Classical String Solutions
Solutions within the light-cone gauge

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May-July 2006

In the following report solutions to the equations of motion for classical relativistic strings are presented. These solutions are found within the light-cone gauge. Several rotating and vibrating solutions are found.

1 Introduction

General solutions to the equations of motion for open and closed strings can be found in the static gauge. However within the static gauge there are constraints to the equations of motion for the classical relativistic string. The solutions found in the static gauge are solutions of rotating open and closed strings. No vibrating strings are found within the static gauge, see [2]. The aim in this report is to find solutions for open and closed strings that do vibrate. To do this the light-cone gauge is used. The advantage of solutions within the light-cone gauge is that the constraints that are present in the static gauge are solved within the light-cone gauge. This means that different solutions will be possible. In the beginning an introduction is given about these constraints and the light-cone gauge. To generate the solutions several theorems will be needed. These theorems are also given. If this is done several general solutions for open and closed strings are presented. As will be seen there are simple and more complicated solutions found.

1.1 The action

After setting $h_{\alpha\beta} = f(\sigma)\eta_{\alpha\beta}$ the Howe and Tucker form of the action becomes

$$S = -\frac{T}{2}\int d^2\sigma\eta^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu, \tag{1}$$

where T denotes the strings tension. There are however constraints, given by

$$g_{\alpha\beta} - \frac{1}{2}(h\cdot g)h_{\alpha\beta} = 0. \tag{2}$$

The solutions for open and closed strings can be derived from this action.

1.2 Open Strings

Solutions to the equations of motion for open strings are found using the Neuman boundary condition

$$\partial_\sigma x^\mu(\tau, 0) = \partial_\sigma x^\mu(\tau, \pi).$$

These solutions are given by

$$x^\mu(\tau,\sigma) = q^\mu + \frac{1}{\pi T}P^\mu\tau + i\ell\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma. \tag{3}$$

with

$$\alpha_0^\mu = \frac{1}{\pi T\ell}P^\mu \tag{4}$$

and the reality condition

$$\alpha_n^\mu = (\alpha_{-n}^\mu)^*. \tag{5}$$

Here P^μ is the total momentum.

The constraints given by (2) can be written like

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}(x')^2 = 0$$

$$\dot{x} \cdot x' = 0$$

This can also be written as, see [1]

$$L_n \equiv \frac{1}{2}\sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0 \tag{6}$$

1.3 Closed Strings

The boundary conditions for closed strings become

$$x^\mu(\tau,\sigma) = x^\mu(\tau,\sigma+\pi)$$

For the closed string the solutions to the equations of motion are now given by

$$x^\mu(\tau,\sigma) = q^\mu + \frac{1}{\pi T}P^\mu\tau + \frac{i}{2}\ell\sum_{n\neq 0}\frac{1}{n}e^{-2in\tau}(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}) \tag{7}$$

with

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2\pi\ell T}P^\mu \tag{8}$$

and the same reality condition as for the open string, see (5)

$$\alpha_n^\mu = (\alpha_{-n}^\mu)^* \qquad \tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^*.$$

The constraints are now given by

$$L_n \equiv \frac{1}{2}\sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0, \tag{9}$$

$$\tilde{L}_n \equiv \frac{1}{2}\sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0. \tag{10}$$

1.4 Light-cone gauge

To get rid of the constraints given by (6), (9) and (10) a new coordinate transformation is made to coordinates $\sigma(\tilde{\sigma})$ in such a way that $\tilde{x}(\tilde{\sigma}) = x(\sigma)$. The new action after this transformation should require $S = \tilde{S}$ and the vanishing of the variation of the action. If these conditions are solved the following is found, see [1]

$$\frac{\partial\tilde{\tau}}{\partial\sigma} = \frac{\partial\tilde{\sigma}}{\partial\tau} \quad and \quad \frac{\partial\tilde{\tau}}{\partial\tau} = \frac{\partial\tilde{\sigma}}{\partial\sigma}. \tag{11}$$

A coordinate transformation that satisfies these conditions is

$$\tilde{\tau} = \frac{n\cdot x}{n\cdot P}\pi T, \qquad \tilde{\sigma} = \frac{\pi}{n\cdot P}\int_0^\sigma d\sigma' n\cdot P^\tau(\tau,\sigma'). \tag{12}$$

Here n_μ is a fixed vector with the property $n^2 \leq 0$, P^μ is the total momentum and P^τ_μ the canonical momentum defined as $P^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$.

