university of groningen
faculty of mathematics and natural sciences

The Sommerfeld Enhancement

S.C.A. Nierop
nieropsam@gmail.com
s1619888

Supervisor: Prof. Dr. M. de Roo

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Abstract

The Sommerfeld enhancement is an elementary effect in nonrelativistic quantum mechanics, which accounts for the effect of a potential on the interaction cross section. First a general formula for the Sommerfeld enhancement is deduced. Next this general formula is illustrated by computing the Sommerfeld enhancement in two well-known cases, the rectangular potential well/barrier and the electromagnetic potential. Thereafter, we compute the Sommerfeld enhancement for the electromagnetic potential for finite interaction regions (instead of pointlike), using a Taylor expansion. It turns out that the correction of this finite interaction region is negligibly small in most cases. To conclude, a simple program is written in Mathematica to compute the Sommerfeld enhancement for the Yukawa potential. The results of this program are found to be consistent with other articles in recent literature.

Contents

1 The Sommerfeld enhancement

1.1 Introduction

The Sommerfeld enhancement is an elementary effect in nonrelativistic quantum mechanics. To give a basic understanding of the Sommerfeld enhancement, let us take the example of an electron-positron annihilation: a positron and an electron collide and produce two photons. We place the positron in the origin and the electron flying towards the positron along some axis, while at this stage we do not take the interaction between the positron and the electron into account. Now there is some quantum mechanical chance that the electron annihilates with the positron or, to picture it classically, there is a chance that the positron annihilates with the electron or that it flies by it. In physics one way to quantify this likelihood of a process happening is the cross section. In this example we therefore have some annihilation cross section for the annihilation process without interaction. Now we proceed to add the interaction in the picture. In this example this is an attractive electromagnetic (EM) potential. Therefore the incident electron is attracted by the positron, thereby enhancing the annihilation cross section. This enhancement is what is called the Sommerfeld enhancement.

In the above example we chose some interaction (annihilation) and potential (EM), but in general we can compute a Sommerfeld enhancement for any interaction and any potential. To summarize: the Sommerfeld enhancement accounts for the effect of a potential on the interaction cross section.

The Sommerfeld enhancement is named after Sommerfeld, who proposed it in 1931 [1]. Recent studies have shown that the Sommerfeld enhancement could be of importance in dark-matter annihilation (see for instance [2]).

1.2 A general formula

We will begin by deducing a general formula for the Sommerfeld enhancement factor using nonrelativistic quantum mechanics [2]. To do so, we make the following assumptions:

1. The incident particle is (essentially) a non-relativistic free particle (when there is no added potential). This means that without the added potential we can describe the incident particle by the following wavefunction:

$$\psi_k^{(0)}(\vec{x}) = e^{ikz} \tag{1.1}$$

Without loss of generality we have taken the z-axis as the axis along which the particle advances. $\psi_k^{(0)}$ denotes the wavefunction of the incident particle without the added potential.

2. The interaction is pointlike and takes place in the origin. This assumption is reasonable for most interacting elementary particles. However in this thesis we will compute the Sommerfeld enhancement for non-pointlike interactions in an electromagnetic potential.

Assumption 2 leads to the following: the interaction cross section is proportional to the squared wavefunction in the origin. In quantum-mechanics, the squared wavefunction at some place can be interpreted as the chance that a particle is at that particular place. Because the interaction takes places only in the origin, we know that the chance for being at the origin has to be proportional to the chance of the interaction happening (i.e. the interaction cross section).

3. The added potential is a central potential (a potential which magnitude only depends on the distance from the origin). Using textbook quantum mechanics, we know that scattering of a central potential can only produce outgoing spherical waves of the form [3]:

$$\psi_k \rightarrow e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \text{ as } r \rightarrow \infty \tag{1.2}$$

where ψ_k denotes the wavefunction of the incident particle with the added potential.

With the above assumptions we can deduce the general formula for the Sommerfeld enhancement. Note that we want to find the difference of the cross section with and without potential, which in general is a multiplicative factor. This factor is what is called the Sommerfeld enhancement factor S_k.

$$\sigma = \sigma_0 S_k \tag{1.3}$$

where

$$S_k = \frac{|\psi_k(0)|^2}{\left|\psi_k^{(0)}(0)\right|^2} = |\psi_k(0)|^2 \tag{1.4}$$

Thus in order to find the Sommerfeld enhancement we need to find the wavefunction $\psi_k(0)$, which in non-relativistic quantum mechanics basically means that we need to solve the Schrödinger equation.

In general, the axially symmetric (about the z-axis) solutions of the Schrödinger equation for wavefunctions of the type in equation (1.2) are of the form

$$\psi_{kl} = \sum_{l=0}^{\infty} A_l P_l(cos(\theta)) R_{kl}(r) \tag{1.5}$$

where A_l is some to be determined parameter, $P_l(cos(\theta))$ denote the associated Legendre functions and $R_{kl}(r)$ is the radial part of the wavefunction.

Because we assumed a central potential, the (angle dependent) parameter A_l will be independent of the choice for the potential and therefore can be written down immediately following standard non-relativistic scattering theory ([4], p.470):

$$\psi_{kl} = \sum_{l=0}^{\infty} \frac{i^l e^{i\delta_l}(2l+1)}{k} P_l(cos(\theta)) R_{kl}(r) \tag{1.6}$$

Following equation (1.6) the remaining work to compute the Sommerfeld enhancement is to find the radial part of the wavefunction $R_{kl}(r)$ by solving the radial Schrödinger equation. This will be done for the rectangular potential well/barrier and the electromagnetic potential in chapters 2 and 3.

Furthermore, we note that the above analysis will be approximately valid for an interaction which is not pointlike, but small and finite i.e. with a radius r_0 where $0 < r_0 << 1$. Equation 1.4 then takes the following form:

$$S_k = \frac{\int_0^{r_0} |\psi_k(r)|^2 \, dr}{\int_0^{r_0} \left|\psi_k^{(0)}(r)\right|^2 \, dr} \tag{1.7}$$

2 The rectangular potential well/barrier

As a first illustration of the computation of the Sommerfeld enhancement, let us review the relatively simple case of the potential well/barrier [2, 5]:

$$V = \begin{cases} \pm V_0 & \text{for } 0 \leq r < a, \\ 0 & \text{for } r > a. \end{cases} \tag{2.1}$$

with the plus sign corresponding to a potential barrier and the minus sign corresponding to a potential well. For simplicity it is assumed that $l = 0$.

