Supersymmetry in Quantum Mechanics

In this thesis is looked at the Schrödinger equation and the Dirac equation in supersymmetry. While both equations can be treated in a supersymmetric manner, they both take a very different approach to supersymmetry.
Supersymmetry can help solve these equations much simpler than the common methods solve these equations.

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Contents

1. Introduction:

Supersymmetry has been developed at first in string-theory. It then developed its way back through quantum field theory and quantum mechanics. This Bachelor thesis will focus primarily on the quantum mechanics case.
The main goal of this research is to extend the supersymmetry applied on the Schrödinger equation to a relativistic one.
Another goal is putting the electromagnetic field in the equations for supersymmetry and looking at examples where supersymmetry was of use.

The structure of this thesis will be as follows:
First the supersymmetry in the non-relativistic Schrödinger equation will be shown. I will also look at the Schrödinger equation with a Coulomb electric field and a pure magnetic field.
Then the supersymmetry in the Dirac equation will be discussed. Just like in the Schrödinger case, the Dirac equation with the Coulomb electric field and a pure magnetic field will be shown.
Parallels between the supersymmetry in the Schrödinger equation and the supersymmetry in the Dirac equation will be drawn in the last section. What are the remarkable equalities between the two cases and what are the differences? Is it useful to use supersymmetry in the two cases? What are the advantages of using supersymmetry?

2. General concept of supersymmetric quantum mechanics

The basic concepts of supersymmetric quantum mechanics begins with the definition of a set of operators $Q_i \, (i = 1, ..., N)$ and P with the following (anti-)commutator relations:

$$[H, Q_i] = 0 \tag{1}$$
$$\{Q_i, Q_j\} = H\delta_{ij} \tag{2}$$
$$[H, P] = \{Q_i, P\} = 0 \tag{3}$$
$$P^2 = 1 \tag{4}$$

From (4) it follows that *P* has eigenvalue 1 and -1. So we have two subspaces: one of which has $P = 1$ and the other has $P = -1$. Because *H* commutes with *P* (3), the eigenstates of *H* ($|\psi\rangle$) are also eigenstates of *P*. So the Hilbert space of *H* will also be split into two subspaces and the eigenstate $|\psi\rangle$ will be divided in two parts:

$$|\psi\rangle = \begin{pmatrix} |\psi_{P=1}\rangle \\ |\psi_{P=-1}\rangle \end{pmatrix} \tag{5}$$

With the eigenstate of *P* and *H* written in this form, the operator *P* can be written in the following matrix-form:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (6)$$

This is the same form as the Pauli matrix (σ_3 or σ_z). The Pauli matrices obey the following anti-commutator relation:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad (7)$$

If we take in account the anti-commutator relation (3) and the anti-commutator relations of the Pauli matrices (7) Q_1 and Q_2 (for N = 2) can be put in the matrix form evolving σ_1 and σ_2:

$$Q_1 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & A_1^+ \\ -A_1 & 0 \end{pmatrix}, \qquad Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & A_2^+ \\ A_2 & 0 \end{pmatrix} \quad (8)$$

From (2) it follows that:

$$2Q_{i_i}^2 = H \quad (9)$$

Combining (8) and (9) we get an equation for the hamiltonian H:

$$H = \begin{pmatrix} A_1^+ A_1 & 0 \\ 0 & A_1 A_1^+ \end{pmatrix} = \begin{pmatrix} A_2^+ A_2 & 0 \\ 0 & A_2 A_2^+ \end{pmatrix} \quad (10)$$

Here A_1 and A_2 are later to be determined operators. So the hamiltonian will be split into two parts: One part (let's call it H_1) for the subspace where $P = 1$ and the other part (from now on H_2) for the subspace where $P = -1$.
Because Q_1 and Q_2 anti-commute (2) there is a restriction on A_1 and A_2 in the following sense:

$$A_1^+ A_2 = A_2^+ A_1 \quad (11)$$

This is for example satisfied when $A_1 = A_2 = A$. We can then put the Hamiltonian in the following (final) form:

$$H = \begin{pmatrix} A^+ A & 0 \\ 0 & AA^+ \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \quad (12)$$

