Generalizations of Born-Infeld Theory

On the restrictiveness of duality invariance on the power series expansion of nonlinear electrodynamics Lagrangians.

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Bachelorthesis

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Introduction

In the beginning of the twentieth century, some physicists were dissatisfied with the picture of charged particles that was present in physics. When calculating the self-energy of these charged particles due to electrostatic repulsion in the classical model, physicists came up with a very high amount of energy, which most certainly would have to cause the charged particle to explode into its smaller parts under the giant Coulomb force.

An attempt of the German physicist Gustav Mie to tackle this problem consisted of reinterpreting the concept of the electron. This attempt of Mie was special in the way that Mie was the first one who tried to construct a theory to describe and understand the electron entirely in electromagnetic terms. To make this work, Mie had to modify the electromagnetic theory developed by Maxwell and therefore Mie's theory gave rise to a construction of an alternative electromagnetic theory. This was really the beginning of nonlinear electrodynamics.

Since Mie's theory came at at a high price, it was dismissed for some 10 years until Max Born and Leopold Infeld changed Mie's theory. They developed a general framework for nonlinear electrodynamics and the special kind of nonlinear electrodynamics they developed is known as Born-Infeld theory, which was published in [2]. The core of this theory is the Born-Infeld Lagrangian. This Lagrangian has some interesting properties: it is Lorentz-invariant and it is invariant under the exchange of electric and magnetic fields: $\mathbf{E} \to -\mathbf{B}$ and $\mathbf{B} \to \mathbf{E}$. These two properties are also shared by Maxwell theory and in fact in the Taylor series of the Born-Infeld Lagrangian, the lowest order term equals the Maxwell Lagrangian. [1]

The problem that gave rise to the research which resulted in this bachelorthesis concerned the uniqueness of the Born-Infeld Lagrangian in the properties of Lorentz-invariance and the invariance under the duality transformation in nonlinear electrodynamics. The question is whether there are more generalizations of Maxwell theory besides standard Born-Infeld theory, which admit both properties just mentioned.

In an attempt to say something useful about this problem, the reader will first be introduced to nonlinear electrodynamics in section 1. After this, duality transformations in Maxwell theory and in nonlinear electrodynamics will be considered in section 2 and a condition which assures invariance of Lagrangians under this duality transformation will be derived in section 3. In section 4 the Born-Infeld Lagrangian will be investigated and a power expansion of this Lagrangian will be made. After this, the important part of the thesis will be about the power expansion of Lagrangians which satisfy the duality transformation. The expansion up to the first few orders will be made in section 5 and a general result about this power expansion will be proven in section 6. These results and their importance will be interpreted in the Discussion and Conclusions section, which will also contain some suggestions for follow-up research.

In this thesis some concepts will be used that are supposed to be known to the reader. In general, these concepts have probably already been introduced to the reader in a 'Classical Electrodynamics' course.

[1]See [1] for more details of the history of nonlinear electrodynamics.

Also, the concept of Lagrangian and Lagrangian density will frequently be used throughout this thesis. In classical mechanics, the action S can be written as an integral of the Lagrangian over time $S = \int \mathcal{L}\,\mathrm{d}t = \int L\,\mathrm{d}^4x$. The Lagrangian $\mathcal{L}$ is in general no Lorentz-invariant quantity, but the Lagrangian density L has to be in order to guarantee Lorentz-invariant action. Since in this thesis only the Lagrangian density will be important, the quantity L will just be called the Lagrangian throughout this thesis.

Another concept that will be used without further introduction is the antisymmetric Levi-Civita tensor. In this thesis it will be denoted by the Greek symbol η and it will be referred to as the η-tensor. More information on the Levi-Civita tensor can be found in almost every book on (classical) electrodynamics, like the ones mentioned in the References ([4] and [5]).

The units which will be used in this thesis coincide with the units used in the article of Gibbons and Rasheed [3] which formed the author's introduction to the matter of this research. The units which are used in this article are units in which $c = \hbar = \varepsilon_0 = \mu_0 = 1$. This means that in the Born-Infeld Lagrangian the parameter b has the dimension of length squared. Furthermore, in these units, charges (both electric and magnetic) are dimensionless.

1 Nonlinear Electrodynamics

1.1 Maxwell theory

In classical electrodynamics, the electromagnetic interaction is governed by a set of equations, which are called the Maxwell equations. In absence of electric charges and currents, these equations read:

$$\nabla \cdot \mathbf{B} = 0 \tag{1a}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1b}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{1c}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0 \tag{1d}$$

where $\mathbf{E}$ is the electric field intensity and $\mathbf{B}$ is the magnetic induction.

Using the Lagrangian formalism, one can define a Lagrangian from which equations of motion can be derived. The general definition for the field strength tensor reads:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2}$$

where the fourvector $A^\mu = (\phi, \mathbf{A})$ consists of the scalar electric potential ϕ and the vector potential $\mathbf{A}$. Working out this definition gives a form for the field stength tensor in terms of the components of the electric field intensity and the magnetic induction:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

It is furthermore well known (see any textbook on classical electrodynamics, e.g. [4] or [5]) that the Lagrangian which governs Maxwell theory is given by:

$$L_{\text{Maxwell}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}.$$

When evaluating this Lagrangian in the Euler-Lagrange equation, one obtains the second pair of Maxwell equations (1c) & (1d), which in relativistic notation reads as:

$$\partial_\mu F^{\mu\nu} = 0. \tag{3}$$

The first two (1a) and (1b)of these Maxwell equations, however, can be obtained from the equation

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0,$$

which alternatively can be written as

$$\partial_\mu \eta^{\mu\nu\sigma\tau} F_{\sigma\tau} = \partial_\mu \mathcal{F}^{\mu\nu} = 0. \tag{4}$$

In this equation an alternative notation is introduced, which will be used throughout this thesis, where $\mathcal{F}^{\mu\nu}$ is called the *dual field strength tensor* or *the dual of* $F^{\mu\nu}$. Working out the definition following from (4) shows that $\mathcal{F}^{\mu\nu}$ can be obtained from $F^{\mu\nu}$ by making the following substitution in the components of the electric and magnetic field:

$$\begin{cases} \mathbf{B} \to \mathbf{E} \\ \mathbf{E} \to -\mathbf{B} \end{cases}$$

So the Maxwell description of the electromagnetic field can be summarized in the two equations (3) & (4).