To compute the Sommerfeld enhancement, we need to find ψ_k. The first step in the computation of the Sommerfeld enhancement is therefore to write down the Schrödinger equation. In this $l = 0$ case, following equation (1.6) we are primarily concerned with the radial part:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) = \begin{cases} -k_{in}^2 R(r) & \text{for } 0 \leq r < a \\ -k^2 R(r) & \text{for } r > a \end{cases} \tag{2.2}$$

where $k_{in}^2 = \frac{2M}{\hbar^2}(E \mp V_0)$, $k^2 = \frac{2M}{\hbar^2}E$ and the plus sign corresponding to the potential well and the minus sign to the potential barrier.

If we make the substitution $\chi = rR(r)$ then equation (2.2) reduces to (with the subscriptions in and out denoting the inner and outer solutions):

$$\begin{cases} \frac{d^2\chi_{in}}{dr^2}(r) - k_{in}^2\chi_{in}(r) = 0 & \text{for } 0 \leq r < a \\ \frac{d^2\chi_{out}}{dr^2}(r) - k^2\chi_{out}(r) = 0 & \text{for } r > a \end{cases} \tag{2.3}$$

Equation (2.3) has the general solution:

$$\begin{cases} \chi_{in}(r) = A\sin(k_{in}r) & \text{if } 0 \leq r < a \\ \chi_{out}(r) = B\sin(kr) & \text{if } r > a \end{cases} \tag{2.4}$$

We found the general solution and now we need to find the constants A and B. First, we need to normalize χ_{out} at infinity, which we choose to do with the condition $B = 1$. Second, we have to match the boundary of the two solutions, i.e. $\chi_{in}(a) = \chi_{out}(a)$ and $\frac{d\chi_{in}}{dr}(a) = \frac{d\chi_{out}}{dr}(a)$, resulting in two equations.

$$A\sin(k_{in}a) = \sin(ka)$$

$$Ak_{in}\cos(k_{in}a) = k\cos(ka) \tag{2.5}$$

Dividing both sides of the second equation by k and then squaring and adding both equations:

$$A^2(\sin^2(k_{in}a) + \frac{k_{in}^2}{k^2}\cos^2(k_{in}a)) = 1 \tag{2.6}$$

determines that

$$A = \pm\frac{1}{\sqrt{\sin^2(k_{in}a) + \frac{k_{in}^2}{k^2}\cos^2(k_{in}a)}} \tag{2.7}$$

So we obtained the radial part of the wavefunction:

$$R_{kl,in}(r) = \pm\frac{1}{\sqrt{\sin^2(k_{in}a) + \frac{k_{in}^2}{k^2}\cos^2(k_{in}a)}}\frac{\sin(k_{in}r)}{r} \tag{2.8}$$

We insert equation (2.8) in equation (1.6) (with $l = 0$) to obtain the wavefunction for $0 \le r < a$:

$$\psi_{kl,in}(r) = \pm\frac{e^{i\delta_0}}{k}\frac{1}{\sqrt{\sin^2(k_{in}a) + \frac{k_{in}^2}{k^2}\cos^2(k_{in}a)}}\frac{\sin(k_{in}r)}{r} \tag{2.9}$$

And to obtain the Sommerfeld enhancemnent, we put equation (2.9) in equation (1.4) . Note that $\frac{\sin(k_{in}r)}{k_{in}r} \to 1$ for $r \to 0$, so we multiply by $\frac{k_{in}}{k_{in}}$ to obtain

$$\begin{aligned} S_k &= \left| \pm\frac{e^{i\delta_0}k_{in}}{k}\frac{1}{\sqrt{\sin^2(k_{in}a) + \frac{k_{in}^2}{k^2}\cos^2(k_{in}a)}}\frac{\sin(k_{in}\cdot 0)}{k_{in}\cdot 0} \right|^2 \\ &= \frac{1}{\frac{k^2}{k_{in}^2}\sin^2(k_{in}a) + \cos^2(k_{in}a)} \end{aligned} \tag{2.10}$$

Equation (2.10) is the final result. Let us check some limits to see how the Sommerfeld enhancement works.

First let us consider when $V_0 \to 0$. In this case, there obviously should be no enhancement. When $V_0 = 0$, then $k_{in} = \frac{2M}{\hbar^2}E = k$ and if we put $k_{in} = k$ in equation (2.10), we indeed get $S_k = 1$. Furthermore we note that when $V_0 > 0$ and $\sin^2(k_{in}a) \neq 0$, then for the potential well $\frac{k^2}{k_{in}^2} < 1$ and $S_k > 1$, while for the potential barrier $\frac{k^2}{k_{in}^2} > 1$ and $S_k < 1$.

Second let us review the case when $k_{in}a = \frac{1}{2}\pi + n\pi$ with $n = 0, 1, 2.....$ In this case $\cos^2(k_{in}a) = 0$ and $\sin^2(k_{in}a) = 1$, resulting in a Sommerfeld enhancement

$$S_k = \frac{k_{in}^2}{k^2} = 1 \mp \frac{V_0}{E} \tag{2.11}$$

Third let us review the case when $k_{in}a = 0 + n\pi$ with $n = 0, 1, 2.....$ In this case $\cos^2(k_{in}a) = 1$ and $\sin^2(k_{in}a) = 0$, resulting in a Sommerfeld enhancement $S_k = 1$.

Let us illustrate the above two cases by a figure. Let us take $\frac{k^2}{k_{in}^2} = \frac{1}{4}$ for the potential well and $\frac{k^2}{k_{in}^2} = 4$ for the potential barrier. If we now take $k_{in}a = x$, we get figure 1. The resonant pattern following the above analysis is clearly visible. The nodes are at $k_{in}a = \frac{1}{2}\pi + n\pi$ and $k_{in}a = 0 + n\pi$ with $n = 0, 1, 2.....$, where we expected them.