3 The Schrödinger Hamiltonian

Let us start with the standard quantum mechanical hamiltonian

$$H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x) \quad (13)$$

Instead of defining the potential $V_1(x)$, we want to start defining an eigenfunction $\psi_0(x)$ (which is nodeless and perfectly vanishing for $x = \pm\infty$) such that it corresponds to a ground state energy E_0 which is equal to zero.
So the Schrödinger equation for this eigenfuction $\psi_0(x)$ becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V_1(x)\psi_0(x) = 0 \qquad (14)$$

It then follows directly that

$$V_1(x) = \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)} \qquad (15)$$

So when one knows the eigenfunction $\psi_0(x)$, one can easily determine the corresponding potential $V_1(x)$.
To apply supersymmetry to the hamiltonian in (13), we need to factorize that hamiltonian in a similar way done in (12).

$$H_1 = A^+ A \qquad (16)$$

It turns out that we can choose A and A^+ in the following matter

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \phi(x) \qquad\qquad A^+ = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \phi(x) \qquad (17)$$

where the following relation between the superpotential $\phi(x)$ and $V_1(x)$ must hold

$$V_1(x) = \phi^2(x) - \frac{\hbar}{\sqrt{2m}}\phi'(x) \qquad (18)$$

When we look at equation (12), we see that there is a second part of the hamiltonian H_2 which can be constructed out of A and A^+. By using the equations (17) we get

$$H_2 = AA^+ = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x) \qquad (19)$$

with

$$V_2 = \phi^2(x) + \frac{\hbar}{\sqrt{2m}}\phi'(x) \qquad (20)$$

We call $V_1(x)$ and $V_2(x)$ partner potentials.
With equation (15) and (18) or (20) we can get the superpotential $\phi(x)$ in terms of the ground state wave function $\psi_0(x)$

$$\phi(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi_0'(x)}{\psi_0(x)} \tag{21}$$

Because Q_i commutes with H (and because of that, A and A^+ also) the energy levels of H_1 and H_2 are degenerate (apart from the *E = 0* energy level which is defined on H_1 only).

shape-invariance

We call two potentials $V_1(x)$ and $V_2(x)$ shape-invariant when they only differ by a function of a collection of parameters *a*

$$V_2(x;a) = V_1(x; f(a)) + C(a) \tag{22}$$

This can be especially useful in supersymmetric quantum mechanics. When the two partner potentials $V_1(x)$ and $V_2(x)$ are shape-invariant, you can get the energylevels and wavefunctions of the two hamiltonians by a very simple iterative procedure. Let us look at the example of the Coulomb potential to illustrate this process.

3.1 The Coulomb field

For the case of a spherically symmetric potential in three dimensions ($V(\vec{x}) = V(r)$) the Schrödinger equation can be written as

$$-\frac{\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} + V(r)u(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} = Eu(r) \tag{23}$$

The third term on the left is the centrifugal force. ℓ is the quantum number for total angular momentum and $u(r)$ is $rR(r)$ where $R(r)$ is the radial part of the wavefunction $\psi(\vec{x})$. We will take for $V(r)$ the Coulomb potential:

$$V(r) = -\frac{W}{r} \qquad\qquad W = \frac{Ze^2}{4\pi\varepsilon_0} \tag{24}$$

The energy levels *E* are given by the following formula

$$E = -\frac{mW^2}{2\hbar^2 (l+1)^2} \tag{25}$$

To get (23) in the form of (14) we have to take $V_1(r, \ell+1)$ as follows

$$V_1(r;\ell+1) = \frac{\hbar^2 \ell(\ell+1)}{2mr^2} - \frac{W}{r} + \frac{mW^2}{2\hbar^2 (\ell+1)^2} \tag{26}$$

When solving the differential equation in (18) we can obtain the super potential $\phi(r)$

$$\phi(r) = -\frac{(\ell+1)\hbar}{r\sqrt{2m}} + \frac{W\sqrt{2m}}{2(\ell+1)\hbar} \tag{27}$$

By using (20) we can then get the second partner potential $V_2(r;\ell+1)$

$$V_2(r;\ell+1) = \frac{\hbar^2(\ell+1)(\ell+2)}{2mr^2} - \frac{W}{r} + \frac{mW^2}{2\hbar^2(\ell+1)^2} \tag{28}$$

