Spiky Strings

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Contents

Chapter 1

Introduction

In this bachelorthesis I am going to look at some solutions to the equations of motion in string theory, called spiky strings. These are closed string solutions which recently got some attention in the string theory field (see for example [7], [6], [8] or [10]).

Originally this was not the goal of this research. The goal started out to be finding some solutions for the classical relativistic string. Solutions with more than just one oscillation mode, which are given in most introductory books on string theory. This proved to be quite difficult since solving the nonlinear constraints that arise in string theory is not an easy thing to do. The attention was also mainly focused on the open string.

This was until Prof. Dr. M. de Roo pointed me at an article of M. Kruczenski [6] about some properties of spiky strings. Spiky strings showed to be fairly simple solutions for the case of a closed string, which are not commonly known I believe (although they were already pointed out to excist in 1985 by C.J. Burden in [3]).

So the goal of the research changed to finding out what these spiky string solutions are and learning something about their properties. A sideremark here is that the research remained classical in the sense that Quantum Mechanical properties of the spiky string were not investigated.

Finally I want to thank Prof. Dr. M. de Roo for guiding me trough the proces of research and writing this thesis.

Chapter 2

A short review of classical string theory

To get a good understanding of the topics that will be covered in this thesis, we will first give a short introduction in string theory. We will generalize the idea of a zero-dimensional point particle to that of a two-dimensional object, called a string. We will find its equation of motion through the action formalism and also derive some constraints on its solutions. Finally we will give the general solutions which obey these constraints.

2.1 The relativistic point particle

First we will cover the point particle. This part will be considered as known, but we will briefly cover it for completeness and reference. For a more complete description of the formalism see for example [11].

Consider a free relativistic point particle, in a d-dimensional Minkowski spacetime. The spacetime coordinates are given by[1]

$$x^\mu(\tau) = (x^0, x^1, \ldots, x^{d-1}) \tag{2.1}$$

The particle traces a path in spacetime, which is called the worldline of the particle. This worldline is parametrized by a parameter τ, called the proper time.
An infinitesimal pathlength swept out by the particle in spacetime is given by

$$dl = \sqrt{-ds^2} = \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} \equiv \sqrt{-dx^\mu dx_\mu} \tag{2.2}$$

In this $\eta_{\mu\nu}$ is the Minkowski metric, given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \tag{2.3}$$

[1]We will use: $c = \hbar = 1$.

The action for the particle is simply the total length of the trajectory the particle follows in spacetime

$$S[x] = -m \int dl(\tau) = -m \int d\tau \sqrt{-\dot{x}^2}, \qquad \dot{x}^2 = \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu, \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} \tag{2.4}$$

The parameter m represents the mass of the particle.

An important property of the action is that it is invariant under arbitrary local reparametrisations

$$\tau \to \tau'(\tau), \qquad \frac{d\tau}{d\tau'} > 0$$

For proof of this, see for example [13].

To find the equation of motion, we demand that the variation of the action vanishes[2]:

$$\delta S = 0$$

We find that the equation of motion for a free relativistic point particle is given by

$$\frac{dp_\mu}{d\tau} = 0 \tag{2.5}$$

Which means that the momentum of the particle is conserved along its wordline.

2.2 The relativistic string

Now we generalize the idea of a pointparticle to a two dimensional object, the string. Where we described the particle and its movement by a wordline, which was parametrized by τ, we describe the string by a worldsheet, which is parametrized by the coordinates

$$(\xi^0, \xi^1) = (\tau, \sigma) \tag{2.6}$$

In this σ is a spatial coordinate which runs from 0 to π and τ is a parameter that describes the evolution of the string in time. The coordinates of the d-dimenional spacetime are now given by

$$x^\mu(\tau, \sigma) = (x^0, x^1, \ldots, x^{d-1}) \tag{2.7}$$

This induces a metric, an embedding metric,

$$g_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} \tag{2.8}$$

The action for the string is

$$S = -T \int d\sigma d\tau \sqrt{-g} = -T \int d\sigma d\tau \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}, \qquad g = \det g_{\alpha\beta} \tag{2.9}$$

[2] As it should by Hamilton's principle.