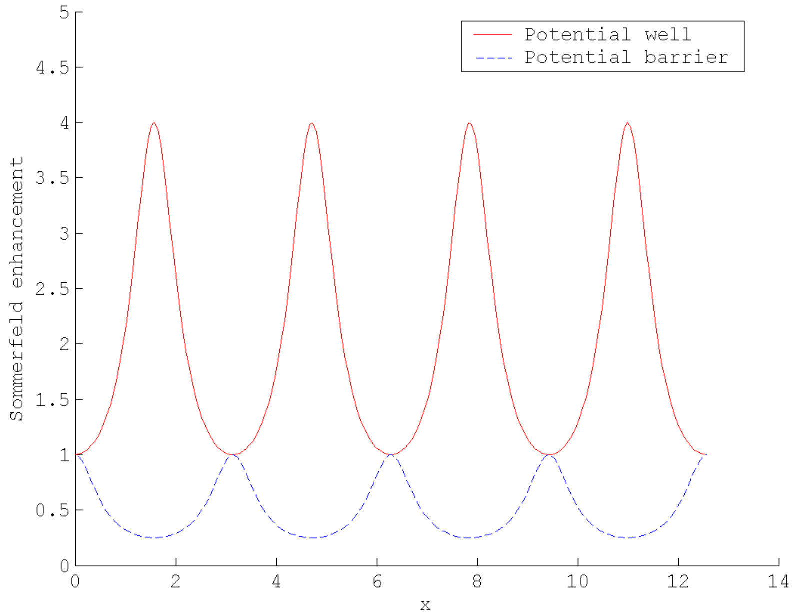

Figure 1: Sommerfeld enhancement for the rectangular potential well/barrier

3 The Coulomb Potential with $r_0 = 0$

Let us begin to compute the Sommerfeld enhancement for the Coulomb potential $V(r) = \pm\frac{\alpha}{r}$ with $r_0 = 0$ (i.e. using equations (1.4) and (1.6)). The plus sign refers to the repulsive case and the minus sign to the attractive case.

We assumed in section 1 that $\psi_k^{(0)}(\vec{x}) = e^{ikz}$. This means that we are only concerned with the solutions where $E > 0$, because we cannot compare bounded (i.e. $E < 0$) solutions of ψ_k with the free-particle case. Solutions where $E < 0$ only occur in the attractive case ($V(r) = -\frac{\alpha}{r}$.)

The equation for the radial part of the Schrödinger equation obtains the following form ([4], p. 102):

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - \frac{l(l+1)}{r^2}R(r) + \frac{2M}{\hbar^2}(E - \pm\frac{\alpha}{r})R(r) = 0 \tag{3.1}$$

where M is the mass of the incident particle.

There exists an analytical solution to the above equation. To obtain this solution, let us introduce some new parameters in atomic units in which to redefine the above equation in a more convenient form:

$$\rho = kr \tag{3.2}$$

$$\eta = \frac{M\alpha}{k} = \frac{\alpha}{v} \tag{3.3}$$

$$\chi(r) = R(r)r \tag{3.4}$$

Expressed in the above parameters, equation (3.1) takes the following form:

$$\frac{d^2\chi(\rho)}{d\rho^2} + \left(1 \mp \frac{2\eta}{\rho} - \frac{l(l+1)}{\rho^2}\right)\chi(\rho) = 0 \tag{3.5}$$

where the minus sign refers to the repulsive case and the plus sign to the attractive case.

The solutions for equation (3.5) are in terms of confluent hypergeometric functions ([6], p.537 et seq). The general mathematical solution is

$$\chi(\rho) = C_1 F + C_2 G \tag{3.6}$$

where F is the regular Coulomb Wave function and G is the irregular Coulomb wavefunction. The irregular Coulomb wavefunction is eliminated in the solution based on physical arguments: when $\rho = 0$ and $l > 0$, G is ∞ and it's derivate is $-\infty$, which cannot be a physical solution. The solution then turns out to be [6]

$$\chi(\rho) = \sqrt{\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta} - 1)}} \frac{(2\rho)^l \rho}{(2l+1)!} e^{i\rho} M(l+1 \pm i\eta, 2l+2, -2i\rho) \prod_{s=1}^{l} \sqrt{s^2 + \eta^2} \tag{3.7}$$

where $M(\alpha, \gamma, z)$ is a confluent hypergeometric function, defined by

$$M(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + ... \tag{3.8}$$

.

The radial wavefunction is then simply obtained by dividing by r.

$$R_{kl}(\rho) = \sqrt{\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta} - 1)}} \frac{(2\rho)^l k}{(2l+1)!} e^{i\rho} M(l+1 \pm i\eta, 2l+2, -2i\rho) \prod_{s=1}^{l} \sqrt{s^2 + \eta^2} \tag{3.9}$$

We have obtained the radial part of the wavefunction and we can compute the Sommerfeld enhancement for $r = 0$ by first substituting equation (1.6) in (1.4):

$$S_k = |\psi_k(0)|^2 = \left| \sum_{l=0}^{\infty} \frac{i^l e^{i\delta_l}(2l+1)}{k} P_l(cos(\theta)) R_{kl}(0) \right|^2 \tag{3.10}$$

The computation is facilitated by the observation that $R_{kl}(0) = 0$ for all $l \neq 0$. Therefore we can put $l = 0$ in equation (3.10) (and equation (3.9)) leading to the following result:

$$S_k = \left| \frac{1}{k} \sqrt{\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta} - 1)}} k \right|^2 = \frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta} - 1)} \tag{3.11}$$

In the next two subsections we will evaluate this solution for the repulsive and the attractive case.

3.1 The repulsive case

First, let us consider the repulsive case $V(r) = \frac{\alpha}{r}$.

Figure 2 plots the speed in atomic units (in SI units $v = \frac{\alpha c}{\eta}$ with c the speed of light) versus the Sommerfeld enhancement with $\alpha = \frac{1}{137}$