Now the light-cone coordinates are introduced as

$$x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^1). \tag{13}$$

The light-cone gauge is given by a choice of $n^\mu = (-1, 1, 0......, 0)$. This choice of n^μ gives, see [1]

$$x^+ = \frac{1}{\pi T} P^+ \tau. \tag{14}$$

The constraints can be written like

$$\dot{x}^- = \frac{\pi T}{2P^+}[(x^i)^2 + (x'^i)^2] \tag{15}$$

$$x'^- = \frac{\pi T}{P^+}\dot{x}^i x'^i. \tag{16}$$

Where $i = (2, ..., D-1)$ labels the transverse coordinates.

These constraints can be solved for α^-_n. For the open string this will give

$$\alpha^-_n = \frac{\pi T \ell}{2P^+}\sum_k \alpha^i_k \alpha^i_{n-k} = \frac{1}{2\alpha^+_0}\sum_k \alpha^i_k \alpha^i_{n-k}. \tag{17}$$

The same holds for the closed string, here it will give

$$\alpha^-_n = \frac{\pi T \ell}{P^+}\sum_k \alpha^i_k \alpha^i_{n-k} = \frac{1}{2\alpha^+_0}\sum_k \alpha^i_k \alpha^i_{n-k}. \tag{18}$$

$$\tilde{\alpha}^-_n = \frac{\pi T \ell}{P^+}\sum_k \tilde{\alpha}^i_k \tilde{\alpha}^i_{n-k} = \frac{1}{2\alpha^+_0}\sum_k \tilde{\alpha}^i_k \tilde{\alpha}^i_{n-k}. \tag{19}$$

This means that the constraints are solved, but only the α^i_n remain as independent oscillators. Further note that expressed in α^+_0 equation (18) is the same formula as for the open string given in equation (17).

2 Solving the α^-_n

The solution for x^- is totally dependent on the α^-_n. Since α^-_n is totally dependent on the α^+_0 and the α^i oscillators, x^- is fixed if values for the α^i and α^+_0 oscillators are known. Here α^+_0 is always non-zero.

2.1 L_n equivalence with α^-_n

The formulas for α^-_n and L_n can be expressed into each other, if this is done the following expressions are found

$$\alpha^-_n = \frac{1}{\alpha^+_0} L^\perp_n \quad \textit{and} \quad \tilde{\alpha}^-_n = \frac{1}{\tilde{\alpha}^+_0}\tilde{L}^\perp_n \tag{20}$$

where the $L_n^{\perp}$ are the transverse virasoro modes.

2.2 Finding α_n^-

Formulas (17) and (18) can be written in a different way for even and odd n. This also holds for (19), but then with tildes on the α's. If one looks more carefully at the sum, at a certain point n will be equal to $n = 2k$. This will happen since k runs from $-\infty$ to $+\infty$. When this value for k is labeled l, the following is found

$$\alpha_{2l}^- = \frac{1}{2\alpha_0^+}[... + \alpha_{l-2}^i \alpha_{l+2}^i + \alpha_{l-1}^i \alpha_{l+1}^i + \alpha_l^i \alpha_l^i + \alpha_{l+1}^i \alpha_{l-1}^i + \alpha_{l+2}^i \alpha_{l-2}^i + ...]. \quad (21)$$

Which can be written as

$$\alpha_{2l}^- = \frac{1}{2\alpha_0^+} \sum_{k>0} \{\alpha_{l-k}^i, \alpha_{l+k}^i\} + \alpha_l^i \alpha_l^i. \quad (22)$$

Where the $\{\}$ denotes the anti-commutator taken for the $\alpha's$.

If the same is done for odd n, $n = 2l + 1$, the following is found

$$\alpha_{2l+1}^- = \frac{1}{2\alpha_0^+}[... + \alpha_{l-1}^i \alpha_{l+2}^i + \alpha_l^i \alpha_{l+1}^i + \alpha_{l+1}^i \alpha_l^i + \alpha_{l+2}^i \alpha_{l-1}^i + ...] \quad (23)$$

Which can be written as

$$\alpha_{2l+1}^- = \frac{1}{2\alpha_0^+} \sum_{k\geq 0} \{\alpha_{l+k+1}^i \alpha_{l-k}^i\}. \quad (24)$$

This together with the following theorems gives the recipe of finding the α_n^-.

Since α_n^- can be expressed in terms of the transverse virasoro modes with just a constant in front of it, some of the theorems for the virasoro modes can be used to get theorems for the α_n^-. These theorems and their proof can be found in [2]. The proof for the α_n^- oscillators will be similar to the proof for the L_n given in [2].

Theorem 1 $\alpha_n^- = 0$ *if and only if* $\alpha_{-n}^- = 0$.

