which the derivatives of X^- can be expressed. By using (2.34) we see that

$$\dot{X}^- + X^{-\prime} = \sqrt{2\alpha'}\sum_{n\in Z} \bar{\alpha}_n^- e^{-in(\tau+\sigma)}, \quad \dot{X}^- - X^{-\prime} = \sqrt{2\alpha'}\sum_{n\in Z} \alpha_n^- e^{-in(\tau-\sigma)}. \tag{3.11}$$

Finally, by comparing (3.10) and (3.11) we can express the minus oscillators in terms of the Virasoro operators

$$\sqrt{2\alpha'}\,\bar{\alpha}_n^- = \frac{2}{p^+}\bar{L}_n^{\perp}, \quad \sqrt{2\alpha'}\,\alpha_n^- = \frac{2}{p^+}L_n^{\perp}. \tag{3.12}$$

In particular, for $n = 0$ we use (2.32) and find

$$\sqrt{2\alpha'}\,\alpha_0^- = \alpha' p^- = \frac{2}{p^+}L_0^{\perp}. \tag{3.13}$$

As we will see below, this formula only holds at the classical level and is not correct in the quantum theory.

We know that $\alpha_0^- = \bar{\alpha}_0^-$ and this gives rise to the *level matching* condition

$$L_0^{\perp} = \bar{L}_0^{\perp}. \tag{3.14}$$

What does it mean that $L_0^{\perp}$ and $\bar{L}_0^{\perp}$ must be equal? Well, they are operators and an operator is defined by how it acts on a state. So the level matching condition says that for any state $|\lambda, \bar{\lambda}\rangle$ of the closed string we *must* have $L_0^{\perp}|\lambda, \bar{\lambda}\rangle = \bar{L}_0^{\perp}|\lambda, \bar{\lambda}\rangle$. States that do not satisfy this constraint do not belong to the state space.

There is however, one subtlety in defining the Virasoro operators. In the derivation, the α modes were treated as commuting classical variables. But we know that they do not commute in the quantum theory. We must ask ourselves whether the ordering matters and if so, whether the ordering in (3.9) is the correct one. Two α operators only fail the commute when their mode numbers add up to zero. This means that the only ambiguous operator is $L_0^\perp$. We will need to pick an ordering and the standard choice of ordering is *normal ordering*. Normal ordering means that we put annihilation operators to the right of creation operators. In this way, the Virasoro operators will always annihilate the vacuum. If we carefully look at $L_0^\perp$, and make sure we write everything normal ordered, it turns out that all we have to do is include a constant. So every time we see $L_0^\perp$ in a classical formula, this needs to be replaced by $(L_0^\perp + a)$ when going to the quantum theory. The ordering constant a can be calculated using zeta function regularization, and it is -1 [3].

We use (3.9), (2.32) and (3.3) to define the operators $L_0^\perp$ and $\bar{L}_0^\perp$ without the ordering constant as

$$\bar{L}_0^\perp = \frac{\alpha'}{4} p^I p^I + \bar{N}^\perp, \quad L_0^\perp = \frac{\alpha'}{4} p^I p^I + N^\perp. \tag{3.15}$$

Here we have introduced $\bar{N}^\perp$ and $N^\perp$, the *number operators* associated with the barred and unbarred operators, respectively:

$$\bar{N}^\perp \equiv \sum_{n=1}^{\infty} n\, \bar{a}_n^{I\dagger}\, \bar{a}_n^I, \quad N^\perp \equiv \sum_{n=1}^{\infty} n\, a_n^{I\dagger}\, a_n^I \tag{3.16}$$

A number operator counts the number of right-moving or left-moving oscillators. Its eigenvalue is the sum of the mode numbers of the creations operators appearing in the state.

When we introduce the normal ordering constant, the expression (3.12) changes to

$$\sqrt{2\alpha'}\, \bar{\alpha}_n^- = \frac{2}{p^+}(\bar{L}_n^\perp - 1), \quad \sqrt{2\alpha'}\, \alpha_n^- = \frac{2}{p^+}(L_n^\perp - 1). \tag{3.17}$$

Note that the level matching constraint does not change due to these constant shifts. The level matching constraint can be written in terms of the number operators by using (3.15):

$$\bar{N}^\perp = N^\perp \tag{3.18}$$

This equation tells us that, for a state to be physical, the number of right-moving operators (the number of α) must equal the number of left-moving operators (the number of $\bar{\alpha}$). So, for example, if we act on the vacuum with an unbarred creation operator $a_n^{I\dagger}$, we *must* also act with a barred creation operator $\bar{a}_m^{I\dagger}$. If we do not do this, we will not satisfy the level matching constraint and our state will not be physical.

We will construct the closed string state space in the next section and only focus on the massless subsector of the spectrum. To know the mass of the states we will of course need a mass formula. Since $\alpha_0^- = \bar{\alpha}_0^-$ we average the expression (3.17) to get

$$\sqrt{2\alpha'}\alpha_0^- \equiv \frac{1}{p^+}(L_0^\perp + \bar{L}_0^\perp - 2) = \alpha' p^-, \tag{3.19}$$

where the last equality follows from (2.32). We have expressed p^- in terms of the Virasoro operators and use this to construct the mass formula:

$$M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{2}{\alpha'}(L_0^\perp + \bar{L}_0^\perp - 2) - p^Ip^I, \tag{3.20}$$

or, in terms of the number operators,

$$M^2 = \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2). \tag{3.21}$$

3.2 Closed string state space

Finally, we arrive at the interesting part of closed string theory, its state space. We will be witness of the remarkable result that string theory gives rise to gravitons. String theory was first thought of to be a theory of the strong force, until it was found that one of the particles in the spectrum of closed strings was a massless spin 2 particle. String theory was believed to have failed since no such particle was observed in the strong force. But the graviton is a massless spin 2 particle! So now string theory is considered to be a theory of quantum gravity.

Different states that satisfy (3.18) will be discussed. First we need a vacuum on which to act. The time-independent states of the quantum theory are labeled by the eigenvalues of a maximal set of commuting operators. Because it is usually convenient to work in momentum space, we will work with the operators p^+ and p^I. The vacuum will be denoted by $|p^+, \vec{p}_T\rangle$, where p^+ is the eigenvalue of the p^+ operator and $\vec{p}_T$ is the transverse momentum, the components of which are eigenvalues of the p^I operators.

3.2.1 Tachyon states

Since we need to satisfy $N^\perp = \bar{N}^\perp$, the first state we can think of has $N^\perp = \bar{N}^\perp = 0$. They are ground states and have $M^2 = -\frac{4}{\alpha'}$, i.e., they have a negative mass! These states are called tachyons and they are unstable. Their instability is expected to be an instability of spacetime itself. They remain largely mysterious and we will not consider them any further.

3.2.2 Massless states

The next state we can think of has $N^\perp = \bar{N}^\perp = 1$. Since we want the lowest possible excited state, we take one oscillator from the left-sector and one from the right-sector, both with the lowest possible mode number, i.e. mode number one. Recall that the eigenvalue of a number operator is the sum of the mode numbers of the creation operators appearing in the state. So if we act with two creation operators with mode number one, the corresponding states are massless. The most general state we can write down is

$$\sum_{I,J} R_{IJ}\, a_1^{I\dagger}\, \bar{a}_1^{J\dagger}\, |p^+, \vec{p}_T\rangle. \tag{3.22}$$

Here I and J are completely arbitrary labels attached to different oscillators. Because they can each take 24 different values (see (2.12)), the number of states is $(D-2)^2$ where D=26. The R_{IJ} are elements of an arbitrary square matrix of size $(D-2)$.

One can ask oneself the question what it is exactly that the creation operators create. They create oscillations on the string so that, by acting with all kinds of operators on the vacuum, we can make the string wiggle in all sorts of ways.

A general square matrix is reducible, it contains subspaces that do not mix with the other subspaces. We can always decompose a square matrix in its symmetric and antisymmetric part

$$R_{IJ} = \frac{1}{2}(R_{IJ} + R_{JI}) + \frac{1}{2}(R_{IJ} - R_{JI}) \equiv S_{IJ} + A_{IJ}, \tag{3.23}$$

where S_{IJ} is the symmetric part and A_{IJ} the antisymmetric part of R_{IJ}. The symmetric part can be decomposed further into a symmetric traceless part and a trace. Let $\hat{S}_{IJ}$ denote the traceless part and let $S' = \delta^{IJ} S_{IJ}/(D-2)$FF. The matrix R_{IJ} now becomes

$$R_{IJ} = \hat{S}_{IJ} + A_{IJ} + S'\delta_{IJ}. \tag{3.24}$$

These terms are independent and they will not mix with each other. They cannot be decomposed any further, so we can now split the states in (3.22) in groups of linearly independent states:

$$\sum_{IJ} \hat{S}_{IJ}\, a_1^{I\dagger}\, \bar{a}_1^{J\dagger}\, |p^+, \vec{p}_T\rangle, \tag{3.25}$$

$$\sum_{IJ} A_{IJ}\, a_1^{I\dagger}\, \bar{a}_1^{J\dagger}\, |p^+, \vec{p}_T\rangle, \tag{3.26}$$

$$S' a_1^{I\dagger}\; \bar{a}_1^{I\dagger}\, |p^+, \vec{p}_T\rangle. \tag{3.27}$$

We now claim that the states in (3.25) represent one-particle graviton states. Why this is the case, can be read in [3]. So gravity appears in string theory!

The states in (3.26) correspond to the one-particle states of the Kalb-Ramond field, an antisymmetric tensor field $B_{\mu\nu}$ with two indices. This field can be thought of as a generalization of the Maxwell field A_μ. The Kalb-Ramond field couples to strings in a way that is analogous to the way that the Maxwell fields couples to particles, so strings carry Kalb-Ramond charge.

The state in (3.27) has no free indices since I is summed over, so it represents only one state. It corresponds to a one-particle state of a massless scalar field. The field is called the *dilaton field*.

To summarize, the massless subsector of closed string theory contains gravity fields, Kalb-Ramond fields, and dilaton fields. We can of course also construct massive states, but they are so massive that we cannot even see them in experiments. We do not have enough energy to make them. They are so massive because from (3.21) we see the mass will have a factor $1/\alpha'$ and α' is a very small number.[5] Such massive states will not interesting for us.

4 T-duality of closed strings

In this section we will talk about T-duality. This is a symmetry which relates two systems that have very different description, but identical physics. We will see that a world where one dimension is curled up into a circle of radius R cannot be distinguished from a world in which the circle has radius α'/R.

We will make only one dimension compact, since this is the most simple case but it does show some interesting features. As soon as we compactify a dimension, the relation (2.29) will not hold any more. We will still have the level matching condition but this time it does not lead to a constraint of the form $N^\perp = \bar{N}^\perp$. We will again construct the space state and interpret the results.

4.1 Winding and dual coordinates

Bosonic string theory tells us we are living in a 26-dimensional spacetime. This is a lot more than the 4 dimensions we see. Now of course we know bosonic string theory is not all there is (like its name suggests, it does not contain fermions) but even the other string theories that we know of have at

[5] The length of the string is in fact $\sqrt{\alpha'}$.

least 10 spacetime dimensions. We must therefore find a way to get rid of the extra dimensions. We do not see them at low energy experiments, but nonetheless they are there. How do we interpret such dimensions? We say they are curled up, or compact.

4.1.1 Compact dimensions

Imagine living in a world with only one spatial dimension. The dimension is infinite, but by making *identifications* we can make it compact. We declare that points with coordinates that differ by $2\pi R$ are exactly the same point, meaning that if we would walk a distance $2\pi R$, we would find ourselves at the same point where we started. This is of course always the case in a compact dimension: you will get back to the same point over and over again as long as you keep on walking. It is therefore the same as walking on a circle. So by making the identification

$$x \sim x + 2\pi R, \tag{4.1}$$

we transformed the an infinite dimension into a circle. The interval $0 \leq x < 2\pi R$ is called the *fundamental domain* for the identification (4.1). A fundamental domain is a subset of the entire space in which no two points are identified. Any point in the entire space is in the fundamental domain or is related by the identification to some point in the fundamental domain.

4.1.2 Winding

In this section we discuss the effects on closed strings when one spatial dimension has been made into a circle. The closed strings we have considered up until now were moving in Minkowski space, a simply connected space. If we have one or more compact dimensions, not all closed strings can be reduced continuously to zero size. We will only talk about one compact dimension here, since it is easiest to visualize and it is enough to introduce winding. When we discuss double field theory however, more dimensions will be compact. It is just a generalization of what we discuss here.

Imagine a world with only two spatial dimensions, one of which is compact. Such a world can be thought of as the surface of an infinitely long cylinder. Let x be the coordinate that has been made compact via the identification (4.1). We thus have $x \in [0, 2\pi R]$ and $y \in [-\infty, \infty]$.

We will now consider different strings living on this two-dimensional surface. Let us look at Figure 2. On the left we see strings on the two-dimensional surface. On the right is the *covering space* of the cylinder, this is the plane with identification. Circles are represented as lines here, of which

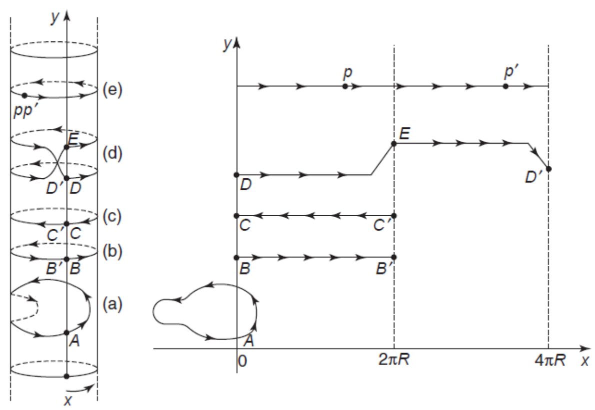

Figure 2: *Left: closed strings living on the surface of a cylinder. Right: the same strings in covering space.*[3]

the endpoints are identified. We will denote the x coordinate of a string as X.

The simplest string is a string that does not wrap around the compact dimension. This is string (a) in Figure 2. We see that it is a closed string in the covering space and it satisfies

$$X(\tau, \sigma = 2\pi) = X(\tau, \sigma = 0). \tag{4.2}$$

This is the same condition as (2.20). This string can be continuously shrunk into a point.

But now look at string (b). This string wraps around the compact dimension once. It is a straight line in covering space and the points B and B' are identified. We can clearly see from covering space that the periodicity condition is now slightly changed. All the strings are parameterized by $\sigma \in [0, 2\pi]$, so that the point B corresponds to $\sigma = 0$ and the point B' corresponds to $\sigma = 2\pi$. We get the following periodicity condition:

$$X(\tau, \sigma = 2\pi) = X(\tau, \sigma = 0) + 2\pi R. \tag{4.3}$$

A consequence of this slightly changed periodicity condition is that the mode expansion of the string coordinate of string (b) will also slightly change. We will discuss this in a minute.

String (c) is also wrapped around the compact dimension once, but now in the opposite direction. This gives a similar condition as (4.3) but now with $-2\pi R$ instead of $+2\pi R$.

String (d) and string (e) are both wrapped twice around the cylinder. They satisfy the condition

$$X(\tau, \sigma = 2\pi) = X(\tau, \sigma = 0) + 2 * 2\pi R. \tag{4.4}$$

Of course a more general case would be that we wrap around the compact dimension as many times as we want. Do not forget that σ always ranges from 0 to 2π. We say that the string has *winding number* m, with m an integer, if it wraps m times around the cylinder in the direction of positive x. This means that if we say string (b) has winding number +1, then string (c) has winding number -1. A generic string now satisfies the boundary condition

$$\text{winding number } m: \quad X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + m2\pi R. \tag{4.5}$$

The winding number appears because we have actually two circles: one with coordinate σ and one with coordinate x. The closed strings are mappings from the σ-circle into the x-circle. The mapping of one circle into another is characterized by an integer, the winding number of the map. But the winding number m actually never shows up explicitly in formulae. We here introduce the *winding* w since it plays an important role in double field theory. It is defined as follows

$$w \equiv \frac{mR}{\alpha'}. \tag{4.6}$$

If winding is defined in this way, it has dimensions of momentum,[6] and it will turn out we can indeed interpret it as a new kind of momentum. We write the periodicity condition in terms of w

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + 2\pi\alpha' w. \tag{4.7}$$

4.1.3 Mode expansion

We will consider strings that are propagating in a 26-dimensional spacetime. For simplicity we only make one dimension compact and we work in light-cone coordinates. The coordinates will be organized in such a way that X^{25} is the one curled up into a circle. We thus have

$$X^+, X^-, \underbrace{X^2, X^3, \ldots, X^{24}}_{X^i}, X^{25}, \tag{4.8}$$

where the X^i are the transverse light-cone coordinates. We will not consider them any more since we already discussed them in sections 2 and 3. They are noncompact and they do not have winding. We will here only consider the compact dimension X^{25}, which we will just call just X from now on.

