# The possible influence of mass dependent flight times on neutrino oscillations

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*July 10, 2008*

**Abstract**

Neutrino's are leptons whose flavour eigenstates are a superposition of their mass eigenstates. This rather special feature is a quantum-mechanical consequence of the experimental observation of neutrino oscillation. Taking the mass dependence of the flight times of the mass eigenstates into account, the value of the mass squared difference changes. Making use of this time dependence we come up with a hypothetical experiment that would give us information about the absolute values off the mass eigenstates.

# Contents

# 1 Introduction

Before understanding the nature of neutrino oscillations and its origin, we must first look at what we already know about neutrinos. Neutrinos are members of an elementary particle family, the *leptons*, the same family which contains a more famous member, the electron. For completeness sake it should be mentioned that the other important family of elementary particles is that of the *quarks*. Each family consists of six members and together both families constitute all matter in the universe. For this article, however, the family of the leptons is the one of interest.

We can think of the leptons as being subdivided into two groups of each three particles: the charged leptons and the neutral leptons. The charged group contains the electron, muon and tauon, respectively denoted by $e$, $\mu$ and $\tau$. They all have the same charge, namely $-e \simeq -1.602 \cdot 10^{-19}C$, but differ in mass. The electron is the lightest with a rest mass of 0.511 $MeV/c^2$, the muon is heavier with a rest mass of 0.1066 $GeV/c^2$ and the tauon is the heaviest of them all with a rest mass of 1.777 $GeV/c^2$ . The neutral group contains the neutral counterparts of these charged leptons, the neutrinos. Like the charged leptons, there are three of them, the electron-neutrino, muon-neutrino and tauon-neutrino, respectively denoted by $\nu_e$, $\nu_\mu$and $\nu_\tau$. It is common to speak of the three *flavours* of the neutrinos, the electron, muon and tauon flavour. Their absolute masses are unknown, but a vast amount of experimental data suggest that it is at least non zero but indeed very small. There are some upper limits known though, which will be addressed later on. Since they posses so little mass and are neutral, they interact very weakly with matter, or in other words have a very small cross section. Per second an enormous amount of neutrinos fly through the earth and our bodies without being noticed, unless you posses the right equipment and then still the number of interaction events is much smaller than we're accustomed to with the "other" particles.

The existence of neutrinos was first proposed by Pauli in 1930 in order to explain the continuous spectrum of beta decay. At a fundamental level neutrinos interact through the weak interaction. A $W$ boson decays into a $l_f$, a lepton of flavour $f$ and into a $\nu_f$, a neutrino of the same flavour $f$. That reaction is denoted by $W^+ \longrightarrow \overline{l_f} + \nu_f$. The flavour of the lepton determines the initial flavour of the neutrino. The flavour of the neutrino during its flight however is not uniquely determined, as we shall see, due to so called neutrino *oscillation*. When after flight one measures a neutrino of flavour $f'$ (which can differ from flavour $f$, because of this neutrino oscillation!) what one actually measures is the existence of the related lepton $l_{f'}$, since the conjugate reaction $\nu_{f'} \longrightarrow l_{f'} + W^+$ ensures this preservation of flavour. So one measures the neutrino flavour indirectly by measuring the lepton flavour in the weak interaction.

Now unlike the charged leptons, the neutrinos posses the special quantum mechanical feature of changing flavour as they travel, the so called *flavour* or *neutrino oscillation* mentioned above. We have got to be a bit more precise with what we mean by "changing flavour". What we actually mean by this is that, because the neutrino travelled a finite distance, quantum mechanics predicts that there is a finite probability of finding a neutrino with a *different* flavour than the flavour with which it was "born". Let's say that the nuclear processes of $W$ boson decay in the sun creates a bundle of $\nu_e$'s and as they travel towards earth, due to neutrino oscillation, there is a certain distribution of $\nu_e$'s, $\nu_\mu$'s and $\nu_\tau$'s measured when they arrive here. Based on the models of the sun and its nuclear processes, we can predict the amount of events of interacting $\nu_e$'s we should measure when they arrive on earth. Instead we measure substantially less interacting $\nu_e$ events and furthermore a certain amount of interacting $\nu_\mu$ and $\nu_\tau$ events, which weren't expected nor predicted by our solar models[1].

This experimental phenomenon is theoretically explained by claiming that if one expresses the state of the neutrino $|\nu\rangle$ as a vector in *flavour space*, that is $|\nu_f\rangle$ , it does not coincide with expressing it in *mass space*, that is $|\nu_m\rangle$ . In other words the flavour eigenstates $|\nu_f\rangle$ are not "equal" to the mass eigenstates $|\nu_m\rangle$ , but rather a *superposition* of mass eigenstates and by inversion vice versa. By this inequality we mean that a neutrino of distinct flavour has not one well defined mass. Actually when one tries to measure the mass of the $\nu_e$ for example, when measuring a bundle of them, one would get a certain distribution of three[1] distinct masses. This superposition is the theoretical and mathematical origin of neutrino oscillation. In the next chapter this is more thoroughly elaborated.

In this article we will try to solve the relation between the so called oscillation*length* (the typical

[1] Assuming here that there are only three distinct flavours and mass eigenstates, ignoring in other words the possibility of *sterile* neutrinos. In Chapter 4 this will be mentioned in slightly more detail.

length associated with the oscillation) and the hypothesis that different neutrino mass eigenstates have different times of flight. We will make some assumptions about this possible relation and see what impact this assumption has on the oscillation length and thus what the related consequences are.

The phenomenon of neutrino oscillation is a topic that by itself is interesting enough, but understanding it serves more than just this purpose. Neutrino oscillations can give us information about possible CP violations[2] in decay reactions, which in turn say something about the asymmetric distribution of matter and anti-matter in the universe and therefore our very existence!