Theorem 2 *For any set of non-zero coefficients* α_n^i *with* $\alpha_{n_{max}}^i$ *the coefficient with the highest value of* n *holds that* $\alpha_k^- = 0$ *if* $k > 2n_{max}$.

Theorem 3 *The term* $\alpha_k^i \alpha_l^i$ *is only contained in* α_{l+k}^-.

To find the α_n^- when the α_k^i's are given a table can be made with all values for k in one column. Now the same values are also placed in another column called l. The sequence of the first column k is kept the same and the values for l in the second column are cycled around until the starting point is reached again. A third column where $k + l = n$ is listed is also in the table. Now the correct α_n^- can be found following the theorems. So if for example α_1^2, α_{-1}^2, α_2^2 and α_{-2}^2 are the non-zero oscillators. α_n^- can be found by the following table.

Table 1

k	l	k+l= n		k	l	k+l= n
2	2	4		2	-1	1
1	1	2		1	-2	-1
-1	-1	-2		-1	2	1
-2	-2	-4		-2	1	-1
2	-2	0		2	1	3
1	2	3		1	-1	0
-1	1	0		-1	-2	-3
-2	-1	-3		-2	2	-0

So here the non-zero α_n^- are

$$\alpha_0^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_{-1}^2 + \alpha_{-1}^2\alpha_1^2 + \alpha_2^2\alpha_{-2}^2 + \alpha_{-2}^2\alpha_2^2)$$

$$\alpha_1^- = \frac{1}{2\alpha_0^+}(\alpha_2^2\alpha_{-1}^2 + \alpha_{-1}^2\alpha_2^2) \qquad \alpha_{-1}^- = \frac{1}{2\alpha_0^+}(\alpha_{-2}^2\alpha_1^2 + \alpha_1^2\alpha_{-2}^2)$$

$$\alpha_2^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_1^2) \qquad \alpha_{-2}^- = \frac{1}{2\alpha_0^+}(\alpha_{-1}^2\alpha_{-1}^2)$$

$$\alpha_3^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_1^2) \qquad \alpha_{-3}^- = \frac{1}{2\alpha_0^+}(\alpha_{-1}^2\alpha_{-2}^2 + \alpha_{-2}^2\alpha_{-1}^2)$$

$$\alpha_4^- = \frac{1}{2\alpha_0^+}(\alpha_2^2\alpha_2^2) \qquad \alpha_{-4}^- = \frac{1}{2\alpha_0^+}(\alpha_{-2}^2\alpha_{-2}^2).$$

3 Open string solutions

In the static gauge there are some solutions found to the equations of motion for the strings. Within the light-cone gauge other solutions can be found because the constraints for the L_n's do not have to hold here. Now values for the α's can be chosen to get a solution.

The open string solution to the equation of motion is described by (3). To get a general solution, α_n^i for $n > 0$ is set equal to

$$\alpha_n^i = a_n e^{-\frac{1}{2}\varphi_n^i} \tag{25}$$

and for $n < 0$ according to the reality condition

$$\alpha_n^i = a_{-n} e^{\frac{1}{2}\varphi^i_{-n}} \tag{26}$$

Where a_n is a real constant number. The value for α_0^+ is set to a real constant

$$\alpha_0^+ = a. \tag{27}$$

For simplicity also take

$$a_n = a_{-n} = a \quad \textit{and} \quad \varphi_n^i = \varphi^i_{-n} = \varphi \tag{28}$$

3.1 Simple open string solutions

A simple solution can be made if just three dimensions are taken in to account, x^+, x^- and x^2. This can be done by taking only α_1^2 and α_{-1}^2 to be non-zero transverse oscillators.

When this is done the expressions for α_n^- can be found by making a table described in section 2.2.

Table 2

k	l	k+l= n
1	1	2
-1	-1	-2
1	-1	0
-1	1	0

Now the following non-zero α_n^- are obtained

$$\alpha_0^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_{-1}^2 + \alpha_{-1}^2\alpha_1^2)$$

$$\alpha_2^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_1^2) \qquad \alpha_{-2}^- = \frac{1}{2\alpha_0^+}(\alpha_{-1}^2\alpha_{-1}^2).$$

The α_n^- oscillators become according to equations (25) until (28)

$$\alpha_0^- = a$$

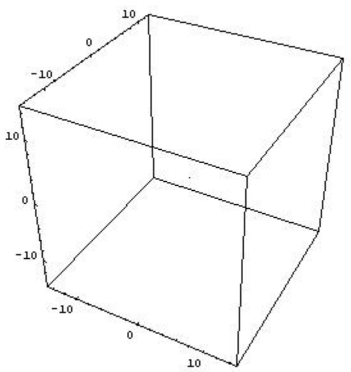
$$\alpha_2^- = \frac{1}{2} a e^{i\varphi} \qquad \alpha_{-2}^- = \frac{1}{2} a e^{-i\varphi}.$$