It is clear that $V_2(r;\ell+1)$ and $V_1(r,\ell+2)$ only differ by a constant which is not dependent of *r*.
So clearly $V_2(r;\ell+1)$ and $V_1(r,\ell+2)$ are shape invariant. To take on the formalism of (22) we get for a, $f(a), C(a)$:

$$\begin{aligned} a &= \ell \\ f(a) &= \ell+1 \\ C(f(a)) &= -\frac{mW^2}{2\hbar^2}\left(\frac{1}{(\ell+2)^2} - \frac{1}{(\ell+1)^2}\right) \end{aligned} \tag{29}$$

Let us take the lowest possible value of ℓ (which is zero) and construct the hamiltonian H_1. Clearly its ground state energy ($E_0^{l=0}$) is zero. Now let us construct the Hamiltonian $H_2^{l=0}$ with the lowest possible value of ℓ:

$$-\frac{\hbar^2}{2m}\psi'' + V_2(r;1)\psi = E\psi \tag{30}$$

By using equation (22) and (29) we can make a related equation for $H_1^{l=1}$

$$-\frac{\hbar^2}{2m}\psi'' + V_1(r;2)\psi = (E - C(1))\psi \tag{31}$$

Clearly the lowest possible value for *E* is $C(1)$ in this equation. So this is also the case in equation (30) (because the *E* in equation (30) and (31) are the same). By supersymmetry the energylevels for $H_2^{l=0}$ and $H_1^{l=0}$ are the same. So we get

$$E_1^{l=0} = C(1) \tag{32}$$

Now let us follow the same procedure for $H_2^{l=1}$

$$-\frac{\hbar^2}{2m}\psi'' + V_2(r;2)\psi = E\psi \tag{33}$$

By the same way as in (31) we get

$$-\frac{\hbar^2}{2m}\psi'' + V_1(r;3)\psi = (E - C(2))\psi \tag{34}$$

So the lowest possible energylevel for E in (33) and (34) is $C(2)$. Because by supersymmetry the energylevels of $H_2^{l=1}$ and $H_1^{l=1}$ are the same, so $C(2)$ is also an energylevel of $H_1^{l=1}$. By equation (30) and (31) we can see that $C(1)+C(2)$ is an energylevel of $H_2^{l=0}$ and thus by supersymmetry of $H_1^{l=0}$. So we get for the third energylevel of $H_1^{l=0}$

$$E_2^{l=0} = C(1) + C(2) \tag{35}$$

It should be obvious now that the other energylevels can be constructed in the same way. We get the following energylevels

$$E_n = \sum_{k=1}^{n} C(k) \tag{36}$$

For the case of the Coulomb potential this is (using (29))

$$E_n = \sum_{k=1}^{n} -\frac{mW^2}{2\hbar^2}\left(\frac{1}{(k+1)^2} - \frac{1}{(k)^2}\right) = -\frac{mW^2}{2\hbar^2}\left(\frac{1}{(n+1)^2} + 1\right) \tag{37}$$

These are exactly the same energylevels as the energylevels for a Coulomb field obtained without using supersymmetry. Thus supersymmetry provides us with a much simpler procedure to calculate energylevels. Of course this is only true when the potential V is shape-invariant.

3.2 Uniform magnetic field and the Pauli equation

Let us look at another example in the case of a magnetic field. We can use the Pauli Hamiltonian for an electron in this case

$$H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2 + \frac{e\hbar}{2mc}\vec{B}\vec{\sigma} \tag{38}$$

To make a further simplification, we can get to the case where the motion of the electron is perpendicular to the magnetic field. Equation (38) then reduces to

$$H_P = \frac{1}{2m}\sum_{i=1}^{2}\left(-i\hbar\nabla_i + \frac{e}{c}A_i\right)^2 + \frac{e\hbar}{2mc}B_3\sigma_3 \tag{39}$$

Here we have taken the magnetic field in the third direction and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If we want to get a supersymmetric version of this equation we have to get it in the form of equation (12). We take for A and A^+ the following equations

$$A = \frac{1}{\sqrt{2m}}\left(-i\hbar\frac{d}{dx} + \frac{e}{c}A_x\right) + i\frac{1}{\sqrt{2m}}\left(-i\hbar\frac{d}{dy} + \frac{e}{c}A_y\right)$$
$$A^+ = \frac{1}{\sqrt{2m}}\left(-i\hbar\frac{d}{dx} + \frac{e}{c}A_x\right) - i\frac{1}{\sqrt{2m}}\left(-i\hbar\frac{d}{dy} + \frac{e}{c}A_y\right) \tag{40}$$