# 2 Superposition and neutrino oscillation

In the introduction we proposed that the very experimental fact of the measured neutrino oscillations gives rise to the theoretical consequence of the $|\nu_f\rangle$ being a *superposition* of $|\nu_m\rangle$ . To see why this follows theoretically, let's first write mathematically what we mean by this superposition. Mathematically we can write the $|\nu_f\rangle$ being a superposition of $|\nu_m\rangle$ as[2]

$$|\nu_f\rangle \ = \sum_m U_{fm} \, |\nu_m\rangle \tag{2.1}$$

(Notice that this represents the state $|\nu_f\rangle$ at the *birth* ($t = 0$) of the neutrino, since after time $t$ evolution operators should be taken into account in describing the neutrino state.) The matrix $U$ in Eq. (2.1) is called the leptonic mixing matrix and is unitary, with $|U_{fm}|^2$ being the *probability* of measuring a neutrino of flavour $f$ having a mass $m$ (still at $t = 0$). Since the total sum of all probabilities of measuring $m$ for each $f$ must add up to 1, it is easy to see why $U$ must be unitary. Because of the unitarity of $U$, the conjugate of Eq. (2.1) can be taken expressing $|\nu_m\rangle$ in terms of $|\nu_f\rangle$ , as we can see in Eq. (2.2)

$$|\nu_m\rangle \ = \sum_f U^*_{mf} \, |\nu_f\rangle \tag{2.2}$$

With $U^\dagger_{fm} = U^*_{mf}$ being the hermitian adjoint of $U_{fm}$ , in this case simply the inverse because of $U$ being unitary.

## 2.1 Amplitude and Probability

Now the base is set and we can search for the mathematical origin of the neutrino oscillations. I find it most illuminating to talk in terms of *amplitudes* of processes because they are easily linked to our matrix entries $U_{fm}$ and $U^*_{mf}$ . To derive the $Amp(\nu_f \to \nu_{f'})$ of the process of a neutrino born with flavour $f$, travelling a distance $L$ and being measured having a flavour $f'$ in the indirect way described in the introduction, one has to take into account the fact that the $|\nu_f\rangle$ is a superposition of $|\nu_m\rangle$ . A close inspection of the whole process of creation, travelling and being detected reveals that $Amp(\nu_f \to \nu_{f'})$ can be factored in three parts.

The first part of the $Amp(\nu_f \to \nu_{f'})$ is the amplitude of a neutrino of flavour eigenstate $|\nu_f\rangle$ being a mass eigenstate $|\nu_m\rangle$ . This amplitude is exactly our matrix entry $U_{fm}$, as we can see from equation (2.1). The second factor of $Amp(\nu_f \to \nu_{f'})$ is the *propagation* amplitude $Prop(\nu_m)$. What this amplitude looks like in detail, we will see in a moment. First let me address the third and last factor of $Amp(\nu_f \to \nu_{f'})$ , namely the amplitude for the travelling mass eigenstate (since during flight we express $|\nu\rangle$ in the base of mass eigenstates $|\nu_m\rangle$ ) of having a flavour $f'$. This amplitude is exactly $U^*_{mf'}$, as follows from Eq. (2.2). Since we only considered one mass eigenstate $|\nu_m\rangle$ in this derivation, to get the total $Amp(\nu_f \to \nu_{f'})$ we must sum over all $m$, giving

$$Amp(\nu_f \to \nu_{f'}) = \sum_m U_{fm} \; Prop(\nu_m) \; U^*_{mf'} \tag{2.3}$$

Now we need to find this $Prop(\nu_m)$. There are several ways to derive this amplitude and one of the simplest ways is to consider the neutrino as a free particle and to look at the neutrino in its *rest* frame. In this rest frame the mass eigenstate vector $|\nu_m\rangle$ at eigentime $\tau_m$ must obey the Schrödinger equation

$$i\hbar\frac{\partial}{\partial\tau_m}\,|\nu_m(\tau_m)\rangle = M_m c^2\,|\nu_m(\tau_m)\rangle$$

The solution for $|\nu_m(\tau_m)\rangle$ of this equation is

$$|\nu_m(\tau_m)\rangle = \exp\left[-i\frac{M_m\tau_m c^2}{\hbar}\right]|\nu_m(0)\rangle \tag{2.4}$$

From Eq. (2.4) it is easy to see that the amplitude $Prop(\nu_m)$ for the process of a neutrino mass eigenstate $|\nu_m(\tau_m)\rangle$ to travel for a time $\tau_m$ is $\exp\left[-i\frac{M_m\tau_m c^2}{\hbar}\right]$, since it is just the inner product $\langle\nu_m(0)|\nu_m(\tau_m)\rangle$. But we normally don't measure these amplitudes, or rather the probabilities, in the rest frame of the neutrino. We first have to translate this $Prop(\nu_m)$ to our laboratory frame in which the neutrino has a certain momentum $\vec{p}_m$. In this new frame we measure a time $t$, the distance $L$ between source and target is fixed. Here we assume that the neutrino flavour eigenstate is a coherent superposition of mass eigenstates and that it takes the same amount of time $t$ for all mass eigenstates $|\nu_m\rangle$ to travel the distance $L$. This is a crucial point and in the next chapter I will explain what happens, if this assumption is altered. We also assume that the energies of the different $|\nu_m\rangle$ are equal, since there is no physical reason to assume otherwise. I will show, after some calculations, that this assumption can be mathematically justified.

From theory of relativity we know that a product of two well defined four-vectors is Lorentz translation invariant and should give us the proper transformation of the rest frame to the laboratory frame. We write $P^\mu$ for the neutrino's four-momentum in the lab frame and $P^\mu_{rest}$ for the four-momentum of the neutrino in the rest frame were $\vec{v} = 0$ and thus $\gamma = 1$, with

$$\begin{aligned} P^\mu &= \left(\frac{E}{c}, \vec{p}\right) \qquad\qquad \text{and so} \\ P^\mu_{rest} &= \left(\frac{E}{c}, 0\right) = \left(\frac{\gamma M c^2}{c}, 0\right) = (Mc, o) \end{aligned} \tag{2.5}$$

Taking the product with the space-time four-vector $x^\mu$, which is defined by

$$x^\mu = (ct, -\vec{x})$$

gives us in the laboratory frame

$$P_\mu x^\mu = Et - \vec{p}_m\vec{x} \tag{2.6}$$

and

$$P_{\mu,rest}x^\mu = M_m\tau c^2 \tag{2.7}$$

in the rest frame. We write $\tau$ in Eq. (2.7) because we're looking at the rest frame of the neutrino where $t = \tau$. Making use of the Lorentz invariance and thus equality of the Eqs. (2.6) and (2.7), and the fact that we measure all neutrinos at the same distance $L$ from the source, a relation between the quantities in the different frames of reference follows easily:

$$M_m\tau c^2 = Et - \vec{p}_m L \tag{2.8}$$

$p_m$ is the only variable in the lab frame of reference. The relativistic energy and momentum are approximately related by

$$p_m = \sqrt{\frac{E^2}{c^2} - M_m^2 c^2} = \sqrt{\frac{E^2}{c^2}\left(1 - \frac{M_m^2 c^4}{E^2}\right)} \approx \frac{E}{c} - \frac{M_m^2 c^3}{2E} \tag{2.9}$$