This gives the following general solutions

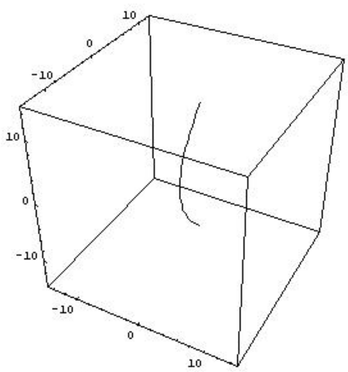
$$\begin{aligned} x^+ &= a\ell\tau \\ x^- &= a\ell\tau + \frac{1}{2} a\ell \sin(2\tau + \varphi)\cos(2\sigma) \\ x^2 &= 2a\ell \sin(\tau + \frac{\varphi}{2})\cos(\sigma). \end{aligned} \tag{29}$$

These equations satisfy the constraints (15) and (16) which can be checked by calculating the constraints yourself or by putting them in mathematica.This is advisable when later on the solutions become more complicated.

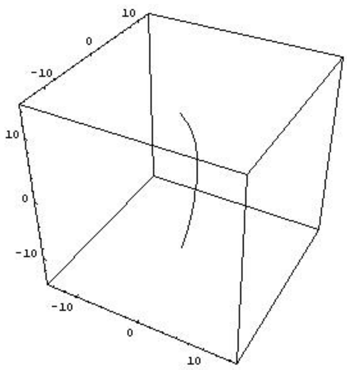
When the values for a, ℓ and φ are set equal to $a = 1$, $\ell = 2\pi$ and $\varphi = 0$, remembering that σ runs from 0 to ℓ. A solution is found of a rotating open string that collapses to a point and stretches out again in time. Which can be seen in the following pictures.



τ=0



τ=1



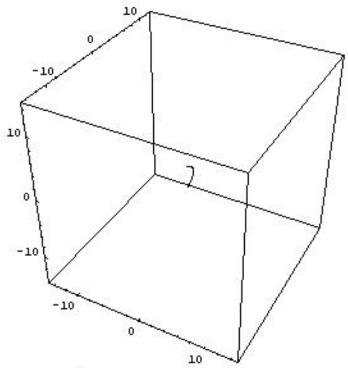


Figure 1: **A rotating open string for times** $\tau = 0,\ \tau = 1,\ \tau = 2$ **and** $\tau = 3$ **in the** x^+**,** x^-**,** x^2 **space.**

If an extra dimension is added there is little change for the α^- oscillators when only α_1^3 and α_{-1}^3 are added. No new α^- oscillators are obtained and the oscillators that already existed just become

$$\alpha_0^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_{-1}^2 + \alpha_{-1}^2\alpha_1^2 + \alpha_1^3\alpha_{-1}^3 + \alpha_{-1}^3\alpha_1^3)$$

$$\alpha_2^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_1^2 + \alpha_1^3\alpha_1^3) \qquad \alpha_{-2}^- = \frac{1}{2\alpha_0^+}(\alpha_{-1}^2\alpha_{-1}^2 + \alpha_{-1}^3\alpha_{-1}^3).$$

If now equations (25) until (28) are taken into account a similar solution compared with (29) is obtained.

$$\begin{aligned} x^+ &= a\ell\tau \\ x^- &= 2a\ell\tau + a\ell\sin(2\tau+\varphi)\cos(2\sigma) \\ x^2 &= 2a\ell\sin(\tau+\frac{\varphi}{2})\cos(\tau) \\ x^3 &= 2a\ell\sin(\tau+\frac{\varphi}{2})\cos(\tau) \end{aligned} \tag{30}$$

This again is a solution of a rotating open string that collapses into a point and stretches out again in time. The plots of the functions in the x^-,x^2 and x^3 space with $a = 1$, $\ell = 2\pi$ and $\varphi = 0$ look a lot like the plots in figure 1, but now with a larger amplitude in the x^- plane.