When combined with equation (12) this results in

$$H_1 = \frac{1}{2m}\left(-i\hbar\frac{d}{dx} + \frac{e}{c}A_x\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{d}{dy} + \frac{e}{c}A_y\right)^2 + \frac{e\hbar}{2mc}B_z$$
$$H_2 = \frac{1}{2m}\left(-i\hbar\frac{d}{dx} + \frac{e}{c}A_x\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{d}{dy} + \frac{e}{c}A_y\right)^2 - \frac{e\hbar}{2mc}B_z \tag{41}$$

This is (suprisingly) exactly the same result as in equation (39)! So this simplification of the Pauli hamiltonian is already supersymmetric but we didn't know it. H_1 and H_2 just corresponds to the two different spin-states of the electron just like the two Schrödinger Hamiltonians correspond to two different potentials.
Furthermore a special choice of $\vec{A}$ can be made. We can look at the following assymmetric gauge

$$A_x = W(y) \qquad\qquad A_y = 0 \tag{42}$$

In this case the Hamiltonian in equation (39) becomes

$$H_P = \frac{1}{2m}\left(-i\hbar\frac{d}{dx} + \frac{e}{c}W(y)\right)^2 + -\frac{1}{2m}i\hbar\frac{d^2}{dy^2} + \frac{e\hbar}{2mc}W'(y)\sigma_3 \tag{43}$$

This *H* does not depend on *x* so we can factorize the eigenfunctions $\tilde{\psi}$ of this Hamiltonian.

$$\tilde{\psi}(x,y) = e^{ikx}\psi(y) \tag{44}$$

Here *k* is an eigenvalue of the operator $-i\hbar\frac{d}{dx}$. The Schrödinger equation then becomes

$$\left[-\frac{1}{2m}i\hbar\frac{d^2}{dy^2} + \frac{1}{2m}\left(\hbar k + \frac{e}{c}W(y)\right)^2 + \frac{e\hbar}{2mc}W'(y)\sigma\right]\psi(y) = E\psi(y) \tag{45}$$

σ can take the values of *+1* and *-1* which are the eigenvalues of σ_3. Again the problem is reduced to supersymmetry with the following superpotential

$$\phi(y) = \frac{1}{\sqrt{2m}}\left(\hbar k + \frac{e}{c}W(y)\right) \tag{46}$$

The only restriction on $W(y)$ is that it cannot depend on *k*. This restriction is so strong that there are only four forms of $W(y)$ which give shape-invariant potentials.

$$\begin{aligned} W(y) &= \omega y + c_1 \\ W(y) &= a \tanh y + c_1 \\ W(y) &= a \tanh y + c_1 \\ W(y) &= c_1 + c_2 \exp(-y) \end{aligned} \tag{47}$$

So only these choices for the vector potential give shape-invariant partner-potentials. With these choices you can easily compute the energylevels as done in the example for the coulomb potential.

4. The Dirac equation

Now that we have defined supersymmetry in the Schrödinger equation, we can see how this looks in a relativistic equation, namely the Dirac formalism. We know that the standard Dirac equation has the following form

$$\left[i\gamma^{\mu}\left(\partial_{\mu} + iA_{\mu}\right) - m\right]\psi = 0 \tag{48}$$

Here we have chosen $e = \hbar = c = 1$ for simplicity. The matrices γ^{μ} must satisfy the standard anticommutator relation

$$\left\{\gamma^{\mu}, \gamma^{\nu}\right\} = \eta^{\mu\nu} \tag{49}$$

Here $\eta^{\mu\nu}$ is the flat Minkowski space metric. Just like in the Schrödinger case, let us look at a simpler case possible. So we don't have a Electromagnetic field, just a scalar potential. Equation (48) then reduces to

$$\left[i\gamma^{\mu}\partial_{\mu} - \phi(\vec{x}) - m\right]\psi = 0 \tag{50}$$

A useful choice for the matrices γ^{μ} satisfying equation (49) is given by

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma_i = \begin{pmatrix} i\sigma_i & 0 \\ 0 & -i\sigma_i \end{pmatrix} \tag{51}$$