1.2 Nonlinear electrodynamics: Maxwell theory generalized

Nonlinear electrodynamics (NED) can be viewed at as a generalization of Maxwell theory. As mentioned in the introduction its origins lay in the work of Gustav Mie, in his attempt to give a purely electromagnetic theory of the electron. Later, in the mid-nineteen-thirties, Born and Infeld, inspired by the work of Mie, came up with a general framework for nonlinear electrodynamics and invented the so-called Born-Infeld Lagrangian for their own reasons.

A first step in generalizing Maxwell theory is to give a generalized version of the Maxwell equations. In NED, this generalized version reads:

$$\nabla \cdot \mathbf{B} = 0 \tag{5a}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{5b}$$

$$\nabla \cdot \mathbf{D} = 0 \tag{5c}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 \tag{5d}$$

Here the quantities $\mathbf{D}$ and $\mathbf{H}$ are introduced, which are respectively the electric induction and the magnetic field intensity.

Analogous to how the field strength tensor $F^{\mu\nu}$ is defined, another field strength tensor $G^{\mu\nu}$ for which $G^{0i} = \mathbf{D}^i$ and $G^{kl} = \frac{1}{2}\eta^{klm}\mathbf{H}_m$ can be introduced. This means that:

$$G^{\mu\nu} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{pmatrix}$$

Then it is easy to verify that the second pair of equations (5c) & (5d) can be summarized in the equation:

$$\partial_\mu G^{\mu\nu} = 0. \tag{6}$$

Summarizing, this means that in general the equations which govern nonlinear electrodynamics can be written as

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0$$

$$\partial_\mu G^{\mu\nu} = 0$$

Furthermore, in the framework of NED, there exists a constitutive relation which links both field strength tensors $G^{\mu\nu}$ and $F^{\mu\nu}$ and the Lagrangian L under consideration. The constitutive relation reads:

$$G^{\mu\nu} = -\frac{\partial L}{\partial F_{\mu\nu}} \tag{7}$$

and follows from the variational principle, as can be seen from [1], and from the assumption in NED that all Lagrangians L are a function of $F^{\mu\nu}$, that is $L = L(F^{\mu\nu})$.

It is, finally, worth noting that when the Maxwell Lagrangian is considered, then $G^{\mu\nu} = F^{\mu\nu}$ and the ordinary Maxwell equations (1) are reobtained.

2 Duality Transformations

2.1 Duality invariance of Maxwell theory

A very interesting and special property of the Maxwell equations is that the equations are invariant under the so-called *duality transformation* or *duality rotation* (see chapter 6 of [4] for more details).

This duality transformation is defined as:

$$\begin{cases} \mathbf{E} \rightarrow \cos\alpha \mathbf{E} - \sin\alpha \mathbf{B} \\ \mathbf{B} \rightarrow \cos\alpha \mathbf{B} + \sin\alpha \mathbf{E} \end{cases} \tag{8}$$

So, if the above transformation is performed on the ordinary source-less Maxwell equations (1), the same equations will be reobtained, as can easily be shown. The transformation (8) can also be written in terms of $F_{\mu\nu}$ and its dual $\mathcal{F}_{\mu\nu}$:

$$F_{\mu\nu} \rightarrow \cos\alpha F_{\mu\nu} + \sin\alpha \mathcal{F}_{\mu\nu} \tag{9}$$

Performing this transformation will leave the set of equations (3) and (4) unchanged and so Maxwell theory is said to be *duality invariant*.

2.2 Duality transformation in nonlinear electrodynamics

At this stage it is necessary to find a generalized duality rotation analogue to (9), which guarantees duality invariance in NED.

From the transformation (9) it comes about as a good guess to state that the pair of equations (4) & (6) is invariant the transformation:

$$\begin{cases} F_{\mu\nu} \rightarrow \cos\alpha F_{\mu\nu} + \sin\alpha \mathcal{G}_{\mu\nu} \\ G_{\mu\nu} \rightarrow \cos\alpha G_{\mu\nu} + \sin\alpha \mathcal{F}_{\mu\nu} \end{cases} \tag{10}$$

where $\mathcal{G}_{\mu\nu}$ is the dual of the tensor $G_{\mu\nu}$. It is again an easy calculation to verify that the pair (4) & (6) is indeed invariant under the transformation (10), so the conclusion is that NED should necessarily be invariant under this duality rotation.

In the next section, this conclusion will be used to obtain a condition on Lagrangians, which assures that these Lagrangians give rise to a (nonlinear) electrodynamics which is duality invariant.

3 Duality Invariance Condition

As mentioned in the introduction, the aim of this thesis is to investigate Lagrangians which are a function of $F_{\mu\nu}$ and which are invariant under the duality transformations (10). In this section first there will be obtained a necessary and sufficient condition which guarantees duality invariance by considering infinitesimal transformations. Secondly, some general remarks about Lagrangians will be made and finally the condition will be rewritten in some form, which will be used throughout the thesis.