We can now follow the same steps as in 2.2.6. We will not go through all the steps again since the changes are minor. The compact coordinate

[6] Since R is a length and α' is a length squared, winding takes dimensions $[\text{length}]^{-1}$, which is a momentum dimension since we work in natural units.

X still satisfies the wave equation, so equations (2.17) and (2.18) still hold. However, since our periodicity condition has now slightly changed, we get

$$X_L(u+2\pi) - X_L(u) = X_R(v) - X_R(v-2\pi) + 2\pi\alpha' w. \tag{4.9}$$

(Recall that $u = \tau + \sigma$ and $v = \tau - \sigma$) This reduces to (2.25) if we have no compact dimensions, i.e. if $w = 0$. The left hand side is still independent of v and the right hand side is still independent of u, so the mode expansions of $X_L(u)$ and $X_R(v)$, (2.27), still hold (only now without the μ index since we consider only one compact dimension) since their derivatives are still 2π-periodic.

But now we fill in (4.9) and see that we do not get $\alpha_0 = \bar{\alpha}_0$ but rather

$$\bar{\alpha_0} = \alpha_0 + \sqrt{2\alpha'}w, \tag{4.10}$$

which of course reduces to (2.29) when w equals zero.

We can also calculate the momentum p of the string along the compact direction:

$$p = \int_0^{2\pi} \mathcal{P}^\tau(\tau,\sigma)d\sigma = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma(\dot{X}_L + \dot{X}_R) = \frac{1}{\sqrt{2\alpha'}}(\alpha_0 + \bar{\alpha}_0). \tag{4.11}$$

To emphasize the resemblance between the momentum p and the winding w we write

$$\begin{aligned} p &= \frac{1}{\sqrt{2\alpha'}}(\bar{\alpha}_0 + \alpha_0), \\ w &= \frac{1}{\sqrt{2\alpha'}}(\bar{\alpha}_0 - \alpha_0). \end{aligned} \tag{4.12}$$

This suggests that the winding w is on the same footing as the momentum p. Their dimension is the same and we can think of both of them as momentum operators. It is convenient to record the values of the zero modes:

$$\begin{aligned} \alpha_0 &= \sqrt{\frac{\alpha'}{2}}(p-w), \\ \bar{\alpha}_0 &= \sqrt{\frac{\alpha'}{2}}(p+w). \end{aligned} \tag{4.13}$$

By using these equations we can write the mode expansions (2.27) (again without the μ index) as

$$X_L(\tau+\sigma) = \frac{1}{2}x_0^L + \frac{\alpha'}{2}(p+w)(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{\alpha}_n}{n}e^{-in(\tau+\sigma)},$$

$$X_R(\tau-\sigma) = \frac{1}{2}x_0^R + \frac{\alpha'}{2}(p-w)(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-in(\tau-\sigma)}. \qquad (4.14)$$

Just as we did in section 2.2.6, we will write down the full solution of our string coordinate and our dual coordinate. Recall that $X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma)$ and $\tilde{X}(\tau,\sigma) = X_L(\tau+\sigma) - X_R(\tau-\sigma)$. As promised, the dual coordinate now *does* represent a physical solution

$$X(\tau,\sigma) = x_0 + \alpha' p\tau + \alpha' w\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\bar{\alpha}_n e^{-in\sigma} + \alpha_n e^{in\sigma}),$$

$$\tilde{X}(\tau,\sigma) = q_0 + \alpha' w\tau + \alpha' p\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\bar{\alpha}_n e^{-in\sigma} - \alpha_n e^{in\sigma}), \qquad (4.15)$$

where we have defined

$$x_0 \equiv \frac{1}{2}(x_0^L + x_0^R),$$

$$q_0 \equiv \frac{1}{2}(x_0^L - x_0^R). \qquad (4.16)$$

The dual coordinate did not represent a physical solution before since it did not satisfy the periodicity condition (2.20) for a string in a noncompact spacetime. But since we are now working with a compact dimension, this periodicity condition has changed into (4.7). It can easily be seen that this condition is satisfied by the normal string coordinate $X(\tau,\sigma)$. But we still see that the dual coordinate violates this condition! How can this be? Instead of (4.7) we get

$$\tilde{X}(\tau,\sigma+2\pi) = \tilde{X}(\tau,\sigma) + 2\pi\alpha' p. \qquad (4.17)$$

The winding w has been replaced by the momentum p! We will discuss in a minute that this is no coincidence. In the dual world, the roles of momentum and winding have interchanged and therefore the dual coordinate *should* satisfy (4.17) rather than (4.7).

Let us now interpret equations (4.15). For the normal string coordinate we get what we expected. Since we added the left-movers and right-movers, the terms containing pw and $\tau\sigma$ dropped out. The centre of mass momentum p of the string (which is in this case just the momentum in the compact direction since we consider only one dimension) multiplies the time τ. It is therefore the momentum associated with the coordinate x_0. The last term represents, just like before, the oscillations on the string itself. If the winding term would not be there, the result would be exactly the same as (2.30) (without the μ index of course). The role of the winding is more apparent if we look at the expansion of the dual coordinate.

To form the dual coordinate, we *substract* the right-movers from the left-movers. As a consequence, the terms containing $p\tau$ and $w\sigma$ drop out. Instead of p, the 'winding' w now multiplies τ! We put this between exclamation marks because w does not play the role of winding any more. The coordinate zero mode has also changed. We interpret this as w being the momentum associated with the coordinate q_0. The winding associated with q_0 is now p. The roles of momentum and winding have thus interchanged.

This is a very important result. It will be the main reason that a world with a compact dimension of radius R *cannot* be physically distinguished from a world where the compact dimension has radius α'/R. This seems absurd, how can we not notice whether we live in a very big dimension or a very small one? The interchangeability of winding and momentum plays a key role in understanding this.

4.2 Quantization of compactified closed strings

Just like before, we will not be bothered by the derivation of the commutation relations. They are in fact the same as in (3.1), only now without the superscript index. We can interpret the winding w as an operator, just like p. When we treat w as an operator, the eigenvalues will correspond to the various possible windings.

4.2.1 A discrete spectrum

Because we have compactified the x-direction, the zero mode x_0 is a coordinate that lives on a circle with radius R. This results in the momentum operator p being quantized. For suppose we make a translation along the x-direction by an amount y. This is done by acting with the operator e^{-iyp}. Since x_0 lives on a circle of radius R, a translation of $2\pi R$ should not make any difference. We get back to exactly the same point. This is the same as saying the condition

$$e^{-ipy} = e^{-ip(y+2\pi R)} \tag{4.18}$$

must hold. This means that $e^{-ip2\pi R} = 1$ which results in

$$p = \frac{n}{R}, \quad n \in \mathbb{Z}. \tag{4.19}$$

This is not something that holds only for strings. For particles the momentum will also be quantized if we work in compact dimensions. It is analogous to the case of a particle in a box. The momentum spectrum of a free particle is continuous, but as soon as we put the particles in an infinite potential well, the momentum will become quantized.

The winding w is also quantized, which we can see from its definition (4.6). Now we will come to an important conclusion. We just saw that the coordinate x_0 lives on a circle of radius R resulting in the momentum p being quantized with eigenvalues n/R. Now we know the winding w is quantized with eigenvalues mR/α' and this suggests that the coordinate q_0 lives on a circle with radius $\tilde{R} = \alpha'/R$. The dual string coordinate $\tilde{X}$ is thus a coordinate living on a circle of radius α'/R! We will come back to this in section 4.4.

4.2.2 Mass formula

We now return to the picture where we have 25 noncompact spacetime dimensions, and one compact spacial dimension, i.e. (4.8).

Since we have not compactified the $-$ direction, we still have $\alpha_0^- = \bar{\alpha}_0^-$ and thus the level matching condition $L_0^\perp = \bar{L}_0^\perp$ still holds. This time however, this will not lead to the equality of the barred and unbarred number operators. This will be important once we start constructing the state space of compactified closed strings. To see why we get $N^\perp \neq \bar{N}^\perp$ we look carefully at the definitions of the Virasoro operators and split out the compact dimension. By using (3.15),(3.9) and (2.32), we find

$$\begin{aligned}\bar{L}_0^\perp &= \frac{1}{2}\bar{\alpha}_0^I\bar{\alpha}_0^I + \bar{N}^\perp \\ &= \frac{1}{2}\bar{\alpha}_0^i\bar{\alpha}_0^i + \bar{\alpha}_0\bar{\alpha}_0 + \bar{N}^\perp \\ &= \frac{\alpha'}{4}p^ip^i + \frac{1}{2}\bar{\alpha}_0\bar{\alpha}_0 + \bar{N}^\perp. \end{aligned} \tag{4.20}$$

A similar expression holds for $L_0^\perp$ but then without the bars. The number operators $N^\perp$ and $\bar{N}^\perp$ include contributions from all the oscillators, those corresponding to the noncompact directions with superscript i and those correspond to the compact coordinate X. We can now calculate

$$L_0^\perp - \bar{L}_0^\perp = \frac{1}{2}(\alpha_0\alpha_0 - \bar{\alpha}_0\bar{\alpha}_0) + N^\perp - \bar{N}^\perp = -\alpha' pw + N^\perp - \bar{N}^\perp, \tag{4.21}$$

where we have used (4.13). By using the level matching condition $L_0^\perp = \bar{L}_0^\perp$, we get

$$N^\perp - \bar{N}^\perp = \alpha' pw. \tag{4.22}$$

This is the constraint that must be satisfied to represent a physical state. We see that if there is neither winding nor momentum (momentum in the compact dimension), the number of left-moving and the number of right-moving excitations must be the same. This is of course the same result as the one we got in section 3 where we did not have a compact dimension. But for states that have both nonzero winding and momentum, the number of left-moving and right-moving excitations will not be the same. We can write the constraint (4.22) in a more useful way by using (4.19) and (4.6):

$$N^{\perp} - \bar{N}^{\perp} = nm. \tag{4.23}$$

To study the string spectrum, we need a formula for the mass of the states. We will construct such a formula from the viewpoint of an observer that lives in the 25-dimensional Minkowski space. The observer does not see the compact dimension, he does not know this extra dimension is present. But of course if there is any momentum in the compact direction, the corresponding energy cannot just disappear. The observer will 'see' this energy as extra mass. So if there is momentum in the compact direction, an observer living in the 25-dimensional Minkowski space will measure a higher rest mass than an observer living in the 26-dimensional spacetime.

Just like we did in section 3, we will start with the formula $M^2 = -p^2$ where p only contains the noncompact components p^+, p^-, en p^i. We thus have

$$M^2 = -p^2 = 2p^+p^- - p^ip^i = \frac{1}{\alpha'}(\alpha_0\alpha_0 + \bar{\alpha}_0\bar{\alpha}_0) + \frac{2}{\alpha'}(N^{\perp} + \bar{N}^{\perp} - 2), \tag{4.24}$$

where we have used (3.19) to write p^- in terms of the Virasoro operators, and (4.20) to rewrite the Virasoro operators. Finally we use (4.13) to write

$$M^2 = p^2 + w^2 + \frac{2}{\alpha'}(N^{\perp} + \bar{N}^{\perp} - 2). \tag{4.25}$$

We already explained why the term p^2 enters in the rest mass. From our viewpoint we just do not see the compact dimension so any momentum in this direction contributes to the rest mass. But what does the term w^2 mean? This is a real stringy term. If the string winds m times around the compact dimension, its length is $2\pi mR$. The energy is given by the product of the string length and the tension in the string. Recall (section 2.2.4) that α' is defined as $\frac{1}{2\pi T}$ where T is the tension in the string. We thus have

$$\text{energy} = \text{length} * \text{tension} = 2\pi Rm\,\frac{1}{2\pi\alpha'} = \frac{Rm}{\alpha'} = w, \tag{4.26}$$

and we find that w^2 indeed has dimensions of [mass]^2. The term w^2 in the mass formula thus tells us that a string gains energy if it is stretched out.

4.3 State space of compactified closed strings

Finally, we are able to construct the state space of compactified closed strings. Again, we will only be interested in the massless states. Just like we did before, we start by defining a vacuum. We use the familiar labels that are associated with 25-dimensional Minkowski space, but the states now carry additional labels that specify the momentum and the winding along the compact dimension. Since the momentum is quantized as $p = n/R$, we can use n as a label for the momentum of the state. Similarly, we will use m as a label for the winding of the state. The vacuum will thus be denoted by $|p^+, \vec{p}_T; n, m\rangle$. Note that not all these states are physical, they still have to satisfy (4.23). Since in the vacuum we have no excitations ($N^\perp = \bar{N}^\perp = 0$) this means that either n or m must be zero. States with both n and m nonzero are not allowed, but they *can* be acted upon by combinations of creation operators to produce allowed states.

Since we are working with 25 noncompact dimensions and one compact dimension, we have to seperate out the oscillators that arise from the compact dimension. They carry no 25-dimensional Lorentz index. We thus use operators of the form $a_n^{i\dagger}$ and $a_n^\dagger$ and their barred versions. For convenience we write (4.25) as

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2). \tag{4.27}$$

4.3.1 States with m=n=0: the (1,1) sector

These states have neither winding nor momentum in the compact direction. Such a state could for example be string (a) in Figure 2 provided that it has no momentum along the x axis. We expect these states to be the same as the ones discussed in section 3.2, with the only difference being that the states now live in a 25-dimensional Minkowski spacetime instead of a 26-dimensional one. The constraint (4.23) tells us that $N^\perp = \bar{N}^\perp$ which means we should act with the same number of left-moving oscillators as right-moving oscillators.

The vacuum itself has no oscillators at all and we see from the mass formula that it is a tachyon state with $M^2 = -\frac{4}{\alpha'}$.

For the next excited states we act with two operators, one from the left and one from the right sector. To get massless states we give them both mode number one, we thus have $(N^\perp, \bar{N}^\perp) = (1, 1)$. For both sectors we can act with two kinds of oscillators (those that belong to the compact direction and those that do not) so there are four ways we can combine the oscillators to form massless states:

$$\begin{aligned}
&a_1^\dagger \bar{a}_1^\dagger \, |p^+, \vec{p}_T; 0, 0\rangle, \\
&a_1^\dagger \bar{a}_1^{i\dagger} \, |p^+, \vec{p}_T; 0, 0\rangle, \\
&a_1^{i\dagger} \bar{a}_1^\dagger \, |p^+, \vec{p}_T; 0, 0\rangle, \\
&a_1^{i\dagger} \bar{a}_1^{j\dagger} \, |p^+, \vec{p}_T; 0, 0\rangle.
\end{aligned} \tag{4.28}$$

The first line contains only one state. It carries no 25-dimensional index, meaning it is a massless scalar field. The states in the second and third line each carry the index i, they are photon states. Each set of states corresponds to a Maxwell field, so we get a total of two Maxwell fields. Finally, the states in the fourth line have exactly the same structure as the massless closed string states of Minkowski space, except that the dimensionality is now reduced to 25. These states therefore comprise a gravity field, a Kalb-Ramond field and a dilaton.

How can these states be the same as the ones discussed in section 3? We did not even get one Maxwell field there, let alone two! We also seem to have an extra scalar field now. Fortunately, there is a nice interpretation for this. Suppose all the 26 spacetime dimensions are noncompact. We use μ to tell us in which dimension we are. We can write $\hat{\mu} = \{\mu, 25\}$ where we have done nothing special, we just write the spacial dimension labeled by 25 separately. This is the same situation which was discussed in section 3.2 so we get the fields

$$\begin{aligned}
&h_{\hat{\mu}\hat{\nu}} \\
&B_{\hat{\mu}\hat{\nu}} \\
&\phi
\end{aligned} \tag{4.29}$$

These are the gravity, Kalb-Ramond and dilaton fields as we discussed them in section 3.2.

We now compactify the direction with label 25. As an observer living in 25-dimensional spacetime we do not see this compact direction, we only see the 25 noncompact directions. We can take $\hat{\mu}, \hat{\nu}$ to both lie along μ, we can take only one of them to lie along μ or none of them, in which case they both take the value 25. This results in the following fields:

We see that $h_{\mu\nu}, B_{\mu\nu}$ and ϕ are the 25-dimensional gravity, Kalb-Ramond and dilaton fields which correspond to the fourth line in (4.28). The fields in the second column have only one Lorentz index. They are vectors from the viewpoint of a 25-dimensional observer. These are the two extra vector fields we found. The last column does not have any Lorentz indices at all, these

$\hat{\mu}, \hat{\nu}$ both along μ	only one of them along μ	none of them along μ
$h_{\mu\nu}$	$h_{\mu 25}$	h_{2525}
$B_{\mu\nu}$	$B_{\mu 25}$	
ϕ		

Table 1: *Kaluza Klein reduction*

states represent scalars from a 25-dimensional viewpoint. We only get one extra scalar field since B_{2525} is zero because B is an antisymmetric tensor (so $B_{2525} = -B_{2525} = 0$).