In this use was made of the notation

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau}, \qquad x'^\mu = \frac{dx^\mu}{d\sigma}$$

Because an action has the units mass × length, the constant T has the units of force. It describes the tension in the string. The form of the action in (2.9) is called the Nambu-Goto action.
Another form of the action, which is classically equivalent (see for example [14]) to the Nambu-Goto action is the Polyakov action:

$$S = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta}\partial_\alpha x \cdot \partial_\beta x \tag{2.10}$$

In this $h_{\alpha\beta}(\sigma,\tau)$ is an auxiliary metric on the worldsheet[3] and $h = \det h_{\alpha\beta}$. This form of the action is often more convenient to work with, than the Nambu-Goto action, because it does not have the overall square root.
The action (2.10) is invariant under the reparametrization

$$\sigma^\alpha \to f^\alpha(\sigma) = \sigma'^\alpha \qquad \text{and} \quad h_{\alpha\beta}(\sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha}\frac{\partial f^\delta}{\partial \sigma^\beta} h_{\gamma\delta}(\sigma') \tag{2.11}$$

And also under the rescalings

$$h_{\alpha\beta} \to e^{\phi(\sigma,\tau)} h_{\alpha\beta} \tag{2.12}$$

which are called Weyl transformations.
Using the symmetries (2.11) and (2.12) $h_{\alpha\beta}$ can be gauge fixed. The auxilary field can be chosen as

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.13}$$

In this gauge, called the conformal gauge, the action (2.10) becomes

$$-\frac{T}{2}\int d\tau d\sigma \eta^{\alpha\beta}\partial_\alpha x \cdot \partial_\beta x \tag{2.14}$$

From this it is clear that the canonical momentum is given by

$$P^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = T\dot{x}^\mu \tag{2.15}$$

By varying the action (2.14) and demanding that $\delta S = 0$, the equation of motion can be derived (e.g. see [4]), which is given by

$$\partial_\alpha \partial^\alpha x^\mu = 0 \tag{2.16}$$

In addition to this, a boundary condition must be imposed, to make the boundary term[4]

$$T\int d\tau \partial_\sigma x^\mu \delta x_\mu \Big|_{\sigma=0}^{\sigma=\pi}$$

[3]The role of this auxilary metric is analogous that of the auxilary field $e(\tau)$, which is sometimes introduced for the particle, see for example [1].

[4]We take the length of the string to be π.

which also arises in the varying of the action, vanish. We give three possible boundary conditions to do that.

The first boundary condition that can be imposed comes down to identifying the beginning of the string with the end of the string.

$$x^\mu(\tau,\sigma) = x^\mu(\tau,\sigma+\pi) \tag{2.17}$$

In this case the string is closed.
Another possibility is imposing Neumann boundary conditions on the string

$$\partial_\sigma x^\mu(\tau,0) = \partial_\sigma x^\mu(\tau,\pi) = 0 \tag{2.18}$$

Which results in an open string. Just like the last possibility we will consider: Dirichlet boundary conditions. This corresponds to taking the endpoint of the string fixed

$$\partial_\tau x^\mu(\tau,0) = \partial_\tau x^\mu(\tau,\pi) = 0 \tag{2.19}$$

In this case $\mu \neq 0$, because Dirichlet boundary conditions cannot be imposed on the time component.

From the above boundary conditions we will predominantly use the closed string boundary condition in this thesis. For reasons that will become clear in section 3.3.