From Eqs. (2.8) and (2.9) it follows that

$$M_m c^2 \tau \approxeq E(t - \frac{L}{c}) + \frac{M_m^2 c^3}{2E} L \tag{2.10}$$

Since $E(t - \frac{L}{c})$ is assumed to be independent of $m$, this becomes irrelevant in the amplitude $Prop(\nu_m)$. So substituting Eq. (2.10) into Eq. (2.4) we get

$$Prop(\nu_m) = \exp\left[-i \frac{M_m^2 c^3}{2E\hbar} L\right] \tag{2.11}$$

As mentioned, we write $E$ instead of $E_m$ in Eqs. (2.9) and (2.10). We can write $E$ instead of $E_m$ on somewhat different type of grounds: If the energy of a mass eigenstate $|\nu_m\rangle$ was *not* equal for different $m$, this would give a phase factor $\exp[\frac{-i(E_m - E_{m'})t}{\hbar}]$ for the interference patern at detection. Now since one always measures an *average* over time, this factor would always give zero, unless $E_m$ equals $E_{m'}$. In other words you would not measure anything unless the different mass eigenstates have the same energy $E$.

The other justification is more of a mathematical reasoning[3]: Let's say that instead of equal energy $E$ of the mass eigenstates, we assumed equal momentum $p$ for all mass eigenstates. Then Eq. (2.9) would transform into

$$E_m = \sqrt{p^2 c^2 + M_m^2 c^4} \approxeq pc\left(1 + \frac{M_m^2 c^2}{2p^2}\right) = pc + \frac{M_m^2 c^3}{2p} \tag{2.12}$$

The amplitude $Prop(\nu_m)$ of equation (2.11) would than change into

$$Prop(\nu_m) = \exp\left[-i \frac{M_m^2 c^3}{2p\hbar} L\right] \tag{2.13}$$

But in the relativistic limit $E \approx p$, giving Eq. (2.11) back. So we conclude that either approximation leaves us the same amplitude for the process to occur in the relativistic limit and are therefore equivalent. For later purposes, however, we prefer to think of the neutrinos having equal energy. Having derived $Prop(\nu_m)$ we are ready to conclude the total amplitude $Amp(\nu_f \to \nu_{f'})$ to be

$$Amp(\nu_f \to \nu_{f'}) = \sum_m U_{fm} \ \exp\left[-i \frac{M_m^2 c^3}{2E\hbar} L\right] U^*_{mf'} \tag{2.14}$$

The probability $P(\nu_f \to \nu_{f'})$ for this process is simply the absolute amplitude $Amp(\nu_f \to \nu_{f'})$ squared, for convenience we put $\hbar = c = 1$:

$$P(\nu_f \to \nu_{f'}) = |Amp(\nu_f \to \nu_{f'})|^2 \tag{2.15a}$$

$$= \left| \sum_m U_{fm} \ \exp\left[-i \frac{M_m^2}{2E} L\right] U^*_{mf'} \right|^2 \tag{2.15b}$$

$$= \left( \sum_{m'} U^*_{m'f} \exp\left[-i \frac{M_{m'}^2}{2E} L\right] U_{f'm'} \right) \times \left( \sum_m U_{fm} \ \exp\left[-i \frac{M_m^2}{2E} L\right] U^*_{mf'} \right) \tag{2.15c}$$

Using the unitarity of $U_{fm}$ , the relations for complex numbers $z + z^* = 2\Re z$ and $z - z^* = 2i\Im z$ and the appropriate trigonometric formulas we have (See Appendix A)

$$\boxed{\begin{aligned} P(\nu_f \to \nu_{f'}) = \delta_{ff'} - 4 \sum_{m>m'} \Re\big(U^*_{m'f} U_{f'm'} U_{fm} \ U^*_{mf'}\big) \sin^2\left(\Delta M^2_{mm'} \frac{L}{4E}\right) \ + \\ 2 \sum_{m>m'} \Im\big(U^*_{m'f} U_{f'm'} U_{fm} \ U^*_{mf'}\big) \sin\left(\Delta M^2_{mm'} \frac{L}{2E}\right) \end{aligned}} \tag{2.16}$$

With $\Delta M^2_{mm'} \equiv M^2_{m'} - M^2_m$.

In the Eq. (2.16) the oscillating character of the probability is easily recognised. It is a purely quantum mechanical effect, entirely due to the flavour eigenstates $|\nu_f\rangle$ being a superposition of the mass eigenstates $|\nu_m\rangle$ , see Eqs. (2.1) and (2.2).

## 2.2 Interpretation and characteristics of $P(\nu_f \to \nu_{f'})$

So Eq. (2.16) gives the general probability of a neutrino with energy $E$ to be born with a flavour $f$, to travel a distance $L$ and to be detected as a neutrino with flavour$f'$. This in turn means that when we have let's say $N$ $\nu_e$'s at birth, at detection these $N$ $\nu_e$'s will arrive as a certain distribution:

1. $P\ (\nu_e \to \nu_e) \times N$ is the amount of detected $\nu_e$'s
2. $P\ (\nu_e \to \nu_\mu) \times N$ is the amount of detected $\nu_\mu$'s
3. $P\ (\nu_e \to \nu_\tau) \times N$ is the amount of detected $\nu_\tau$'s

From Eq. (2.15c) it is easy to verify that in general for every flavour $f$ the sum of all probabilities for the detection of different flavours $f'$ adds up to 1:

$$\begin{aligned}
\sum_{f'} P(\nu_f \to \nu_{f'}) &= \sum_{f'} \left( \sum_{m'} U^*_{m'f} \exp\left[-i\frac{M^2_{m'}}{2E}L\right] U_{f'm'} \right) \times \left( \sum_m U_{fm}\ \exp\left[-i\frac{M^2_m}{2E}L\right] U^*_{mf'} \right) \\
&= \sum_{m,m'} U^*_{m'f} U_{fm}\ \exp\left[i\frac{\Delta M^2_{mm'}}{2E}L\right] \sum_{f'} U_{f'm'} U^*_{mf'} \\
&= \sum_{m,m'} U^*_{m'f} U_{fm}\ \exp\left[i\frac{\Delta M^2_{mm'}}{2E}L\right] \delta_{mm'} \\
&= \sum_m |U_{fm}|^2 = 1
\end{aligned}$$

This certainly must be the case, since we assume that there are no particles lost and thus the probability of measuring *any* flavour at the detector must be 1.