3.1 Infinitesimal transformations

To obtain the duality invariance condition, the article of Gibbons and Rasheed [3], will be followed very closely. In this article *infinitesimal* transformations are considered, which by looking at (10), can easily be seen to equal:

$$\delta F_{\mu\nu} = \left[\frac{\partial F_{\mu\nu}}{\partial\alpha}\right]_{\alpha=0} = \mathcal{G}_{\mu\nu} \tag{11a}$$

$$\delta G_{\mu\nu} = \left[\frac{\partial G_{\mu\nu}}{\partial\alpha}\right]_{\alpha=0} = \mathcal{F}_{\mu\nu} \tag{11b}$$

These infinitesimal transformations give rise to an important equality which in turn gives the duality condition. Using (11) and some rewriting gives:

$$\mathcal{F}_{\mu\nu} = \delta G_{\mu\nu} = \frac{\partial G_{\mu\nu}}{\partial F_{\sigma\tau}}\delta F_{\sigma\tau} = \mathcal{G}_{\sigma\tau}\frac{\partial}{\partial F_{\sigma\tau}}\left(-\frac{\partial L}{\partial F^{\mu\nu}}\right) \tag{12}$$

Rewriting the dual forms and using the constitutive relation (7) gives:

$$\eta_{\mu\nu\lambda\rho}F^{\lambda\rho} = \eta_{\sigma\tau\alpha\beta}\left(-\frac{\partial L}{\partial F_{\sigma\tau}}\right)\frac{\partial}{\partial F_{\sigma\tau}}\left(-\frac{\partial L}{\partial F^{\mu\nu}}\right) \tag{13}$$

This condition is the one that is needed in order to obtain duality invariant Lagrangians. After some rewriting and integrating, completely analogous to what is done in [3], the following result is obtained:

$$\eta^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = \eta_{\sigma\tau\alpha\beta}\frac{\partial L}{\partial F_{\sigma\tau}}\frac{\partial L}{\partial F_{\alpha\beta}} \tag{14}$$

Actually, the integration step in the derivation of (14) gives an integration constant, but since the boundary condition is imposed that at least Maxwell theory should satisfy this condition, the integration constant has to equal zero. Another thing that is seen, is that the condition is manifestly Lorentz-invariant and it can be rewritten to give:

$$F_{\mu\nu}\mathcal{F}^{\mu\nu} = G_{\mu\nu}\mathcal{G}^{\mu\nu}.$$

It is now easy to see that alternatively this condition can also be rewritten in terms of $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{D}$ as

$$\mathbf{E}\cdot\mathbf{B} = \mathbf{D}\cdot\mathbf{H}$$

3.2 Lagrangians: some remarks

From the general statement that the action S, which is defined as the integral over the Lagrangian $S = \int L \, \mathrm{d}^4x$, should be a Lorentz invariant quantity, it follows that also the Lagrangian should be invariant.

In both [4] and [5] it is stated clearly that there are only two Lorentz invariant quantities in electrodynamics, which will be defined as P and Q:

$$P = F^{\alpha\beta}F_{\alpha\beta} = F^2 = 2(\mathbf{B}^2 - \mathbf{E}^2) \tag{15a}$$

$$Q = \frac{1}{2}\eta^{\alpha\beta}{}_{\mu\nu}F^{\mu\nu}F_{\alpha\beta} = \mathcal{F}F = 4\mathbf{E}\cdot\mathbf{B} \tag{15b}$$

Since one can always express the Lagrangian L in terms of these P and Q, it might be favorable to express condition (14) in these terms.

A second remark considers the term Q in the Lagrangian. Suppose that there is an odd power of Q present in the Lagrangian, then this can be written as: $Q^{2n+1} = (F\mathcal{F})^{2n+1}$. When performing the duality transformation (10), it follows that $F\mathcal{F} \to -\mathcal{G}G$, where the minus sign appears as a result of taking the dual of the dual tensor $\mathcal{F}$. As a result of the duality condition, the odd power of Q transforms to give $Q^{2n+1} \to -(\mathcal{G}G)^{2n+1} = -(\mathcal{F}F)^{2n+1} = -Q^{2n+1}$, from which it is immediate that terms with odd powers of Q are *not* invariant under duality transformations and therefore will not appear in the Lagrangian and we might as well leave these terms out when constructing the power expansion for L.

Analogous to this argument, one can reason that even powers of Q, *do* admit duality invariance, as do all (i.e. odd and even) powers of P.

3.3 Duality invariance condition in terms of P and Q

As stated before, each Lagrangian under consideration can be expressed in terms of P and Q, or to put is more correctly: in P and Q^2. This means that it must be possible to rewrite the condition (14) in terms of P and Q. That this form for the duality invariance condition provides a better insight, is obvious from the fact that P and Q^2 are both scalar quantities, while $F^{\mu\nu}$ are tensors.

So rewriting the duality invariance condition is what will be done. First consider:

$$\frac{\partial L}{\partial F_{\alpha\beta}} = \frac{\partial L}{\partial P}\frac{\partial P}{\partial F_{\alpha\beta}} + \frac{\partial L}{\partial Q}\frac{\partial Q}{\partial F_{\alpha\beta}} \tag{16}$$

Now there are two terms which have to be calculated:

$$\frac{\partial P}{\partial F_{\alpha\beta}} = \frac{\partial}{\partial F_{\alpha\beta}}\left(F_{\mu\nu}F^{\mu\nu}\right) = 2F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial F_{\alpha\beta}} = 2F^{\mu\nu}(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu) = 4F^{\alpha\beta}$$

$$\begin{aligned}\frac{\partial Q}{\partial F_{\alpha\beta}} &= \frac{\partial}{\partial F_{\alpha\beta}}\left(\frac{1}{2}\eta^{\lambda\rho}{}_{\mu\nu}F^{\mu\nu}F_{\lambda\rho}\right) = \frac{1}{2}\eta^{\lambda\rho\mu\nu}\left(F_{\lambda\rho}\frac{\partial F_{\mu\nu}}{\partial F_{\alpha\beta}} + F_{\mu\nu}\frac{\partial F_{\lambda\rho}}{\partial F_{\alpha\beta}}\right) = \\ &= \frac{1}{2}\eta^{\lambda\rho\mu\nu}\left(F_{\lambda\rho}(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu) - F_{\mu\nu}(\delta^\alpha_\lambda\delta^\beta_\rho - \delta^\beta_\lambda\delta^\alpha_\rho)\right) \\ &= 2\eta^{\alpha\beta\lambda\rho}F_{\lambda\rho}\end{aligned}$$

Where in the last steps in the above equations the anti-symmetry property of the field strength tensor $F^{\alpha\beta} = -F^{\beta\alpha}$ and the property of the η-tensor that $\eta^{\lambda\rho\mu\nu} = \eta^{\mu\nu\lambda\rho}$ are used.