Another equation of motion, namely the one for $h_{\alpha\beta}$, obtained by varying the action (2.10) with respect to $h_{\alpha\beta}$, implies the vanishing of the energy-momentum tensor of the worldsheet (see [2]):

$$T_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x \cdot \partial_\delta x = 0 \tag{2.20}$$

Together with the gauge fixing of the auxilary metric (2.13) this gives rise to the constraints

$$T_{00} = T_{11} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(x')^2 = 0 \qquad \text{and} \qquad T_{01} = T_{10} = \dot{x}\cdot x' = 0$$

which can be summarized by

$$(\dot{x} \pm x')^2 = 0 \tag{2.21}$$

Now the general solution for the open string with Neumann boundary conditions can be written like

$$x^\mu(\tau,\sigma) = q^\mu + \frac{1}{\pi T} P^\mu \tau + i\ell \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma, \qquad \alpha_n^\mu = (\alpha_{-n}^\mu)^* \tag{2.22}$$

where q^μ is the position of the center of mass and P^μ is the total momentum of the string, defined as

$$P^\mu \equiv \int_0^\pi d\sigma P^{\mu\tau} = T \int_0^\pi d\sigma \dot{x}^\mu \tag{2.23}$$

The last term in (2.22) corresponds to the oscillations of the string. The α_n^μ are coefficients of the oscillation modes, if they all vanish, the equations represents a point particle.

The momentum is also conserved in time because

$$\dot{P}^\mu = T \int d\sigma \ddot{x}^\mu = T \int d\sigma x''^\mu = T x'^\mu \Big|_0^\pi = 0$$

where the second equality follows from the equation of motion, and the last equality holds because of the Neumann boundary condition[5].

The general solution for the closed string is given by a left moving part and a right moving part

$$x_L^\mu = \frac{1}{2}q^\mu + \frac{1}{2\pi T}P^\mu (\tau + \sigma) + \frac{i}{2}\ell \sum_{n \neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2i(\tau+\sigma)n} \tag{2.24}$$

$$x_R^\mu = \frac{1}{2}q^\mu + \frac{1}{2\pi T}P^\mu (\tau - \sigma) + \frac{i}{2}\ell \sum_{n \neq 0} \frac{1}{n}\alpha_n^\mu e^{-2i(\tau-\sigma)n} \tag{2.25}$$

so that

$$x^\mu (\tau, \sigma) = x_L^\mu + x_R^\mu = q^\mu + \frac{1}{\pi T}P^\mu \tau + \frac{i}{2}\ell \sum_{n \neq 0} \frac{1}{n} e^{-2in\tau} \left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma} \right) \tag{2.26}$$

Again the total momentum is given by (2.23). The constraints (2.21) can now be rewritten in terms of the mode coefficients. For the open string it holds that

$$\dot{x} \pm x' = \ell \sum_n \alpha_n^\mu e^{-in(\tau \pm \sigma)} \tag{2.27}$$

Where the sum is over all values of n, including zero. In this α_0^μ is defined as

$$\alpha_0^\mu \equiv \frac{1}{\pi T \ell} P^\mu \tag{2.28}$$

So that now the constraints for the open string can be written like

$$L_n \equiv \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0 \tag{2.29}$$

In the case of the closed string the constraints take the form

$$L_n \equiv \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0, \qquad \alpha_0^\mu \equiv \frac{1}{2\pi T \ell} P^\mu \tag{2.30}$$

$$\tilde{L}_n \equiv \frac{1}{2} \sum_m \tilde{\alpha}_m^\mu \tilde{\alpha}_{n-m,\mu} = 0, \qquad \tilde{\alpha}_0^\mu = \alpha_0^\mu \tag{2.31}$$

[5]For a closed string the equality also holds because 0 and π represent the same point on the string. For Dirichlet boundary conditions the momentum is not always conserved. We will not cover this here.

Chapter 3

Spiky string solutions

In this chapter we find solutions for the case of the closed string. These solutions were already known in 1985 in the context of cosmic strings (see [3]).
We will find the solutions in two ways. First we will derive them directly from the equation of motion and the constraints. Secondly we will use the general solution (2.26) and find the correct oscillation mode coefficients to get to the solution.

3.1 Deriving the solutions (1)

We try to find solutions that satisfy (2.16), (2.21) and (2.17). In doing this we mainly follow the steps pointed out in [9].