Closer inspection of Eq. (2.16) reveals that *if* the neutrinos were mass-less, then $\Delta M^2_{mm'} \equiv M^2_m - M^2_{m'} = 0$ and therefore $P(\nu_f \to \nu_{f'}) = \delta_{ff'}$. In other words no oscillation would have occurred. So the experimental fact that these oscillations have been measured, proves that indeed the neutrinos have non zero mass! However, Eq. (2.16) only tells us something about the relative squared mass *difference*, it says nothing about the *absolute* masses or eigenvalues of the neutrino mass eigenstates.

To be able to interpret the arguments of the sin and $\sin^2$ of Eq. (2.16) physically, we multiply these arguments again by the appropriate factors of $c$ and $\hbar$. $P(\nu_f \to \nu_{f'})$ becomes

$$\begin{aligned}
P(\nu_f \to \nu_{f'}) = \delta_{ff'} - 4 \sum_{m>m'} \Re\big(U^*_{m'f} U_{f'm'} U_{fm}\ U^*_{mf'}\big) \sin^2\left(\Delta M^2_{mm'} \frac{L}{4E}\frac{c^3}{\hbar}\right) + \\
2 \sum_{m>m'} \Im\big(U^*_{m'f} U_{f'm'} U_{fm}\ U^*_{mf'}\big) \sin\left(\Delta M^2_{mm'} \frac{L}{2E}\frac{c^3}{\hbar}\right)
\end{aligned} \tag{2.17}$$

giving a typical periodic length $l_{mm'}$ , called the *oscillation length*, for the oscillation to occur:

$$\boxed{l_{mm'} = \frac{2\pi}{\Delta M^2_{mm'} \frac{c^3}{2E\hbar}} = \frac{4\pi E\hbar}{\Delta M^2_{mm'} c^3}} \tag{2.18}$$

This $l_{mm'}$ is the spatial period of $P(\nu_f \to \nu_{f'})$. This means that when measuring a certain distribution of neutrino flavours at a distance $L$ from the source, measuring at a distance $L + l_{mm'}$ would give us exact the same distribution. Now after finding this distance $l_{mm'}$ of equal distributions, we are able to calculate $\Delta M_{mm'}$, given that the energy $E$ is fixed.

After inserting the values of $c$ and $\hbar$ in Eq. (2.17), we can approximate $P(\nu_f \rightarrow \nu_{f'})$ by

$$P(\nu_f \rightarrow \nu_{f'}) = \delta_{ff'} - 4 \sum_{m>m'} \Re(U^*_{fm'} U_{fm} U_{f'm'} U^*_{f'm}) \sin^2\left(1.27 \Delta M^2_{mm'}\left(eV^2\right) \frac{L(km)}{E(GeV)}\right) + \\ 2 \sum_{m>m'} \Im(U^*_{fm'} U_{fm} U_{f'm'} U^*_{f'm}) \sin\left(2.54 \Delta M^2_{mm'}\left(eV^2\right) \frac{L(km)}{E(GeV)}\right) \tag{2.19}$$

Since $\sin^2(x) \approx x^2 + h.o.$ for small $x$, this oscillation becomes significant if $x$ is in the order of 1 or higher. This means that for a given $L$ and $E$, the experiment is sensitive for values of $\Delta M^2_{mm'}$ up to $[L(km)/E(GeV)]^{-1}$. For example if $L = 10^4$ km and $E = 1$ GeV, then $\Delta M^2_{mm'}$ can be measured op to $10^{-4}$ eV$^2$. Since the diameter of the earth is in the order of $10^4$ km, experiments performed here on earth with the right energies could get information about very small neutrinos mass differences[2].

# 3 Time of flight

In this section we are interested whether or not we can say something about the time of flight of the neutrinos, expressed as a superposition of mass eigenstates. Do the different mass eigenstates arrive at different times? Is it at all correct to look at the process in this way? What does superposition actually mean when one measures the neutrinos? And what *does* one actually measure, mass or flavour eigenstates? These questions need to be answered.

Like mentioned in the previous section we consider the neutrino flavour eigenstates at birth to be having a certain sharp fixed energy. It seems a reasonable assumption to state that since the neutrino flavour eigenstates are a superposition of mass eigenstates, we consider those mass eigenstates to be having the same initial energy at birth. We saw in Eqs. (2.12) and (2.13), instead of equal energy of the mass eigenstates, assuming equal momenta of the mass eigenstates does not change the $P(\nu_f \rightarrow \nu_{f'})$ in the relativistic limit. Since neutrinos travel close to the speed of light and are therefore highly relativistic, this is a reasonable assumption. Furthermore since one measures the distributions directly proportional to this $P(\nu_f \rightarrow \nu_{f'})$, there will be no difference in measurements for the different assumptions and we can therefore choose the equal energy approach. In the derivation of the phase of $P(\nu_f \rightarrow \nu_{f'})$ we also assumed that the time $t$ it takes for a mass eigenstate to travel a distance $L$ was equal for all masses, see Eqs (2.6–2.11). It seems a bit odd at most that two mass eigenstates $|\nu_m\rangle$ and $|\nu_{m'}\rangle$ with equal energy $E$, but different masses $M_m$ and $M_{m'}$ would take the same time $t$ to travel a fixed distance $L$ and in this section we are going to see the resulting change in $P(\nu_f \rightarrow \nu_{f'})$ when this assumption is dropped.

## 3.1 The mass dependence of the time of flight

Let's say that the relativistic energy of the mass eigenstates is $E$. Then for a mass eigenstate with mass $M_m$ the energy is,

$$E = \gamma M_m c^2 = \frac{M_m c^2}{\sqrt{1 - \frac{v_m^2}{c^2}}} \tag{3.1}$$

with $v_m$ the speed of the mass eigenstate $|\nu_m\rangle$ .