The 1's in γ_0 are two-by-two identity matrices. The σ_i are standard Pauli matrices. We now write the wave equation ψ from (50) as usual as a product of a time-dependent part and a space dependent part

$$\psi = \exp\left(-iEx^0\right)\psi(\vec{x}) \tag{52}$$

With this ansatz, equation (50) reduces to

$$\left(\gamma^0 E + i\vec{\gamma}\vec{\nabla} - \phi(\vec{x}) - m\right)\psi(\vec{x}) = 0 \qquad (53)$$

When writing the four-component vector $\psi(\vec{x})$ as

$$\psi(\vec{x}) = \begin{bmatrix} \xi(\vec{x}) \\ \chi(\vec{x}) \end{bmatrix} \qquad (54)$$

We can write (47) as two separated coupled equations

$$\begin{aligned} \left(\vec{\sigma}\vec{\nabla} + \phi(\vec{x}) + m\right)\xi(\vec{x}) &= E\chi(\vec{x}) \\ \left(-\vec{\sigma}\vec{\nabla} + \phi(\vec{x}) + m\right)\chi(\vec{x}) &= E\xi(\vec{x}) \end{aligned} \qquad (55)$$

We already see that this looks like the operators defined in equation (17). So we define

$$\begin{aligned} A &= \vec{\sigma}\vec{\nabla} + \phi(\vec{x}) + m \\ A^+ &= -\vec{\sigma}\vec{\nabla} + \phi(\vec{x}) + m \end{aligned} \qquad (56)$$

Combining (49) and (50) we see the supersymmetric structure of equation (12)

$$\begin{aligned} A^+ A\xi(\vec{x}) &= E^2\xi(\vec{x}) \\ AA^+\chi(\vec{x}) &= E^2\chi(\vec{x}) \end{aligned} \qquad (57)$$

In this case the superpotential from equation (21) is $\phi(\vec{x}) + m$. A^+A and AA^+ are just the two Hamiltonians H_1 and H_2 while $\xi(\vec{x})$ and $\chi(\vec{x})$ are the two eigenstates corresponding to H_1 and H_2.
So we began with the Dirac equation, we ultimately get an equation that looks very similar to the Schrödinger supersymmetry. But while we created two linked Hamiltonians in the Schrödinger case, the Dirac equation provides us with those two Hamiltonians in its original form.

4.1 The Coulomb potential

We now look at the Dirac equation with a Coulomb potential. Again the start is with the Dirac equation and an electromagnetic field

$$\left[i\gamma^\mu\left(\partial_\mu + iA_\mu\right) - m\right]\psi = 0 \qquad (48)$$

where $\vec{A}(\vec{x}) = 0$ and $A_0(\vec{x}) = V(r)$. For the matrices γ^μ the standard representation are taken

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{58}$$

Equation (48) can be rewritten in the following form

$$i\frac{d\psi}{dt} = H\psi = \left[-i\gamma^0 \vec{\gamma}\vec{\nabla} + \gamma^0 m + V(r) \right]\psi \tag{59}$$

We only need the radial part of this equation because the hamiltonian *H* does only depend on *r*. When this is done, we get two coupled equations which I will not further derive

$$\begin{aligned} G'(r) + \frac{kG}{r} - (m + E - V(r))F = 0 \\ F'(r) - \frac{kF}{r} - (m - E + V(r))G = 0 \end{aligned} \tag{60}$$

The derivation of this can be found in reference [7]. Here F and G are the eigenfunctions after seperation of the angular part. The *k* in this equation represents a eigenvalue of the operator $-\left(\vec{\sigma}\vec{L} + 1\right)$ where $\vec{L}$ is the angular momentum operator. In general these coupled equations cannot be solved. However there is one exception which is the well known Coulomb potential. So we choose for *V*

$$V(r) = -\frac{W}{r} \qquad W = \frac{Ze^2}{4\pi\varepsilon_0} \tag{24}$$

This should be solvable with super symmetry and shape-invariance. Equation (60) can then be rewritten in matrix form using equation (24)

$$\begin{pmatrix} G'(r) \\ F'(r) \end{pmatrix} + \frac{1}{r}\begin{pmatrix} k & -W \\ W & -k \end{pmatrix}\begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} 0 & m+E \\ m-E & 0 \end{pmatrix}\begin{pmatrix} G \\ F \end{pmatrix} \tag{61}$$