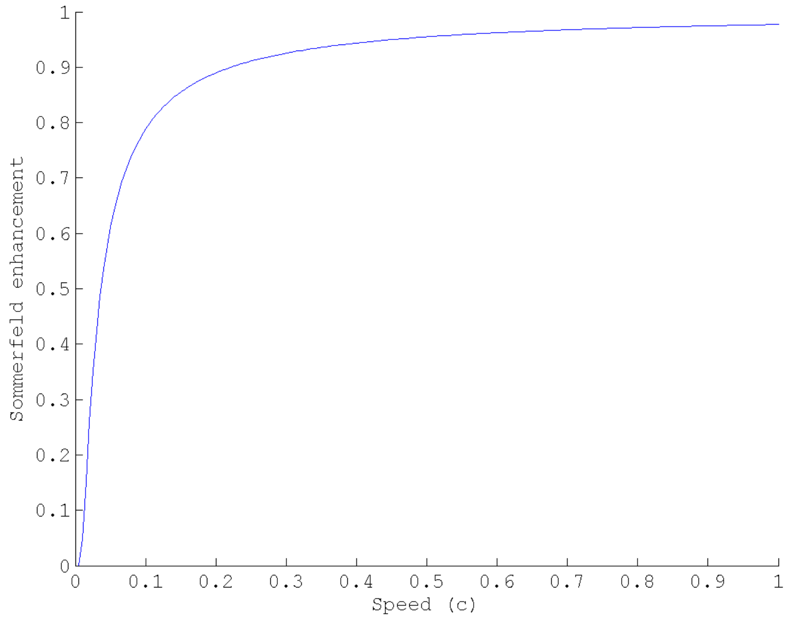

Figure 2: Sommerfeld enhancement for the repulsive case

In the limit $v \to \infty$, $S_k \to 1$. This makes sense since a very high speed of one of the particles causes a very small interaction time with the potential. Because this is a non-relativistic treatment, $S_k \neq 1$ as $v \to c$. Naturally we expect that $S_k \to 1$ as $v \to c$ for a relativistic treatment. We note however that $S_k \approx 1$ as $v \to c$, thus our treatment can be a good approximation.

In the limit $\alpha >> v$, $S_k \to 0$; because this is the repulsive case, the Sommerfeld enhancement must lead to a suppression of the interaction cross section.

3.2 The attractive case

Second, let us consider the attractive case $V(r) = -\frac{\alpha}{r}$.

Figure 3 plots the speed (in SI units, $v = \frac{\alpha c}{\eta}$ with c the speed of light) versus the Sommerfeld enhancement.

In the limit v $\to \infty, S_k \to 1$, for the same reasoning as in the repulsive case.

In the limit $\alpha >> v, S_k \to \frac{\pi\alpha}{v}$. This is an frequently cited result, which tells us that for small velocities the Sommerfeld enhancement is inversely proportional to the velocity for EM-potentials. Furthermore it is linearly dependent on α, which makes sense because a bigger coupling constant results in a stronger potential.

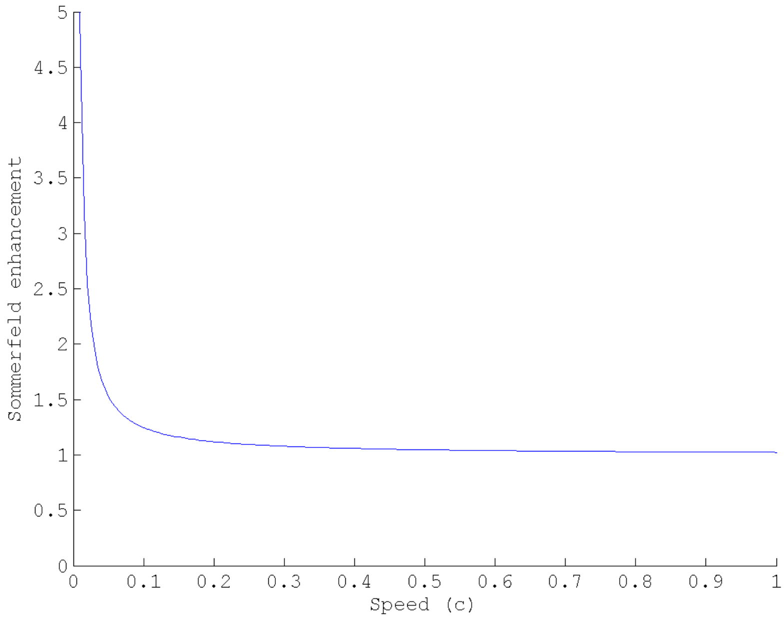

Figure 3: Sommerfeld enhancement for the attractive case

4 The Coulomb potential with $0 < r_0 << 1$

As was noted before, our analysis remains approximately valid for an interaction radius r_0 with $0 < r_0 << 1$. This corresponds to the physical situation that an interaction is not pointlike, but has some finite interaction region with radius r_0. It is worth researching what kind of correction this gives to the Sommerfeld enhancement.

The first part of the analysis remains the same as in the previous section: we obtain the same radial wavefunction (equation (3.9)). Now we need to obtain the wavefunction for small r. In the remainder it proves more convenient to work with ρ, where $0 < \rho << 1$. This substitution is valid for finite k.

We will use a Taylor series in ρ to reformulate equation (3.9):

$$R_{kl}(\rho) = \sum_{n=0}^{\infty} \left[\frac{d^n R(\rho)}{d\rho}\right]_{\rho=0} \frac{\rho^n}{n!} \tag{4.1}$$

To compute $\frac{d^n R(0)}{d\rho}$, we first compute all the separate derivatives:

$$\left[\frac{d^n M(l+1+\pm i\eta, 2l+2, -2i\rho)}{d\rho}\right]_{\rho=0} = \frac{i}{2}\prod_{s=1}^{n} \frac{l+1 \pm i\eta + n - 1}{2l+2+n-1} \tag{4.2}$$

(If $n = 0, \prod_{s=1}^{n} \frac{l+1\pm i\eta+n-1}{2l+2+n-1} = 1$)

$$\left[\frac{d^n (2\rho)^l}{d\rho}\right] = 2\frac{l!}{(l-n)!}(2\rho)^{l-n} \tag{4.3}$$

Equation (4.3) has the interesting feature that $\left[\frac{d^n e^{i\rho}(2\rho)^l}{d\rho}\right]_{\rho=0} = 0$, when $n \neq l$.

$$\left[\frac{d^n e^{i\rho}}{d\rho}\right]_{\rho=0} = i^n \tag{4.4}$$

We have a product of functions, so we have to use Leibniz rule (the general product rule):

$$(fg)^{(n)} = \sum_{q=0}^{n} \binom{n}{q} f^{(q)} g^{(n-q)} \tag{4.5}$$

Now we have (using here $x^{(n)}$ as a short notation for the n-th derivate of x)

$$(((2\rho)^l)(e^{i\rho}M(l+1+\pm i\eta, 2l+2, -2i\rho)))^{(n)} =$$
$$\sum_{q=0}^{n} \binom{n}{q} ((2\rho)^l)^{(q)} (e^{i\rho}M(l+1+\pm i\eta, 2l+2, -2i\rho))^{(n-q)} \tag{4.6}$$

where

$$(e^{i\rho}M(l+1+\pm i\eta, 2l+2, -2i\rho))^{(n-q)} =$$
$$\sum_{w=0}^{n-q} \binom{n-q}{w} (M(l+1+\pm i\eta, 2l+2, -2i\rho))^{(w)} (e^{i\rho})^{(n-q-w)} \tag{4.7}$$