Now the righthand side of equation (14) is calculated, using equation (16):

$$\begin{aligned}
\eta_{\sigma\tau\alpha\beta}\frac{\partial L}{\partial F_{\sigma\tau}}\frac{\partial L}{\partial F_{\alpha\beta}} &= \eta_{\sigma\tau\alpha\beta}\left(\frac{\partial L}{\partial P}\frac{\partial P}{\partial F_{\sigma\tau}} + \frac{\partial L}{\partial Q}\frac{\partial Q}{\partial F_{\sigma\tau}}\right)\left(\frac{\partial L}{\partial P}\frac{\partial P}{\partial F_{\alpha\beta}} + \frac{\partial L}{\partial Q}\frac{\partial Q}{\partial F_{\alpha\beta}}\right)\\
&= \eta_{\sigma\tau\alpha\beta}\frac{\partial P}{\partial F_{\sigma\tau}}\frac{\partial P}{\partial F_{\alpha\beta}}\left(\frac{\partial L}{\partial P}\right)^2 +\\
&\quad + \eta_{\sigma\tau\alpha\beta}\left[\frac{\partial P}{\partial F_{\alpha\beta}}\frac{\partial Q}{\partial F_{\sigma\tau}} + \frac{\partial P}{\partial F_{\sigma\tau}}\frac{\partial Q}{\partial F_{\alpha\beta}}\right]\left(\frac{\partial L}{\partial P}\right)\left(\frac{\partial L}{\partial Q}\right) +\\
&\quad + \eta_{\sigma\tau\alpha\beta}\frac{\partial Q}{\partial F_{\sigma\tau}}\frac{\partial Q}{\partial F_{\alpha\beta}}\left(\frac{\partial L}{\partial Q}\right)^2
\end{aligned}$$

First the factors preceding the partial derivatives with respect to L are calculated to give an equivalent expression for the righthand side of the condition (14) in terms of P and Q:

$$\eta_{\sigma\tau\alpha\beta}\frac{\partial P}{\partial F_{\sigma\tau}}\frac{\partial P}{\partial F_{\alpha\beta}} = 16\eta_{\sigma\tau\alpha\beta}F^{\alpha\beta}F^{\sigma\tau} = 32Q \tag{17a}$$

$$\begin{aligned}
\eta_{\sigma\tau\alpha\beta}\left(\frac{\partial Q}{\partial F_{\sigma\tau}}\frac{\partial P}{\partial F_{\alpha\beta}} + \frac{\partial P}{\partial F_{\sigma\tau}}\frac{\partial Q}{\partial F_{\alpha\beta}}\right) &= 8\eta_{\sigma\tau\alpha\beta}\left(\eta^{\sigma\tau\lambda\rho}F_{\lambda\rho}F^{\alpha\beta} + F^{\sigma\tau}\eta^{\alpha\beta\lambda\rho}F_{\lambda\rho}\right)\\
&= 16\eta_{\sigma\tau\alpha\beta}\eta^{\sigma\tau\lambda\rho}F_{\lambda\rho}F^{\alpha\beta}\\
&= -64F_{\alpha\beta}F^{\alpha\beta} = -64P
\end{aligned} \tag{17b}$$

$$\begin{aligned}
\eta_{\sigma\tau\alpha\beta}\frac{\partial Q}{\partial F_{\sigma\tau}}\frac{\partial Q}{\partial F_{\alpha\beta}} &= 4\eta_{\sigma\tau\alpha\beta}\eta^{\sigma\tau\lambda\rho}F_{\lambda\rho}\eta^{\alpha\beta\lambda\rho}F_{\lambda\rho}\\
&= -16\eta^{\alpha\beta\lambda\rho}F_{\alpha\beta}F_{\lambda\rho} = -32Q
\end{aligned} \tag{17c}$$

Where in obtaining (17b) and (17c) the fact that $\eta_{\sigma\tau\alpha\beta}\eta^{\sigma\tau\lambda\rho} = -2(\delta^\lambda_\alpha\delta^\rho_\beta - \delta^\rho_\alpha\delta^\lambda_\beta)$ and the anti-symmetry property of the field-strength tensor are being used.

It is trivial to see that the lefthand side of condition (14) equals $\eta^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = 2Q$, such that one obtains the equivalent of (14) in terms of P and Q:

$$Q = 16Q\left(\frac{\partial L}{\partial P}\right)^2 - 32P\left(\frac{\partial L}{\partial P}\right)\left(\frac{\partial L}{\partial Q}\right) - 16Q\left(\frac{\partial L}{\partial Q}\right)^2 \tag{18}$$

It is immediate that the Maxwell Lagrangian, which can be expressed as $L_{\text{Maxwell}} = -P/4$, satisfies this condition.

At first sight, condition (18) looks terrible, because of the fact that the partial derivatives come in squared forms and multiplications of each other and there does not seem to be a way to factorize this condition. Although the condition looks this bad, there are two ways in which its restrictiveness will be investigated in the coming sections. First, the Born-Infeld Lagrangian will be under consideration, of which it is commonly known that this Lagrangian admits the duality transformation. Secondly, and this is the core of the research in this thesis, a power expansion of the Lagrangian will be constructed and it will be investigated how restrictive condition (18) is on this general power expansion, when imposed that a duality invariant Lagrangian should be obtained. Furthermore, this power expansion will be compared to the power expansion of the Born-Infeld Lagrangian.

