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The Road to Loop Quantum Gravity

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1 Introduction

This article is the final version of my bachelor thesis for the bachelor Theoretical Physics at the University of Groningen, Holland. It focusses on the road to Loop Quantum Gravity, a theory that attempts to unify General Relativity Theory with the laws of Quantum Mechanics without the use of perturbation theory.

The full theory of Loop Quantum Gravity is too extended for a bachelor thesis. Therefore this article mainly focusses on the road toward Loop Quantum Gravity. This road covers a number of different subjects.

In section 3 the canonical description of general relativity using the ADM-formalism will be discussed, the constraints given by the canonical formulation and the problems when they are quantized by Dirac quantization.

Section 4 focusses on the introduction of Ashtekar's variables and their application to the constraint equations. The Ashtekar variables will be introduced with help of the tetrad formulation of general relativity. After the introduction of Ashtekar's variables I will give a short qualitative introduction to physics in terms of loops, but no attempt will be made to actually do the quantization in terms of these loops.

In the last section a short analysis of the present state of LQG will be carried out. The reason to stop at that point is that there is quite some controversy around the steps after the introduction of the variables. The limited time of this bachelor thesis would not be enough to extensively analyze all the problems the theory is faced with.

In appendix A gives an introduction to the LQG-formalism in 2+1 dimensions, in which the theory is further developed. This is meant as an birdview of this subject and does not contain much hard mathematical proof. It is however a good introduction to the reasoning to choose 'loops' as the basic variables of the theory.

2 Index Notation

Throughout this article quite a lot of different indices are used, which can differ from other articles on the subject. Here follows a summary to give a simple overview.

$$
\begin{aligned}
g_{\mu\nu} &= \text{the spacetime metric} \\
h_{ij} &= \text{the spatial metric} \\
\bar{e}_{\mu i} &= \text{a tetrad} \\
e_{ai} &= \text{a triad} \\
\mu, \nu, ... &= \text{Greek indices from the middle of the alphabet indicate} \\
&\quad \text{spacetime coordinates } \mu = 0, 1, 2, 3. \\
&\quad \text{In section 5 they notate tetrad coordinates } \mu = 0, 1, 2, 3 \\
i, j, ... &= \text{Latin indices from the middle of the alphabet indicate} \\
&\quad \text{spatial coordinates } i = 1, 2, 3 \\
a, b, ... &= \text{Latin indices from the start of the alphabet indicate} \\
&\quad \text{spatial coordinates } a = 1, 2, 3. \\
&\quad \text{In section 5 they notate triad coordinates } a = 1, 2, 3 \\
\nabla_\mu &= \text{the covariant derivative, also denoted with } {}_{;\mu} \\
\partial_\mu &= \text{the partial derivative, also denoted with } {}_{,\mu}
\end{aligned}
$$

3 The origin of Loop Quantum Gravity

3.1 The unification of General Relativity and Quantum Mechanics

One of the key problems in 21^{st}-century physics is to find a connection between Quantum Mechanics, which describes the electromagnetic force, the weak force and the strong force, and General Relativity Theory (denoted with GRT in the remainder of this article), which describes the laws of gravity. Both theories are very well developed and supported by a large amount of experimental evidence. It is therefore only natural to try to unify these theories to obtain one theory that describes all four fundamental forces of nature.

The most logical point to begin is to obtain a quantum theory of gravity by using a pertubation expansion of canonical GRT and quantize that using ordinary rules of quantum field theory. This approach is analogous to the way QED was arrived from electrodynamics. It turns out that terms in this perturbation expansion are divergent. This in itself is not a major obstacle, because a finite amount of divergent terms can be cancelled against counterterms of equal magnitude and opposite sign to obtain a finite perturbation series. In the pertubation expansion of GRT it has been proven by Goroff and Sagnotti [1] that there are infinitely many divergences. Thus infinitely many counterterms are needed to produce any physical results. This is called the non-renormalizability of GRT.

There is also a more intuitive reason why GRT is a non-renormalizable theory. Newton's constant is not dimensionless and to serve as the coupling constant of GRT it should be multiplied by energy. Therefore at high energies the coupling constant becomes very large. This results in an infinite amount of divergent Feynmann diagrams.