To summarize, we see that the states we found in section 3.2 and the states in (4.28) are actually the same states, they are just reorganized. This is called *Kaluza Klein reduction*

4.3.2 States with n=0 or m=0

These states still satisfy $N^{\perp} - \bar{N}^{\perp} = 0$. The ground states are

$$\begin{aligned} &|p^+, \vec{p}_T; n, 0\rangle, && M^2 = \frac{n^2}{R^2} - \frac{4}{\alpha'}, \\ &|p^+, \vec{p}_T; 0, m\rangle, && M^2 = \frac{m^2 R^2}{\alpha'^2} - \frac{4}{\alpha'}. \end{aligned} \tag{4.30}$$

Both sets of states correspond to scalar fields since they have no Lorentz indices. The states can be tachyonic, massless, or massive, depending on the value of the radius R. This is the case from the viewpoint of an observer that lives in 25-dimensional Minkowski spacetime. Acting with excitations operators on these vacua produces heavier states. Such states have $N^{\perp} + \bar{N}^{\perp} \geq 2$ and, as a result, they are massive for all values of the radius R.

4.3.3 States with $\mathbf{n} = \mathbf{m} = \pm 1$ or $\mathbf{n} = -\mathbf{m} = \pm 1$

Here we only get massless states at a particular radius $R^* = \sqrt{\alpha'}$. This is called the self-dual radius for reasons that will become clear later. We will not be interested in these massless fields since they only exist at this dual radius, not in general situations.

4.3.4 The (2,0) and (0,2) sectors

The mass formula given in (4.27) gives the mass in 25 dimensions. A state that is massive in 25 dimensions, can perfectly well be massless in 26 dimensions. The mass formula from the viewpoint of an observer living in 26 dimensions is given by

$$\mathcal{M}^2 = \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2) \tag{4.31}$$

The only difference with (4.27) is that we have no contribution of the winding and momentum in the compact direction. This makes sense, we do not treat the compact dimension differently from all the other dimensions in this case. If a string is moving in the compact dimension, this will not contribute to the rest mass any more.

If we use (4.31) to calculate the mass of a state, we get extra massless fields. These fields are massless in 26 dimensions, but not in lower dimensional situations. If we want the states to be massless in both 26 (10 in the case of super string theory) and 25 (or less) dimensions, the states must have $n = m = 0$ to get $\mathcal{M}^2 = M^2 = 0$. This means we cannot have nonzero winding and nonzero momentum at the same time. Since $N^\perp - \bar{N}^\perp = nm$ must always be satisfied, the only sector we then have, is the sector where $(N^\perp, \bar{N}^\perp) = (1, 1)$. As we discussed, this sector gives rise to the massless fields $h_{\mu\nu}, B_{\mu\nu}$, and ϕ. But the sector also gives massive states, since if $N^\perp = \bar{N}^\perp = 1$ we have $nm = 0$ and if we have no winding (m=0), n can take any value. Also, if we have no momentum (n=0), m can take any value and the level matching condition will still be satisfied. So the (1,1) sector has one unique solution to give massless states in lower dimensions, which is $n = m = 0$. It also gives an infinite tower of massive states, known as the *Kaluza-Klein* tower.

But double field theory does not restrict itself to sectors which give massless fields in the lower dimensional cases. Here we want to consider the fields that give massless states in 10 dimensions, irrespective of the mass the states will have in 9 or less dimensions. This means we do not need to restrict ourselves to the (1,1) sector. By looking at (4.31) we see that we get extra massless fields if $(N^\perp, \bar{N}^\perp) = (2, 0), (0, 2)$.

For future use, we will discuss these sectors here. We get eight different vacua since we can construct eight different combinations of n and m which satisfy (4.23) and $(N^\perp, \bar{N}^\perp) = (2, 0), (0, 2)$. They are listed in the table below.

$(N^\perp, \bar{N}^\perp) = (2, 0)$	$(N^\perp, \bar{N}^\perp) = (0, 2)$
$(n, m) = (1, 2), (2, 1), (-1, -2), (-2, -1)$	$(n, m) = (-1, 2), (1, -2), (2, -1), (-2, 1)$

Table 2: *The possible values of* (n, m)

Let us consider the possibility $(n, m) = (1, 2)$. We get the following states:

$$
\begin{aligned}
|p^+, \vec{p}_T, 1, 2\rangle \qquad & \rightarrow \text{vacuum} \\
a_1^{I\dagger} a_1^{J\dagger} |p^+, \vec{p}_T, 1, 2\rangle \qquad & \rightarrow \text{gravity field + scalar field} \\
a_2^{I\dagger} |p^+, \vec{p}_T, 1, 2\rangle \qquad & \rightarrow \text{vector field}
\end{aligned} \tag{4.32}
$$

In the second line, the operators $a_1^{I\dagger}$ and $a_1^{J\dagger}$ commute. Instead of a general matrix of size $(D-2)$ that splits in a symmetric traceless, anti-symmetric and a trace part, we now have a symmetric matrix that splits in a symmetric traceless matrix and a trace. That is why we do not get a Kalb-Ramond field this time, but we do get a graviton and a scalar field. The third line just gives a vector fields since the creation operator has one Lorentz-index.

All the other possibilities listed in Table 2 give the same fields. In total we thus get eight extra gravitons, scalar fields and vector fields. We will come back to this in section 6.2.

4.4 T-duality as an exact symmetry

In this section we will discuss a remarkable property of the spectrum of the compactified string. We will show that the spectrum for a compactification with radius R is identical to one with radius $\tilde{R} = \frac{\alpha'}{R}$, which is called the dual radius. But not only the mass spectrum is identical, it turns out that the two compactifications are physically indistinguishable! This is called T-duality of closed string theory.

4.4.1 Identical Mass spectra

To show this property we must take a closer look into equation (4.27), which reads

$$
M^2 = \Big(\frac{n}{R}\Big)^2 + \Big(\frac{mR}{\alpha'}\Big)^2 + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2), \qquad N^\perp - \bar{N}^\perp = nm. \tag{4.33}
$$

We see that if we replace R by $\frac{\alpha'}{R}$, the formula almost takes the same form

$$
\begin{aligned}
M^2\Big(R; n, m\Big) &= \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2), \\
M^2\Big(\tilde{R}; n, m\Big) &= \frac{n^2 R^2}{\alpha'^2} + \frac{m^2}{R^2} + \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2).
\end{aligned} \tag{4.34}
$$

One might say these formulae are not the same, but the difference is merely superficial. As n and m run over all possible integers, the *lists* of masses are the same in both cases. So explicitly, we have

$$M^2\Big(R; n, m\Big) = M^2\Big(\tilde{R}; m, n\Big), \qquad \forall\ n, m \in \mathbb{Z}. \tag{4.35}$$

Of course this equation only holds if we are comparing states with identical oscillator structure. Note that the exchange of n and m does not affect the constraint in (4.33). This proves that the mass spectra of theories with dual radii are identical. The exchange of n and m is of course just the exchange of winding and momentum.

But now we have only proved that the mass spectra are equal. We still have to show that the physics is identical at dual radii. But of course we remember the discussion in section 4.2.1, where we noted that the dual coordinate given in (4.15) lives on a circle of radius $\frac{\alpha'}{R}$! This implies that a theory where we use the dual coordinate $\tilde{X}$ to describe the strings living on a circle of radius $\frac{\alpha'}{R}$ is identical to a theory where we use the 'normal' coordinate X living on a circle of radius R (assuming of course that m and n are exchanged plus some additional exchanges we will now discuss). To establish the equivalence of the two theories we need to make a map between them.

First we list the operators, quantization conditions and commutations relations for both theories in Table 3 and Table 4 below.

Theory with radius R
$H(R) = \frac{1}{2}\alpha' p^i p^i + \frac{1}{2}\alpha'(p^2 + w^2) + N^\perp + \bar{N}^\perp - 2$
x_0 lives on a circle of radius R
p has eigenvalues n/R
q_0 lives on a circle of radius α'/R
w has eigenvalues mR/α'
$[x_0, p] = [q_0, w] = i$
$[\bar{\alpha}_m, \bar{\alpha}_n] = [\alpha_m, \alpha_n] = m\delta_{m+n,0}$
$[\alpha_m, \bar{\alpha}_n] = 0$

Table 3: *A theory where the X coordinate lives on a circle of radius R*

We have placed a tilde over all the operators of the theory with radius α'/R to distinguish them from the operators of the other theory. The formula for the Hamiltonian is also given and although we did not discuss how to derive this formula[7] we show it here to make the equivalence of both theories clear.

[7]The derivation of the Hamiltonian is actually quite simple. The Hamiltonian should generate τ translations. Recall the light-cone gauge $X^+ = \alpha' p^+ \tau$, so that we have $\frac{\partial}{\partial\tau} = \frac{\partial X^+}{\partial\tau}\frac{\partial}{\partial X^+} = \alpha' p^+ p^-$ where we used that p^- generates X^+ translations which is

Theory with radius α'/R
$\tilde{H}(\tilde{R}) = \frac{1}{2}\alpha' p^i p^i + \frac{1}{2}\alpha'(\tilde{p}^2 + \tilde{w}^2) + \tilde{N}^\perp + \bar{\tilde{N}}^\perp - 2$
$\tilde{x}_0$ lives on a circle of radius R
$\tilde{p}$ has eigenvalues n/R
$\tilde{q}_0$ lives on a circle of radius α'/R
$\tilde{w}$ has eigenvalues mR/α'
$[\tilde{x}_0, \tilde{p}] = [\tilde{q}_0, \tilde{w}] = i$
$[\bar{\tilde{\alpha}}_m, \bar{\tilde{\alpha}}_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n,0}$
$[\tilde{\alpha}_m, \bar{\tilde{\alpha}}_n] = 0$

Table 4: *A theory where the X coordinate lives on a circle of radius α'/R*

The mapping we make between the theories is as follows:

$$\begin{Bmatrix} x_0 & \rightarrow & \tilde{q}_0 \\ q_0 & \rightarrow & \tilde{x}_0 \end{Bmatrix} \quad \begin{Bmatrix} p & \rightarrow & \tilde{w} \\ w & \rightarrow & \tilde{p} \end{Bmatrix} \quad \begin{Bmatrix} \bar{\alpha}_n & \rightarrow & \bar{\tilde{\alpha}}_n \\ \alpha_n & \rightarrow & -\tilde{\alpha}_n. \end{Bmatrix} \tag{4.36}$$

This mapping is suggested by comparing the two lines of (4.15). For all operators associated with the 25-dimensional Minkowski spacetime, the map is the identity map. A theory on the dual radius is mapped into itself, hence the name dual radius.

We explicitly see that the operators are mapped into others that live on similar spaces and have identical spectra. For example, both x_0 and $\tilde{q}_0$ live on identical circles. Both p and $\tilde{w}$ have the same spectrum. The sign factor in the map of the oscillators does not affect $N^\perp$. The map establishes the physical equivalence of the theories under consideration and it proves that T-duality is an exact symmetry of free closed string theory compactified on a circle.

5 Double field theory

Finally we are ready to discuss double field theory. We will first show that by doubling the coordinates and demanding diffeomorphism invariance, we cannot have a theory of gravity alone. We have to include two extra fields: a Kalb-Ramond field and a dilaton field. So we get exactly the same fields as we have in the massless subsector of closed string theory! We will rewrite the level matching condition so that it becomes the constraint ΔA=0, where

a consequence of the off-diagonal metric in light-cone coordinates. The Hamiltonian is thus given by $H = \alpha' p^+ p^-$ which can be written in terms of the number operators as is done in Table 3 and 4

A is any field or gauge parameter. For the action to cubic order in the fields this constraint is enough to ensure gauge invariance and no extra constraints are needed. However, to prove gauge invariance for the action to all orders, a stronger constraint is needed. We will show why the situation is different if we go beyond cubic interactions and we will show that the strong constraint leads to a consistent theory. Finally, we show that this strong constraint implies that all fields and gauge parameters are restricted. They depend only on the coordinates of a totally null subspace N, so that the theory is related by an $O(D, D)$ transformation to one in which all fields and gauge parameters depend on x but do not depend on $\tilde{x}$.

5.1 Linearised double diffeomorphism symmetry

In this section we show that linearised double diffeomorphism invariance requires the massless multiplet of closed string theory, i.e., the gravity field $h_{\mu\nu}$, the Kalb-Ramond field $b_{\mu\nu}$ and the dilaton ϕ. But first we discuss O(D,D) symmetries and introduce a new notation.

5.1.1 The doubled torus and some words on notation

In section 4 we have discussed the situation where only one of the dimensions was made compact. Of course if we believe super string theory we need to compactify at least six dimensions. One compact dimension it looks like a circle, two compact dimensions form a donut, and d compact dimensions form a d-dimensional torus T^d. Here we will consider the situation in which spacetime is a product of n-dimensional Minkowski space $\mathbb{R}^{n-1,1}$ with a d-dimensional torus T^d, i.e., $\mathbb{R}^{n-1,1} \times T^d$. We thus work in D-dimensional flat space[8] where $D = n + d$.

We will use coordinates $x^i = (x^\mu, x^a)$ with $i = 0, 1, 2, \ldots, D - 1$ where we have split the D coordinates into coordinates x^μ on the n-dimensional Minkowski space $\mathbb{R}^{n-1,1}$ and coordinates x^a on the d-torus T^d. States are labeled by momentum $p_i = (k_\mu, p_a)$ and the string windings w^a (which we of course only need in the compact dimensions, so there is no w^μ). For the coordinates x^a we have the identification $x^a \sim x^a + 2\pi R$ (which is the same as (4.1) only now we have attached a label since we have more than one compact dimension) and the operators p_a and w^a have integer eigenvalues.

By Fourier transforming, dependence on the momenta k_μ, p_a becomes dependence on the spacetime coordinates x^μ, x^a while dependence on w^a becomes dependence on $\tilde{x}_a$. So because we have winding, we can include the

[8] Flat space means that parallel lines never intersect. So a donut is a flat space whereas a sphere is not.

dual coordinates $\tilde{x}$ and we let our fields depend on both the normal and the dual coordinates. This is the main idea of double field theory, the motivation of which was discussed in the introduction.

The fields are fields on $\mathbb{R}^{n-1,1} \times T^{2d}$ where T^{2d} is the doubled torus containing the original torus T^d together with another torus T^d parameterized by the winding coordinates. The doubled torus actually contains the original torus and *all* the tori related to it by T-duality. So T-duality is changing which T^d subspace of the doubled torus is regarded as being part of the spacetime $\mathbb{R}^{n-1,1} \times T^d$.

All physical strings must satisfy the level matching condition, i.e., they must be annihilated by $L_0^\perp - \bar{L}_0^\perp$. This gives $L_0^\perp - \bar{L}_0^\perp = N^\perp - \bar{N}^\perp - p^a w_a = 0$ which is a generalization of (4.22) where now we sum over a since we have more compact dimensions.

5.1.2 O(D,D) symmetries

T-duality is actually an $O(d,d,\mathbb{Z})$ transformation acting on the torus coordinates $x^a, \tilde{x}^a$. An $O(d,d;\mathbb{Z})$ transformation is defined as a transformation that leaves the metric $\hat{\eta}$ invariant. For $g \in O(d,d;\mathbb{Z})$ we have

$$g^T \hat{\eta} g = \hat{\eta}, \quad \hat{\eta} = \begin{pmatrix} -\mathbb{1}_d & 0 \\ 0 & \mathbb{1}_d \end{pmatrix} \quad \text{or} \quad \hat{\eta} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{5.1}$$

where g is a $2d \times 2d$ matrix. The two different forms of the metric are related by a basis transformation and they can both be used. The $\mathbb{Z}$ just means that we are considering matrices g with integer values. This is needed because we want p and w to have integer eigenvalues. The situation we discussed in section 4 was actually an $O(1,1;\mathbb{Z})$ symmetry since we had only one compact dimension. In that case T-duality transforms the x-coordinate into the $\tilde{x}$-coordinate, which is actually an example of *inversion*. Inversion is the transformation which transforms all x to $\tilde{x}$. If we have more than one compact dimension, more general transformations are possible. For example if we compactify two dimensions we get a donut. By doing a T-duality transformation we can change both radii to their dual onces, or only one of them. We thus get four donuts related by T-duality. The T-duality symmetry includes a $\mathbb{Z}_2$ symmetry for each direction a that interchanges x^a with $\tilde{x}^a$. Performing a $\mathbb{Z}_2$ transformation on each of the toroidal dimensions takes a theory in the original spacetime $\mathbb{R}^{n-1,1} \times T^d$ with coordinates x^μ, x^a to a theory in the dual spacetime $\mathbb{R}^{n-1,1} \times \tilde{T}^d$ with coordinates $x^\mu, \tilde{x}^a$, this is inversion.