We consider strings in flat Minkowski spacetime with $d = 3$ and use the conformal gauge (2.13). Furthermore we identify t by $\ell\tau$ (t is just x^0). Now the constraints become

$$\dot{\mathbf{x}}^2 + \mathbf{x}'^2 = \ell^2 \qquad \text{and} \qquad \dot{\mathbf{x}} \cdot \mathbf{x}' = 0 \tag{3.1}$$

And the equation of motion becomes

$$\ddot{\mathbf{x}} - \mathbf{x}'' = 0 \tag{3.2}$$

In (3.1) and (3.2) $\mathbf{x}$ denotes the $d-1$ spatial component vector. As pointed out in [3], the general solution to the equation of motion is

$$\mathbf{x} = \frac{1}{2}[\mathbf{a}(\sigma_-) + \mathbf{b}(\sigma_+)] \tag{3.3}$$

where $\sigma_- = \sigma - \tau$, $\sigma_+ = \sigma + \tau$ and $\mathbf{a}$, $\mathbf{b}$ are arbitrary funcions. From (3.1) we see that $\mathbf{a}$ and $\mathbf{b}$ have to satisfy the conditions

$$(\mathbf{a}^{(1)})^2 = \ell^2 \qquad (\mathbf{b}^{(1)})^2 = \ell^2. \tag{3.4}$$

In this the superscript (1) denotes the first derivative with respect to the argument.

The boundary condition for a closed string (2.17) gives that $\int_0^\pi d\sigma \mathbf{x}' = 0$. On top of that, we choose to work in the center of mass frame of the string, which implies that the total spatial momentum of the string is zero: $\int_0^\pi d\sigma \dot{\mathbf{x}} = 0$. This gives that

$$\int_0^\pi d\sigma \mathbf{a}^{(1)} = 0 \qquad \int_0^\pi d\sigma \mathbf{b}^{(1)} = 0 \tag{3.5}$$

A solution that meets all these conditions and was already known in 1985 (see [3] and [9]), is called the spiky string solution and is given by

$$\mathbf{a}^{(1)}(\sigma_-) = \ell \cos(2\pi M \frac{\sigma_-}{L})\mathbf{e}_1 - \ell \sin(2\pi M \frac{\sigma_-}{L})\mathbf{e}_2 \tag{3.6}$$
$$\mathbf{b}^{(1)}(\sigma_+) = \ell \cos(2\pi N \frac{\sigma_+}{L})\mathbf{e}_1 - \ell \sin(2\pi N \frac{\sigma_+}{L})\mathbf{e}_2 \tag{3.7}$$

In this M and N are integers and L is the length of the string. So if we give the solutions in the cartesian coördinates of the string, they look like

$$x = \ell \frac{L}{4\pi} \left\{ \frac{1}{N} \cos(2\pi N \frac{\sigma_+}{L}) + \frac{1}{M} \cos(2\pi M \frac{\sigma_-}{L}) \right\} \tag{3.8}$$
$$y = \ell \frac{L}{4\pi} \left\{ \frac{1}{N} \sin(2\pi N \frac{\sigma_+}{L}) + \frac{1}{M} \sin(2\pi M \frac{\sigma_-}{L}) \right\} \tag{3.9}$$

$\mathbf{a}^{(1)}$ and $\mathbf{b}^{(1)}$ are two curves which are centered at the same origin. If they cross each other $\mathbf{x}'$ will become zero. $|\dot{\mathbf{x}}|$ will be equal to ℓ, which means that the velocity of the string in spacetime $|\dot{x}|$ vanishes. At the points on the string where this happens, the string moves with the speed of light. These are the spikes of the string. When $\frac{N}{M}$ is integer the string will not intersect itself. We also want N and M to be relatively prime, which means that their greatest common divisor is 1. If they are not relatively prime the string is traced out multiple times and we can define a N and M, which are relatively prime and give the same string. These last two properties of N and M imply that $M = 1$.
The number of spikes will be equal to $N + M$[1], so we take $N = n - 1$ to make the number of spikes equal to n. On top of this we multiply the solutions by $4(n-1)$ to get a nicer looking solution[2]. Finally we take the length of the string to be π, as we did earlier. The complete solutions for the closed string in a flat space with two spatial dimensions become[3]

$$x = \ell \cos(2(n-1)\sigma_+) + \ell(n-1)\cos(2\sigma_-) \tag{3.10a}$$
$$y = \ell \sin(2(n-1)\sigma_+) + \ell(n-1)\sin(2\sigma_-) \tag{3.10b}$$
$$t = 4\ell(n-1)\tau = 2\ell(n-1)(\sigma_+ + \sigma_-) \tag{3.10c}$$

In figure 3.1 the solution is plotted for $\tau = 0$ and $n = 3, 7, 11$.