From the conclusion that GRT is non-renormalizable there are different ways to go. One can omit GRT as a fundamental theory and think of it as a low energy limit in which the divergences are not yet significant. This choice has led to the development of String Theory. Another option is to assume that the perturbation expansion in Newton's constant is not well defined, but that GR can still be quantized correctly. The solution is then to quantize GRT non-perturbatively. This will lead to the theory which is the subject of this article: Loop Quantum Gravity (denoted with LQG in the remainder of the article)

3.1.1 Arguments for a non-perturbative approach

The most encountered argument for a non-perturbative approacht is that it leads to a background independent theory. This means that the laws of GRT hold no matter what background metric you apply them to. The equations given by such a theory should themselve determine the background metric (e.g. space and time variables). Simply put: a background dependent theory will presuppose a metric and then start defining physical laws. These physical laws are only valid for the presupposed metric. A background independent theory will lead to a set of equations (in GRT Einstein's equations) which contains an undetermined metric. The form of this metric is then given by the solutions to the equations. So if you suppose GRT is a valid theory, your follow-up theory should also be background independent. This can also explain the problems when quantizing GRT with help of the quantum field theory formalism, because this formalism relies on a presupposed background metric.

Another argument is that Loop Quantum Gravity uses a lot less new mathematical and physical structures than for example string theory (no extra dimensions and no supersymmetry). This argument was mainly heard in the mid 90s. Now the structure of LQG with its unusual Hilbert space of spin networks seems to require some additional structures to obtain physical results.

3.1.2 Development of the formalism, a short overview

To quantize GRT non-perturbutavely one assumes that the Einstein Hilbert-action from which GRT is derived is exact and not a low-energy limit of an underlying theory. Quantizing GRT without using a perturbation series yields a lot of difficulties. Three constraint equations (the Hamiltonian, Gauss and diffeomorphism constraint) follow from this approach. The constraint equations form after quantization the so called Wheeler-de Witt equations. These equations are highly singular and so far there are no known solutions to these equations. To circumvent this problem Abbhay Ashtekar introduced a new set of variables in [3], which today are named after him. These variables turn the constraint equations into simple polynomials. The initial hope that they would simplify the constraint equations was damped due to the neccessity of introducing a parameter in the new variables: the Barbero-Immirzi parameter. When this parameter is chosen to be complex, it indeed gives polynomial constraint equations. The downside of this choice is that it leads to a complex phase space of GRT. To obtain real solutions reality conditions must be imposed. For the classical case this is not a problem, but after quantizing the theory it turns out to be a major problem to find such reality conditions. Therefore this complex form is abandoned and the polynomial form of the constraint equations is lost. The numerical value of the Barbero Immirzi parameter poses another problem. At this moment its value is fixed by demanding a correct prediction of the entropy of the Hawking-Berkenstein black hole. There is no physical reason for this value to be logical.

Even in the complex form problems arise when the theory is quantized. The metric is no longer a simple operator and deriving it turns out to be very complicated. This is a first indication that a theory, such as LQG, that uses the Ashtekar variables to quantize GRT will have difficulties finding semi-classical states. Also the quantum constraints, however simpler because of the change of the metric to the new variables, still does not yield any results. Therefore another change of variables has to be made. This brings us to the loop representation. The argument for this was that certain functionals, loops, do annihilate the Hamiltonian constraint. They depend only on the Ashtekar variables through the trace of the holonomy, a measure of the change of the direction of a vector when it's parallel transported over a closed circle (a loop). This loop representation will be discussed more extensively in section 4.3

4 Canonical General Relativity: The ADM-formalism

In this section the procedure for describing General Relativity in a canonical way will be discussed. To do so space and time are seperated with a method called 3+1 decomposition. After that the Lagrangian density and the Hamiltonian density can be obtained. The constraint equations follow from the latter. This calculation was done for the first time by Arnowitt, Deser en Misner in 1962 [3]. We will focus on the constraint equations and the problems they bring along.