If we would have D noncompact directions, we would have a continuous $O(D,D)$ symmetry. This time the $\mathbb{Z}$ is not needed since the momentum in all directions would be continuous, not discrete and we would not have winding. As soon as we compactify d dimensions the symmetry breaks into $O(d,d;\mathbb{Z}) \times O(n,n)$ and if we restrict the fields to be independent of the doubled noncompact coordinates, this further breaks to $O(d,d;\mathbb{Z}) \times O(n-1,1)$. An O(D,D) symmetry ensures the Lorentz and T-duality symmetries of the compactified cases relevant to string theory, and it will often be convenient to simply refer to the O(D,D) symmetry in what follows.

5.1.3 The massless multiplet

We start with Einstein's action for gravity but we consider only weak gravity fields. This means we only use the action to quadratic order in the fluctuation field $h_{ij}(x) = g_{ij}(x) - \eta_{ij}$ [1]:

$$(2\kappa^2)S_0 = \int dx \Big[\frac{1}{4}h^{ij}\,\partial^2 h_{ij} - \frac{1}{4}h\partial^2 h + \frac{1}{2}(\partial^i h_{ij})^2 + \frac{1}{2}h\,\partial_i\,\partial_j h^{ij}\Big]. \tag{5.2}$$

The metric η_{ij} is the constant metric we use in Minkowski space. The gravity field h_{ij} is a fluctuation of this constant metric, so we can see a graviton as being a disturbance of flat spacetime. By including more orders of the fluctuation field h_{ij} spacetime will become more and more curved. We also used $h = \eta^{ij}h_{ij}$.

This action is invariant under the following linearised diffeomorphisms

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i. \tag{5.3}$$

This is just saying that the action must not depend on which coordinates we use (for example, physics cannot depend on whether we use Cartesian coordinates or polar coordinates to describe our fields).

Now we start doing double field theory. We will let the gravity field not only depend on the normal coordinates, but also on the dual coordinates, i.e. $h_{ij} = h_{ij}(x^\mu, x^a, \tilde{x}_a)$. The action must now of course also be invariant under coordinate transformations of these dual coordinates! The action must thus be invariant under linearised 'dual diffeomorphisms'

$$\tilde{\delta} h_{ij} = \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i. \tag{5.4}$$

The action (5.2) is the action we get if the gravity field would only depend on x and not on $\tilde{x}$. To include the dependence on $\tilde{x}$ the most natural thing to do is just to double all the terms in the action but now with tilde derivatives:

$$\begin{aligned}(2\kappa^2)S_0 = \int [dx d\tilde{x}] \Big[&\frac{1}{4}h^{ij}\,\partial^2 h_{ij} - \frac{1}{4}h\partial^2 h + \frac{1}{2}(\partial^i h_{ij})^2 + \frac{1}{2}h\,\partial_i\,\partial_j h^{ij} \\ &+\frac{1}{4}h^{ij}\,\tilde{\partial}^2 h_{ij} - \frac{1}{4}h\tilde{\partial}^2 h + \frac{1}{2}(\tilde{\partial}^i h_{ij})^2 + \frac{1}{2}h\,\tilde{\partial}_i\,\tilde{\partial}_j h^{ij}\Big].\end{aligned} \tag{5.5}$$

The first line in (5.5) is of course invariant under (5.3) and the second line is invariant under (5.4). But the first line is *not* invariant under the dual diffeomorphisms (5.4)! After varying under $\tilde{\delta}$ we get

$$(2\kappa^2)\tilde{\delta}S = \int [dxd\tilde{x}]\Big[(\tilde{\partial}_j h^{ij})\partial^k(\partial_i\tilde{\epsilon}_k - \partial_k\tilde{\epsilon}_i) + (\partial_i\partial_j h^{ij} - \partial^2 h)\tilde{\partial}\cdot\tilde{\epsilon} \\ +(\partial^i h_{ij} - \partial_j h)(\partial\cdot\tilde{\partial})\tilde{\epsilon}^j\Big]. \tag{5.6}$$

To cancel this variation we can introduce extra fields. The gauge transformations of these extra fields are fixed since we need to exactly cancel the terms in (5.6). We will introduce only two extra fields and they will cancel the first two terms in the variation. To cancel the first term we can use a field b_{ij} in the following way:

$$(2\kappa)^2 S_1 = \int [dxd\tilde{x}](\tilde{\partial}_j h^{ij})\partial^k b_{ik}, \quad \text{with} \quad \tilde{\delta}b_{ij} = -(\partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i). \tag{5.7}$$

The second term in (5.6) can be canceled by introducing a field ϕ in a similar way:

$$(2\kappa)^2 S_1 = \int [dxd\tilde{x}](-2)(\partial_i\partial_j h^{ij} - \partial^2 h)\phi, \quad \text{with} \quad \tilde{\delta}\phi = \frac{1}{2}\tilde{\partial}\cdot\tilde{\epsilon}. \tag{5.8}$$

Now, lo and behold, the transformation of the field b_{ij} given in (5.7) is exactly the transformation of an antisymmetric tensor field, meaning we can interpret b_{ij} as a Kalb-Ramond field! Also, the field ϕ is interpreted as a dilaton field. Thus by demanding that our action is independent under double diffeomorphisms (the normal and the dual ones) we *have* to add extra fields which turn out to be *exactly* the same fields string theory predicted in the massless sector!

But what about the third term in (5.6)? We put the third term to zero by demanding that the gauge parameter $\tilde{\epsilon}$ satisfies the constraint $\partial\cdot\tilde{\partial} = 0$. This is an important constraint. We will see in a moment that this means that the gauge parameters should satisfy the so-called weak constraint. You can of course wonder why we do not simply introduce another field to cancel this third term in the variation of the action. This can be done, but a non-trivial theory that is invariant under both δ and $\tilde{\delta}$ transformations without the constraint has not yet been found [1].

5.2 Toroidal backgrounds

Here we will write the level matching condition from closed string theory in a different form. For this, we need to discuss closed string theory in toroidal backgrounds. We use G_{ij} and B_{ij} as the constant background metric and antisymmetric tensor, respectively. But first we will discuss why we do this and what it means.

5.2.1 The Kalb-Ramond field

Here we try to get a better understanding of how to interpret the Kalb-Ramond field.

Imagine we have a charged particle. We can put the particle in an electric field which can be seen as a background that influences the movements of the particle. The particle can couple to the vector potential A_μ. This coupling term in the action will be of the form

$$-q\int dx^\mu A_\mu, \tag{5.9}$$

where q is the charge of the particle. The electromagnetic potential is integrated over the one-dimensional word-line of the particle, and it has a gauge symmetry transformation $A_\mu \to A_\mu + \partial_\mu \Lambda(x)$. It does not matter what function we choose for $\Lambda(x)$, the physics we get will be the same for every choice. This is because the term actually appearing in the action is the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which is gauge invariant. Note that the gauge parameter Λ has no indices, it is a scalar.

This story can be generalized to strings and Kalb-Ramond fields. Just as the vector potential A_μ can be seen as a background for a charged particle, the Kalb-Ramond field $B_{\mu\nu}$ can be seen as a background for a string that has a certain 'Kalb-Ramond charge'. A string can couple to a Kalb-Ramond field in the same way a particle couples to the vector potential. However, this time the Kalb-Ramond field is integrated over a two-dimensional world-sheet instead of a one-dimensional word-line. The coupling between the string and the Kalb-Ramond field has the form

$$-\int dx^\mu dx^\mu B_{\mu\nu}. \tag{5.10}$$

The Kalb-Ramond field $B_{\mu\nu}$ is an antisymmetric Lorentz tensor which has two indices and a gauge symmetry transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$ where now the gauge parameter ϵ_μ has one Lorentz index. The tensor which is invariant under these gauge transformations is $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. We have just inserted an extra Lorentz index everywhere. The way in which a particle couples to the Maxwell field is the same as the way in which a string couples to the Kalb-Ramond field. That is why the Kalb-Ramond field can be seen as a higher-dimensional Maxwell field.

We will use G_{ij} and B_{ij} as constant backgrounds, and h_{ij} and $\hat{b}_{ij}$ represent fluctuations of the constant background metric and antisymmetric tensor, respectively.

Since we have split our indices in noncompact and compact directions we get

$$G_{ij} = \begin{pmatrix} \hat{G}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} \hat{B}_{ab} & 0 \\ 0 & 0. \end{pmatrix} \tag{5.11}$$

Note that for noncompact directions, the metric is just the Minkowski metric $\eta_{\mu\nu}$ but as soon as we compactify a direction, the metric will become the metric on a torus $\hat{G}_{ab}$. We also define

$$E_{ij} \equiv G_{ij} + B_{ij}. \tag{5.12}$$

5.2.2 The weak constraint

To write the level matching condition in a more convenient form, we need to introduce some new symbols. This is done a lot in string theory and it can be confusing but it helps us in writing down nice and simple formulae. We start by giving the zero modes [1]:

$$\begin{aligned} \alpha_{0i} &= -\frac{i}{\sqrt{2}}\Big(\frac{\partial}{\partial x^i} - E_{ik}\,\frac{\partial}{\partial \tilde{x}_k}\Big) = -i\sqrt{\frac{\alpha'}{2}}D_i, \\ \bar{\alpha}_{0i} &= -\frac{i}{\sqrt{2}}\Big(\frac{\partial}{\partial x^i} + E_{ki}\,\frac{\partial}{\partial \tilde{x}_k}\Big) = -i\sqrt{\frac{\alpha'}{2}}\bar{D}_i, \end{aligned} \tag{5.13}$$

where we have introduced the derivatives

$$\begin{aligned} D_i &= \frac{1}{\alpha'}\Big(\frac{\partial}{\partial x^i} - E_{ik}\,\frac{\partial}{\partial \tilde{x}_k}\Big), \\ \bar{D}_i &= \frac{1}{\alpha'}\Big(\frac{\partial}{\partial x^i} + E_{ki}\,\frac{\partial}{\partial \tilde{x}_k}\Big). \end{aligned} \tag{5.14}$$

The equations (5.13) are just generalizations of (4.13) since $p_j = \frac{1}{i}\partial_j$ and $w^k = \frac{1}{i}\tilde{\partial}^k$ which is easily obtained by Fourier expanding. Also note that $w^i = \{w^a, 0\}$, $\tilde{x}_i = \{\tilde{x}_a, 0\}$ and $\frac{\partial}{\partial \tilde{x}_i} = \{\frac{\partial}{\partial \tilde{x}_a}, 0\}$ since there are no windings nor dual coordinates along the noncompact directions. If we put the tilde derivatives equal to zero, the zero modes reduce to (2.32)[9]. The derivatives D and $\bar{D}$ are real and independent derivatives with respect to right- and left-moving coordinates $(\tilde{x}_i - E_{ij}x^j)$ and $(\tilde{x}_i + E_{ji}x^j)$, respectively.

[9] The placing of the factors of α' is just convention.

We now introduce the operator Δ, which is quadratic in the α_0 and $\bar{\alpha}_0$ operators:

$$-\frac{\alpha'}{2}\Delta \equiv \frac{1}{2}\alpha_0^i G_{ij}\alpha_0^j - \frac{1}{2}\bar{\alpha}_0^i G_{ij}\bar{\alpha}_0^j. \tag{5.15}$$

By using equation (3.15) we write

$$L_0^\perp - \bar{L}_0^\perp = N^\perp - \bar{N}^\perp - \frac{\alpha'}{2}\Delta, \tag{5.16}$$

so that the level matching condition for fields with $N^\perp = \bar{N}^\perp$ becomes the constraint $\Delta = 0$. In terms of the introduced derivatives, we get

$$\Delta = \frac{1}{2}(D^2 - \bar{D}^2), \tag{5.17}$$

where we used $D^2 = D^i D_i$ and $\bar{D}^2 = \bar{D}_i \bar{D}^i$. It is now easy to show that

$$\Delta = -\frac{2}{\alpha'}\sum_i \frac{\partial}{\partial \tilde{x}_i}\frac{\partial}{\partial x^i} = -\frac{2}{\alpha'}\sum_a \frac{\partial}{\partial \tilde{x}_a}\frac{\partial}{\partial x^a}. \tag{5.18}$$

So for fields with $N^\perp = \bar{N}^\perp$ the level matching condition implies that all fields must be annihilated by $\partial_i\tilde{\partial}^i$. This is called the *weak constraint*. In fact, all gauge parameters should also obey the weak constraint, we already saw in section 5.1.3 that we needed to impose the weak constraint on the gauge parameter $\tilde{\epsilon}$ to make the action invariant under the dual diffeomorphisms.

5.3 Constraint and Null subspaces

Here we take a closer look at the weak constraint that requires fields and gauge parameters to lie in the kernel of the second-order differential operator Δ. We also define a projector $[[\cdot]]$ that takes an arbitrary field into this kernel. Finally, we discuss why this projection is needed if we start including terms beyond cubic order in the fields in the action.

5.3.1 Null momenta and the projector

Consider states with $N^\perp = \bar{N}^\perp = 1$, i.e., states in the (1,1) sector which was discussed in section 4.3.1. The projection to the physical space with $\Delta = 0$ is most easily discussed in momentum space. Given a field $\phi(x^\mu, x^a, \tilde{x}_a)$, a Fourier series for the compact dimensions yields

$$\phi(x^\mu, x^a, \tilde{x}_a) = \sum_{n,m\in\mathbb{Z}^d} \hat{\phi}(x^\mu, n_a, m^a) e^{im^a\tilde{x}_a + in_a x^a}, \tag{5.19}$$

where we used $(w^a, p_a) = (m^a, n_a)$ with $a = 1, 2, \ldots, d$ (Recall that winding and momentum can be labeled by the integers m^a and n_a, respectively).

By using equation (5.18) we see that

$$\Delta\phi = 0 \quad \leftrightarrow \quad \sum_a n_a m^a \equiv nm = 0. \tag{5.20}$$

We can combine the winding m^a and the momentum n_a of the field ϕ into a $2d$-column vector v:

$$v = \begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^{2d}, \tag{5.21}$$

and define the inner product with respect to the $O(d,d)$ invariant metric $\hat{\eta}$

$$v \circ v' \equiv v^T \hat{\eta}\, v' = (m, n) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m' \\ n' \end{pmatrix} = mn' + nm'. \tag{5.22}$$

Since $v \circ v = 2n_a m^a$, the weak constraint now takes the form

$$\Delta\phi = 0 \quad \leftrightarrow \quad v \circ v = 0, \tag{5.23}$$

that is, the vector v is null with respect to $\hat{\eta}$. So if a field satisfies the weak constraint (which must be the case for a field to be physical) the momentum vector v corresponding to each Fourier component of the field is a null vector.

For a general field ϕ like the one in (5.19) we have

$$\Delta\phi = \frac{1}{\alpha'} \sum_{v \in \mathbb{Z}^{2d}} v \circ v \;\; \hat{\phi}(x^\mu, v)\, e^{iv^T \mathbb{X}}, \tag{5.24}$$

where we combined x^a and $\tilde{x}_a$ into the $2d$-column vector $\mathbb{X} = \left(\begin{smallmatrix} \tilde{x}_a \\ x^a \end{smallmatrix}\right)$ and we used (5.18) and $v \circ v = 2nm$.

If we want the field ϕ to be physical we need (5.24) to be zero. This can be done by introducing a projector $[[\cdot]]$ that projects a general field ϕ into a field that satisfies the $\Delta = 0$ constraint

$$[[\phi]] \equiv \sum_{v \in \mathbb{Z}^{2d}} \delta_{v \circ v, 0} \hat{\phi}(x^\mu, v) e^{iv^T \mathbb{X}}. \tag{5.25}$$

It is now clear that

$$\Delta[[\phi]] = 0. \tag{5.26}$$

The operation $[[\cdot]]$ is a linear map from the space of functions on the doubled torus to the kernel of Δ. It is a projector because applying it twice has the same effect as applying it once.

A general superposition of allowed fields (fields that are annihilated by Δ) is also allowed since Δ is a linear operator. However, the product of two fields will in general not be annihilated by Δ (i.e., $[[\phi]][[\phi']] \neq [[\phi\phi']]$). Suppose we have the fields ϕ and ϕ' with null momenta v and v':

$$\begin{aligned} \phi(x^\mu, x^a, \tilde{x}_a) &= \sum_{v \in \mathbb{Z}^{2d}} \hat{\phi}(x^\mu, v) e^{iv^T \mathbb{X}}, \\ \phi'(x^\mu, x^a, \tilde{x}_a) &= \sum_{v' \in \mathbb{Z}^{2d}} \hat{\phi}'(x^\mu, v') e^{iv'^T \mathbb{X}}. \end{aligned} \tag{5.27}$$

The product $(\phi\phi')$ will have momentum $(v + v')$. It is seen from (5.24) that this product is only annihilated by Δ if

$$(v + v') \circ (v + v') = 0. \tag{5.28}$$

Since v and v' are themselves null, this is satisfied if

$$v \circ v' = 0. \tag{5.29}$$

So the product of two fields only satisfies the weak constraint if the inner product of the momenta of the fields is null with respect to $\hat{\eta}$.