Following the analysis of equation (4.3), we know that if $\rho = 0$, then $l = q$, so that

$$\left[\sum_{q=0}^{n} ((2\rho)^l)^{(q)}\right]_{\rho=0} = 2(l!) \tag{4.8}$$

Now we can evaluate equation (4.6) for $\rho = 0$ (and $l = q$ set):

$$\left[(((2\rho)^l)(e^{i\rho}M(l+1+\pm i\eta, 2l+2, -2i\rho)))^{(n)}\right]_{\rho=0} =$$
$$2(l!)\binom{n}{l} \sum_{w=0}^{n-l} \binom{n-l}{w} i^{n-l-w} \frac{i}{2} \prod_{s=1}^{w} \frac{l \pm i\eta + w}{2l+1+w} \tag{4.9}$$

With the above equations

$$\left[\frac{d^n R(\rho)}{d\rho}\right]_{\rho=0} = \sqrt{\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta} - 1)}} \frac{k(l!)}{(2l+1)!} \binom{n}{l} \left[\prod_{s=1}^{l} \sqrt{s^2+\eta^2}\right] \cdot$$
$$\sum_{w=0}^{n-l} \binom{n-l}{w} i^{n-l-w+1} \prod_{s=1}^{w} \frac{l \pm i\eta + w}{2l+1+w} \tag{4.10}$$

So using equation (4.1) our final result is

$$R_{kl}(\rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \sqrt{\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta} - 1)}} \frac{k(l!)}{(2l+1)!} \binom{n}{l} \left[\prod_{s=1}^{l} \sqrt{s^2+\eta^2}\right] \cdot$$
$$\sum_{w=0}^{n-l} \binom{n-l}{w} i^{n-l-w+1} \prod_{s=1}^{w} \frac{l \pm i\eta + w}{2l+1+w} \tag{4.11}$$

Inserting this result in equation (1.6), we obtain the desired wavefunction:

$$\psi_{kl}(\rho) = i\sqrt{\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta}-1)}}\sum_{l=0}^{\infty}\frac{i^l e^{i\delta_l} l!}{(2l)!}P_l(cos(\theta))\left[\prod_{s=1}^{l}\sqrt{s^2+\eta^2}\right]\cdot$$

$$\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\binom{n}{l}\sum_{w=0}^{n-l}\binom{n-l}{w}i^{n-l-w}\prod_{s=1}^{w}\frac{l\pm i\eta+w}{2l+1+w} \tag{4.12}$$

Obviously, expression (4.12) is not easily evaluated. The point however of the Taylor expansion is clear: we can limit the summation over n to $n=1$ or $n=2$, so that we only evaluate the linear and quadratic terms in ρ. This will give a good approximation since $\rho << 1$. Since there is a term $\binom{n}{l}$ in equation (4.12), only terms with $l \leq n$ are nonzero:

l = 0 Let us take for $l=0$, $n=0\rightarrow 1$:

$$\psi_{kl}(\rho) = i\sqrt{\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta}-1)}}e^{i\delta_0}(1+\rho(i+1\pm i\eta)) \tag{4.13}$$

l = 1 Let us take for $l=1$, $n=1\rightarrow 2$:

$$\psi_{kl}(\rho) = -\sqrt{\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta}-1)}}e^{i\delta_1}\sqrt{1+\eta^2}\frac{cos\theta}{2}(\rho+\rho^2(i(1\pm\frac{\eta}{4})+\frac{1}{2})) \tag{4.14}$$

To compute equation (1.7), we furthermore need $\psi_k^{(0)}(\rho)$ This is standard textbook quantum mechanics. The exact result is ([4], p.112)

$$\psi_k^{(0)}(\rho) = \sum_{l=0}^{\infty} i^l(2l+1)P_l(cos(\theta))j_l(\rho) \tag{4.15}$$

And we make the approximation that for $\rho << 1$, $j_l(\rho) = \frac{\rho^l}{(2l+1)!}$:

$$\psi_k^{(0)}(\rho) = \sum_{l=0}^{\infty} i^l P_l(cos(\theta))\frac{\rho^l}{(2l)!} \tag{4.16}$$

Using equation (1.7), we can now compute the Sommerfeld enhancement (because $P_l(cos(\theta))$ is independent of $r=\frac{\rho}{k}$, it comes out of both integrals and is therefore eliminated):

l = 0

$$\begin{aligned} S_k &= \frac{\int_0^{\rho_0}\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta}-1)}((1+\rho)^2+(\rho\pm\rho\eta)^2)\,d\rho}{\rho_0} \\ &= \frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta}-1)}(1+\rho_0+\frac{1}{3}\rho_0^2(1+(1\pm\eta)^2)) \end{aligned} \tag{4.17}$$

l = 1

$$S_k = \frac{\int_0^{\rho_0}\frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta}-1)}\frac{1+\eta^2}{4}((\rho+\frac{1}{2}\rho^2)^2+(\rho^2((1\pm\frac{\eta}{4}))^2)),d\rho}{\int_0^{\rho_0}\frac{\rho^2}{4}\,d\rho}$$

$$S_k = \frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta}-1)}(1+\eta^2)(1+\frac{3}{4}\rho_0+\frac{3}{5}\rho_0^2(\frac{1}{4}+(1\pm\frac{\eta}{4})^2)) \tag{4.18}$$

Let us first check the limits of equation (4.17) to see if it is correct. Fill in $r_0 = \frac{\rho_0}{k} = 0$, then it reduces to

$$S_k = \frac{2\pi\eta}{\pm(e^{\pm 2\pi\eta} - 1)} \tag{4.19}$$

which is the correct Sommerfeld enhancement for $r = 0$ and $l = 0$ (see equation (3.11)).

4.1 Evaluation of the results

In figures 4 and 5 the Sommerfeld enhancement is plotted versus the speed. Figure 4 corresponds to the repulsive case, figure 5 to the attractive case. Note that our analysis was only valid for $\rho << 1$, so the reason that large values for ρ_0 have been chosen in the graphs is to exaggerate the effect in order to clearly show it. In both cases, the Sommerfeld enhancement is multiplied by some factor. In the repulsive case this multiplicative factor is smaller than in the attractive case.