4.1 The Einstein field equations using the Palatini Lagrangian

In General Relativity the Einstein vacuum equations can be derived via the action-principle using the following action, called the Einstein-Hilbert Action:

$$S = \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} R \tag{4.1}$$

where $g = \det g_{\mu\nu}$ and R is the Ricci scalar. The equations of motion are derived by variation of the metric. Since R already includes derivatives of the metric, the equations of motion will be second-order differential equations. To obtain a canonical form of this equations of motion they have to be first-order. Therefore a Lagrangian is used which is linear in first derivatives. This Lagrangian is called the Palatini Lagrangian. It is necessary to view the Christoffel symbols in this Lagrangian to be independent quantities in the variational principle, i.e. not dependent on the metric or derivatives of the metric. The action is rewritten as follows:

$$S = \int d^4x \mathfrak{g}^{\mu\nu} R_{\mu\nu}(\Gamma) \tag{4.2}$$

Here $\mathfrak{g}^{\mu\nu}$ is the density metric $\mathfrak{g} = \sqrt{-g} g^{\mu\nu}$. Also

$$R_{\mu\nu}(\Gamma) = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu} \Gamma^\kappa_{\lambda\kappa} - \Gamma^\lambda_{\mu\kappa} \Gamma^\kappa_{\nu\lambda} \tag{4.3}$$

Important here is that the components of $R_{\mu\nu}$ now not involve the metric.

This action should also give the Einstein vacuum equations if $g^{\mu\nu}$ is varied.

Because $R_{\mu\nu}$ does not involve the metric, varying $g^{\mu\nu}$ only has an effect on $\mathfrak{g}^{\mu\nu}$. Therefore it is sufficient to calculate:

$$(\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}g^{\kappa\lambda}))R_{\kappa\lambda} = ((\frac{\partial}{\partial g^{\mu\nu}}\sqrt{-g})g^{\kappa\lambda} + \sqrt{-g}(\frac{\partial}{\partial g^{\mu\nu}}g^{\kappa\lambda}))R_{\kappa\lambda} \tag{4.4}$$

Using the standard rules for differentiating a determinant the following is obtained:

$$\begin{aligned} 0 &= (\frac{1}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}\sqrt{-g})g^{\kappa\lambda}R_{\kappa\lambda} + R_{\mu\nu} \\ &= -\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu} \end{aligned} \tag{4.5}$$

Which are indeed the Einstein field equations. The difference between the field equations obtained via (4.1) and the equations of (4.5) is that in the latter there is not yet a connection between $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$. This can be obtained by varying $\Gamma^{\alpha}_{\mu\nu}$ to find the usual relation $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}(\partial_\nu g_{\mu\kappa} + \partial_\mu g_{\nu\kappa} - \partial_\kappa g_{\mu\nu})$.

4.2 3+1 decomposition of the Einstein field

The equations of motion can be solved explicitly for the time derivatives used in the Hamiltonian formulation ($\dot{q}$ and $\dot{p}$) using the 3+1 decomposition of the Einstein field. This means all 4 dimensional quantities break up to obtain a 3 dimensional part (space) and a 1 dimensional part (time). This is only possible if the spacetime has a causal structure, so there will not be copies of the same event/observer on different spacelike hypersurfaces. Such a spacetime is called globally hyperbolic and can be foliated into 3D hypersurfaces of constant time Σ_t, where t is a vector field that parametrizes the proper time. The question is now how one defines transformations from one hypersurface to another. To do so a spatial metric on each hypersurface is defined:

$$h_{ij} = k_{ij} + n_i n_j \tag{4.6}$$

With n^i representing the normal unit vector to the hypersurface Σ_t and k_{ij} is an arbitrary Lorentz metric which ensures that the spacetime in fact is globally hyperbolic.