Since we can define the time $t_m$ it takes for a mass eigenstate $|\nu_m\rangle$ to travel a distance $L$ by $t_m = L/v_m$, we can derive $t_m$[4] (See Appendix B)

$$t_m \approxeq \frac{L}{c}\left(1 + \frac{M_m^2 c^4}{2E^2}\right) \quad . \tag{3.2}$$

With equation (3.2) we have an expression for the flight time of a neutrino mass eigenstate of mass $M_m$

[2]We have not yet taken the influence of matter on the oscillation length $l_{mm'}$ into account. When an experiment is performed on earth and neutrinos fly through matter, this in general changes $P(\nu_f \rightarrow \nu_{f'})$[2].

and energy $E$ travelling a distance $L$. It's now easy to derive the *separation* $\Delta t_{mm'}$ in time between the arrival times of two different neutrino mass eigenstates $|\nu_m\rangle$ and $|\nu_{m'}\rangle$ having respectively mass $M_m$ and $M_{m'}$:

$$\boxed{\Delta t_{mm'} = \frac{Lc^3}{2}\frac{M_m^2 - M_{m'}^2}{E^2} = \frac{Lc^3}{2}\frac{\Delta M_{mm'}^2}{E^2}} \tag{3.3}$$

The question now is how can we measure this $\Delta t_{mm'}$ and how must we interpret the result?

## 3.2 $P^*(\nu_f \to \nu_{f'})$ as a consequence of the mass dependence of $t_m$

Now that we have a relation between the time it takes a relativistic particle with energy $E$ and mass $M_m$ to travel distance $L$ with Eq. (3.2), it is interesting to see what happens when we take this relation into account when deriving $P(\nu_f \to \nu_{f'})$. We actually have to go all the way back to Eq. (2.8). Arguing that the eigentime of the mass eigenstates $|\nu_m\rangle$ is depends on the value of $M_m$ seems very plausible, for particles with the same energy $E$ but different mass $M_m$. In this case also the time of flight $t$ measured in the lab frame depends on $M_m$. So Eq. (2.8) becomes

$$M_m \tau_m c^2 = E t_m - \vec{p}_m L \tag{3.4}$$

The momentum of each mass eigenstate $p_m$ is still approximately given by Eq. (2.9) so Eq. (2.10) becomes

$$M_m c^2 \tau_m \approxeq E t_m - \frac{EL}{c} + \frac{M_m^2 c^3}{2E} L \tag{3.5}$$

with off course the big difference with respect to Eq. (2.10) in the fact now the term $Et_m$ is *not* equal anymore for all $|\nu_m\rangle$ , but $EL/c$ still is, so like before now only this part is left out of the amplitude. This is represented in the new propagation amplitude for the process $Prop^*(\nu_m)$ :

$$Prop^*(\nu_m) = \exp\left[\frac{-i}{\hbar}\left(Et_m + \frac{M_m^2 c^3}{2E}L\right)\right] \tag{3.6}$$

The other amplitudes are not affected by this new perspective and so the total amplitude $Amp^*(\nu_f \to \nu_{f'})$ now becomes

$$Amp^*(\nu_f \to \nu_{f'}) = \sum_m U_{fm} \exp\left[\frac{-i}{\hbar}\left(Et_m + \frac{M_m^2 c^3}{2E}L\right)\right] U^*_{mf'} \tag{3.7}$$

We can again calculate the new oscillation probability $P^*(\nu_f \to \nu_{f'})$ in exact the same way as we did in Eqs. (2.15–2.16), and we end up with

$$\begin{aligned} P^*(\nu_f \to \nu_{f'}) = \delta_{ff'} - 4 \sum_{m>m'} \Re\big(U^*_{fm'} U_{fm} U_{f'm'} U^*_{f'm}\big) \sin^2\left(\frac{E\Delta t_{mm'}}{2\hbar} + \frac{\Delta M_{mm'}^2 c^3 L}{4E\hbar}\right) + \\ 2 \sum_{m>m'} \Im\big(U^*_{fm'} U_{fm} U_{f'm'} U^*_{f'm}\big) \sin\left(\frac{E\Delta t_{mm'}}{\hbar} + \frac{\Delta M_{mm'}^2 c^3 L}{2E\hbar}\right) \end{aligned} \tag{3.8}$$

From here on we are just interested in the arguments of $\sin^2$ and sin. We see that with Eq. (3.3) the first term of the argument argument can be rewritten as follows

$$\frac{E\Delta t_{mm'}}{2\hbar} = \frac{\Delta M_{mm'}^2 c^3 L}{4E\hbar} \tag{3.9}$$

thus when substituted in Eq. (3.8) doubles the argument of both $\sin^2$ and sin. Giving $P^*(\nu_f \to \nu_{f'})$in terms of $\Delta M^2_{mm'}$:

$$\boxed{\begin{aligned} P^*(\nu_f \to \nu_{f'}) = \delta_{ff'} - 4 \sum_{m>m'} \Re\big(U^*_{fm'} U_{fm} U_{f'm'} U^*_{f'm}\big) \sin^2\left(\frac{\Delta M_{mm'}^2 c^3 L}{2E\hbar}\right) + \\ 2 \sum_{m>m'} \Im\big(U^*_{fm'} U_{fm} U_{f'm'} U^*_{f'm}\big) \sin\left(\frac{\Delta M_{mm'}^2 c^3 L}{E\hbar}\right) \end{aligned}} \tag{3.10}$$

Importantly the oscillation length $l_{mm'}$ has changed into $l^*_{mm'}$ , with

$$\boxed{l^*_{mm'} = \frac{2\pi}{\Delta M^2_{mm'}\frac{c^3}{E\hbar}} = \frac{2\pi E\hbar}{\Delta M^2_{mm'}c^3}} \tag{3.11}$$

Which means that $l^*_{mm'} = l_{mm'}/2$. So if an oscillation length is measured, the calculation of the value of the mass squared difference $\Delta M^2_{mm'}$ depends on which assumptions you make, since clearly the value of $\Delta M^2_{mm'}$ if calculated using $l_{mm'}$ is twice the value if calculated using $l^*_{mm'}$ .