Now the matrix $\begin{pmatrix} k & -W \\ W & -k \end{pmatrix}$ can be diagonalized be multiplying with a matrix *D* on the left and D^{-1} to the right. It turns out that D and D^{-1} must be

$$D = \begin{pmatrix} k+s & -W \\ -W & k+s \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} \frac{1}{2s} & \frac{W}{2s(s+k)} \\ \frac{W}{2s(s+k)} & \frac{1}{2s} \end{pmatrix} \tag{62}$$

Here $s = \sqrt{-W^2 + k^2}$.
By multiplying equation (61) from the left with *D*, the equation becomes

$$D\begin{pmatrix} G'(r) \\ F'(r) \end{pmatrix} + \frac{1}{r}\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} D\begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} \frac{EW}{s} & m + \frac{k}{s}E \\ m - \frac{k}{s}E & -\frac{EW}{s} \end{pmatrix} D\begin{pmatrix} G \\ F \end{pmatrix} \tag{63}$$

When the following useful choices are substituted

$$\rho = Er \qquad \begin{pmatrix} \tilde{G}(r) \\ \tilde{F}(r) \end{pmatrix} = D\begin{pmatrix} G(r) \\ F(r) \end{pmatrix} \tag{64}$$

we get the following two coupled equations

$$\begin{aligned} -\frac{d}{d\rho}\tilde{G}(r) - \frac{s}{\rho}\tilde{G}(r) + \frac{W}{s}\tilde{G}(r) &= -\left(\frac{m}{E} + \frac{k}{s}\right)\tilde{F}(r) \\ \frac{d}{d\rho}\tilde{F}(r) - \frac{s}{\rho}\tilde{F}(r) + \frac{W}{s}\tilde{F}(r) &= \left(\frac{m}{E} - \frac{k}{s}\right)\tilde{G}(r) \end{aligned} \tag{65}$$

Again a supersymmetric structure arises. By following the usual steps we did in the previous sections, equations for A and A^{+} can be found

$$\begin{aligned} A &= \frac{d}{d\rho} - \frac{s}{\rho} + \frac{W}{s} \\ A^{+} &= -\frac{d}{d\rho} - \frac{s}{\rho} + \frac{W}{s} \end{aligned} \tag{66}$$

By copying the structure of equation (12) we find the Hamiltonian

$$\begin{aligned} H_1\tilde{F} &= A^{+}A\tilde{F} = \left(\frac{k^2}{s^2} - \frac{m^2}{E^2}\right)\tilde{F} \\ H_2\tilde{G} &= AA^{+}\tilde{G} = \left(\frac{k^2}{s^2} - \frac{m^2}{E^2}\right)\tilde{G} \end{aligned} \tag{67}$$

So the superpotential in this problem is using (66)

$$\phi(\rho) = -\frac{s}{\rho} + \frac{W}{s} \tag{68}$$

Now we can easily construct the partner potential $V_1(\rho)$ and $V_2(\rho)$

$$\begin{aligned} V_1(\rho; s; W) &= \left(-\frac{s}{\rho} + \frac{W}{s}\right)^2 - \frac{s}{\rho^2} \\ V_2(\rho; s; W) &= \left(-\frac{s}{\rho} + \frac{W}{s}\right)^2 + \frac{s}{\rho^2} \end{aligned} \tag{69}$$

$V_1(\rho)$ and $V_2(\rho)$ are shape-invariant as can be seen

$$V_2(\rho;s;W)=V_1(\rho;s+1;W)+\frac{W^2}{s^2}-\frac{W^2}{(s+1)^2} \tag{70}$$

So the parameters of equation (29) are

$$\begin{aligned} a &= s \\ f(a) &= s+1 \\ C(s) &= \frac{W^2}{s^2}-\frac{W^2}{(s+1)^2} \end{aligned} \tag{71}$$

The energylevels can be constructed using equation (71) and equation (36)

$$\begin{aligned} &\frac{k^2}{s^2}-\frac{m^2}{E_n{}^2}=\tilde{E}_n=\sum_{k=0}^{n}C(k)=\frac{W^2}{s^2}-\frac{W^2}{(s+n)^2}\Rightarrow \\ &E_n=\frac{m}{\sqrt{1+\frac{W^2}{(s+n)^2}}} \end{aligned} \tag{72}$$

These are the Energy values for both H_1 and H_2 because the two hamiltonians are degenerate. We notice that de energylevels are not negative. This is because we shifted the energylevels so that H_1 has an extra groundstate level *E=0*.