Suppose an infinitesimal amount of distance on a hypersurface is given by $h_{ij}(t, x^i)dx^i dx^j$. The proper time is differing from the coordinate time by the

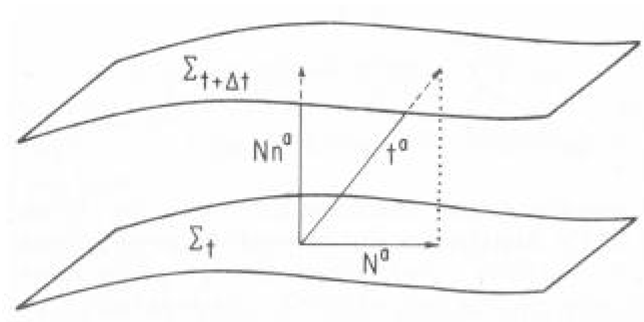

Figure 1: Graphical representation of the lapse and shift function

lapse-function N. $d\tau = N(t, x^i)dt$. The distance between to coordinates separated an infinitesimal amount of distance is given by $x_2^i = x_1^i - N^i(t, x^i)dt$, where N^a is called the shift function. The physical interpretation of these two functions is that the lapsefunction represents the rate of flow of proper time with respect to t. N_a represents the movement tangential to the surface Σ_t after an infinitesimal change in time. This is sketched in figure 1.

In 4D spacetime an infinitesimal amount of distance is given by:

$$ds^2 = (coordinate\ distance)^2 - (proper\ time)^2 \tag{4.7}$$

Filling in our previous results and taking in account the lapse- and shift-functions the following expression for line-element is obtained:

$$\begin{aligned} ds^2 &= h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - (N dt)^2 \\ &= g_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{4.8}$$

From this the components of $g_{\mu\nu}$ can be derived.

$$\begin{aligned} g_{00} &= h_{ij} N^i N^j - N^2 \\ &= N_j N^j - N^2 \\ g_{0b} &= h_{ij} N^i = N_j \\ g_{a0} &= h_{ij} N^j = N_i \\ g_{ab} &= h_{ij} \end{aligned} \tag{4.9}$$

Also it is easy to see that:

$$\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} = \sqrt{N^2 \det h_{ij}} = N\sqrt{h} \tag{4.10}$$

Spacetime is now split into spacelike hypersurfaces of constant time, where you can move to a hypersurface further in time by a lapse-function and on the hypersurface itself by a shift-function.

h_{ij}, N^i and N are the new field variables defining the field since they contain the same information as the original spacetime metric. The Lagrangian has to be re-expressed in terms of these variables.

4.3 The Lagrangian in terms of h_{ij}, N^i and N

The field variables lead to the following relations:

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}^{ij}} \tag{4.11}$$

$$\pi_N = \frac{\partial \mathcal{L}}{\partial \dot{N}} \tag{4.12}$$

$$\pi_N^i = \frac{\partial \mathcal{L}}{\partial \dot{N}^i} \tag{4.13}$$

Important to realize when looking at this equations is that the dot does not indicate a time-derivative. Because of the canonical framework the diffeomorphism invariance of GR has to be maintained, i.e. the system should be coordinate-independent. Time is defined differently in every coordinate-system and is therefore not suited for this approach. It is necessary to differentiate to the 'local' time that at each point of a hypersurface is perpendicular to that hypersurface. This perpendicular direction is given by the field that describes time. Therefore a derivative is used which is defined for differentiating vector fields. This is the Lie-derivative. It is defined with respect to a vector field V as follows:

$$\mathfrak{L}_V T_{\mu\nu} = \lim_{\delta x \to 0} \left(\frac{T'_{\mu\nu} x' - T_{\mu\nu}(x)}{\delta x} \right) \tag{4.14}$$

To find the 'time-derivative' $\mathfrak{L}_t h_{ij}$ the extrinsic curvature is introduced. It is defined as:

$$K_{ij} \equiv h_i^k \nabla_k n_j = \frac{1}{2} \mathfrak{L}_n h_{ij} = \frac{1}{2} N^{-1} (\mathfrak{L}_t h_{ij} - \mathfrak{L}_N h_{ij}) \tag{4.15}$$

The physical interpretation of this extrinsic curvature is quite simple as can be seen in 2. When an arrow normal to a line is parallel transported along a line,