# 4 Limiting case

In our derivations we did not explain how all mentioned quantities actually could be measured. In this section I will discuss the limiting case of only two flavour oscillations and try to think of an experiment which could say something about the flight times and the related values of the mass eigenstates.

## 4.1 A two flavour approximation

To make it a little bit easier and more insightful, let us assume that only two of the three flavours, $\nu_e$ and $\nu_\mu$, are mixed and thus also just two mass eigenstates $\nu_1$ and $\nu_2$ participate in the oscillations. We still don't involve the effect that matter has on the oscillations, i.e. we are still in the vacuum. The most insightful way to represent $U$ in this case is by a simple unitary $2 \times 2$ matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.1}$$

Where $\theta$ is called the mixing angle. The amplitude for the process becomes

$$A(\nu_e \to \nu_\mu) = \sum_{m=1,2} U_{em} \exp\left[-i\frac{M_m^2 c^3}{2E\hbar}L\right] U^*_{m\mu} \tag{4.2}$$

Squaring $A(\nu_e \to \nu_\mu)$ would give us the more common expression for the probability (See Appendix C).

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta)\sin^2\left(\frac{\Delta M^2_{12}c^3}{4E\hbar}L\right) \tag{4.3}$$

with $\Delta M^2_{12}$ being the difference between $M_1$ and $M_2$ the masses of $|\nu_1\rangle$ and $|\nu_2\rangle$ the only two mass eigenstates. The $l_{mm'}$ is still given by Eq. (2.18), but there is some difference in the behaviour of $P(\nu_e \to \nu_\mu)$. In the three flavour case there was no travelling $L$ distance where $P(\nu_f \to \nu_{f'})$ would only allow one single flavour, in every point in space there was always a superposition of at least two flavours. Now in the two flavour case, this does not hold. Starting with a $\nu_e$ and detecting the neutrino exactly after distance $L = l_{mm'}$ , the probability of the $\nu_e$ oscillating into a $\nu_\mu$ is zero and one is sure to measure the $\nu_e$ again. If we would detect at $L = l_{mm'}/2$, we would have maximum mixing between the flavours and the probability of measuring a $\nu_\mu$ would then be $\sin^2(2\theta)$, depending on the mixing angle $\theta$. If $\theta$ would turn out to be 45°, the probability of measuring a $\nu_\mu$would be 1 and therefore $\theta = 45°$ is called the maximum mixing angle. So doing an experiment where these distributions could be measured would give us $l_{mm'}$ and $\theta$.

Again, like in the three flavour case, if we assume a time of flight depending on the mass eigenstates $t = t_m$, Eq. (4.3) would become

$$P^*(\nu_e \to \nu_\mu) = \sin^2(2\theta)\sin^2\left(\frac{\Delta M^2_{12}c^3}{2E\hbar}L\right) \tag{4.4}$$

with the known consequences for $l_{mm'}$ and therefore also for $\theta$ and $\Delta M^2_{12}$.

## 4.2 The experiment

Suppose say we have a controllable source of neutrino's, at a fixed distance $L$ from a detector. By controllable we mean that we control whether it's on or off and thus we are able to define what we mean by $t = 0$, plus we control the energy $E$ of the neutrinos and in this experiment it is fixed. The source produces only neutrinos of one distinct flavour, let's take $\nu_e$'s. And furthermore, for simplicity, we only consider the two flavour scenario. Thus we assume $\nu_e$ and $\nu_\mu$ are the only two flavour eigenstates and the only two mass eigenstates we call $\nu_1$ and $\nu_2$. When we stated that a flavour eigenstate is a superposition of mass eigenstates (see Eq. (2.1)), we meant that when trying to measure the mass $M$ of the flavour eigenstates, we won't find *one* particular value belonging to that particular flavour, in our case the electron flavour. Instead we would find a *distribution* of two different values of $M$, which we call $M_1$ and $M_2$. For the $\nu_e$ the superposition looks like

$$|\nu_e\rangle = \sum_{m=1,2} U_{em} |\nu_m\rangle \tag{4.5}$$

Now if $L$ and $E$ are fixed, on could jump to conclusions and state that with Eq. (3.2) the time $t$ of the travelling $\nu_e$ could be measured and thereby the mass $M$ of the $\nu_e$ would be fixed. This is in contradiction to the earlier statement that the flavours have not one particular mass. When you think of it, we have actually three distinct problems:

- We never actually measure a mass eigenstate, only flavour eigenstates. This is because the neutrino is indirectly measured by detecting its associated charged lepton. We consider however the neutrino in flight as a superposition of mass eigenstates, since this explains the flavour oscillations.
- A $\nu_e$ has not one particular mass, but is a superposition of mass eigenstates, with each a probability $|U_{em}|^2$ to measure a mass $M_m$, if mass measurements where possible. This means with Eq. (3.2) that a bundle of $\nu_e$ has not one fixed time $t$, but also a distribution of different $t$'s.
- There is a probability $P(\nu_e \to \nu_\mu)$ that the $\nu_e$ will change flavour. This means that we are dealing with a particle with a different flavour, not just the $\nu_e$ anymore. This particle with flavour $f'$ has different matrix elements and therefore different probabilities, when detecting it. We have to take this somehow into account.

Our measurement consists of two quantities being measured simultaneously: the time $t$ it takes for the neutrinos to arrive and the flavour $f'$ of the detected neutrino. Suppose our neutrino beam consists of $N$ $\nu_e$'s of energy $E$, with $N$ large. When we measure the flavour of the detected neutrinos, we expect the following distribution:

1. $P\ (\nu_e \to \nu_e) \times N$ is the amount of detected $\nu_e$'s
2. $P\ (\nu_e \to \nu_\mu) \times N$ is the amount of detected $\nu_\mu$'s

with $\sum_{f'} P\ (\nu_e \to \nu_{f'}) = 1$ since there are no neutrinos vanishing. So this gives the distribution with respect to flavours.