4 The Born-Infeld Lagrangian

In 1934 the famous German physicist Max Born and his student Leopold Infeld published an article [2] in which they came up with what is now know as the Born-Infeld Lagrangian. This Lagrangian has the interesting property of turning into Maxwell-theory for low electric and magnetic fields and what is even more: although the Lagrangian is clearly nonlinear, it maintains to be invariant under the duality transformation (10). The Born-Infeld Lagrangian reads:

$$L = \frac{1}{b^2}\left\{\sqrt{-g} - \sqrt{-\det(g_{\mu\nu} + bF_{\mu\nu})}\right\} \tag{19}$$

To see if it indeed satisfies duality invariance, first work out the determinant and write L in terms of P and Q:

$$\begin{aligned} L &= \frac{1}{b^2}\left\{1 - \sqrt{1 + b^2(\mathbf{B}^2 - \mathbf{E}^2) - b^4(\mathbf{E}\cdot\mathbf{B})^2}\right\} \\ &= \frac{1}{b^2}\left\{1 - \sqrt{1 + b^2\frac{P}{2} - b^4\frac{Q^2}{16}}\right\} \end{aligned} \tag{20}$$

Where the last step follows from the definitions (15). Now it is investigated if this Lagrangian satisfies condition (18). Therefore first both partial derivatives are calculated:

$$\frac{\partial L}{\partial P} = \frac{-1}{4\sqrt{1 + b^2\frac{P}{2} - b^4\frac{Q^2}{16}}}$$

$$\frac{\partial L}{\partial Q} = \frac{b^2 Q}{16\sqrt{1 + b^2\frac{P}{2} - b^4\frac{Q^2}{16}}}$$

Putting these into condition (18) gives:

$$Q = \frac{Q + \frac{b^2QP}{2} - \frac{b^4Q^3}{16}}{1 + \frac{b^2P}{2} - \frac{b^4Q^2}{16}},$$

which is obviously true, so now the fact that the Born-Infeld Lagrangian admits electric-magnetic duality is proven.

In the next section a power expansion of L will be made, where L is imposed to satisfy the duality invariance condition (18). In order to compare the power expansion of the next section to Born-Infeld theory, the power expansion will be explicitly put here up to order b^6. Using the mathematics software package 'Mathematica 5', this power expansion of the Born-Infeld Lagrangian (20) can be seen to equal:

$$L = -\frac{P}{4} + \frac{b^2}{32}\left(P^2 + Q^2\right) - \frac{b^4}{128}P\left(P^2 + Q^2\right) + \frac{b^6}{2048}(P^2 + Q^2)\left((P^2 + Q^2) + 4P^2\right) + \dots \tag{21}$$

5 Power Series Expansion of Duality Invariant Lagrangians

In order to investigate the properties of duality invariant Lagrangians, a power series expansion for the Lagrangian will be constructed as follows:

$$L = L_0 + aL_1 + a^2L_2 + a^3L_3 + \dots$$

where, in order to reobtain Maxwell theory if $a = 0$, for L_0 there has to hold:

$$L_0 = -\frac{P}{4}.$$

In the power expansion, each higher order of a means that the corresponding Lagrangian has to include terms of one higher order of P, Q or of the product of both. Therefore L_1 will in general be a linear combination of the terms of order P^2 and Q^2 and L_2 will be a linear combination of the terms P^3 and PQ^2.

5.1 Power expansion up to order a^2

Explicitly, the following general forms for L_1 and L_2 will be put in the duality invariance condition:

$$L_1 = c_1P^2 + c_2Q^2 \tag{22a}$$
$$L_2 = d_1P^3 + d_2PQ^2 \tag{22b}$$

The expansion of the Lagrangian then reads:

$$L = -\frac{P}{4} + aL_1 + a^2L_2 \tag{23}$$

and this form will be used to solve condition (18) for a and a^2.

Substituting the Lagrangian from (23) into (18) and solving the condition for each power of a gives two conditions (one for a and one for a^2) which determine L_1 and L_2. The conditions are:

$$a: \qquad Q\left(\frac{\partial L_1}{\partial P}\right) = P\left(\frac{\partial L_1}{\partial Q}\right) \tag{24a}$$

$$a^2: \qquad P\left(\frac{\partial L_2}{\partial Q}\right) - Q\left(\frac{\partial L_2}{\partial P}\right) = 2Q\left[\left(\frac{\partial L_1}{\partial P}\right)^2 + \left(\frac{\partial L_1}{\partial Q}\right)^2\right] \tag{24b}$$

Substituting L_1 from (22a) into condition (24a) gives:

$$2c_1PQ = 2c_2PQ$$

$$\Rightarrow c_1 = c_2$$

So there is free parameter, which can be put in the prefactor a to obtain:

$$L_1 = P^2 + Q^2 \tag{25}$$

Substituting (25) and (22b) into condition (24b) gives:

$$(2d_2 - 3d_1)P^2Q - d_2Q^3 = 8P^2Q + 8Q^3$$

$$\Rightarrow d_1 = d_2 = -8$$

So for L_2 the condition demands:

$$L_2 = -8P(P^2 + Q^2) \tag{26}$$

Thus far, the power expansion that is made contains no free parameters *in* the Lagrangians L_1 and L_2 which give some freedom in the power expansion (23) of the Lagrangian and which therefore admit a deviation from the power expansion of the Born-Infeld Lagrangian. To be more precise, the constant a in the power expansion equals $b^2/32$ in the Born-Infeld Lagrangian.

5.2 Up to order a^4

Now using the same procedure which was followed in obtaining L_1 and L_2, obtaining the condition for order a^3 and from this condition an expression for L_3 is what will now be done.