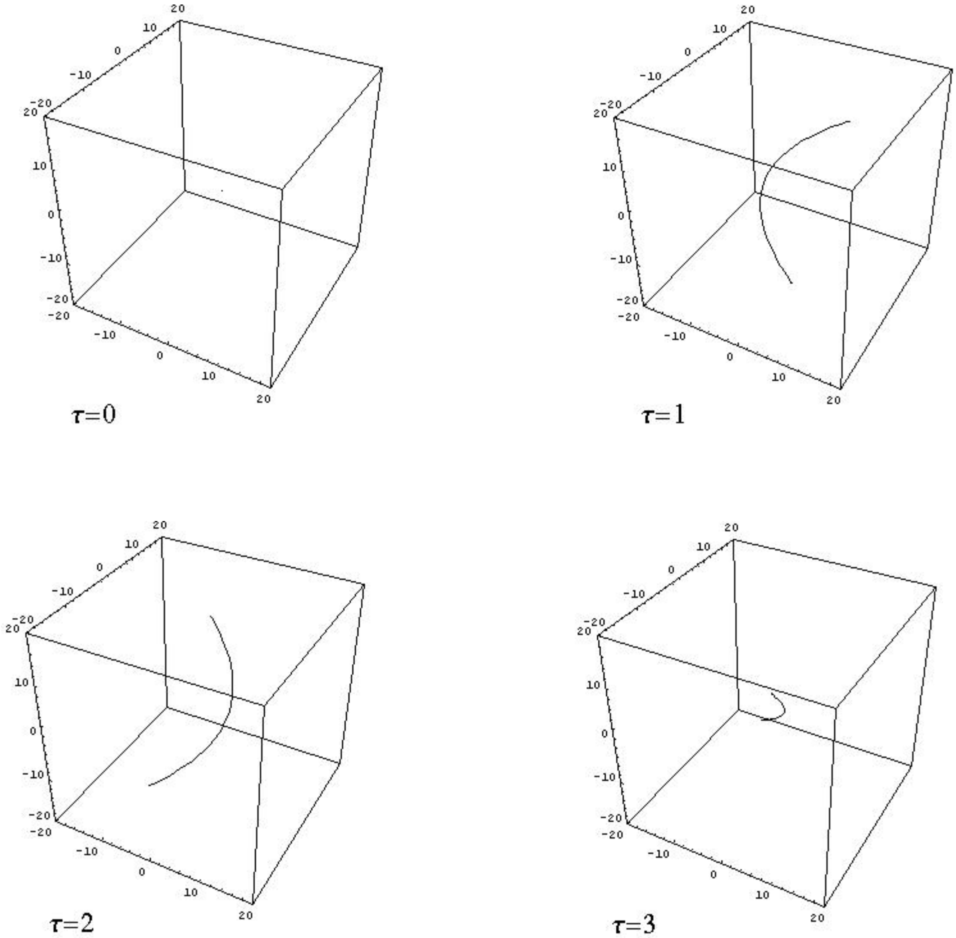


Figure 2: **A rotating open string for times** $\tau = 0,\ \tau = 1,\ \tau = 2$ **and** $\tau = 3$**. in the** x^-**,** x^2**,** x^3 **space.**

3.2 More complicated open string solutions

The solutions to the equations of motion for the open string become more complicated if some α_n^i oscillators of higher order, $n > 1$, are taken to be non-zero. In section 2.2 the solutions for the α_n^- oscillators are already found when α_1^2, α_{-1}^2, α_2^2 and α_{-2}^2 are non-zero. Now a general solution which also includes α_1^3 and α_{-1}^3 oscillators can be found looking at equations (25) until (28).

The solutions become

$$x^+ = a\ell\tau$$

$$\begin{aligned} x^- &= 3a\ell\tau + 2a\ell\sin(\tau)\cos(\sigma) + a\ell\sin(2\tau+\varphi)\cos(2\sigma) \\ &+\frac{2}{3}a\ell\sin(3\tau+\varphi)\cos(3\sigma) + \frac{1}{4}a\ell\sin(4\tau+\varphi)\cos(4\sigma) \\ x^2 &= 2a\ell\sin(\tau+\frac{\varphi}{2})\cos(\sigma) + a\ell\sin(2\tau+\frac{\varphi}{2})\cos(2\sigma) \\ x^3 &= 2a\ell\sin(\tau+\frac{\varphi}{2})\cos(\sigma) \end{aligned} \tag{31}$$

Again the string collapses and stretches out in time. If again $a = 1$, $\ell = 2\pi$ and $\varphi = 0$ are chosen, this is the solution for a rotating open string with a vibration on it.