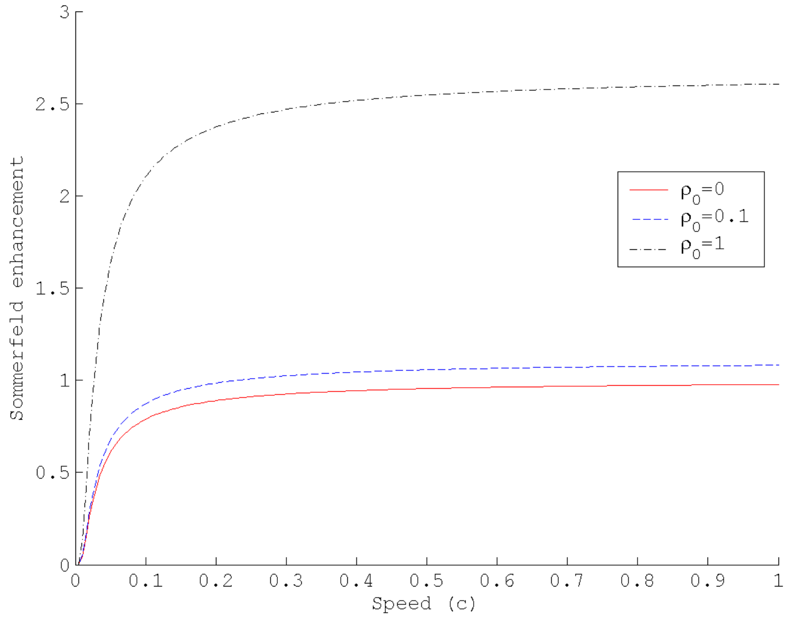

Figure 4: Sommerfeld enhancement for the repulsive case with $l = 0$

A quick view at equation (4.17) tells us the dependence on ρ_0 is positive and that this ρ_0 dependence does not include a speed dependence. Both features are also seen in figures 4 and 5. A consequence of the second feature is that the Sommerfeld enhancement is not equal to 1 for high speeds, which may seem odd intuitively. It is however easily understood algebraically when we take a look at our primary radial wavefunction with potential, equation (3.9), compared to the one without potential, equation (4.16). We see that (for small

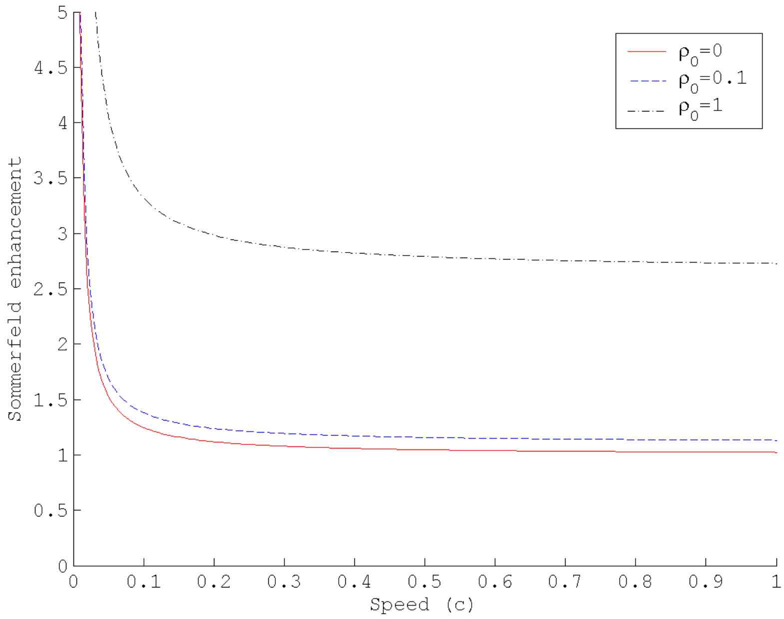

Figure 5: Sommerfeld enhancement for the attractive case with $l = 0$

ρ) the wavefunction with potential has an additional positive ρ dependence, which means that it obtains a larger value for bigger ρ. Therefore the chance of the incident particle being near the origin with EM-potential is boosted with respect to the chance without potential, independent from its speed. Thus the fact that the EM potential is added results to an overall increase of the Sommerfeld enhancement. Due to the positive ρ dependence, it is also logical that a bigger interaction radius leads to a bigger overall boosting factor than a smaller interaction radius.

The actual interaction radius cannot however be arbitrary large, because it is governed by the physical dimensions of the interaction. In this case the chosen values for ρ_0 are way too big in most physical situations. For example, let us take some interaction with an atomic nucleus, where $r \approx 10^{-14}m$, $M \approx 10^{-26}kg$ and $v \approx 10^{-3}\frac{m}{s}$, which gives $\rho_0 \approx 10^{-37}$. This effect can therefore be neglected in most practical applications.

In the $l = 1$ case, the figures have the same shape (see Appendix B). In comparison to the $l = 0$ enhancement, the $l = 1$ enhancement has an extra η dependence (see equation (4.18).) This gives it a little boost with respect to $l = 0$, but this extra factor is more than compensated by the other extra factors in the $l = 1$ case.

5 The Yukawa potential

In the electromagnetic potential the force exchanging particles of the potential are massless photons. In general, the force exchanging particles of the potential can have mass. This is accounted for in the Yukawa potential:

$$V(r) = \pm\frac{\alpha}{r}e^{-m_\phi r} \tag{5.1}$$

where the minus sign corresponds to the attractive case and the plus sign to the repulsive case. m_ϕ is the mass of the exchange particle. The above potential reveals clearly that the Yukawa potential is a generalization of the EM potential, because as $m_\phi \to 0$, Yukawa potential $\to$ EM Potential.

Now, using natural units (the same simplification as in equation (3.5)), the Schrödinger equation becomes:

$$\frac{d^2\chi(\rho)}{d\rho^2} + \left(1 \mp \frac{2\eta e^{-\zeta\rho}}{\rho} - \frac{l(l+1)}{\rho^2}\right)\chi(\rho) = 0 \tag{5.2}$$

wherein

$$\zeta = \frac{m_\phi}{\alpha M} \tag{5.3}$$

Note that $\zeta = 0$ for the EM case.

The above Schrödinger equation (equation (5.2)) does not possess analytical solutions. A numerical solution does exist and has been studied extensively in the literature (see in recent literature for instance [7] and [8]). Here we do not try to obtain the most accurate or fastest solution to the Yukawa potential. Instead, we give a simple program in Mathemathica for the $l = 0$ case (see Appendix A), which in the following we use to illustrate the main features of the Yukawa potential. Also, because we did not encounter an actual instant usable program for solving the Yukawa potential in the literature, hopefully this will also enable others to easily get a solution to the Yukawa potential.