First it is noted that the general form for L_3 from which is started is:

$$L_3 = e_1P^4 + e_2P^2Q^2 + e_3Q^4 \tag{27}$$

The condition of order a^3 can easily be calculated to be:

$$P\left(\frac{\partial L_3}{\partial Q}\right) - Q\left(\frac{\partial L_3}{\partial P}\right) = 4P\left(\frac{\partial L_1}{\partial P}\right)\left(\frac{\partial L_2}{\partial Q}\right) + 4Q\left(\frac{\partial L_1}{\partial Q}\right)\left(\frac{\partial L_2}{\partial Q}\right) \tag{28}$$

The righthand side of this condition can with the aid of (25) and (26) easily be calculated, which with the lefthand side gives:

$$(2e_2 - 4e_1)P^3Q + (4e_3 - 2e_2)PQ^3 = -128(P^3Q + PQ^3)$$

$$\Rightarrow e_2 = 2e_1 - 64 = 2e_3 + 64$$

So a new free parameter r can be defined in L_3. Defining $e_2 = 2r$ gives $e_1 = r + 32$ and $e_3 = r - 32$, which in turn results in an equation for L_3, containing the free parameter r:

$$L_3 = (r + 32)P^4 + 2rP^2Q^2 + (r - 32)Q^4 \tag{29}$$

Alternatively (29) can be written as:

$$\begin{aligned} L_3 &= r(P^2 + Q^2)^2 + 32(P^2 + Q^2)(P^2 - Q^2) \\ &= (P^2 + Q^2)\left[r(P^2 + Q^2) + 32(P^2 - Q^2)\right] \end{aligned}$$

Now consider the part of the condition of order a^4, which after some calculation turns out to be:

$$P\left(\frac{\partial L_4}{\partial Q}\right) - Q\left(\frac{\partial L_4}{\partial P}\right) = 2Q\left[\left(\frac{\partial L_2}{\partial Q}\right)^2 - \left(\frac{\partial L_2}{\partial Q}\right)^2\right] + 4P\left(\frac{\partial L_2}{\partial P}\right)\left(\frac{\partial L_2}{\partial Q}\right) + \\ +4P\left(\frac{\partial L_1}{\partial P}\right)\left(\frac{\partial L_3}{\partial Q}\right) + 4Q\left(\frac{\partial L_1}{\partial Q}\right)\left(\frac{\partial L_3}{\partial Q}\right) \tag{30}$$

The general form for L_4 looks like:

$$L_4 = f_1 P^5 + f_2 P^3 Q^2 + f_3 P Q^4 \tag{31}$$

Evaluating the lefthand side (LHS) of (30) with (31) gives:

$$\text{LHS} = P^4 Q(2f_2 - 5f_1) + P^2 Q^3(4f_3 - 3f_2) - Q^5 f_3 \tag{32}$$

Calculating the righthand side (RHS) of (30) gives:

$$\text{RHS} = P^4 Q(384 + 32r) + P^2 Q^3(-768 + 64r) + Q^5(-1152 + 32r) \tag{33}$$

From equating the LHS and RHS it follows that $f_3 = 1152 - 32r$, and this gives $f_1 = 640 - 32r$ and $f_2 = 1792 - 64r$.

Finally an expression for L_4 is obtained, which only depends on the free parameter r which was first found in L_3:

$$L_4 = 32(20 - r)P^5 + 32(56 - 2r)P^3 Q^2 + 32(36 - r)PQ^4 \tag{34}$$

It is furthermore easy to see that L_4 can alternatively be written as

$$L_4 = 32P(P^2 + Q^2)\left[(P^2 + Q^2)(20 - r) + 512Q^2\right]$$

Comparing the power expansion thus far obtained to the power expansion (21) of the Born-Infeld Lagrangian, it is clear that for the specific choice $r = 48$ for the free parameter r the power expansion up to order a^3 equals the power expansion of the Born-Infeld Lagrangian.

5.3 Order a^5 and higher

Totally analogous to the previous derivations, but increasing in calculational complexity (i.e. there are more algebraic steps to be taken each time a next order of a is considered), conditions and solutions to higher order in a can be obtained. Since the calculations are totally analogous to the ones that were performed earlier this section and since the result is the only interesting thing, from now on the calculations will be skipped and only the results will be noted and commented on.

After some calculation it is seen that

$$L_5 = h_1 P^6 + h_2 P^4 Q^2 + h_3 P^2 Q^4 + h_4 Q^6$$

with

$$\begin{aligned} h_1 &= s + \frac{128}{3}(-176 + 15r) \\ h_2 &= 3s + 128(-144 + 7r) \\ h_3 &= 3s \\ h_4 &= s + 256(32 - r) \end{aligned}$$

It is seen that a new free parameter s had to be introduced. Calculation (the exact result will be omitted here) of L_6 gives no new free parameters, while L_7 *does* give a new parameter. The same holds for the pair L_8 and L_9, where the former does not give a free parameter, while the latter does give a new parameter.

These results lead to a theorem which will be formulated and proven in the next section.

6 Free Parameters in the Power Series Expansion of Duality Invariant Lagrangians

As mentioned, the results which have been obtained in the previous section lead to believe that for every L_s with an odd s a free parameter in the power expansion of L is obtained, while for even s the coefficients in the power expansion are totally determined. This conjecture will more closely be considered in what follows and a theorem will be formulated and proven.

The power expansion for arbitrarily large k can be written as:

$$L = -\frac{P}{4} + aL_1 + a^2L_2 + a^3L_3 + ... + a^{2k-1}L_{2k-1} + a^{2k}L_{2k} \tag{35}$$

By construction, L_{2k-1} and L_{2k} can be written as:

$$L_{2k-1} = \sum_{n=0}^{k} a_n P^{2(k-n)} Q^{2n} \tag{36a}$$

$$L_{2k} = \sum_{n=0}^{k} a_n P^{2(k-n)+1} Q^{2n} \tag{36b}$$

Now the following theorem can be formulated.