[1] See section 4.2 for proof.
[2] We can do that, it will not influence the correctness of the solutions.
[3] In the literature about spiky strings in flat space (for example [6]), the length of the string is often taken equal to 2π and therefore the factors of two inside the cosines and sines and a factor of two in t drop out.

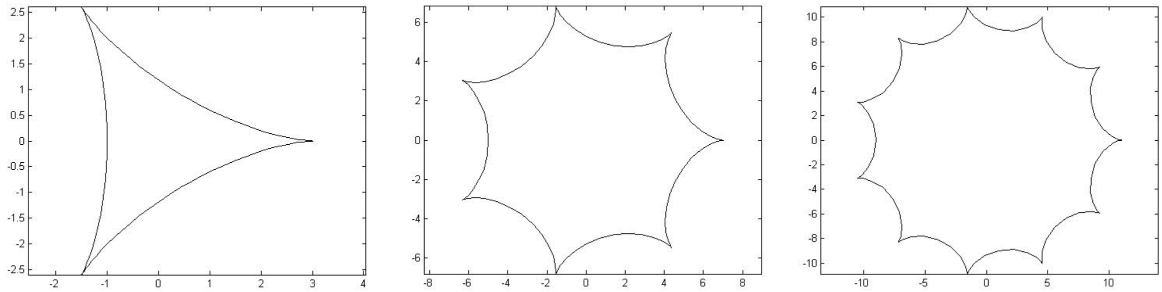

Figure 3.1: Plots of the spiky string solutions for $\tau = 0$, $\ell = 1$ and $n = 3, 7, 11$.

3.2 Deriving the solutions (2)

In the previous section we derived the spiky string solutions, using the general solution given in [3]. In this section we will derive the same solutions, but use expressions (2.24) and (2.25). We will again choose to work in the the center of mass frame of the string, so for the spatial components only the last terms will remain[4]:

$$\begin{aligned} x_L^j &= \frac{i}{2}\ell \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^j e^{-2in\sigma_+} = \frac{i}{2}\ell \sum_{n=1}^{\infty} \frac{1}{n}\left\{\tilde{\alpha}_n^j e^{-2in\sigma_+} - \tilde{\alpha}_{-n}^j e^{2in\sigma_+}\right\} \\ x_R^j &= \frac{i}{2}\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^j e^{-2in\sigma_-} = \frac{i}{2}\ell \sum_{n=1}^{\infty} \frac{1}{n}\left\{\alpha_n^j e^{-2in\sigma_-} - \alpha_{-n}^j e^{2in\sigma_-}\right\} \end{aligned}$$

where again $\sigma_+ = \tau + \sigma$ and $\sigma_- = \tau - \sigma$. Because for a complex quantity z it holds that $z - z^* = 2i\Im(z)$, x_L^j and x_R^j become[5]

$$\begin{aligned} x_L^j &= -\ell \sum_{n=1}^{\infty} \frac{1}{n}\Im(\tilde{\alpha}_n^j e^{-2in\sigma_+}) = -\ell \sum_{n=1}^{\infty} \frac{1}{n}\left\{\Re(\tilde{\alpha}_n^j)\Im(e^{-2in\sigma_+}) + \Im(\tilde{\alpha}_n^j)\Re(e^{-2in\sigma_+})\right\} \\ x_R^j &= -\ell \sum_{n=1}^{\infty} \frac{1}{n}\Im(\alpha_n^j e^{-2in\sigma_-}) = -\ell \sum_{n=1}^{\infty} \frac{1}{n}\left\{\Re(\alpha_n^j)\Im(e^{-2in\sigma_-}) + \Im(\alpha_n^j)\Re(e^{-2in\sigma_-})\right\} \end{aligned}$$