How about the distribution in time? For example, do the $\nu_\mu$'s arrive earlier or later than the $\nu_e$'s? The key argument here is that what determines the time of flight $t$ is actually *not* the flavour, but the *mass*, the eigenvalue of the mass eigenstate for neutrinos with energy $E$ travelling distance $L$. For each mass $M_1,\ M_2$ there corresponds one time by Eq. (3.2), resp. $t_1,\ t_2$, with

$$t_1 = \frac{L}{c}\left(1 + \frac{M_1^2 c^4}{2E^2}\right) \quad \text{and} \quad t_2 = \frac{L}{c}\left(1 + \frac{M_2^2 c^4}{2E^2}\right) \tag{4.6}$$

But during the travel we cannot and must not think of the neutrino as one single mass eigenstate, but as a superposition of mass eigenstates. This superposition oscillates along the way between the (in our case) two flavours, each with it's own configuration of mass eigenstates. At $t = 0$ if all matrix elements of $U$ were known, the configuration would be fixed since the source only produces $\nu_e$'s. But as the neutrino travels it is no longer just a $\nu_e$ anymore but, as mentioned, a superposition of two flavours. Therefore the configuration isn't fixed anymore either at $t \neq 0$, except when the neutrino is detected after a distance of $L = l_{mm'}$ , here we are certain to measure a $\nu_e$. The question now is....at what time does it arrive at this point? The answer is that it does not arrive at *one* distinct time, but in an interval. The one thing we *can* be sure of, is the detected flavour, $\nu_e$ in this case. So at detection we will know it's exact configuration if $U$ is known. Along the way however, we don't. So it's a mistake to state that since we are sure to measure a $\nu_e$ at detection, we also know the exact time of arrival, since on all times between birth and detection we don't know what particle we are dealing with!

There are however boundaries to the possible times of detection. Imagine that it is hypothetically possible at every distance from the source to measure the mass of the neutrino. One would measure strictly one of the two values of the mass eigenstates with a certain probability. Now this probability oscillates as explained, so not only is there some probability of finding a certain mass, this probability itself changes along the way. It means that there is a probability, say a very small one, that if would have lined up infinitely many mass detectors along the path of the particle, they all would measure the smallest value of the two possible mass eigenstates. Meaning that there is a very small chance of the neutrino behaving like the mass eigenstate with the smallest eigenvalue $M_1$. This particle then would be detected as a $\nu_e$at time $t_1$. The same thought experiment can be done for the heaviest of the mass eigenstates $M_2$, it would arrive at $t_2$. All the rest of the $N$ particles will be detected somewhere between these two boundaries. Doing the experiment and having found $t_1$ and $t_2$, the absolute values of the mass eigenstates could be calculated with use of Eq. (3.2)

### 4.2.1 Remarks on the experiment

This seems quite plausible, however we do encounter a few problems that cannot be neglected: To start, if the source would produce a continuous beam of $\nu_e$'s, how do we distinguish the two different times of flight? We couldn't because the neutrinos would arrive between the predicted time boundaries, but since we have a continuous beam the events of arrival would overlap in such a way that there is no clear distinction between the different flight time possible. We thus want to produce a *pulse* of time-length $\Delta T$ in such a way that it is smaller than the time-separation $\Delta t_{12}$ between the flight times, i.e. $\Delta T \ll \Delta t_{12}$.

Secondly we also want our neutrinos to have the same energy $E$, because only in that case the flight times can be compared. If the energies differ too much, so will the momentum and therefore the flight times of the neutrinos, causing overlap and therefore obscuring the different times of flight. Having both $E$ sharply peaked and $\Delta t$ very small, could get into contradiction with the Heisenberg uncertainty principle.

Then there is the problem of detection of all neutrinos. Like mentioned they have a very small cross section and can hardly be detected. So there are always neutrinos not being detected, no matter how large the detector. As mentioned in the introduction, there is always a theoretical possibility of *sterile* neutrinos. These are neutrinos that simply do not interact with matter or for that case detectors. There is a possibility that certain part of all $\nu_e$'s could become sterile at detection and we would be unable to show it. Luckily there is suggestive data telling us that the existence of a sterile neutrino is not likely.

Let alone detecting the difference in flight time. For example, if we were to perform an experiment here on earth with a neutrino mass squared splitting of $M_2^2 - M_1^2 \sim 10^{-4}\text{eV}^2$ and the travelling distance in the order of the diameter of te earth, that is $L \sim 10^7$m and neutrinos had an energy $E \sim 1$GeV, the $\Delta t_{12}$ would be $\sim 10^{-24}$s! So the question is also *if* under these circumstances $\Delta t_{12}$ is measurable at all. For intergalactic neutrinos however the distances become much bigger and therefore $\Delta t_{12}$ might become significant and measurable.

Off course we also did not include the influence of matter, although small, on the oscillations, which must be take into account for realistic predictions.

# 5 Conclusion

Assuming that the flight time $t_m$ of the mass eigenstate depends on the mass $M_m$, the new $l^*_{mm'}$ is half of the old $l_{mm'}$ and therefore a calculated value of $\Delta M^2_{mm'}$ depending on $l^*_{mm'}$ is half the value when assuming $l_{mm'}$ .

We found it hard to say something about the flight times of the individual neutrinos, because of the complexity that the superposition of mass eigenstates takes along with it. However, through the thought experiment, we came up with a hypothetical upper and lower bound for the flight times, which in turn could tell us something about the absolute values $M_m$ of the mass eigenstates. We also saw that this thought experiment needs a lot of practical and maybe fundamental adjustments to become realisable.

## References

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# A Calculation of $P(\nu_f \to \nu_{f'})$

$$P(\nu_f \to \nu_{f'}) = |Amp(\nu_f \to \nu_{f'})|^2 \tag{A.1a}$$

$$= \left| \sum_m U_{fm} \; \exp\left[-i\frac{M_m^2}{2E}L\right] U^*_{mf'} \right|^2 \tag{A.1b}$$

$$= \left( \sum_{m'} U^*_{m'f} \exp\left[-i\frac{M_{m'}^2}{2E}L\right] U_{f'm'} \right) \times \left( \sum_m U_{fm} \; \exp\left[-i\frac{M_m^2}{2E}L\right] U^*_{mf'} \right) \tag{A.1c}$$