Theorem *In the power expansion (35) of duality invariant Lagrangians (i.e. Lagrangians which satisfy (18)), there will occur a free parameter in each L_{2k-1} as defined in (36a) and there will occur* no *free parameter in each L_{2k} as defined in (36b).*

Proof. To obtain a differential equation for a certain L_s from the duality condition (18), first the general form

$$L = -\frac{P}{4} + aL_1 + a^2L_2 + \ldots + a^{s-1}L_{s-1} + a^sL_s$$

will be put into the condition for order a^s. The duality condition then reads:

$$0 = Q\left\{\left(\frac{\partial L}{\partial P}\right)^2\right\}_{\mathcal{O}(a^s)} - 2P\left\{\left(\frac{\partial L}{\partial P}\right)\left(\frac{\partial L}{\partial Q}\right)\right\}_{\mathcal{O}(a^s)} - Q\left\{\left(\frac{\partial L}{\partial Q}\right)^2\right\}_{\mathcal{O}(a^s)},$$

where the subscript $\mathcal{O}(a^s)$ denotes that only the terms of order a^s in the product of partial derivatives are considered. Thus the condition for order a^s is obtained and this can be calculated in parts:

$$\left\{\left(\frac{\partial L}{\partial P}\right)^2\right\}_{\mathcal{O}(a^s)} = \left\{\left(-\frac{1}{4} + a\frac{\partial L_1}{\partial P} + \ldots + a^{s-1}\frac{\partial L_{s-1}}{\partial P} + a^s\frac{\partial L_s}{\partial P}\right)^2\right\}_{\mathcal{O}(a^s)} =$$
$$= 2a^s\left(-\frac{1}{4}\frac{\partial L_s}{\partial P} + \frac{\partial L_1}{\partial P}\frac{\partial L_{s-1}}{\partial P} + \ldots\right)$$

$$\left\{\left(\frac{\partial L}{\partial P}\right)\left(\frac{\partial L}{\partial Q}\right)\right\}_{\mathcal{O}(a^s)} = \left\{\left(-\frac{1}{4} + a\frac{\partial L_1}{\partial P} + \ldots + a^s\frac{\partial L_s}{\partial P}\right)\left(a\frac{\partial L_1}{\partial Q} + \ldots + a^s\frac{\partial L_s}{\partial Q}\right)\right\}_{\mathcal{O}(a^s)}$$
$$= a^s\left(-\frac{1}{4}\frac{\partial L_s}{\partial Q} + \frac{\partial L_1}{\partial P}\frac{\partial L_{s-1}}{\partial Q} + \frac{\partial L_{s-1}}{\partial P}\frac{\partial L_1}{\partial Q} + \ldots\right)$$

$$\left\{\left(\frac{\partial L}{\partial Q}\right)^2\right\}_{\mathcal{O}(a^s)} = \left\{\left(a\frac{\partial L_1}{\partial Q} + a^2\frac{\partial L_2}{\partial Q} + \ldots + a^{s-2}\frac{\partial L_{s-2}}{\partial Q} + a^{s-1}\frac{\partial L_{s-1}}{\partial Q} + a^s\frac{\partial L_s}{\partial Q}\right)^2\right\}_{\mathcal{O}(a^s)} =$$
$$= 2a^s\left(\frac{\partial L_1}{\partial Q}\frac{\partial L_{s-1}}{\partial Q} + \frac{\partial L_2}{\partial Q}\frac{\partial L_{s-2}}{\partial Q} + \ldots\right)$$

Now an important observation is made: since all terms in the power expansion of lower order than s are already known, the part of the differential equation which contains the unknown prefactors can be taken to the lefthand side and the other part is kept on the righthand side. This way the lefthand side is a polynomial in P and Q, with the unknown prefactors in the coefficients. Moreover, the righthand side also is a polynomial, but here the coefficients are known. So the coefficients on the lefthand side are determined by the coefficients on the righthand side and equations for the unknown prefactors can be obtained.

Using the condition and factors that were calculated above, the following is obtained for the lefthand side of the condition:

$$\text{LHS} = P\frac{\partial L_s}{\partial Q} - Q\frac{\partial L_s}{\partial P} \tag{37}$$

Remember that L_s can be rewritten as L_{2k} in (36b) if s is even and as L_{2k-1} in (36a) if s is odd. From (36a), the (LHS) in (37) is calculated for L_{2k-1}:

$$P\frac{\partial L_{2k-1}}{\partial Q} - Q\frac{\partial L_{2k-1}}{\partial P} = \sum_{n=0}^{k} 2a_n\left((n-k)P^{2(k-n)-1}Q^{2n+1} + nP^{2(k-n)+1}Q^{2n-1}\right) \tag{38}$$

The same can be done for L_{2k}, which gives for the (LHS):

$$P\frac{\partial L_{2k}}{\partial Q} - Q\frac{\partial L_{2k}}{\partial P} = \sum_{n=0}^{k} a_n\left(\left[2(n-k)-1\right]P^{2(k-n)}Q^{2n+1} + 2nP^{2(k-n)+2Q^{2n-1}}\right) \tag{39}$$

The odd terms L_{2k-1}in the power expansion

Now consider the terms in (38) for which $n = 0$ and $n = 1$. It follows that for L_{2k-1} for $n = 0$ the term gives:

$$-2a_0kP^{2k-1}Q$$

and for $n = 1$ the term reads

$$2a_1\left[(1-k)P^{2k-3}Q^3 + P^{2k-1}Q\right].$$

So, the coefficient in the polynomial of the product $P^{2k-1}Q$ links the unknown prefactors a_0 and a_1.

Consider the term in the summation for $n = i$, which reads:

$$2a_i\left[(i-k)P^{2(k-i)-1}Q^{2i+1} + iP^{2(k-i)+1}Q^{2i-1}\right].$$

Then the term for which $n = i+1$ reads:

$$2a_{i+1}\left[(i+1-k)P^{2(k-i)-3}Q^{2i+3} + (i+1)P^{2(k-i)-1}Q^{2i+1}\right].$$

Again, it is clear that in the coefficient of $P^{2(k-i)-1}Q^{2i+1}$ both a_i and a_{i+1} are linked to each other, a statement which holds for $i < k$. More precisely, this result means that the coefficients lead to equations from which follows that a_0 determines a_1, which in turn determines a_2 etc. al the way up to a_{k-1}

Now there are two special cases that have to be considered. If $n = k-1$, the term in the summation reads:

$$2a_{k-1}\left[-PQ^{2k-1} + (k-1)P^3Q^{2k-3}\right].$$

If $n = k$, the term in the summation reads:

$$2a_kkPQ^{2k-1}.$$

So from this it is clear that the prefactor a_{k-1} and only that prefactor determines a_k. So clearly all the prefactors a_0 up to a_{k-1} all determine the prefactors with the next higher index. This means that there is the freedom of choosing one of these prefactors to be a free parameter which then determines all other prefactors. So indeed in each L_{2k-1} there comes a new free parameter into the power expansion of duality invariant Lagrangians.