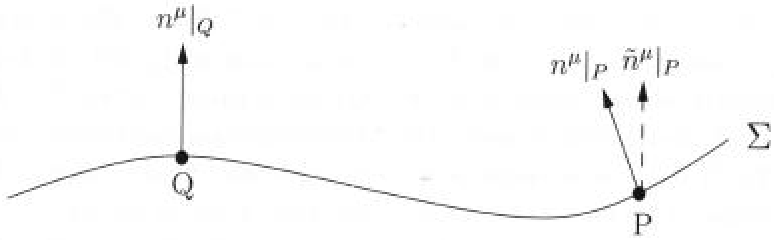

Figure 2: Graphical representation of the extrinsic curvature

the extrinsic curvature is the difference in angle between the transported arrow and the normal arrow at that point.

The Einstein equations in terms of the extrinsic curvature tensor can now be derived.

4.4 Einstein's equations of motion in terms of the new variables

Because of the introduction of a 3-dimensional metric, the spatial metric, a number of quantities in 3 dimensions, which were already defined in 4 dimensional spacetime, need to be defined. First the 3-dimensional Riemann curvature tensor is expressed in terms of a dual vector field and of the derivative associated with h_{ab}. A dual vector field is a vector field consisting of all linear functionals on a vector field V. The result is very similar to the definition of the 4-dimensional curvature tensor:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R^c{}_{dab} V^d \tag{4.16}$$

With identifying the following changes:

$$g_{\mu\nu} \to h_{ij}$$
$$\nabla_a \to D_a$$
$$V^d \to \omega_d$$
$$R^d{}_{abc} \to^{(3)} R^d{}_{abc}$$

And defining:

$$^{(3)}R_{abc}{}^d \omega_d = (D_a D_b - D_b D_a)\omega_c \tag{4.17}$$

The operation of the derivative operator D_a associated with h_{ab}, also called the exterior derivative, is defined as:

$$D_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = h^{a_1}{}_{d_1} \dots h_{b_l}{}^{e_l} h_c h^f \nabla_f T^{d_1 \dots d_k}{}_{e_1 \dots e_l} \tag{4.18}$$

where ∇_a is the derivative operator associated with g_{ab}.

From this it follows that:

$$\begin{aligned} D_a D_b \omega_c &= D_a (h_b{}^d h_c{}^e \nabla_d \omega_c) \\ &= h_a{}^f h_b{}^g h_c{}^k \nabla_f (h_g^d h_k^e \nabla_d \omega_e) \\ &= h_a{}^f h_b{}^d h_c{}^e \nabla_f \nabla_d \omega_e \\ &\quad + h_c{}^e K_{ab} n^d \nabla_d \omega_e \\ &\quad + h_b{}^d K_{ac} n^e \nabla_d \omega e \end{aligned} \tag{4.19}$$

where the relation is used that:

$$h_a{}^b h_c{}^d \nabla_b h_d{}^e = h_a{}^b h_c{}^d \nabla_b (g_d{}^e + n_d n^e) = K_{ac} n^e \tag{4.20}$$

Because by definition:

$$\nabla_b g_d{}^e = 0$$

$$h_a{}^b \nabla_b n_d = K_{ad}$$

Also the following holds:

$$h_b{}^d n^e \nabla_d \omega_e = h_b{}^d \nabla_d (n^e \omega_e) - h_b{}^d \omega_e \nabla_d n^e = -K_b{}^e \omega_e \tag{4.21}$$

The second term on the right-hand side of equation (4.19) is symmetric in a and b and will therefore vanish in the expression (4.17).

Finally, this gives:

$$^{(3)}R_{abc}{}^d = h_a{}^f h_b{}^g h_c{}^k h^d{}_j R_{fgk}{}^j - K_{ac} K_b{}^d + K_{bc} K_a{}^d \tag{4.22}$$