Let us now look at the attractive Yukawa potential and take the values of the constants the same as in [5] i.e. $\alpha = \frac{1}{30}$, $m_\phi = 90GeV$. We obtain figure 6, where the mass of the incident particle is plotted versus the Sommerfeld enhancement. This is the exact same figure as figure 2 in [5], confirming that our method for solving the Yukawa potential is in accordance with the literature.

Figure 6 contains all relevant information about the Yukawa potential. The first thing to notice is the series of resonances, similar to the resonances in a potential well. At small velocites and at particular values of the particle mass, Sommerfeld enhancements can be as big as 10^6. Furthermore, note that if a specific particle mass is chosen, the Sommerfeld enhancement rises as the velocity drops, as expected. For an in depth-discussion of figure 6 we refer to [5].

As a concluding remark we note that in principle the computation of the Sommerfeld enhancement for the Yukawa potential can also be extended to non-pointlike interaction regions. If the result was analagous to the result of the EM-potential i.e. a multiplicative factor, then the absolute effect of this non-pointlike interaction region is a lot bigger in the Yukawa potential (due to the bigger Sommerfeld enhancement). Under the right circumstances this could possibly be a way to find this effect. We defer the actual computation to future work due to its high complexity.

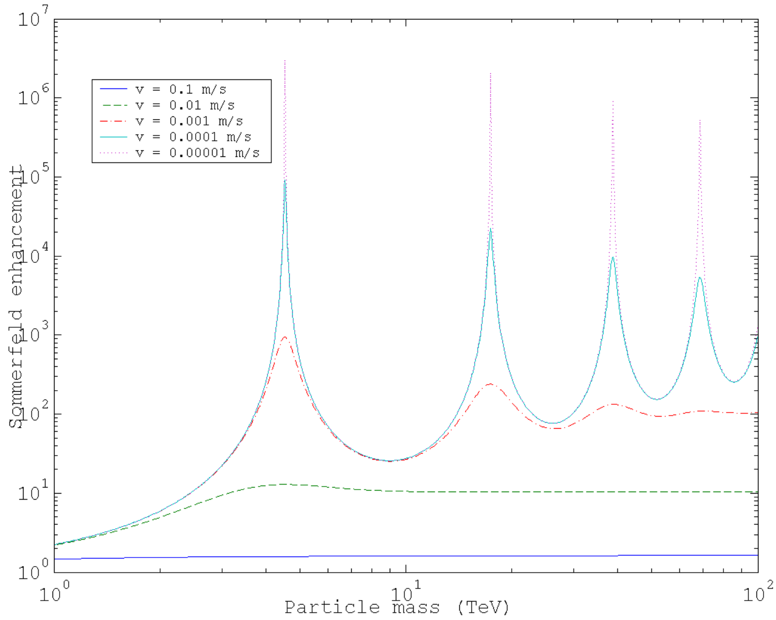

Figure 6: Sommerfeld enhancement for the Yukawa potential

6 Conclusion

The Sommerfeld enhancement is an elementary effect in nonrelativistic quantum mechanics, which accounts for the effect of a potential on the interaction cross section. To compute this Sommerfeld enhancement we used standard nonrelativistic quantum mechanics theory to derive a general formula, which in essence breaks down to finding the wavefunction in the point where the interaction takes place, using the Schrödinger equation.

Using this formula, we first computed the Sommerfeld enhancement for the rectangular potential well/barrier, where we found a resonant pattern. We continued with the computation of the Sommerfeld enhancement for the electromagnetic potential with $r_0 = 0$, both the repulsive and the attractive case. For both cases $S_k \to 1$ as $v \to \infty$, while for the repulsive case $S_k \to 0$ as $\alpha >> v$ and for the attractive case $S_k \to \frac{\pi\alpha}{v}$ as $\alpha >> v$.

With the result for the Sommerfeld enhancement for the electromagnetic potential with $r_0 = 0$ in hand, we attempted to compute the Sommerfeld enhancement for the electromagnetic potential with $0 < r_0 << 1$ We used an Taylor expansion and then took only the first and second order terms. The result was an overall multiplicative factor dependending on the size of the interaction region. This factor is however negligibly small in most practical applications.

To conclude, a simple program was written in Mathematica to compute the Sommerfeld enhancement for the Yukawa potential, which produced consistent results with results in recent literature.

Acknowledgements

The author wishes to express his gratitude towards prof. M. de Roo for his guidance.

7 Appendix A

To obtain the solution to the Yukawa potential, we apply the procedure outlined in [9] for $l = 0$.

We use Wolfram Mathematica 7.0 (©Copyright 1988-2009 Wolfram Research, Inc.), in which we designed the following program (where a is a constant and the solution is computed from $b = \zeta * X$ to $b = \zeta * Y$ with X and Y positive integers):

$$\text{For}[i = X, i < Y, i{+}{+}, \text{With}[\{a = \eta/2, b = \zeta * i\},$$

$$\text{sol} = \text{NDSolve}[\{y\text{''}[x] + (2/x)y'[x] + (1 + (2a/x)e\hat{}(-b * x))(y[x]) == 0,$$

$$y[0.000001]{=}{=}1, y'[0.000001]{=}{=}{-}a\}, y, \{x, 0.000001, 60\}, \text{MaxSteps} \to \text{Infinity}]]$$

$$\text{Print}[1/\text{Evaluate}[((30 * y[30]))\hat{}2 + (((30 - 0.5\text{Pi}) * y[30 - 0.5\text{Pi}]))\hat{}2/.\text{sol}[[1]]]]]$$

For instance, to plot the region from 1 to 10 TeV in figure 6 for $v = 10^{-5}$, one enters:

$$\text{For}[i = 100, i < 1000, i{+}{+}, \text{With}[\{a = (1/60){*}(10\hat{}(5)), b = 90{*}(10\hat{}(5))/(i{*}10)\},$$

$$\text{sol} = \text{NDSolve}[\{y\text{''}[x] + (2/x)y'[x] + (1 + (2a/x)e\hat{}(-b * x))(y[x]) == 0,$$

$$y[0.000001]{=}{=}1, y'[0.000001]{=}{=}{-}a\}, y, \{x, 0.000001, 60\}, \text{MaxSteps} \to \text{Infinity}]];$$

$$\text{Print}[1/\text{Evaluate}[((30 * y[30]))\hat{}2 + (((30 - 0.5\text{Pi}) * y[30 - 0.5\text{Pi}]))\hat{}2/.\text{sol}[[1]]]]]$$

Note that you need to use the 'Merge cells' function in Mathematica to obtain a vector, which can be copy and pasted in a plotting program to make the figures.