Where we used that $\Im(z_1 z_2) = \Re(z_1)\Im(z_2) + \Im(z_1)\Re(z_2)$ to get the second equality.
Now we take only one mode (N) for the left moving wave and one mode (M) for the right moving wave. We get

$$x_L^j = \frac{\ell}{N}\left\{\Re(\tilde{\alpha}_N^j)\sin(2N\sigma_+) - \Im(\tilde{\alpha}_N^j)\cos(2N\sigma_+)\right\} \qquad (3.11)$$

$$x_R^j = \frac{\ell}{M}\left\{\Re(\alpha_M^j)\sin(2M\sigma_-) - \Im(\alpha_M^j)\cos(2M\sigma_-)\right\} \qquad (3.12)$$

where we also used that $\Im(e^{i\theta}) = \sin\theta$ and $\Re(e^{i\theta}) = \cos\theta$.

The next thing to do is fix the values of α_M^μ and $\tilde{\alpha}_N^\mu$. They have to obey the constraints

[4]We use the superscript j to indicate spatial components (in contrast with μ for all spacetime components).
[5]$\Im(z)$ denotes the imaginary part of z, while $\Re(z)$ denotes the real part.

(2.30) and (2.31). The possibility that will lead us to the spiky string solution (3.10) is

$$\begin{aligned}\alpha_0^\mu &= (\tilde{\alpha}_0^0, \tilde{\alpha}_0^1, \tilde{\alpha}_0^2) = (2,0,0)\\ \tilde{\alpha}_0^\mu &= \alpha_0^\mu\\ \alpha_M^\mu &= (0,-i,1)\\ \tilde{\alpha}_N^\mu &= (0,-i,1)\end{aligned}$$

With this choice of coefficients the left- and rightmoving waves become

$$\begin{aligned}\mathbf{x}_L &= \frac{\ell}{N}\left\{\cos(2N\sigma_+)\mathbf{e}_1 + \sin(2N\sigma_+)\mathbf{e}_2\right\}\\ \mathbf{x}_R &= \frac{\ell}{M}\left\{\cos(2M\sigma_-)\mathbf{e}_1 + \sin(2M\sigma_-)\mathbf{e}_2\right\}\end{aligned}$$

Note that these are just the $\mathbf{a}(\sigma_-)$ and $\mathbf{b}(\sigma_+)$ given in section 3.1.

Finally the solutions in cartesian coordinates are

$$\begin{aligned}&x = \ell\left\{\frac{1}{N}\cos(2N\sigma_+) + \frac{1}{M}\cos(2M\sigma_-)\right\}\\ &y = \ell\left\{\frac{1}{N}\sin(2N\sigma_+) + \frac{1}{M}\sin(2M\sigma_-)\right\}\\ &t = 4\ell\tau = 2\ell(\sigma_+ + \sigma_-)\end{aligned}$$

which become equal to the ones in (3.10) if we multiply by N and take $N = n-1$ and $M = 1$. Note that the factor 4 by which we multiplied in the previous section, in this case was already incorporated in our choice of the oscillation coefficients.

3.3 Open spiky strings

The question that must have arised when reading this is: can this also be done for open strings? The answer is quite simple: not inside the scope of this thesis.

The problem is most easily explained in the context of the mode expansions (2.22) and (2.26). For closed strings we have two waves, a left moving and a right moving one, whose coefficients are almost independent. They are only connected through the constraint on α_0^μ and $\tilde{\alpha}_0^\mu$. So for the closed string we can take two different modes, which together give rise to the spikes, by making $\mathbf{x}'$ and $|\dot{x}|$ vanish.

For the open string we cannot just take two different oscillation modes, because the coefficients are not independent. If we were to take two different modes their coefficients have to satisfy the constraints. But these are nonlinear and hard to solve (as already pointed out in the introduction).

So for now we will just focus our attention on the closed string.