4.2 Magnetic field

Again the problem begins with the Dirac equation as in equation (48)

$$\left[i\gamma^{\mu}\left(\partial_{\mu}+iA_{\mu}\right)-m\right]\psi=0 \tag{48}$$

Here the standard representation of the matrices γ^{μ} is used. It is convenient to make a simplification here. We begin with defining an other eigenfunction χ which is related to ψ in the following manner

$$\psi=\left[i\gamma^{\mu}\left(\partial_{\mu}+iA_{\mu}\right)+m\right]\chi \tag{73}$$

By combining equation (48) and (73) we get an second order differential equation for χ

$$\left[\left(i\partial_{\mu}-A_{\mu}\right)\left(i\partial_{\mu}-A_{\mu}\right)-\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\right]\chi=m^2\chi \tag{74}$$

here $\sigma_{\mu\nu}=\frac{1}{2}i\left(\gamma_{\mu}\gamma_{\nu}-\gamma_{\nu}\gamma_{\mu}\right)$. Let us define another 4x4 matrix γ_5

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_0=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \tag{75}$$

Because γ_5 commutes with $\sigma_{\mu\nu}$, $i\gamma_5\chi$ is also an eigenfunction of equation (74). Therefore there are solutions of (74) for which the following holds

$$i\gamma_5\chi = \chi \qquad (76)$$

If we write the 4-component vector χ as two 2-component vectors

$$\chi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad (77)$$

it can be see that with equation (76) this φ_1 and φ_2 are correlated

$$\varphi_1 = -\varphi_2 = \varphi \qquad (78)$$

With equation (74) for χ this results in the following two-component equation

$$\left[\left(i\partial_\mu - A_\mu\right)^2 - \vec{\sigma}\left(\vec{B} + i\vec{E}\right)\right]\varphi = m^2\varphi \qquad (79)$$

which can eventually be rewritten in de following form

$$\left[\left(-i\vec{\nabla} + \vec{A}\right)^2 + m^2 + \vec{\sigma}\left(\vec{B} + i\vec{E}\right)\right]\varphi = \left(E - A_0\right)^2\varphi \qquad (80)$$

To go back to the Dirac equation we simply use equation (73) which can be put in a more confienient form for φ

$$\psi = \begin{pmatrix} \left(\vec{\sigma}\left(-i\vec{\nabla} + \vec{A}\right) + E - A_0 + m\right)\varphi \\ \left(\vec{\sigma}\left(-i\vec{\nabla} + \vec{A}\right) + E - A_0 - m\right)\varphi \end{pmatrix} \qquad (81)$$

Here E corresponds to the energylevels of φ. So ultimately this equation is much simpler to solve than the original Dirac equation. Because we are looking at a magnetic field, another simplification can be made. We can put A_0 (and thus E) to zero which results in

$$\left[\left(-i\vec{\nabla} + \vec{A}\right)^2 + m^2 + \vec{\sigma}\vec{B}\right]\varphi = E^2\varphi \qquad (82)$$

As can be seen this looks very much like the Pauli equation which is earlier discussed. When the mass is put to zero it really is the Pauli equation. Now let us choose zero mass and the same approximation as in the Pauli case (motion in a two dimensional plane with a magnetic field perpendicular on it).
Since

$$\left(H_{Dirac}\right)^2 = \left[\gamma^0\vec{\gamma}\left(-i\vec{\nabla} + \vec{A}\right)\right]^2 = H_{Pauli} \qquad (83)$$

when the mass is zero, there is supersymmetry in the massles Dirac problem. We then can immediately copy the results of the super symmetry in the Pauli equation. Through equation (81) we then get the wavefunction of the Dirac equation back. We can thus immediately write down the energylevels of the massles Dirac equation with the four asymmetric choices of $\vec{A}$ discussed in the section about the Pauli equation. The solutions of the massive Dirac equation are then easily obtained by shifting the energylevels with a factor m^2. Equation(82) then looks like

$$\left[\left(-i\vec{\nabla} + \vec{A}\right)^2 + \vec{\sigma}\vec{B}\right]\varphi = \left(E^2 + m^2\right)\varphi \qquad (84)$$

And thus the energylevels of the Pauli equation are related with the energy levels of the simplified Dirac equation by

$$E_{Pauli} = E_{dirac}^2 + m^2 \qquad (85)$$

We can now conclude that in a pure magnetic field the relativistic equation (Dirac) and the non-relativistic equation are very closely related.