The even terms L_{2k} in the power expansion

For the even terms L_{2k} in the Lagrangian the same line of reasoning as for the odd terms can be followed. It can be easily seen that for a_0 in the summation that only the product $P^{2k}Q$ is involved and this term is also involved in the term of the summation for a_1. So, again, a_0 determines a_1.

Also for L_{2k} it can be analogously shown that for $i < k$ it is true that a_i determines a_{i+1}. Since this fact is obvious from (39) and doesn't provide any important insight, the exact derivation of this statement will be omitted.

The interesting case are $n = k - 1$ and $n = k$. In the term $n = k - 1$ the terms $P^2 Q^{2k-1}$ and $P^4 Q^{2k-3}$ are involved and in the term $n = k$ there are also two terms involved, which are $P^2 Q^{2k-1}$ and Q^{2k+1}. Now there is an important difference with respect to the odd term L_{2k-1}: the parameter a_k is now totally determined by the coefficient of Q^{2k+1} in the polynomial. Then this a_k in turn determines a_{k-1} which this way determines the parameters all the way down to a_0. So now it is clear that all the prefactors are determined and no free parameters occur in the even terms L_{2k}. This finishes the proof of the theorem.

7 Another Attempt

In examining some standard forms for the Lagrangians, the duality condition admits no nice solutions for duality invariant Lagrangians. There is however some general form for the Lagrangian which is not admitted by the duality invariance condition, which will be shown in this section.

Assume that a Lagrangian which satisfies the duality condition (18) is of the form:

$$L = -\frac{P}{4} + cf(Q), \tag{40}$$

where c is a constant (which can be set to zero to give Maxwell theory) and $f(Q)$ is a function that only depends on Q (actually of course only on Q^2) and explicitly *not* on P. If (40) is indeed a solution, it should be possible to find a differential equation for $f(Q)$ which gives a condition for $f(Q)$.

It is easy to see that putting (40) into condition (18) gives the following condition for $f(Q)$:

$$0 = 4Pc\left(\frac{\partial f(Q)}{\partial Q}\right) - 8Qc^2\left(\frac{\partial f(Q)}{\partial Q}\right)^2$$

$$\Rightarrow \frac{\partial f(Q)}{\partial Q} = \frac{P}{2cQ}$$

But now a contradiction is reached, since the assumption was that $f(Q)$ is explicitly no function of P, this latter equation cannot hold by definition (of $f(Q)$). This contradiction leads to the conclusion that there exists no such Lagrangian (40) which satisfies the duality condition.

Discussion and Conclusions

In this thesis first the general framework of nonlinear electrodynamics was introduced as a generalization of Maxwell theory, which made it possible to generalize the duality transformation under which Maxwell theory is invariant to a duality transformation under which nonlinear electrodynamics is invariant. Using this duality invariance of nonlinear electrodynamics and the constitutive relation for $G^{\mu\nu}$, it was possible to derive a necessary and sufficient condition for (Lorentz-invariant) Lagrangians, such that Lagrangians which satisfy this condition are duality invariant. This condition is of great importance, since the main goal of this thesis, as formulated in the Introduction, was to investigate the uniqueness of the Born-Infeld nonlinear theory of electrodynamics. The Born-Infeld Lagrangian was indeed found to be duality invariant.

The next step in this thesis was to make a 'general' power expansion of the Lagrangians, which was then imposed to satisfy the duality invariance condition. From the results followed that there occur free parameters in this general power expansion and this power expansion coincides with the power expansion of the Born-Infeld Lagrangian for specific choices of the free parameters. In fact there are infinitely many free parameters, since for arbitrarily large odd powers of a in the expansion there still occur new free parameters in the general expansion. This means that there can now be given an answer to the question posed in the Introduction. There are indeed more Lagrangians which satisfy duality invariance next to the Born-Infeld theory. This is the important conclusion of this thesis and it means that there indeed exists a generalization of Born-Infeld theory.

As always when a question is answered in physics or any other discipline, there occur new interesting questions, which then can give rise to new topics for research. One such question is how these free parameters must be interpreted in the light of the existence of other analytical solutions. Are there, for example, special choices for the parameters in the power expansion which give rise to some analytical form for the Lagrangian? Or, what can we say about the Lagrangian when infinitesimal deviations from the Born-Infeld Lagrangian are considered?

Another interesting question might be what this possible generalization of Born-Infeld theory means in string theory, since the Born-Infeld Lagrangian occurs as some approximation in string theory.

Something that one may wish to do is to guess some smart solution of the duality invariance condition. This is something the author tried to do as well, but various attempts (except the one mentioned in section 7) did not give rise to any result, but instead resulted in an even more difficult modification of the condition which made the problem more difficult to deal with. However, since the author did not spend much time on it, a closer examination of the condition or using some smart mathematical technique might give rise to some interesting results.

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References

[1] I. Bialynicki-Birula. Nonlinear Electrodynamics: Variations on a theme by Born and Infeld. In B. Jancewicz and J. Lukierski, editors, *Quantum Theory of Particles and Fields*, pages 31–48. World Scientific, 1983.

[2] M. Born and L. Infeld. Foundations of the New Field Theory. In *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, volume 144, pages 425–451. Royal Society of London, 1934.

[3] G. W. Gibbons and D. A. Rasheed. Electric-magnetic duality rotations in non-linear electrodynamics. *Nuclear Physics B*, 454:185, 1995.

[4] J. D. Jackson. *Classical Electrodynamics*, chapter 6.12 & 11.9, pages 251–253 & 547–552. John Wiley & Sons, Second edition, 1975.

[5] L. D. Landau and E. M. Lifschitz. *Mechanics and Electrodynamics*, volume 1. Pergamon Press, First edition, 1972.