5.3.2 Trouble beyond cubic order

Suppose that the full string action (which we construct in section 5.5.2) contains products of fields with other fields or with gauge parameters. We want the action to describe physical processes but we just saw that only imposing the weak constraint is not enough to obtain this. We need the projection on such products to make them physical, i.e., we need something like $[[A(x, \tilde{x})B(x, \tilde{x})]]$ where A and B are fields or gauge parameters.

To show we do not yet need to use this stronger constraint for the action to cubic order, we use some results of string field theory. String field theory tells us that the cubic action has terms of the form [1]

$$\int \phi_1 [[\phi_2 \phi_3]], \tag{5.30}$$

where ϕ_1, ϕ_2 and ϕ_3 have momenta v_1, v_2 and v_3, respectively. When integrating, only the component with $v_1 + v_2 + v_3 = 0$ gives a contribution. So we get only contributions if $v_2 + v_3 = -v_1$. Since the field ϕ_1 must be physical, the momentum vector v_1 is null. But if v_1 is null, the only contribution we get from the integral is the one in which the sum $v_2 + v_3$ is also null.

So the constraint (5.28) is then automatically satisfied, we do not need the projection. We thus have

$$\int \phi_1[[\phi_2\phi_3]] = \int \phi_1\phi_2\phi_3. \tag{5.31}$$

We can also have products of fields and gauge parameters in the gauge transformations of the fields. If this is the case, we should in principle also use the projection here for as a field is allowed, is should still be allowed after a gauge transformation. But again string field theory tells us that a projection is not needed. The gauge transformations to cubic orders will have terms of the form $\int \phi_1[[\lambda\phi_2]]$ and by the same arguments as before we have $\int \phi_1[[\lambda\phi_2]] = \int \phi_1\lambda\phi_2$.

However, if we go to higher orders in the fields, the action can contain terms of the form

$$\int \phi_1\phi_2[[\phi_3\phi_4]]. \tag{5.32}$$

Again, the only terms that contribute to the integral are those for which $v_1 + v_2 + v_3 + v_4 = 0$, i.e., $v_1 + v_2 = -v_3 - v_4$. Whereas the momenta are all null by themselves, neither $v_1 + v_2$ nor $v_3 + v_4$ need to be null! To see this, look at Figure 3. Because all the four momentum vectors are null, they lie on the light-cone. If the vectors are oriented in the way as they are in Figure 3, then also their total sum is null. However, the sum of two vectors lying on the light-cone, does not lie on the light-cone any more, i.e. $v_1 + v_2$ is not null if v_1 and v_2 are null.

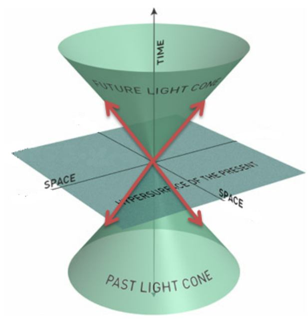

Figure 3: *Null vectors* [8].

This implies

$$\int \phi_1\phi_2[[\phi_3\phi_4]] \neq \int \phi_1\phi_2\phi_3\phi_4, \tag{5.33}$$

and we really *do* need the projection. This means that if we construct the action beyond cubic orders in the fields, the weak constraint alone does not ensure physical processes. We need a constraint that is somewhat stronger.

5.4 The strong constraint and its consequences

We will now discuss a constraint that ensures that such products lie in the kernel of Δ. But bear in mind that this constraint is just one possible stronger constraint, it is certainly not unique. In fact, there are reasons to believe that this so-called *strong constraint* is too strong. We will discuss this in section 6.

5.4.1 The strong constraint

The strong constraint states that *all* products of fields and gauge parameters are annihilated by Δ:

$$\Delta(AB) = \partial_i \tilde{\partial}^i (AB) = 0, \tag{5.34}$$

for any fields A and B. Note that if this constraint is satisfied, automatically also $\partial_i \tilde{\partial}^i (A^\alpha B^\beta) = 0$ for any α, β which can by checked by using the chain rule.

5.4.2 O(D,D) covariant notation

Here we will develop an O(D,D) covariant notation which will be very useful to explain to consequences of imposing the strong constraint. To make such a notation, we introduce extra coordinates $\tilde{x}_\mu$, so that we have $2D$ coordinates X^M with

$$X^M \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}, \quad \text{where} \quad \tilde{x}_i = \begin{pmatrix} \tilde{x}_a \\ \tilde{x}_\mu \end{pmatrix}, \quad x^i = \begin{pmatrix} x^a \\ x^\mu \end{pmatrix}. \tag{5.35}$$

We have now doubled the coordinates in the noncompact directions as well as those in the compact directions. We will consider only fields that are independent of the extra coordinates $\tilde{x}_\mu$, so that these coordinates play no role.

Similarly, we define

$$\partial_M \equiv \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix}. \tag{5.36}$$

The O(D,D) invariant metric which is used to raise and lower indices, is given by $\eta_{MN} = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, so that we have

$$X_M = \begin{pmatrix} x^i \\ \tilde{x}_i \end{pmatrix}, \quad \text{and} \quad \partial^M = \begin{pmatrix} \partial_i \\ \tilde{\partial}^i \end{pmatrix}. \tag{5.37}$$

Note that in this notation (and by using (5.18))

$$\Delta = -\eta^{MN} \partial_M \partial_N = -\partial^M \partial_M = -2\partial_i \tilde{\partial}^i, \tag{5.38}$$

where we set $\alpha' = 1$ for convenience. The strong constraint can now be written as

$$\partial^M \partial_M (AB) = 0 \quad \rightarrow \quad \partial^M A \, \partial_M B = 0, \tag{5.39}$$

where the last step follows from the fact that both A and B must satisfy the weak form of the constraint: $\partial^M \partial_M A = \partial^M \partial_M B = 0$.

5.4.3 Consequences of the strong constraint

Imposing this strong constraint gives rise to a consistent theory, but it also implies that we can always perform an O(D,D) transformation after which all the field and gauge parameters will depend on either x_i or $\tilde{x}^i$ but not on both. We will now prove this statement.

Consider a field A written in terms of its Fourier modes

$$A(\tilde{x}_i, x^i) = \sum_{\tilde{p}^i, p_i} A(\tilde{p}^i, p_i) \, e^{i(\tilde{p}^i \tilde{x}_i + p_i x^i)}. \tag{5.40}$$

Now by demanding that $\partial^M \partial_M A = 0$ we get

$$\tilde{\partial}^i \partial_i A = -\sum_{\alpha} A_\alpha(\tilde{p}^i, p_i) \, p_{i\alpha} \, \tilde{p}^i_\alpha \; e^{i(\tilde{p}^i_\alpha \tilde{x}_{\alpha i} + p_{i\alpha} x^i_\alpha)} = 0 \tag{5.41}$$

where the α refer to different Fourier modes. For this equation to be zero in general, we need $p_{i\alpha} \tilde{p}^i_\alpha = 0$ for all α. As $(\tilde{x}, x)$ transforms as a vector under O(D,D), the vector

$$P_{M\alpha} \equiv \begin{pmatrix} \tilde{p}^i_\alpha \\ p_{i\alpha} \end{pmatrix} \tag{5.42}$$

also transforms as a vector under O(D,D). So the constraint $\partial^M \partial_M A = 0$ implies

$$\tilde{\partial}^i \partial_i A = 0 \quad \leftrightarrow \quad p_{i\alpha} \, \tilde{p}^i_\alpha = 0 \quad \leftrightarrow \quad P_\alpha \cdot P_\alpha = \eta^{MN} P_{M\alpha} P_{N\alpha} = 0. \tag{5.43}$$

That is, the weak constraint implies that the momentum vector P corresponding to *each* Fourier mode (each α) is a null vector and therefore each P_α lies on the light-cone.

Now consider the strong constraint (5.39). We can again write the fields A and B in their Fourier component, and then after applying the strong constraint we get

$$\partial_i A \tilde{\partial}^i B + \partial_i B \tilde{\partial}^i A = -\sum_{\alpha,\beta} A_\alpha B_\beta (p^A_{i\alpha} \tilde{p}^{Bi}_\beta + p^B_{i\beta} \tilde{p}^{Ai}_\alpha) \, e^{i(P^T_\alpha \cdot X)^A + i(P^T_\beta \cdot X)^B} = 0, \tag{5.44}$$

where we have used the labels A, B to specify to which fields the momentum vectors refer. If we want this equation to be zero in general, we need

$$P_\alpha^A \cdot P_\beta^B = 0, \qquad \forall\, \alpha, \beta, A, B, \tag{5.45}$$

where $P_\alpha^A = \begin{pmatrix} \tilde{p}^i_\alpha \\ p_{i\alpha} \end{pmatrix}^A$, etc.

In other words, the momentum vector of each Fourier mode is orthogonal to the momentum vector of any other Fourier mode of any field. This means that *all* momentum vectors P_α live in an isotropic[10] subspace of $\mathbb{R}^{2D}$.

When we say two momentum vectors are orthogonal, we mean $P_\alpha \cdot P_\beta = 0$. This does not mean that the vectors are orthogonal in the usual sense, e.g. two vectors with a 90° angle between them. On the contrary, all vectors P_α lie in the same direction! To understand this, imagine a light-cone and two vectors P_α and P_β. Since both vectors are null themselves, they both lie on the light-cone. But if $P_\alpha \cdot P_\beta = 0$ we also have $(P_\alpha + P_\beta) \cdot (P_\alpha + P_\beta) = 0$ and therefore the sum of the two vectors must also be null. The only situation in which this is the case, is the one in which P_α and P_β point in the same direction. Only then will their sum still lie on the light-cone.

Any vector P_α on that is null (i.e. it lies on the light-cone) can be transformed by an O(D,D) transformation to one in which the $\tilde{p}^i$ component is zero. Since the strong constraint implies that *all* momentum vectors point in the *same* direction, a transformation can be made after which *all* momentum vectors are independent of $\tilde{p}^i$. That is, any Fourier mode of any field has $\tilde{p}^i = 0$ which means that *all* of our fields only depend on x, and not on $\tilde{x}$.

Note that it is really necessary that A and B can be any field or gauge parameter, they can for example represent the same field or two different fields. If the constraint $\partial^M \partial_M (AB) = 0$ would not hold for two different fields but only for products of fields with themselves or with gauge parameters, for example $\partial^M \partial_M (d\, b_{ij}) \neq 0$, then is not true that $P_\alpha \cdot P_\beta = 0$ for all α, β and therefore not all momentum vectors point in the same direction. In this case there will always be some Fourier mode that *does* depend on $\tilde{p}^i$ when the other modes do not. It is not possible to have all modes of all fields independent of $\tilde{p}^i$ any more. This also means that there will always be a field which still depends on $\tilde{x}$.

To summarize, the strong constraint implies that we can always perform an O(D,D) transformation after which all our fields and gauge parameters depend on x but not on $\tilde{x}$. This means that the dual coordinates do not

[10] An isotropic subspace is one in which any two vectors in the space are both null and mutually orthogonal.

represent physical degrees of freedom, we can eliminate the dependence of fields on these dual coordinates by solving constraints. This would imply that double field theory is actually just a single field theory! The strong constraint seems to come a bit out of the blue. One might wonder if it is possible to find alternatives for the strong constraint. We will do this in section 6.

5.5 Background independent action to all orders

Our final goal is to construct the action to all orders and see what extra conditions (besides the weak constraint) we need to make the theory consistent. We can then compare these extra conditions with the strong constraint and discuss whether the strong constraint is actually needed or that it is too strong. We will do this in section 6.1, but first we need to construct the action.

We will not explicitly derive the action to all orders here since we are just interested in the results. We will however, explain how this action was constructed.

5.5.1 The gauge transformations in a background independent form

To derive the action to all orders, it is useful to work in a notation which is manifestly background independent. The cubic action derived in [1] is actually background independent, this was proven in [2]. But it was written in an ugly form, a form in which the background independence is not manifest. The fields that were used are e_{ij} and d where $e_{ij} = h_{ij} + \hat{b}_{ij}$ up to quadratic order and the usual scalar dilaton ϕ is related to the field d by

$$\sqrt{-g}e^{-2\phi} = e^{-2d}, \tag{5.46}$$

so that e^{-2d} is a scalar density. Here $g = \det(g_{\mu\nu})$ where $g_{\mu\nu}$ is the metric tensor.

The cubic action and its gauge transformations will be needed to construct the full action. We will therefore give the cubic action which was constructed in [1]:

$$
\begin{aligned}
(2\kappa^2)S = \int [dx d\tilde{x}] \Big[&\frac{1}{4} e_{ij} \Box e^{ij} + \frac{1}{4}(\bar{D}^j e_{ij})^2 + \frac{1}{4}(D^i e_{ij})^2 - 2d D^i \bar{D}^j e_{ij} - 4 d \Box d \\
&+ \frac{1}{4}\Big((D^i e_{kl})(\bar{D}^j e^{kl}) - (D^i e_{kl})(\bar{D}^l e^{kj}) - (D^k e^{il})(\bar{D}^j e_{kl}) \Big) \\
&+ \frac{1}{2} d \Big((D^i e_{ij})^2 + (\bar{D}^j e_{ij})^2 + \frac{1}{2}(D_k e_{ij})^2 + \frac{1}{2}(\bar{D}_k e_{ij})^2 + 2 e^{ij}(D_i D^k e_{kj} + \bar{D}_j \bar{D}^k e_{ik}) \Big) \\
&+ 4 e_{ij} d D^i \bar{D}^j d + 4 d^2 \Box d \Big]. \qquad (5.47)
\end{aligned}
$$

where $\Box = \frac{1}{2}(D^2 + \bar{D}^2)$. The action is invariant under the following gauge transformations [1]:

$$
\begin{aligned}
\delta e_{ij} = &\bar{D}_j \lambda_i + \frac{1}{2}\Big[(D_i \lambda^k) e_{kj} - (D^k \lambda_i) e_{kj} + \lambda_k D^k e_{ij} \Big] \\
&+ D_i \lambda_j + \frac{1}{2}\Big[(\bar{D}_j \bar{\lambda}^k) e_{ik} - (\bar{D}^k \bar{\lambda}_j) e_{ik} + \bar{\lambda}_k \bar{D}^k e_{ij} \Big], \qquad (5.48)
\end{aligned}
$$

and

$$
\delta d = -\frac{1}{4}(D \cdot \lambda + \bar{D} \cdot \bar{\lambda}) + \frac{1}{2}(\lambda \cdot D + \bar{\lambda} \cdot \bar{D}) d. \qquad (5.49)
$$

In this equation λ_i and $\bar{\lambda}_i$ are two independent vectorial gauge parameters, and the derivatives are the ones as defined in (5.14).

The action also has a $\mathbb{Z}_2$ symmetry

$$
\mathbb{Z}_2 \text{ transformations}: \quad e_{ij} \to e_{ji}, \quad D_i \to \bar{D}_i, \quad \bar{D}_i \to D_i, \quad d \to d. \qquad (5.50)
$$

This $\mathbb{Z}_2$ symmetry is the same $\mathbb{Z}_2$ symmetry as the one discussed in 5.1.2. It says that using the coordinate x^a or the coordinate $\tilde{x}^a$ does not matter, these situations are physically equivalent. We thus have such a $\mathbb{Z}_2$ symmetry in every compact direction. The action to full orders will also have this discrete symmetry and it will in fact play an important role in discussing the strong constraint.

The gauge transformations given above are to first order in the fields. These transformations are used in [4] to construct the gauge transformations to all orders. It turns out that the transformations of the dilaton field (5.49) are actually already the full transformations, exact to all orders. We do not have to include more terms in this equation. The transformation of e_{ij} however, does need some extra terms:

$$
\begin{aligned}
\delta e_{ij} = &D_i \bar{\lambda}_j + \bar{D}_j \lambda_i \\
&+ \frac{1}{2}(\lambda \cdot D + \bar{\lambda} \cdot \bar{D}) e_{ij} + \frac{1}{2}(D_i \lambda^k - D^k \lambda_i) e_{kj} - e_{ik} \frac{1}{2}(\bar{D}^k \bar{\lambda}_j - \bar{D}_j \bar{\lambda}^k) \\
&- \frac{1}{4} e_{ik}(D^l \bar{\lambda}^k + \bar{D}^k \lambda^l) e_{lj}. \qquad (5.51)
\end{aligned}
$$

Note that only one term quadratic in the fields is needed to get the full transformations.