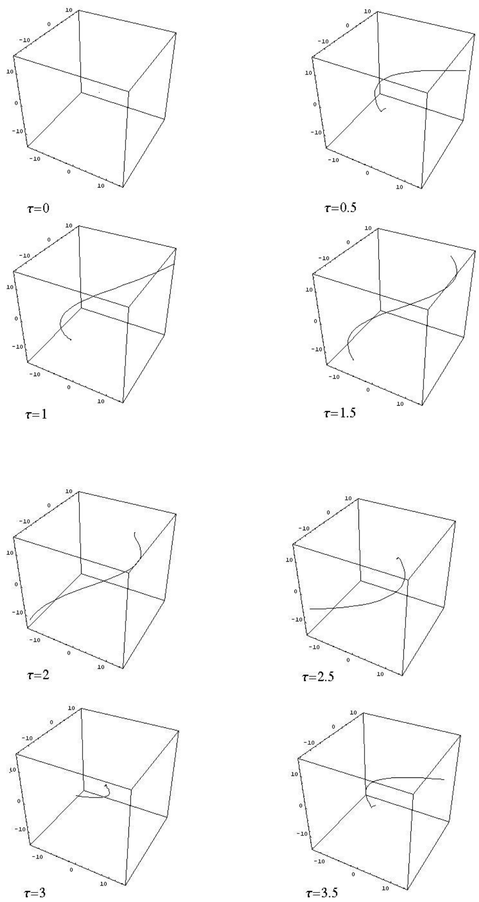


Figure 3: **A rotating open string for times $\tau = 0$ until $\tau = 3.5$ in steps of 0.5 in x^-, x^2 and x^3 space.**

4 Closed string solutions

Solutions for the closed string can also be made, but there is a constraint concerning the α_n^i and the $\tilde{\alpha}_n^i$. Since equation (8) also has to hold for α_0^- and $\tilde{\alpha}_0^-$. The following theorem can be made next to the theorems in section 2.2.

Theorem 4

$$\sum_k \alpha_k^i \alpha_{-k}^i = \sum_k \tilde{\alpha}_k^i \tilde{\alpha}_{-k}^i$$

Proof Since (8) also has to hold for α_0^- and $\tilde{\alpha}_0^-$ the equality $\alpha_0^- = \tilde{\alpha}_0^-$ is obtained. From equations (18) and (19) and remembering that $\alpha_0^+ = \tilde{\alpha}_0^+$ it is found that

$$\sum_k \alpha_k^i \alpha_{-k}^i = \sum_k \tilde{\alpha}_k^i \tilde{\alpha}_{-k}^i \qquad \bullet$$

Not only α_n^i oscillators are of importance, but for the closed string also the $\tilde{\alpha}_n^i$ are non-zero. Suppose equations (25) until (28) still hold for the α_n^i oscillators, but for the $\tilde{\alpha}_n^i$ oscillators holds the following for $n > 0$

$$\tilde{\alpha}_n^i = a_n e^{-\frac{1}{2}\phi_n^i} \tag{32}$$

and for $n < 0$ according to the reality condition

$$\tilde{\alpha}_n^i = a_{-n} e^{\frac{1}{2}\phi^i_{-n}} \tag{33}$$

Where a_n is a real constant number. The value for $\tilde{\alpha}_0^+ = \alpha_0^+$ is set to a real constant

$$\tilde{\alpha}_0^+ = \alpha_0^+ = a. \tag{34}$$

For simplicity also

$$a_n = a_{-n} = a \tag{35}$$

The condition given in theorem 4 still holds in this way, note φ does not have to be equal to ϕ. Further equation (35) differs slightly from equation (28) since here the last part is missing.

4.1 Simple closed string solutions

Simple solutions here also mean that the α_n^i and $\tilde{\alpha}_n^i$ should be of the first order, $n = 1$. If now α_1^2, α_{-1}^2, α_1^3 and α_{-1}^3 and their analogue $\tilde{\alpha}$'s are taken to be non-zero, a simple solution can be generated for the closed string. In section 3.1 the non-zero α_n^- are already found. In this case also the $\tilde{\alpha}_n^-$ are non-zero. The non-zero solutions are

$$\alpha_0^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_{-1}^2 + \alpha_{-1}^2\alpha_1^2 + \alpha_1^3\alpha_{-1}^3 + \alpha_{-1}^3\alpha_1^3)$$

$$\alpha_2^- = \frac{1}{2\alpha_0^+}(\alpha_1^2\alpha_1^2 + \alpha_1^3\alpha_1^3) \qquad \alpha_{-2}^- = \frac{1}{2\alpha_0^+}(\alpha_{-1}^2\alpha_{-1}^2 + \alpha_{-1}^3\alpha_{-1}^3)$$

and

$$\tilde{\alpha}_0^- = \frac{1}{2\tilde{\alpha}_0^+}(\tilde{\alpha}_1^2\tilde{\alpha}_{-1}^2 + \tilde{\alpha}_{-1}^2\tilde{\alpha}_1^2 + \tilde{\alpha}_1^3\tilde{\alpha}_{-1}^3 + \tilde{\alpha}_{-1}^3\tilde{\alpha}_1^3)$$

$$\tilde{\alpha}_2^- = \frac{1}{2\tilde{\alpha}_0^+}(\tilde{\alpha}_1^2\tilde{\alpha}_1^2 + \tilde{\alpha}_1^3\tilde{\alpha}_1^3) \quad \tilde{\alpha}_{-2}^- = \frac{1}{2\tilde{\alpha}_0^+}(\tilde{\alpha}_{-1}^2\tilde{\alpha}_{-1}^2 + \tilde{\alpha}_{-1}^3\tilde{\alpha}_{-1}^3).$$