Chapter 4

Properties of spiky strings

In this chapter we look at some properties of spiky strings, including the momentum and the position of the spikes.

4.1 Momentum and Angular Momentum

The total momentum of the string is given by (2.23):

$$P^\mu = T\int_0^\pi \dot{x}^\mu$$

Because we work in the center of mass frame of the string, which means that the total spatial momentum is zero and we choose $x^0 = \ell\tau$, this is equal to

$$P^\mu = \pi\ell T(1,0,0) \tag{4.1}$$

The same result can also be obtained by looking at the mode-coefficient

$$\alpha_0^\mu = \frac{1}{2\pi\ell T}P^\mu$$

If we take the value in (3.13) for α_0^μ the total momentum becomes

$$P^\mu = 4\pi\ell T(1,0,0) \tag{4.2}$$

where the difference with (4.1) again comes from the fact that in this case we took $x^0 = 4\ell\tau$, instead of $x^0 = \ell\tau$.
The mass of the string in this case is equal to $4\pi\ell T$ because of the mass shell condition.

Another conserved quantity is the angular momentum of the string. It is defined through the current

$$\mathcal{J}^\alpha_{\mu\nu} = T(x_\mu\partial^\alpha x_\nu - x_\nu\partial^\alpha x_\mu)$$

as

$$M_{12} = \int_0^\pi d\sigma \mathcal{J}^\tau_{12} \tag{4.3}$$

For our spiky string with equations (3.10) this is equal to

$$M_{12} = 2\ell^2 \left(\frac{1}{N} + \frac{1}{M}\right)\left(1 + \int_0^\pi d\sigma \cos(2N\sigma_+)\cos(2M\sigma_-) + \sin(2N\sigma_+)\sin(2M\sigma_-)\right)$$

The integral evaluates to zero and therefore the angular momentum becomes

$$M_{12} = 2\ell^2 \left(\frac{1}{M} + \frac{1}{N}\right) \tag{4.4}$$

So we see that the angular momentum is also a conserved quantity.

4.2 The spikes

We want to find out where the spikes are developed[1]. In section 3.1 it turned out that they develop when $\mathbf{x}' = 0$. This happens when

$$2M\sigma_- = 2N\sigma_+ + 2\pi k$$

So the worldsheet-coördinate σ of the spikes is

$$\sigma = \frac{M-N}{N+M}\tau - \frac{\pi k}{N+M} \qquad k = 0, \ldots, M+N-1 \tag{4.5}$$

From this we see that the number of spikes is $M+N$.
If we insert (4.5) in the equations for the spatial coordinates of the string, we get, after some rewriting, that

$$\begin{aligned} x_{\text{spike}} &= \frac{\ell}{4}\left\{\frac{1}{N}\cos\left(\frac{4MN}{M+N}\tau - \frac{2Nk\pi}{M+N}\right) + \frac{1}{M}\cos\left(\frac{4MN}{M+N}\tau + \frac{2Mk\pi}{M+N}\right)\right\} \\ y_{\text{spike}} &= \frac{\ell}{4}\left\{\frac{1}{N}\sin\left(\frac{4MN}{M+N}\tau - \frac{2Nk\pi}{M+N}\right) + \frac{1}{M}\sin\left(\frac{4MN}{M+N}\tau + \frac{2Mk\pi}{M+N}\right)\right\} \end{aligned}$$

Since the fase difference between both cosines and sines is a integer multiple of 2π, the equations simplify to[2]

$$x_{\text{spike}} = \frac{\ell}{4}\left(\frac{1}{N} + \frac{1}{M}\right)\cos\left(\frac{4MN}{M+N}\tau + \frac{2Mk\pi}{M+N}\right) \tag{4.6}$$

$$y_{\text{spike}} = \frac{\ell}{4}\left(\frac{1}{N} + \frac{1}{M}\right)\sin\left(\frac{4MN}{M+N}\tau + \frac{2Mk\pi}{M+N}\right) \tag{4.7}$$

(4.8)

We see that while they move in time, the tips of the spikes lie on a circle with radius

$$r_{\text{spike}} = \frac{\ell}{4}\left(\frac{1}{N} + \frac{1}{M}\right) \tag{4.9}$$

[1] Results similar to the ones in this section can be found in [7].