In [2] a new field $\mathcal{E}_{ij}$ is introduced. This field allows us to write the gauge transformations (5.51) in a manifestly background independent form. This field is defined as $\mathcal{E}_{ij} = E_{ij} + \check{e}_{ij}$ where $\check{e}_{ij}$ combines the fluctuation fields, i.e., $\check{e}_{ij} = h_{ij} + \hat{b}_{ij}$. It was shown in [4] that the field $\check{e}_{ij}$ is related to (e_{ij}, d) by $\check{e}_{ij} = f_{ij}(e, d)$ where

$$f = \Big(1 - \frac{1}{2}e\Big)^{-1} e, \tag{5.52}$$

so that

$$\check{e} = Fe, \tag{5.53}$$

where we use matrix notation and defined F as

$$F \equiv \Big(1 - \frac{1}{2}e\Big)^{-1}. \tag{5.54}$$

The field $\mathcal{E}_{ij}$ thus includes both the full metric $g_{ij} = G_{ij} + h_{ij}$ and antisymmetric tensor gauge field $b_{ij} = B_{ij} + \hat{b}_{ij}$, where G_{ij} and B_{ij} are constant background fields.

To write the gauge transformations in a manifestly O(D,D) covariant form, two new gauge parameters ξ^i and $\tilde{\xi}_i$ must be introduced, which are defined as:

$$\lambda_i = -\tilde{\xi}_i + E_{ij}\xi^j, \qquad \bar{\lambda} = \tilde{\xi}_i + E_{ji}\xi^j. \tag{5.55}$$

We can combine ξ^i and $\tilde{\xi}_i$ into a fundamental O(D,D) vector

$$\xi^M = \begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix}. \tag{5.56}$$

Since we have a relation between $\mathcal{E}_{ij}$ and e_{ij}, the gauge transformations of the field e_{ij} can be used to obtain those of the field $\mathcal{E}_{ij}$. In fact, it was shown in [2] that for any variation or derivative

$$\delta\mathcal{E} = F\delta e F. \tag{5.57}$$

We can now write the gauge transformations (5.51) in the following manifestly background-independent form [2]

$$\delta\mathcal{E}_{ij} = \mathcal{D}_i\tilde{\xi}_j - \bar{\mathcal{D}}_j\tilde{\xi}_i + \xi^M\partial_M\mathcal{E}_{ij} + \mathcal{D}_i\xi^k\mathcal{E}_{kj} + \bar{\mathcal{D}}_j\xi^k\mathcal{E}_{ik}, \tag{5.58}$$

where the calligraphic derivatives are defined as

$$\partial_i = \frac{1}{2}(\mathcal{E}_{ji}\mathcal{D}^j + \mathcal{E}_{ij}\bar{\mathcal{D}}^j), \qquad \tilde{\partial}^i = \frac{1}{2}(-\mathcal{D}^i + \bar{\mathcal{D}}^i). \tag{5.59}$$

For later use, it is convenient to write the gauge transformations in a slightly different form. By using the definition of the calligraphic derivatives, we can rewrite (5.58) as

$$\delta_\xi \mathcal{E}_{ij} = \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i + \mathcal{L}_\xi \mathcal{E}_{ij} + \mathcal{E}_{ik} \left(\tilde{\partial}^q \xi^k - \tilde{\partial}^k \xi^q \right) \mathcal{E}_{qj} + \mathcal{L}_{\tilde{\xi}} \mathcal{E}_{ij}, \tag{5.60}$$

where we used the standard Lie derivative and introduced a dual Lie derivative:

$$\begin{aligned} \mathcal{L}_\xi &= \xi^k \partial_k \mathcal{E}_{ij} + \partial_i \xi^k \mathcal{E}_{kj} + \partial_j \xi^k \mathcal{E}_{ik}, \\ \mathcal{L}_{\tilde{\xi}} &= \tilde{\xi}_k \tilde{\partial}^k \mathcal{E}_{ij} - \tilde{\partial}^k \tilde{\xi}_i \mathcal{E}_{kj} - \tilde{\partial}^k \tilde{\xi}_j \mathcal{E}_{ik}. \end{aligned} \tag{5.61}$$

The transformation of the dilaton (5.49) can be written as

$$\delta d = -\frac{1}{2} \partial_M \xi^M + \xi^M \partial_M d. \tag{5.62}$$

5.5.2 Constructing the full action

An action written only in terms of the calligraphic derivatives, g^{ij}, $\mathcal{E}_{ij}$ and d will be manifestly background independent. The strategy is to seek such an action that agrees with the action (5.47) to cubic order. But how should we start? It is convenient to now use the dilaton theorem. This theorem states that a constant shift of the dilaton is equivalent to a change of the coupling constant. The theorem is manifest in actions where the dilaton appears in an exponential prefactor that multiplies all terms in the Lagrangian, and all other occurrences of the dilaton involve its derivatives. This helps us in finding a form in which to write our terms in the action. We aim for an action where the overall multiplicative factor takes the conventional form e^{-2d} and elsewhere the dilaton appears with derivatives.

Now we begin the computation. The $(-4d\Box d)$ term in the action (5.47) could come from a term of the form $4e^{-2d} g^{ij} \mathcal{D}_i d \mathcal{D}_j d$. Indeed, if we expand the calligraphic derivatives using (5.59) we get the desired quadratic term and even some cubic terms that we want:

$$4e^{-2d} g^{ij} \mathcal{D}_i d \mathcal{D}_j d = -4d\Box d + 4e^{ij} d D_i \bar{D}_j d + 4d^2 \Box d - 2d^2 D^i \bar{D}^j e_{ij} + (\text{td}), \tag{5.63}$$

where (td) stands for total derivatives of ∂_i and $\tilde{\partial}^i$[11]. These terms can be ignored since we integrate to form the action.

We see that the first three terms in (5.63) are indeed present in the cubic action. The last term is not, but it will cancel against other terms that we get

[11] Thus (td)=$\partial_M v^M$ for some v^M.

by repeating this strategy to finally obtain all terms in (5.47) after expansion. This is done in [2] and the result is the following background-independent action:

$$S = \int dx d\tilde{x}\, e^{-2d}\Big[-\frac{1}{4}g^{ik}g^{jl}\mathcal{D}^p\mathcal{E}_{kl}\mathcal{D}_p\mathcal{E}_{ij} + \frac{1}{4}g^{kl}(\mathcal{D}^j\mathcal{E}_{ik}\mathcal{D}^i\mathcal{E}_{jl} + \bar{\mathcal{D}}^j\mathcal{E}_{ki}\bar{\mathcal{D}}^i\mathcal{E}_{lj}) + (\mathcal{D}^i d\bar{\mathcal{D}}^j\mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\mathcal{D}^j\mathcal{E}_{ji}) + 4\mathcal{D}^i d\mathcal{D}_i d\Big]. \quad (5.64)$$

5.5.3 Extra constraints needed

Where exactly do we need extra constraints to make the theory consistent? We will discuss all the places where we need a constraint stronger than the weak constraint. Then in the next section we will calculate what kind of extra conditions we need exactly. We will then compare this to what the strong constraint implies.

We did not need the strong constraint to construct the action, but in [2] it was stated that we *do* need the strong form of the constraint to guarantee the $\mathbb{Z}_2$ symmetry of the action. This is the same $\mathbb{Z}_2$ symmetry as the one in (5.50) with the only difference that now we have the field $\mathcal{E}_{ij}$ instead of e_{ij} and we have calligraphic derivatives instead of normal ones. So the action (5.64) should be the same after exchanging the indices in $\mathcal{E}_{ij}$, exchanging barred and unbarred derivatives and leaving the dilaton invariant. We see that the second and third term indeed stay the same after these transformations. This is however not so obvious for the first and the last term. And by demanding the action should be invariant under the $\mathbb{Z}_2$ transformation, a stronger constraint than the weak constraint is indeed needed.

If the fields are allowed (if they satisfy the weak constraint) we want them to still be allowed after a gauge transformation. The gauge transformations (5.58) and (5.62) contain products of fields with gauge parameters and fields with themselves. So here we also need a stronger constraint than the weak constraint.

Finally, it was stated in [2] and [6] that the strong constraint is required to prove gauge invariance of the action.

We will discuss these three situations in the next section. They seem to be the only places where extra conditions are needed.

6 Alternatives for the strong constraint

To make a consistent theory to all orders, we can impose the strong constraint. But we already saw that the strong constraint leads to the fact that our double field theory is actually a single field theory.

In this section we will discuss two possible solutions to this problem. One of them is finding a lighter constraint. A constraint that still leads to a consistent theory, but lets us keep the dual coordinates. Could there be such a constraint? The other one involves the $(0,2)$ and $(2,0)$ sectors which were discussed in section 4.3.4.

6.1 Finding a less strong constraint

We start analysing all three situations discussed in section 5.5.3. What kind of terms do we need to set to zero to make sure everything turns out the way we want it to? We list some terms that need to be put to zero and in the end discuss what it means.

6.1.1 $\mathbb{Z}_2$ symmetry of the action

As discussed before, only the first and the last term in (5.64) give problems. We start with the first term. We demand that

$$g^{ik}g^{jl}\,\mathcal{D}^p\,\mathcal{E}_{kl}\,\mathcal{D}_p\,\mathcal{E}_{ij} = g^{ik}g^{jl}\,\bar{\mathcal{D}}^p\,\mathcal{E}_{lk}\,\bar{\mathcal{D}}_p\,\mathcal{E}_{ji}. \tag{6.1}$$

This equation can be expanded by using $\mathcal{E}_{kl} = g_{kl} + b_{kl}$:

$$\begin{aligned} &g^{ik}g^{jl}\big(\mathcal{D}^p g_{kl}\mathcal{D}_p g_{ij} + \mathcal{D}^p g_{kl}\mathcal{D}_p b_{ij} + \mathcal{D}^p b_{kl}\mathcal{D}_p g_{ij} + \mathcal{D}^p b_{kl}\mathcal{D}_p b_{ij}\big) = \\ &= g^{ik}g^{jl}\big(\bar{\mathcal{D}}^p g_{kl}\bar{\mathcal{D}}_p g_{ij} + \bar{\mathcal{D}}^p g_{kl}\bar{\mathcal{D}}_p b_{ji} + \bar{\mathcal{D}}^p b_{lk}\bar{\mathcal{D}}_p g_{ij} + \bar{\mathcal{D}}^p b_{lk}\bar{\mathcal{D}}_p b_{ji}\big), \end{aligned} \tag{6.2}$$

where we used that g_{ij} is symmetric in the exchange of its indices. For this equation to hold, the term on the left with two derivatives working on the metric should equal the term on the right with two derivatives on the metric. The same holds for the terms with two derivatives working on the b field. We thus get

$$\begin{aligned} g^{ik}g^{jl}\big[\mathcal{D}^p g_{kl}\mathcal{D}_p g_{ij} - \bar{\mathcal{D}}^p g_{kl}\bar{\mathcal{D}}_p g_{ij}\big] = 0, \\ g^{ik}g^{jl}\big[\mathcal{D}^p b_{kl}\mathcal{D}_p b_{ij} - \bar{\mathcal{D}}^p b_{kl}\bar{\mathcal{D}}_p b_{ij}\big] = 0, \end{aligned} \tag{6.3}$$

We can rewrite these equations if we use the fact that the constraint $\partial_M A\,\partial^M B = 0$ (or equivalently $\partial_i A\tilde{\partial}^i B + \tilde{\partial}^i A\partial_i B = 0$) takes a simple form using calligraphic derivatives. A short calculation shows that $\partial_M A\,\partial^M B = 0$ is equivalent to $\mathcal{D}^i A\mathcal{D}_i B = \bar{\mathcal{D}}^i A\bar{\mathcal{D}}_i B$. So we can write (6.3) as

$$g^{ik}g^{jl}\big[\partial_i\, g_{kl}\,\tilde{\partial}^i\, g_{ij} + \tilde{\partial}^i\, g_{kl}\,\partial_i\, g_{ij}\big] = 0, \tag{6.4}$$

$$g^{ik}g^{jl}\big[\partial_i\, b_{kl}\,\tilde{\partial}^i\, b_{ij} + \tilde{\partial}^i\, b_{kl}\,\partial_i\, b_{ij}\big] = 0, \tag{6.5}$$

which is satisfied if we set

$$\boxed{\begin{aligned}\partial_M\partial^M\big(g_{kl}g_{ij}\big)&=0,\\ \partial_M\partial^M\big(b_{kl}b_{ij}\big)&=0.\end{aligned}} \tag{6.6}$$

As for the mixed terms in (6.3), they are all zero. This is not hard to see since we have

$$g^{ik}g^{jl}\mathcal{D}^p g_{kl}\mathcal{D}_p b_{ij} = -g^{jk}g^{il}\mathcal{D}^p g_{kl}\mathcal{D}_p b_{ij} = -g^{jl}g^{ik}\mathcal{D}^p g_{kl}\mathcal{D}_p b_{ij} = 0, \tag{6.7}$$

where the equation is zero because it equals minus itself.

We now work out the last term in (5.64). This time we want to have

$$\mathcal{D}^i d\mathcal{D}_i d = \bar{\mathcal{D}}^i d\bar{\mathcal{D}}_i d, \tag{6.8}$$

and it is easy to show this is equivalent to

$$\boxed{\partial_M\partial^M\big(dd\big)=0.} \tag{6.9}$$

So to keep $\mathbb{Z}_2$ symmetry, (6.6) and (6.9) need to be satisfied. But of course these equations should still hold after a gauge transformation of the fields. So we should also have

$$\begin{aligned}\delta\Big(\partial_M\partial^M\big(dd\big)\Big)&=0,\\ \delta\Big(\partial_M\partial^M\big(b_{ij}b_{kl}\big)\Big)&=0,\\ \delta\Big(\partial_M\partial^M\big(g_{ij}g_{kl}\big)\Big)&=0.\end{aligned} \tag{6.10}$$

This leads to more conditions that involve products of (derivatives of) gauge parameters with (derivatives of) fields and products of (derivatives of) fields with (derivatives of) other fields.

6.1.2 Gauge invariance

If we would vary the action using (5.62) and (5.58) we will get very long equations. Fortunately we can simplify our calculations by working in a derivative expansion in $\tilde{\partial}$. We write the action (5.64) as

$$S = S^{(0)} + S^{(1)} + S^{(2)}, \tag{6.11}$$

where the superscript denotes the number of $\tilde{\partial}$ derivatives in the action. We can do the same for the Lagrangian

$$S^{(k)} = \int dx d\tilde{x}\, \mathcal{L}^{(k)}, \qquad k = 0, 1, 2, \tag{6.12}$$

and the gauge transformations

$$\delta_\xi = \delta_\xi^{(0)} + \delta_\xi^{(1)}, \tag{6.13}$$

where

$$\begin{aligned}
\delta_\xi^{(0)}\mathcal{E}_{ij} &= \partial_i\tilde{\xi}_j - \partial_j\tilde{\xi}_i + \mathcal{L}_\xi\mathcal{E}_{ij}, \\
\delta_\xi^{(1)}\mathcal{E}_{ij} &= -\mathcal{E}_{ik}\Big(\tilde{\partial}^k\xi^l - \tilde{\partial}^l\xi^k\Big)\mathcal{E}_{lj} + \mathcal{L}_{\tilde{\xi}}\mathcal{E}_{ij}, \\
\delta_\xi^{(0)}d &= \xi^i\partial_i d - \frac{1}{2}\partial_i\xi^i, \\
\delta_\xi^{(1)}d &= \tilde{\xi}_i\tilde{\partial}^i d - \frac{1}{2}\tilde{\partial}^i\tilde{\xi}_i,
\end{aligned} \tag{6.14}$$

which can easily be verified by looking at (5.60) and (5.62).

To prove gauge invariance, we need to prove that the whole action is invariant under all gauge transformations. In other words, we need to show that $\big(\delta_\xi^{(0)} + \delta_\xi^{(1)}\big)\big(S^{(0)} + S^{(1)} + S^{(2)}\big) = 0$. But as discussed in [2] this is equivalent[12] to showing

$$\delta^{(0)}S^{(1)} + \delta^{(1)}S^{(0)} = 0. \tag{6.15}$$

It is also explained in [2] that we only have to focus on the terms in the in the gauge transformations that are not of the form of a Lie derivative. Because if the terms do have that form, they will eventually combine into total derivatives[13]. Since we integrate to get the action, these surface terms do not matter.

Since the gauge parameters ξ_i and $\tilde{\xi}_i$ are independent of each other, the action must be invariant under variations under only ξ_i and variations under only $\tilde{\xi}_i$ separately. This gives us the opportunity to split the calculation in two parts. First we check gauge invariance with only $\tilde{\xi}_i$ nonzero. So we calculate $\delta^{(1)}\mathcal{L}^{(0)} + \delta^{(0)}\mathcal{L}^{(1)}$ with ξ_i set to zero and we focus only on the variations that are not of the form of a Lie derivative. Then we check gauge invariance with only ξ_i nonzero. The calculation is done in appendix A. It shows that to prove total gauge invariance we need to set the following terms to zero:

---

[12] If (6.15) is zero, then also $\delta^{(1)}S^{(1)} + \delta^{(0)}S^{(2)} = 0$ since these two equations are T-dual versions of each other [2]. Also, $\delta^{(0)}S^{(0)} = 0$ is just saying that the conventional action is gauge invariant and $\delta^{(1)}S^{(2)} = 0$ is again the T-dual version of this. So by only proving (6.15) it is guaranteed that $\big(\delta_\xi^{(0)} + \delta_\xi^{(1)}\big)\big(S^{(0)} + S^{(1)} + S^{(2)}\big) = 0$.