For the simple solutions also take $\phi_n^i = \varphi_n^i = \varphi$. This gives the general solution

$$\begin{aligned} x^+ &= 2a\ell\tau \\ x^- &= 4a\ell\tau + a\ell\sin(4\tau+\varphi)\cos(4\sigma) \\ x^2 &= 2a\ell\sin(2\tau+\frac{\varphi}{2})\cos(2\tau) \\ x^3 &= 2a\ell\sin(2\tau+\frac{\varphi}{2})\cos(2\tau) \end{aligned} \tag{36}$$

Notice the similarity with the solution for the open string in the same case as given in equation (30). Here the angle dependency of τ and σ has been doubled and the speed at which the string moves in space has also been doubled. This means that the string is moving faster in space, but still has the same movement as can be seen in Figure 2. This means that the open string has collapsed into a line which is moving in space.

4.2 More complicated closed string solutions

To get more interesting solutions the solutions need to become more complicated. This involves taking α_n^i and $\tilde{\alpha}_n^i$ oscillators of higher order, $n > 1$. Another option is to make a different choice for some ϕ_n^i.

When the α_2^2 and α_{-2}^2 are added the correct values for α_n^- and $\tilde{\alpha}_n^-$ can be found in section 2.2. If the condition $\phi_n^i = \varphi_n^i = \varphi$ still holds, the following solution is found taking equations (25) until (28) and (32) until (35) into account.

$$\begin{aligned} x^+ &= 2a\ell\tau \\ x^- &= 6a\ell\tau + 2a\ell\sin(2\tau)\cos(2\sigma) + a\ell\sin(4\tau+\varphi)\cos(4\sigma) \\ &\quad + \frac{2}{3}a\ell\sin(6\tau+\varphi)\cos(6\sigma) + \frac{1}{4}a\ell\sin(8\tau+\varphi)\cos(8\sigma) \\ x^2 &= 2a\ell\sin(2\tau+\frac{\varphi}{2})\cos(2\sigma) + a\ell\sin(4\tau+\frac{\varphi}{2})\cos(4\sigma) \\ x^3 &= 2a\ell\sin(2\tau+\frac{\varphi}{2})\cos(2\sigma) \end{aligned} \tag{37}$$

If in this case a, l and φ are taken to be $a = 1,\ l = 2\pi$ and $\varphi = 0$ the plot for the equation of motion in the $x^-,\ x^2$ and x^3 plane is again a closed string that is collapsed into a line and has a vibration on it. The solution looks a lot like the solution for the open string given in (31).

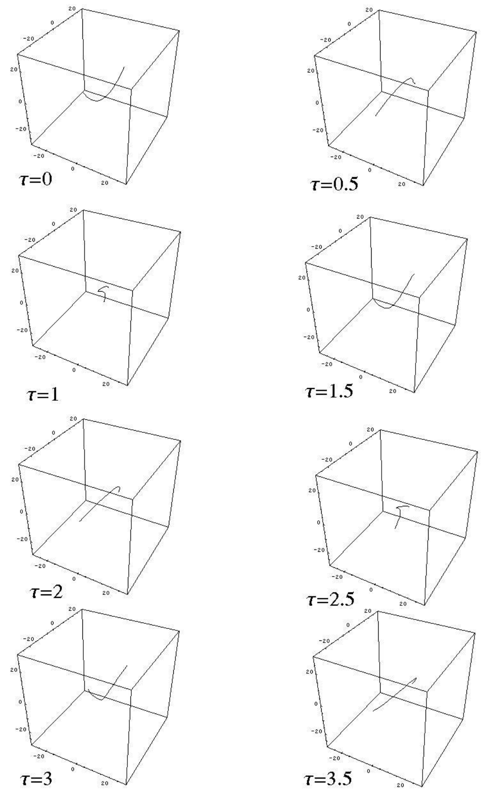


Figure 4: **A vibrating closed string (collapsed into a line) for times $\tau = 0$ until $\tau = 3.5$ in steps of 0.5 in x^-, x^2 and x^3 space.**

All solutions so far are solutions for closed strings that have collapsed into a line. To find other solutions the simplification made by setting $\phi^i_n = \varphi^i_n = \varphi$ is dropped.