Subdividing de product of the sums into sums over $m = m',\; m > m'$ and $m < m'$, gives

$$\begin{aligned}
P(\nu_f \to \nu_{f'}) = & \sum_{\substack{m,m' \\ m=m'}} U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'} \exp\left[i\frac{\Delta M^2_{mm'}}{2E}L\right] \; + \\
& \sum_{\substack{m,m' \\ m>m'}} U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'} \exp\left[i\frac{\Delta M^2_{mm'}}{2E}L\right] \; + \\
& \sum_{\substack{m,m' \\ m<m'}} U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'} \exp\left[i\frac{\Delta M^2_{mm'}}{2E}L\right] \\
= & \sum_{\substack{m \\ m=m'}} U_{f'm} U^*_{mf} \sum_{\substack{m' \\ m=m'}} U_{fm'} U^*_{m'f'} \times 1 \; + \\
& \sum_{\substack{m,m' \\ m>m'}} U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'} \exp\left[i\frac{\Delta M^2_{mm'}}{2E}L\right] \; + \\
& \sum_{\substack{m,m' \\ m>m'}} U^*_{mf} U_{f'm} U_{fm'} U^*_{m'f'} \exp\left[-i\frac{\Delta M^2_{mm'}}{2E}L\right]
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
= \; & \delta_{ff'} \quad + \\
& \sum_{\substack{m,m' \\ m>m'}} U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'} \left( \cos\left(\Delta M^2_{mm'} \frac{L}{2E}\right) + i \sin\left(\Delta M^2_{mm'} \frac{L}{2E}\right) \right) \quad + \\
& \sum_{\substack{m,m' \\ m>m'}} \left(U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'}\right)^* \left( \cos\left(\Delta M^2_{mm'} \frac{L}{2E}\right) - i \sin\left(\Delta M^2_{mm'} \frac{L}{2E}\right) \right).
\end{aligned}$$

$$\begin{aligned}
= \; & \delta_{ff'} + 2 \sum_{m>m'} \Re\big(U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'}\big) \left( \cos\left(\Delta M^2_{mm'} \frac{L}{2E}\right) \right) + \\
& 2 \sum_{m>m'} \Im\big(U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'}\big) \sin\left(\Delta M^2_{mm'} \frac{L}{2E}\right)
\end{aligned}$$

$$\begin{aligned}
= \; & \delta_{ff'} - 2 \sum_{m>m'} \Re\big(U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'}\big) \left( 1 - 2\sin^2\left(\Delta M^2_{mm'} \frac{L}{4E}\right) \right) + \\
& 2 \sum_{m>m'} \Im\big(U^*_{m'f} U_{f'm'} U_{fm} \; U^*_{mf'}\big) \sin\left(\Delta M^2_{mm'} \frac{L}{2E}\right)
\end{aligned}$$

giving

$$P(\nu_f \to \nu_{f'}) = \delta_{ff'} - 4 \sum_{m>m'} \Re\big(U^*_{m'f} U_{f'm'} U_{fm}\ U^*_{mf'}\big) \sin^2\left(\Delta M^2_{mm'} \frac{L}{4E}\right) + \\ 2 \sum_{m>m'} \Im\big(U^*_{m'f} U_{f'm'} U_{fm}\ U^*_{mf'}\big) \sin\left(\Delta M^2_{mm'} \frac{L}{2E}\right) \tag{A.3}$$

# B Time of flight

Take

$$E^2 = \frac{M_m^2 c^4}{1 - \frac{v_m^2}{c^2}}.$$

We can then write

$$\begin{aligned} -\frac{v_m^2}{c^2} &= \frac{M_m^2 c^4}{E^2} - 1 \\ \Rightarrow v_m^2 &= c^2 \left(1 - \frac{M_m^2 c^4}{E^2}\right) \\ \Rightarrow \frac{L}{t_m} &= c\sqrt{1 - \frac{M_m^2 c^4}{E^2}} \approxeq c\left(1 - \frac{M_m^2 c^4}{2E^2}\right) \\ \Rightarrow t_m &= \frac{L}{c}\left(1 - \frac{M_m^2 c^4}{2E^2}\right)^{-1} \approxeq \frac{L}{c}\left(1 + \frac{M_m^2 c^4}{2E^2}\right) \end{aligned} \tag{B.1}$$

were we used the appoximation $(1+x)^N \approxeq 1 + nx$ for $x \ll 1$ and in the relativistic limit we can take $E \gg M_m$ thus the approximation holds.

# C Two flavour probability $P(\nu_e \to \nu_\mu)$

Take

$$A\left(\nu_e \to \nu_\mu\right) = \sum_{m=1,2} U_{em} \exp\left[-i\frac{M_m^2 c^3}{2E\hbar}L\right] U_{m\mu}^*$$

and

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{C.1}$$

then squaring and substituting the matrix entries we get:

$$\begin{aligned}
P(\nu_e \to \nu_\mu) = & \left(-\cos\theta \exp\left[-i\frac{M_1^2 c^3}{2E\hbar}L\right]\sin\theta + \cos\theta\exp\left[-i\frac{M_2^2 c^3}{2E\hbar}L\right]\sin\theta\right) \times \\
& \left(-\cos\theta\exp\left[i\frac{M_1^2 c^3}{2E\hbar}L\right]\sin\theta + \cos\theta\exp\left[i\frac{M_2^2 c^3}{2E\hbar}L\right]\sin\theta\right) \\
= & \sin^2\theta\cos^2\theta\cdot 1 - \\
& \sin^2\theta\cos^2\theta\exp\left[i\frac{\Delta M_{12}^2 c^3}{2E\hbar}L\right] - \sin^2\theta\cos^2\theta\exp\left[i\frac{-\Delta M_{12}^2 c^3}{2E\hbar}L\right] + \\
& \sin^2\theta\cos^2\theta\cdot 1 \\
= & 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta\left(\exp\left[i\frac{\Delta M_{12}^2 c^3}{2E\hbar}L\right] + \exp\left[i\frac{-\Delta M_{12}^2 c^3}{2E\hbar}L\right]\right) \\
= & \sin^2\theta\cos^2\theta\left(2 - 2\cos\left(\frac{\Delta M_{12}^2 c^3}{2E\hbar}L\right)\right) \\
= & \sin^2\theta\cos^2\theta\left(2 - 2\left(1 - 2\sin^2\left(\frac{\Delta M_{12}^2 c^3}{4E\hbar}L\right)\right)\right) \\
= & 4\sin^2\theta\cos^2\theta\sin^2\left(\frac{\Delta M_{12}^2 c^3}{4E\hbar}L\right) \\
= & \sin^2(2\theta)\sin^2\left(\frac{\Delta M_{12}^2 c^3}{4E\hbar}L\right)
\end{aligned} \tag{C.2}$$