[13] More precisely, given $\delta_\xi\big(e^{-2d}\big) = \partial_i\big(\xi^i e^{-2d}\big)$ and $\delta_\xi L = \xi^i\partial_i L$ we find $\delta\xi\big(e^{-2d}L\big) = \partial_i\big(\xi^i e^{-2d}L\big)$, where L denotes a Lagrangian that transforms as a scalar under diffeomorphisms.

$$\boxed{\begin{aligned} &\partial_M\partial^M\big(\tilde{\xi}_q g_{kl}\big)=0, \quad \partial_M\partial^M\big(\xi^r g_{pi}\big)=0, \quad \partial_M\partial^M\big(\partial_i\xi_j g_{kl}\big)=0,\\ &\partial_M\partial^M\big(\tilde{\xi}_q d\big)=0, \quad \partial_M\partial^M\big(\xi^r d\big)=0, \quad \partial_M\partial^M\big(\partial_k b_{pj}\xi^j\big)=0,\\ &\partial_M\partial^M\big(\tilde{\xi}_q b_{jp}\big)=0, \quad \partial_M\partial^M\big(\xi^r b_{jp}\big)=0,\\ &\partial_M\partial^M\big(\tilde{\xi}_j g^{ij}\big)=0, \quad \partial_M\partial^M\big(\xi^j b_{pj}\big)=0. \end{aligned}}$$

Again, these equations should also hold after a gauge transformation. More precisely, the condition

$$\delta\Big(\partial_M\partial^M\big(AB\big)\Big)=0, \tag{6.16}$$

should hold where A and B are such, that their combination corresponds to any term in the boxed equation above. This gives new conditions but it is important to note that it leads to a sum of terms to be annihilated by $\partial_M\partial^M$, so in the most general case we do not get conditions in the from that the operator $\partial_M\partial^M$ should annihilate products of fields with other fields/gauge parameters separately.

6.1.3 Finding dependence on both normal and dual coordinates

Here we discuss whether is it possible to have both x dependence *and* $\tilde{x}$ dependence (without having the possibility to perform an O(D,D) transformation after which the dependence on $\tilde{x}$ has disappeared) given that the conditions discussed in sections 6.1.1 and 6.1.2 must be satisfied.

First note that if the strong constraint is imposed, indeed all the constraints are satisfied since the strong constraint demands that every product between fields and gauge parameters we can think of should be annihilated by $\partial_M\partial^M$. However, the strong constraint is not needed. As noted before, to make sure the conditions listed in sections 6.1.1 and 6.1.2 still hold after gauge transformations, new conditions arise but they do not involve products between fields and gauge parameters to be annihilated by $\partial_M\partial^M$ separately. We can therefore impose a different, more general constraint but it will look ugly. In fact, in [5] it was found that consistency of gauge invariance of double field theory requires two closure constraints and one invariance constraint. These constraints select subsets of fields and gauge parameters for which the gauge symmetries are consistent. The weak and strong constraints are sufficient to satisfy the closure and invariance constraints, but not necessary.

The closure constraints take the following form [5]:

$$\xi^Q_{[1}\partial^P\xi_{2]Q}\partial_P V^M{}_N + 2\partial_P\xi^Q_{[1}\partial^P\xi_{2]N}V^M{}_Q + 2\partial_P\xi_{[1Q}\partial^P\xi^M_{2]}V^Q{}_N = 0, \tag{6.17}$$

and

$$\frac{3}{2}\partial^R\left(\xi^P_{[1}\xi^Q_2\partial_P\xi_{3]Q}\right)\partial_R V^M{}_N = 0, \tag{6.18}$$

where $V^M{}_N$ is a generic tensor and ξ_1, ξ_2, ξ_3 represent different transformations. Finally, the invariance constraint takes the form [5]:

$$\int d^K X\ e^{-2d}\ G(\xi, \mathcal{E}, d) = 0, \tag{6.19}$$

which says that the action should transform as a scalar under the gauge transformations. The function $G(\xi, \mathcal{E}, d)$ is given in appendix B.

These constraints were derived by demanding the closure of the so-called C-bracket which governs the gauge algebra. In principle, the same conditions should be derivable after writing out (6.10) and (6.16) and collecting every condition that needs to be satisfied.

6.2 Include extra massless fields

The motivation for this solution is as follows. One of the massless fields we found in section 4.3 is the gravity field $h_{\mu\nu}$. But we saw in section 5.1.3 that this field alone was not enough for double field theory. We *needed* to include both the Kalb-Ramond field and the dilaton field to make the theory invariant under double diffeomorphisms. These three fields together form an irreducible representation of O(D,D). But the theory becomes inconsistent again if we go beyond cubic order in the fields, and if we do not impose the strong constraint. It seems natural to include the massless fields belonging to the $(0, 2)$ and $(2, 0)$ sectors discussed in section 4.3.4. Including extra massless fields worked before, so why shouldn't we try it again? This means we will include eight more gravitons, vector fields and scalar fields. It is certainly worth a try. But the extra massless fields, together with the ones we already had ($h_{\mu\nu}$, $b_{\mu\nu}$ and d), should again form a representation of O(D,D). We can check this by adding the number of degrees of freedom of all the massless fields and compare this to the dimension of the adjoint representation of O(D,D). This is the dimension of a $2D \times 2D$ antisymmetric matrix, which is $\frac{1}{2}\,2D\,(2D-1) = 2D^2 - 1$. We will count the off-shell number of degrees of freedom for the massless fields we obtained in the (2,0) and (0,2) sectors. The number of off-shell degrees of freedom is larger than the number of on-shell degrees of freedom, since we also count non-physical degrees of freedom which can be eliminated using certain gauge choices. We get the following

$$\begin{array}{cr} \tfrac{1}{2}D\big(D+1\big)-1 & \text{(graviton field)} \\ D & \text{(vector field)} \\ 1 & \text{(scalar field)} \quad + \\ \hline \multicolumn{2}{c}{\tfrac{1}{2}D\big(D+3\big)} \end{array} \tag{6.20}$$

These are the numbers of degrees of freedom (off-shell) of one gravity field, one vector field and one scalar field. If we would include the whole (2,0) sector we should multiply this by four, and if we would also include the (0,2) sector we should multiply it by eight. We have to add the number of degrees of freedom of the (1,1) sector we already had, which is D^2. It will soon be clear it is not possible to set this equal to $2D^2 - 1$, since we get

$$D^2 + k\frac{1}{2}\big(D^2+3\big) \neq 2D^2 - D, \quad \text{for any } k \in \mathbb{Z} \tag{6.21}$$

This means that if we include the extra massless fields obtained from the (2,0) and (0,2) sectors, all the fields together do not fit in an O(D,D) representation. This should have been the case if we want to obtain a theory which has an O(D,D) symmetry.

7 Conclusion

In double field theory, all fields and gauge parameters should satisfy the weak constraint, which states that they must be annihilated by the operator $\partial_M\partial^M$. A stronger constraint is needed to make the theory consistent to all orders in the fields. We took a closer look into what conditions are exactly needed to accomplish this. A possible constraint is the so-called strong constraint. However, it can be proven that by imposing this strong constraint a transformation can be performed after which dependence on the dual coordinate $\tilde{x}$ is absent. This implies the double field theory is equivalent to a single field theory up to a T-duality rotation. If a constraint can be found which is less strong than the strong constraint, dependence on both x and $\tilde{x}$ can exist in a form where it is not possible to 'rotate the $\tilde{x}$ dependence away'.

A set of conditions was found that is needed to make the action gauge invariant and to keep T-duality as a symmetry. These conditions must of course still hold after gauge transformations and demanding this gives rise to more complicated conditions. However, these extra conditions do not imply that the operator $\partial_M\partial^M$ annihilates *all* possible products of fields and gauge parameters (as the strong constraint does). It only demands certain combinations of certain products between gauge parameters and fields to be annihilated.

In this thesis, an explicit form of an intermediate constraint was unfortunately not found due to lack of time. However, such an intermediate constraint was found in [5] where this result was derived requiring the closure of the so-called C-bracket. The intermediate constraint takes a complicated form which makes it cumbersome to solve it in full generality.

We also briefly discussed another way of trying to do double field theory beyond the strong constraint: including extra massless fields, coming from the $(N^{\perp}, \bar{N}^{\perp}) = (2,0), (0,2)$ sectors of closed string theory. However, these extra fields, together with the usual Kalb-Ramond 2-form, graviton and dilaton field, do not seem to fit in a representation of O(D,D). Therefore, such a theory would not have T-duality symmetry.

8 Acknowledgements

First of all, I would like to thank my supervisor Diederik Roest for proposing this interesting subject for helping me with the problems I had to face making this thesis. I have learned a lot of new things while working on this thesis.

Secondly, I would like to thank G. Dibitetto, R. Andringa, J. Rosseel and A. Guarino for always taking the time to answer my questions and for making me better understand some concepts. Without them, this thesis would not be what it is now.

Last but not least, I would like to thank the Dutch weather for not being too nice and therefore not distracting me from writing my thesis.

A Explicit check of gauge invariance

The expression for $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(0)}$ are given in [2] and read

$$\begin{aligned}\mathcal{L}^{(1)} = e^{-2d}\Big[&\frac{1}{2}g^{ik}g^{jl}g^{pq}\Big(b_{ir}\tilde{\partial}^{r}b_{jp}H_{klq} + b_{pr}\tilde{\partial}^{r}g_{kl}\partial_{q}g_{ij} - 2b_{lr}\tilde{\partial}^{r}g_{ip}\partial_{k}g_{jq}\Big)\\ &- g^{ik}g^{pq}\tilde{\partial}^{j}b_{ip}\partial_{k}g_{jq} + 2b_{ir}\partial_{j}\tilde{\partial}^{r}g^{ij} + 2\tilde{\partial}^{k}b_{ik}\partial_{j}g^{ij}\\ &+ 2g^{ij}\partial_{i}\tilde{\partial}^{k}b_{jk} + \tilde{\partial}^{k}g^{ij}\partial_{i}b_{jk} - 8g^{ij}b_{ik}\tilde{\partial}^{k}d\partial_{j}d\Big] + (\text{td}), \qquad (A.1)\end{aligned}$$

and

$$\mathcal{L}^{(0)} = e^{-2d}\Big(-\frac{1}{4}g^{ik}g^{jl}\tilde\partial^p g_{kl}\partial_p g_{ij} + \frac{1}{2}g^{kl}\tilde\partial^j g_{ik}\partial^i g_{jl} \\ - \partial_i\partial_j g^{ij} + 4\partial^i d\partial_i d - \frac{1}{12}H^{ijk}H_{ijk}\Big) + (\text{td}). \tag{A.2}$$

A.1 Invariance under $\tilde\xi_i$

We first check the gauge invariance under $\tilde\xi_i$. It can be seen from (6.14) that $\delta^{(0)}_{\tilde\xi}$ is trivial on all fields, except for $\delta^{(0)}_{\tilde\xi}b_{ij} = \partial_i\tilde\xi_j - \partial_j\tilde\xi_i$. So we only have to vary the b's inside (A.1).

The variation of $\mathcal{L}^{(0)}$ is somewhat lengthier. We use the following variations [2]:

$$\begin{aligned}
\delta^{(1)}_{\tilde\xi}H_{ijk} &= \mathcal{L}_{\tilde\xi}H_{ijk} + 3\tilde\partial^p\tilde\xi_{[i}\partial_{|p|}b_{jk]} + 3\partial_{[i}\tilde\xi_{|p|}\tilde\partial^p b_{jk]} + 6\partial_{[i}\tilde\partial^p\tilde\xi_j b_{k]p},\\
\delta^{(1)}_{\tilde\xi}\big(\partial_i g_{jk}\big) &= \mathcal{L}_{\tilde\xi}\big(\partial_i g_{jk}\big) + \tilde\partial^p\tilde\xi_i\partial_p g_{jk} + \partial_i\tilde\xi_p\tilde\partial^p g_{jk} - 2\partial_i\tilde\partial^p\tilde\xi_{(j}g_{k)p},\\
\delta^{(1)}_{\tilde\xi}\big(\partial_i b_{jk}\big) &= \mathcal{L}_{\tilde\xi}\big(\partial_i b_{jk}\big) + \tilde\partial^p\tilde\xi_i\partial_p b_{jk} + \partial_i\tilde\xi_p\tilde\partial^p b_{jk} + 2\partial_i\tilde\partial^p\tilde\xi_{(j}b_{k)p},\\
\delta^{(1)}_{\tilde\xi}\big(\partial_i g^{jk}\big) &= \mathcal{L}_{\tilde\xi}\big(\partial_i g^{jk}\big) + \tilde\partial^p\tilde\xi_i\partial_p g^{jk} + \partial_i\tilde\xi_p\tilde\partial^p g^{jk} + g^{p(j}\partial_i\tilde\partial^{k)}\tilde\xi_p,\\
\delta^{(1)}_{\tilde\xi}\big(\partial_i d\big) &= \mathcal{L}_{\tilde\xi}\big(\partial_i d\big) + \tilde\partial^k\tilde\xi_i\partial_k d + \partial_i\tilde\xi_k\tilde\partial^k d - \frac{1}{2}\partial_i\tilde\partial^k\tilde\xi_k,\\
\delta^{(1)}_{\tilde\xi}\big(\partial_i\partial_j g^{ij}d\big) &= \mathcal{L}_{\tilde\xi}\big(\partial_i\partial_j g^{ij}\big) + 2\partial_j\partial_j g^{ip}\tilde\partial^j\tilde\xi_p + 2\partial_i\tilde\xi_p\tilde\partial^p\partial_j g^{ij} + 2\partial_j g^{ip}\partial_i\tilde\partial^j\tilde\xi_p + \partial_i\partial_j\tilde\xi\tilde\partial^p g^{ij}.
\end{aligned} \tag{A.3}$$

which can be calculated using (6.14) and (5.61). For the variations, we do not use the terms of the form of a Lie derivative. These terms combine together with the transformation of e^{-2d} into a total derivative. We need to vary only terms involving partial derivatives since $\delta^{(1)}_{\tilde\xi}g_{ij}$ is just the Lie derivative on g_{ij} which can be seen by looking at (6.14). The same holds for the variation of b_{ij}, and the dilaton d transforms as a density.

Let us first look at the terms quadratic in derivatives on d. These are the last term in $\mathcal{L}^{(1)}$ and the fourth term in $\mathcal{L}^{(0)}$. Since none of the other terms contain a dilaton field, after variating only these two terms the d's should vanish. Indeed, we find

$$\begin{aligned}
\delta^{(0)}\mathcal{L}^{(1)}_{\text{last}} + \delta^{(1)}\mathcal{L}^{(0)}_{\text{fourth}} &= \delta^{(0)}\Big(-8e^{-2d}g^{ij}b_{ik}\tilde\partial^k d\partial_j d\Big) + \delta^{(1)}\Big(4e^{-2d}g^{ij}\partial_i d\partial_j d\Big)\\
&= 8e^{-2d}\partial^i d\big(\partial_k\tilde\xi_i\tilde\partial^k d + \tilde\partial^k\tilde\xi_i\partial_k d\big) - 4e^{-2d}\partial^i d\,\partial_i\tilde\partial^k\tilde\xi_k\\
&= -2e^{-2d}\big(\partial_j g^{ij}\partial_i\tilde\partial^k\tilde\xi_k + g^{ij}\partial_i\partial_j\tilde\partial^k\tilde\xi_k\big),
\end{aligned} \tag{A.4}$$

where in the last step we performed a partial integration and we set the following term to zero:

$$\partial^i d\Big(\partial_k \tilde{\xi}_i \tilde{\partial}^k d + \tilde{\partial}^k \tilde{\xi}_i \partial_k d\Big) = 0. \tag{A.5}$$

This is indeed zero according to the strong constraint. The terms that are left do not contain a d any more and they will eventually cancel against other terms.