This can be done by still looking at equations (25) until (28) and (32) until (35) again. This means that the condition $\varphi^i_n = \varphi$ still holds, but that $\phi^i_n = \varphi$ does not have to hold any more. A choice of doing this is to set

$$\phi^2_1 = \phi$$

and leave

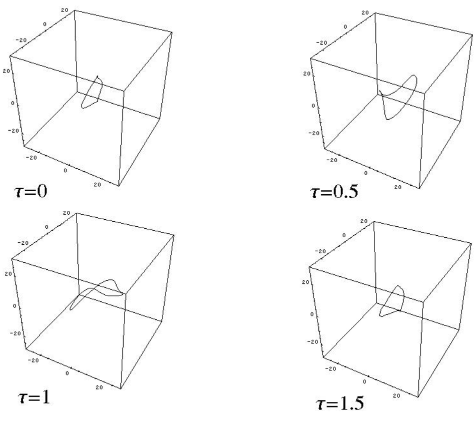
$$\phi^2_2 = \phi^3_1 = \varphi.$$

This is only one choice, many others are also possible. The solution to the equation of motion for the closed string now becomes

$$
\begin{aligned}
x^+ &= 2a\ell\tau \\
x^- &= 6a\ell\tau + a\ell\sin[2(\tau-\sigma)] + a\ell\sin[2(\tau+\sigma)+\theta] \\
&+\frac{1}{2}a\ell\sin[4(\tau-\sigma)+\varphi] + \frac{1}{4}a\ell\sin[4(\tau+\sigma)+\phi] + \frac{1}{4}a\ell\sin[4(\tau+\sigma)+\varphi] \\
&+\frac{1}{3}a\ell\sin[6(\tau-\sigma)+\varphi] + \frac{1}{3}a\ell\sin[6(\tau+\sigma)+\chi] \\
&+\frac{1}{8}a\ell\sin[8(\tau-\sigma)+\varphi] + \frac{1}{8}a\ell\sin[8(\tau+\sigma)+\varphi] \qquad (38) \\
x^2 &= a\ell\sin[2(\tau-\sigma)+\frac{\varphi}{2}] + a\ell\sin[2(\tau+\sigma)+\frac{\phi}{2}] \\
&+\frac{1}{2}a\ell\sin[4(\tau-\sigma)+\frac{\varphi}{2}] + \frac{1}{2}a\ell\sin[4(\tau+\sigma)+\frac{\varphi}{2}] \\
x^3 &= a\ell\sin[2(\tau-\sigma)+\frac{\varphi}{2}] + a\ell\sin[2(\tau+\sigma)+\frac{\varphi}{2}]
\end{aligned}
$$

where $\chi = \frac{1}{2}(\varphi+\phi)$ and $\theta = \frac{1}{2}(\varphi-\phi)$.

When now a, ℓ, φ and ϕ are chosen to be $a=1$, $\ell=2\pi$, $\varphi=0$ and $\phi=\pi$ the equation of motion gives a closed string which has not collapsed into a line and is rotating and vibrating in time.



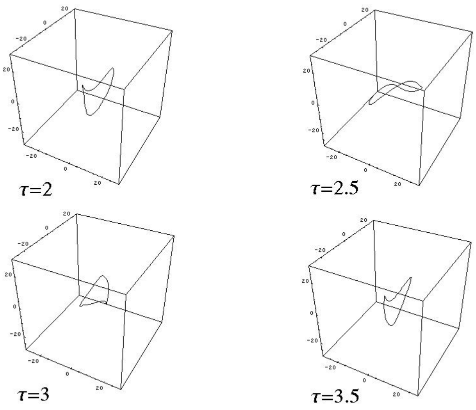


Figure 5: **A vibrating closed string for times $\tau = 0$ until $\tau = 3.5$ in steps of 0.5 in x^-, x^2 and x^3 space.**

5 Conclusions

Within the light-cone gauge there are many solutions possible for open and closed strings. Rotating open strings are found that collapse into a point again. Closed string solutions are found that actually look a lot like open string solutions. The closed string solutions tend to collapse into a line, leaving the simplification $\phi_n^i = \varphi_n^i = \varphi$ was necessary to find a solution that does not collapse into a line. Many more solutions can be found, this can be done by dropping some of the simplifications that are made. Also solutions of higher order, $n > 2$, for the α_n^i oscillators can be chosen. Also more dimensions can be included.

6 Acknowledgment

I would like to thank Dr. Mees de Roo for his support by making this report.

7 References

[1] M. de Roo, *Basic String Theory*, February 2003.

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