[2] Here we take the argument of the second term as the angle, this is equivalent to taking the argument of the first term, since we designed them exactly with that property.

And the spikes have an angle of

$$\theta_{\text{spike}} = \frac{4MN}{M+N}\tau + \frac{2Mk\pi}{M+N} \tag{4.10}$$

So the speed at which the spikes move is[3]

$$\begin{aligned} r\frac{d\theta}{dt} &= r\frac{d\theta}{d\tau}\frac{d\tau}{dt} \\ &= \frac{\ell}{4}\left(\frac{1}{N}+\frac{1}{M}\right)\frac{4MN}{M+N}\frac{1}{\ell} \\ &= 1 \end{aligned}$$

Which means that the spikes move with a velocity equal to the speed of light.

If we now take $N = n - 1$ and $M = 1$, as we did in section 3.1, we get the following numbers for our spiky string:

$$\#_{\text{spikes}} = n \tag{4.11}$$

$$\sigma_{\text{spike}} = \frac{2-n}{n}\tau - \frac{k}{n}\pi, \qquad k = 0, \ldots, n-1 \tag{4.12}$$

$$r_{\text{spike}} = \frac{\ell}{4}\left(\frac{n}{n-1}\right) \tag{4.13}$$

$$\theta_{\text{spike}} = \frac{4(n-1)}{n}\tau + \frac{2k\pi}{n} \tag{4.14}$$

These numbers do not exactly match the ones one would find from equations (3.10), because there we multiplied by a factor $4(n-1)$. So to find the correct values for those equations one has to multiply r_{spike} by that same factor. This can also by seen in figure 3.1.

[3] remember that we took $t = \ell\tau$.

Chapter 5

Relevance of spiky strings

Finally we want to briefly discuss the position of spiky strings in string theory, although this was not part of the research.

Spiky strings are often mentioned in connection with the giant magnon and the Ads/CFT correspondence.
The giant magnon is a quantum of spin wave with a short wavelength (see [5]). Spin waves are propagating disturbances in the ordering of magnetic material. One can compare this with phonons, which are quanta of vibration. The major difference between the two is their dispersion relation, $\omega \propto k$ for the phonon and $\omega \propto k^2$ for the magnon.
It was shown that in certain limits the spiky string corresponds to this giant magnon (see [8]), because it has the same dispersion relation

The AdS/CFT correspondence [12] in short is an conjectured equivalence between string theory on one space and quantum field theory on the boundary of this space. The first space is the product of an anti-de-Sitter space with an closed manifold (e.g. a sphere). This is the reason that people often investigate the spiky string on a sphere and from there go on to for example $AdS_5 \times S^5$. In understanding the AdS/CFT correspondence classical solutions have played an important role an therefore also spiky strings are studied extensively. Note in particular the paper [7] by M. Kruczenski in which he argues that certain single trace operators in gauge theories 'have a dual description in terms of strings with spikes, one spike for each field appearing in the operator'.

Chapter 6

Conclusion

The final question now is: what have we learned about spiky strings?

We learned that spiky strings are classical solutions to the equation of motion for closed strings. They are built from two different modes N and M for the left- and the rightmoving part of the solution. Because these modes are different the solution is not equal to a collapsed closed string which lookes like a open string.

These modes and the fact that one is leftmoving and the other rightmoving, together cause $\mathbf{x}'$ and $|\dot{x}|$ to vanish at certain points on the string with wordsheet coordinate σ given by (4.5). At these points the spikes appear. We calculated their spatial coordinates (4.6) and found that they move at the speed of light.

We calculated the momentum and angular momentum for the spiky string and found that they are both conserved quantities.

In most of the literature on the subject of spiky strings one takes $M = 1$ and $N = n - 1$, because then the number of spikes is equal to n and the spike is traced out only once. In this thesis we also investigated the solutions with general M and N and found that the answers also come out right.

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