The first term of $\mathcal{L}^{(1)}$ and the last term of $\mathcal{L}^{(0)}$ are the only terms that still contain a b after variation. They should therefore cancel against each other. A straightforward calculation shows that these terms indeed cancel if we set the following term to zero

$$e^{-2d} g^{ik} g^{jl} g^{pq} \partial_{[k} b_{lq]} \Big(\tilde{\partial}^\alpha \tilde{\xi}_i \partial_\alpha b_{jp} + \partial_\alpha \tilde{\xi}_i \tilde{\partial}^\alpha b_{jp}\Big) = 0. \tag{A.6}$$

We now compute the variation of $\mathcal{L}^{(1)}$, but without the first and last terms, since we already used them. A straightforward calculation shows that we get

$$\begin{aligned} \delta^{(0)} \mathcal{L}^{(1)} = {} & e^{-2d} g^{ik} g^{jl} g^{pq} \Big[\partial_{[q} \tilde{\xi}_{\alpha]} \tilde{\partial}^\alpha g_{kl} - 2\partial_{[l} \tilde{\xi}_{\alpha]} \tilde{\partial}^\alpha g_{qk} - 2\tilde{\partial}_l \partial_{[q} \tilde{\xi}_{k]}\Big] \partial_p g_{ij} \\ & + 2e^{-2d}\Big(2\partial_{[i} \tilde{\xi}_{\alpha]} \partial_j \tilde{\partial}^\alpha g^{ij} + 2\tilde{\partial}^\alpha \partial_{[i} \tilde{\xi}_{\alpha]} \partial_j g^{ij} \\ & + 2g^{ij} \partial_i \tilde{\partial}^\alpha \partial_{[j} \tilde{\xi}_{\alpha]} + \tilde{\partial}^\alpha g^{ij} \partial_i \partial_{[j} \tilde{\xi}_{\alpha]}\Big) \end{aligned} \tag{A.7}$$

We can do the same for the variation of $\mathcal{L}^{(0)}$. If we do not consider the fourth and last term for a moment we get

$$\delta^{(1)} \mathcal{L}^{(0)} = -\frac{1}{2} e^{-2d} g^{ik} g^{jl} g^{pq} \Big[\delta^{(1)}\Big(\partial_q g_{kl}\Big) - 2\delta^{(1)}\Big(\partial_l g_{qk}\Big)\Big] \partial_p g_{ij} - e^{-2d} \delta^{(1)}\Big(\partial_i \partial_j g^{ij}\Big). \tag{A.8}$$

By using (A.3) this is easy to calculate. Finally, we add (A.7), (A.8) and the remaining two terms of (A.4). It can be shown this vanishes only if we set the following terms to zero

$$e^{-2d} g^{ik} g^{jl} g^{pq} \partial_p g_{ij} \Big(\partial_\alpha \tilde{\xi}_q \tilde{\partial}^\alpha g_{kl} + \tilde{\partial}^\alpha \tilde{\xi}_q \partial_\alpha g_{kl}\Big) = 0, \tag{A.9}$$

$$e^{-2d} \partial_i d \Big(\partial_\alpha \tilde{\xi}_j \tilde{\partial}^\alpha g^{ij} + \tilde{\partial}^\alpha \tilde{\xi}_j \partial_\alpha g^{ij}\Big) = 0, \tag{A.10}$$

$$e^{-2d} g^{ik} g^{jl} \Big(\partial_i \partial_\alpha \tilde{\xi}_j \tilde{\partial}^\alpha g_{kl} + \partial_i \tilde{\partial}^\alpha \tilde{\xi}_j \partial_\alpha g_{kl}\Big) = 0, \tag{A.11}$$

and of course the weak constraint always holds, so $\tilde{\partial}^i \partial_i$ annihilates every field or gauge parameter.

A.2 Invariance under ξ_i

We now check gauge invariance under ξ_i. This time, to vary $\mathcal{L}^{(1)}$ we only need to vary terms that involve partial derivatives and therefore transform non-covariantly. We use the following variations [2]:

$$\delta_\xi^{(0)}\big(\tilde{\partial}^r b_{jp}\big) = \mathcal{L}_\xi\big(\tilde{\partial}^r b_{jp}\big) + \tilde{\partial}^k b_{jp}\,\partial_k\xi^r + \tilde{\partial}^r\xi^k\partial_k b_{jp} - 2\tilde{\partial}^r\partial_{[j}\xi^k\, b_{p]k}\ , \tag{A.12}$$

$$\delta_\xi^{(0)}\big(\tilde{\partial}^k g^{ij}\big) = \mathcal{L}_\xi\big(\tilde{\partial}^k g^{ij}\big) + \tilde{\partial}^p g^{ij}\,\partial_p\xi^k + \tilde{\partial}^k\xi^p\,\partial_p g^{ij} - 2\tilde{\partial}^k\partial_p\xi^{(i}\, g^{j)p}\ , \tag{A.13}$$

$$\delta_\xi^{(0)}\big(\tilde{\partial}^k d\big) = \mathcal{L}_\xi\big(\tilde{\partial}^k d\big) + \tilde{\partial}^p d\,\partial_p\xi^k + \tilde{\partial}^k\xi^j\,\partial_j d - \frac{1}{2}\tilde{\partial}^k\partial_j\xi^j\ , \tag{A.14}$$

$$\delta_\xi^{(0)}\big(\tilde{\partial}^r g_{kl}\big) = \mathcal{L}_\xi\big(\tilde{\partial}^r g_{kl}\big) + \tilde{\partial}^p g_{kl}\,\partial_p\xi^r + \tilde{\partial}^r\xi^p\,\partial_p g_{kl} + 2\tilde{\partial}^r\partial_{(k}\xi^p\, g_{l)p}\ , \tag{A.15}$$

$$\delta_\xi^{(0)}\big(\partial_i b_{jk}\big) = \mathcal{L}_\xi\big(\partial_i b_{jk}\big) - 2\partial_i\partial_{[j}\xi^p\, b_{k]p}\ , \tag{A.16}$$

$$\begin{aligned}\delta_\xi^{(0)}\big(\tilde{\partial}^r\partial_j g^{ij}\big) = {}& \mathcal{L}_\xi\big(\tilde{\partial}^r\partial_j g^{ij}\big) + \tilde{\partial}^p\partial_j g^{ij}\,\partial_p\xi^r + \tilde{\partial}^r\xi^p\,\partial_p\partial_j g^{ij} \\ & - \partial_j g^{pj}\,\tilde{\partial}^r\partial_p\xi^i - \tilde{\partial}^r g^{pj}\,\partial_j\partial_p\xi^i - g^{pj}\tilde{\partial}^r\partial_j\partial_p\xi^i - \tilde{\partial}^r g^{ip}\,\partial_p\partial_j\xi^j \\ & - g^{ip}\tilde{\partial}^r\partial_p\partial_j\xi^j\ ,\end{aligned} \tag{A.17}$$

$$\begin{aligned}\delta_\xi^{(0)}\big(\partial_i\tilde{\partial}^k b_{jk}\big) = {}& \mathcal{L}_\xi\big(\partial_i\tilde{\partial}^k b_{jk}\big) + \partial_i\partial_k\xi^p\,\tilde{\partial}^k b_{jp} + \partial_k\xi^p\,\partial_i\tilde{\partial}^k b_{jp} \\ & + \partial_i\partial_j\xi^p\,\tilde{\partial}^k b_{pk} + \partial_i\tilde{\partial}^k\xi^p\,\partial_p b_{jk} + \tilde{\partial}^k\xi^p\,\partial_i\partial_p b_{jk} + \partial_i\tilde{\partial}^k\partial_j\xi^p\, b_{pk} \\ & - \tilde{\partial}^k\partial_j\xi^p\,\partial_i b_{kp}\ ,\end{aligned} \tag{A.18}$$

$$\delta_\xi^{(0)}\big(\partial_p g_{kl}\big) = \mathcal{L}_\xi\big(\partial_p g_{kl}\big) + \partial_p\partial_k\xi^q\, g_{lq} + \partial_p\partial_l\xi^q\, g_{kq}\ , \tag{A.19}$$

$$\delta_\xi^{(0)}\big(\partial_i d\big) = \mathcal{L}_\xi\big(\partial_i d\big) - \frac{1}{2}\partial_i\partial_j\xi^j\ . \tag{A.20}$$

Next, we look at $\delta^{(1)}$. It acts trivially on d, while on g and b we find the non-linear transformations [2]

$$\delta_\xi^{(1)} g_{ij} = 2\big(\tilde{\partial}^k\xi^l - \tilde{\partial}^l\xi^k\big) g_{k(i}\, b_{j)l}\ , \tag{A.21}$$

$$\delta_\xi^{(1)} g^{ij} = -\big(\tilde{\partial}^i\xi^k - \tilde{\partial}^k\xi^i\big) g^{jl} b_{lk} + (i \leftrightarrow j)\ , \tag{A.22}$$

$$\delta_\xi^{(1)} b_{ij} = g_{ik}\big(\tilde{\partial}^l\xi^k - \tilde{\partial}^k\xi^l\big) g_{lj} + b_{ik}\big(\tilde{\partial}^l\xi^k - \tilde{\partial}^k\xi^l\big) b_{lj}\ . \tag{A.23}$$

Since these transformations do not take the form of a Lie derivative, we have to vary everything in $\delta^{(1)}\mathcal{L}^{(0)}$, not only the terms involving partial derivatives. It is therefore convenient to slightly rewrite $\mathcal{L}^{(0)}$ with less appearances of metrics and inverse metrics,

$$\begin{aligned}\mathcal{L}^{(0)} = e^{-2d}\Big(&\frac{1}{4}g^{pq}\partial_p g^{ij}\,\partial_q g_{ij} - \frac{1}{2}g^{ij}\partial_j g^{kl}\,\partial_l g_{ik} - \partial_i\partial_j g^{ij} \\ & + 4g^{ij}\partial_i d\,\partial_j d - \frac{1}{12}g^{il}g^{jp}g^{kq}H_{ijk}H_{lpq}\Big)\ .\end{aligned} \tag{A.24}$$

So we vary all terms of $\mathcal{L}^{(1)}$ involving partial derivatives by using (A.12)-(A.20). Then we vary every term of $\mathcal{L}^{(0)}$ under $\delta^{(1)}$ using (A.21)-(A.23).

This is a long calculation and it is easiest to split it in three parts. One part with terms proportional to $g^{ik}g^{jl}g^{pq}$, another part with terms proportional to $g^{jl}g^{pq}$ and finally a part with terms proportional to g^{pq}. In doing this, we keep the indices of ξ^i and the tilde derivative $\tilde{\partial}^i$ up and the rest of the indices down. Also, by using $g^{kl}g^{ij}\partial_q g_{ik} = -\partial_q g^{lj}$, we rewrite certain terms in such a way that derivatives work on a metric with indices down. The three parts should now all cancel by themselves.

The terms proportional to $g^{ik}g^{jl}g^{pq}$ cancel if we set the following terms to zero:

$$e^{-2d}g^{ik}g^{jl}g^{pq}b_{ir}\partial_{[k}b_{lq]}\Big(\partial_\alpha\,\xi^r\,\tilde{\partial}^\alpha\,b_{jp} + \tilde{\partial}^\alpha\,\xi^r\,\partial_\alpha\,b_{jp}\Big) = 0, \tag{A.25}$$

$$e^{-2d}g^{ik}g^{jl}g^{pq}b_{lr}\partial_j g_{kq}\Big(\partial_\alpha\,\xi^r\,\tilde{\partial}^\alpha\,g_{pi} + \tilde{\partial}^\alpha\,\xi^r\,\partial_\alpha\,g_{pi}\Big) = 0. \tag{A.26}$$

The terms proportional to $g^{jl}g^{pq}$ cancel if the following terms are set to zero:

$$e^{-2d}g^{jl}g^{pq}\partial_j g_{ql}\Big(\partial_\alpha\,\xi^r\,\tilde{\partial}^\alpha\,b_{pr} + \tilde{\partial}^\alpha\,\xi^r\,\partial_\alpha\,b_{pr}\Big) = 0, \tag{A.27}$$

$$e^{-2d}g^{jl}g^{pq}\partial_p b_{lr}\Big(\partial_\alpha\,\xi^r\,\tilde{\partial}^\alpha\,g_{qj} + \tilde{\partial}^\alpha\,\xi^r\,\partial_\alpha\,g_{qj}\Big) = 0, \tag{A.28}$$

$$e^{-2d}g^{jl}g^{pq}\partial_j d b_{pr}\Big(\partial_\alpha\,\xi^r\,\tilde{\partial}^\alpha\,g_{ql} + \tilde{\partial}^\alpha\,\xi^r\,\partial_\alpha\,g_{ql}\Big) = 0. \tag{A.29}$$

And finally, to cancel the terms proportional to g^{pq} we need

$$e^{-2d}g^{pq}\partial_q d\Big(\partial_\alpha\,\xi^j\,\tilde{\partial}^\alpha\,b_{pj} + \tilde{\partial}^\alpha\,\xi^j\,\partial_\alpha\,b_{pj}\Big) = 0, \tag{A.30}$$

$$e^{-2d}g^{pq}\partial_q d b_{pk}\Big(\partial_\alpha\,\xi^k\,\tilde{\partial}^\alpha\,d + \tilde{\partial}^\alpha\,\xi^k\,\partial_\alpha\,d\Big) = 0. \tag{A.31}$$

A.3 Gathering all the terms

We will collect the terms that ensure gauge invariance of the action after setting them to zero. Some of the terms derived above have the same structure so it is not necessary to list them all.

B The invariance constraint

The invariance constraint which was derived in [5] states

$$\int d^K X\ e^{-2d}\ G(\xi, \mathcal{E}, d) = 0, \tag{B.1}$$

$$\boxed{\begin{aligned}
&\partial_M\partial^M\left(\tilde{\xi}_q g_{kl}\right)=0, \quad \partial_M\partial^M\left(\xi^r g_{pi}\right)=0, \quad \partial_M\partial^M\left(\partial_i\xi_j g_{kl}\right)=0,\\
&\partial_M\partial^M\left(\tilde{\xi}_q d\right)=0, \quad \partial_M\partial^M\left(\xi^r d\right)=0, \quad \partial_M\partial^M\left(\partial_k b_{pj}\xi^j\right)=0,\\
&\partial_M\partial^M\left(\tilde{\xi}_q b_{jp}\right)=0, \quad \partial_M\partial^M\left(\xi^r b_{jp}\right)=0,\\
&\partial_M\partial^M\left(\tilde{\xi}_j g^{ij}\right)=0, \quad \partial_M\partial^M\left(\xi^j b_{pj}\right)=0.
\end{aligned}}$$

where

$$\begin{aligned}
G(\xi,\mathcal{E},d) = &-\partial^P\partial_M\xi_N\partial_P\mathcal{H}^{MN} - 2\partial^P\xi_N\partial_P\partial_M\mathcal{H}^{MN} + 4\partial_P d\partial_M\partial^P\xi_N\mathcal{H}^{MN}\\
&+ 4\partial_P d\partial^P\xi_N\partial_M\mathcal{H}^{MN} + 4\partial_N d\partial^P\xi_M\partial_P\mathcal{H}^{MN}\\
&+ \frac{1}{4}\mathcal{H}^{MN}\partial^P\xi_M\partial_P\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \mathcal{H}^{MN}\partial^P\xi_M\partial_P\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}\\
&+ 8\mathcal{H}^{MN}\partial^P\xi_M\partial_P\partial_N d - 8\mathcal{H}^{MN}\partial_M d\partial^P\xi_N\partial_P d\\
&- 2\partial_M\left(\partial^P\partial_P\xi_N\mathcal{H}^{MN}\right) + 4\partial^P\partial_P\xi_M\partial_N d\mathcal{H}^{MN}\\
&+ \partial_P\xi^Q\partial_Q\mathcal{E}^a{}_M\partial^P\mathcal{E}^b{}_N S_{ab}\eta^{MN} + \partial_P\partial^N\xi^M\mathcal{E}^a{}_M\partial^P\mathcal{E}^b{}_N S_{ab}\\
&- \partial_P\partial^M\xi^N\mathcal{E}^a{}_M\partial^P\mathcal{E}^b{}_N S_{ab}\ .
\end{aligned} \tag{B.2}$$

We will not explain in detail what all the terms in this equation mean, for a thorough understanding see [5]. The purpose of showing this formula is to emphasize that the intermediate constraint takes on a very ugly form. So although it is not necessary to impose the strong constraint, the intermediate constraint is complicated and this makes it cumbersome to solve it in full generality.

References

[1] C. Hull and B. Zwiebach, "*Double Field Theory*,"(2009), arXiv:0904.4664v2 [hep-th].

[2] O.Hohm, C. Hull and B. Zwiebach, "*Background independent action for double field theory*,"(2010), arXiv:1003.5027v2 [hep-th].

[3] B. Zwiebach (2009), *A first Course in String Theory*, Cambridge.

[4] C. Hull and B. Zwiebach, "*The Gauge Algebra of Double Field Theory and Courant Brackets*,"(2009), arXiv:0908.1792v1 [hep-th].

[5] M. Graña and D. Marques, "*Gauged Double Field Theory*,"(2012), arXiv:1201.2924v2 [hep-th].

[6] O. Hohm, "*T-duality versus Gauge Symmetry*,"(2011), arXiv:1101.3484v1 [hep-th].

[7] http://universe-review.ca/R15-18-string.htm, 07/07/2012, *Review of the Universe - Structures, Evolutions, Observations, and Theories.*

[8] http://www.compression.org/expanding-your-light-cone/, 07/07/2012, Robert W. Hall, *Light Cones and Model Myopia*, compression institute.