5. Conclusions:

5.1 Parallels between the Schrodinger and Dirac equation in supersymmetry

If we compare section 3 and section 4 we see some remarkable coincidences. Both the Schrödinger Hamiltonian as well as the Dirac Hamiltonian with a Lorentz scalar field can be factorized. So in both equations the problem can be solved through shape-invariant partner potentials. This is remarkable because the Schrödinger and the Dirac equation are totally different. In the Schrödinger equation for example exists a second order derivative while there is a first order derivative in the Dirac equation. How can these two different problems be solved with the same concept?
An explanation lies in the differences between the two problems. We need to construct two Schrödinger Hamiltonians in order to get two shape-invariant partner potentials. This is in contrast with the Dirac hamitonian. We only have one Dirac Hamiltonian with one Lorentz scalar potential. In fact it turns out that the Lorentz scalar potential is the superpotential $\phi(x)$ in the Schrödinger equation. The factorization of the Hamiltonian which is commom in susy is already in the Dirac equation. The matrix form of the Dirac equation provides two related equation which are exactly the two factorization equations.
We have to actively factorize the Schrödinger Hamiltonian in order to get a supersymmetry structure. The Schrödinger equation in itself is not factorized while the Dirac equation is.
While both problems can be solved using the same concepts of supersymmtry, they both need a very different approach to the same ideas.

5.1.1 Parallels between both equations and a Coulomb field

In the coulomb field supersymmetry becomes concrete. The equations can both actively be solved using the concept of shape-invariance. In the end the energy spectrum of the Coulomb field for both equations can be found. This is however where the parallels stop. The same differences discussed earlier in section 5.1 can be

applied here, however in this case it is not so clear. The solving of the Dirac problem is troubled by a lot of rewritings which is confusing.
Equation (65) shows us however that the factorization is already embedded in the problem and we need not to actively pull it out of the hamiltonian.
So the general idea of section 5.1 remains standing.

5.1.2. Parallels between both equations and a magnetic field

In section 3.2 the Pauli Hamiltonian is used, which is slightly different from the Schrödinger hamiltonian. This can be seen immediately when the Hamiltonian is factorized. The Hamiltonian needs to be factorized like the Schrödinger equation but the creation of two partner Hamiltonians isn't needed. Both Hamiltonians are already in the Pauli-Hamiltonian. So the Pauli problem already looks like the Dirac problem. It also has some properties of the supersymmetry in the Schrödinger equation.
When looking at section 4.2 we can see that the Dirac equation and Pauli equation are really closely related. In fact the Dirac equation with a magnetic field can be written such that it almost looks like the Pauli problem.
We may note here that some very bold simplifications have been made when discussing the magnetic field. The similarities between supersymmetry in the Pauli and the Dirac equation may not be in general true.

5.2 The practicle use of Supersymmetry

The first practicle use we see in supersymmetry is shape-invariance. If two potentials are shape-invariant by the definition of equation (22) and they can both be connected through a superpotential, the energylevels of those potentials can be very easily constructed. So this is a great advantage of supersymmetry over the common used mathematics in quantum mechanics.
Another great use of supersymmetry is that it provides us with a general concept for different equations. So we don't need to make a computational method for each of equation, but we can instead use supersymmetry on all equations.

6. Afterword

So what goals mentioned in the introduction are accomplished? The first goal, namely extending the supersymmetry to a relativistic equivalence, has been accomplished. I have put a analogy to the Dirac equation with succes.
The second goal however has not been accomplished. I have tried extensively to put the general electromagnetic field in the equations of supersymmetry but I didn't got it. In stead, I have made a few a examples of an electric and a magnetic field which however were already known. Not quite as satisfactory.
The mathematics behind this thesis have mostly been done by others. I made some adaptations in the mathematics which I think are easier to understand and fit better in the goals of this thesis but the general concepts are not new.
I have done the comparison however by myself and I have tried to emphasize the extension to a relativistic equation more clearly.

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