A few approximations/simplifications have been made:

1. We use a for loop, limiting our range in so far that if a large range is chosen the steps in some regions are relatively bigger than in other regions (for instance the 100Gev region and the 1 Tev region). This problem can be solved manually by computing the solution for the different regions individually.

2. Technically, the initial conditions are y[0] == 1 and y'[0] == -a. Due to the involved singularities, we have chosen to take x very small, thereby making a good appromixation.

3. The above procedure tries to correct for oscillations by implementing $(x * y[x]))^2 + (((x - 0.5Pi) * y[x - 0.5Pi]))^2$, involving a 0.5 Pi difference. The choice to evaluate the solution in the point 30 is arbitrary. However, as is seen in figure 7, $(x * y[x]))^2 + (((x - 0.5Pi) * y[x - 0.5Pi]))^2$ remains constant within our accuracy range, so that in principle we can choose any point $x > 10$.

For figure 7 the same values have been used as in figure 1 of [9] (i.e. $\eta = 10^3$ and $\zeta = 10$, assuming that the $v = 2 * 10^5$ in the article is a clerical, instead we use $v = 2 * 10^{-5}$). Consequently, we get (except for the fact that we chose a smaller x range, in part due to simplification 2) exactly the same value as in the above article, confirming that our implementation is consistent with it.

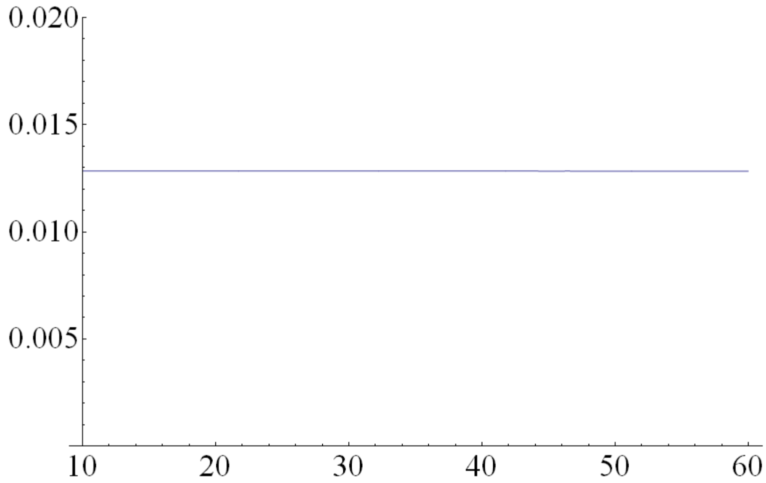

Figure 7: $((x * y[x]))^2 + (((x - 0.5Pi) * y[x - 0.5Pi]))^2$

8 Appendix B

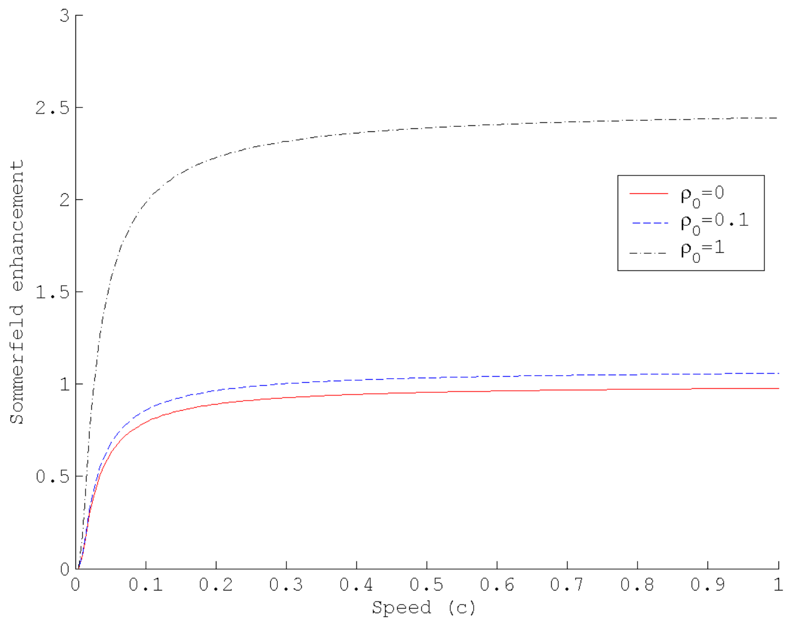

Figure 8: Sommerfeld enhancement for the repulsive case with $l = 1$

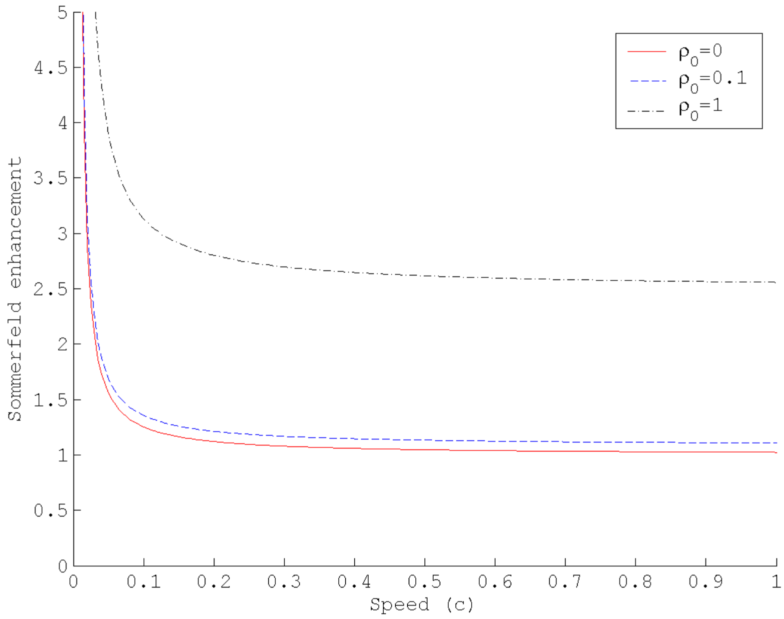

Figure 9: Sommerfeld enhancement for the attractive case with $l = 1$

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