Also:

$$\begin{aligned} R_{abcd} h^{ac} h^{bd} &= R_{abcd} (g^{ac} + n^a n^c)(g^{bd} + n^b n^d) \\ &= R + 2 R_{ac} n^a n^c \\ &= 2 G_{ac} n^a n^c \end{aligned} \tag{4.23}$$

$$\begin{aligned}
G_{ab}n^a n^b &= \frac{1}{2}R_{acbd}h^{ab}h^{cd} \\
&= \frac{1}{2}g_{dm}R_{acb}{}^{m}h^{ab}h^{cd} \\
&= \frac{1}{2}(h_{dm} - n_d n_m)R_{acb}{}^{m}h^{ab}h^{cd} \\
&= \frac{1}{2}(h_{dm} - n_d n_m)h^f_a h^g_c h^k_b h^m_j({}^{(3)}R_{fgk}{}^{j} + K_{fk}K_g{}^{j} - K_{gk}K_f{}^{j})h^{ab}h^{cd} \\
&= \frac{1}{2}(h_{dm} - n_d n_m)h^{fk}h^{gd}h^m_j({}^{(3)}R_{fgk}{}^{j} + K_{fk}K_g{}^{j} - K_{gk}K_f{}^{j}) \\
&= \frac{1}{2}h^{fk}\delta^g_j({}^{(3)}R_{fgk}{}^{j} + K_{fk}K_g{}^{j} - K_{gk}K_f{}^{j}) \\
&= \frac{1}{2}({}^{(3)}R + K^k{}_k K^j{}_j - K_{jk}K^{jk}) = 0
\end{aligned} \tag{4.24}$$

From the second line in (4.23) it can be seen that:

$$R = 2(G_{ab}n^a n^b - R_{ab}n^a n^b) \tag{4.25}$$

If now $R_{ab}n^a n^b$ is calculated in terms of the extrinsic curvature the Lagrangian density can be written in terms of the extrinsic curvature via $L = \sqrt{-g}R$.

$$\begin{aligned}
R_{ab}n^a n^b &= R_{acb}{}^{b}n^a n^b \\
&= -n^a(\nabla_a\nabla_c - \nabla_c\nabla_a)n^c \\
&= (\nabla_a n^a)(\nabla_c n^c) - (\nabla_c n^a)(\nabla_a n^c) - \nabla_a(n^a\nabla_c n^c) + \nabla_a(n^a\nabla_a n^c) \\
&= K^2 - K_{ac}K^{ac} - \text{Divergence terms}
\end{aligned} \tag{4.26}$$

To summarize the following results have been derived:

$$G_{ij}n^i n^j = \frac{1}{2}({}^{(3)}R - K_{ij}K^{ij} + K^2) \tag{4.27a}$$

$$R_{ij}n^i n^j = K^2 - K_{ij}K^{ij} \tag{4.27b}$$

Combining these two equations and filling them in in the Einstein-Hilbert Lagrangian leads to

$$\begin{aligned}
\mathcal{L} &= \sqrt{h}N({}^{(3)}R + K_{ij}K^{ij} - K^2) \\
&= \sqrt{h}N({}^{(3)}R + (\frac{1}{4}N^{-2}(\mathfrak{L}_t h_{ij} - \mathfrak{L}_N h_{ij})(\mathfrak{L}_t h^{ij} - \mathfrak{L}_N h^{ij}) - K^i{}_i K^i{}_i))
\end{aligned} \tag{4.28}$$

From this the equations of motion can be derived:

$$\begin{aligned}\pi^{ij} &= \frac{\partial \mathcal{L}}{\partial \mathfrak{L}_t h_{ij}} \\ &= \sqrt{h}N(\frac{1}{2}N^{-2}(\mathfrak{L}_t h^{ij} - \mathfrak{L}_N h^{ij}) - \frac{\partial}{\partial \mathfrak{L}_t h_{ij}} K^2) \\ &= \sqrt{h}N(N^{-1}K^{ij} - \frac{1}{4}N^{-2}(\mathfrak{L}_t h_{ij}h^{ij} - \mathfrak{L}_N h_{ij}h^{ij})^2) \\ &= \sqrt{h}N(N^{-1}K^{ij} - N^{-1}Kh^{ij}) \\ &= \sqrt{h}(K^{ij} - Kh^{ij})\end{aligned} \tag{4.29}$$

And furthermore:

$$\pi_N = \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 \tag{4.30}$$

$$\pi_N^i = \frac{\partial \mathcal{L}}{\partial \dot{N}_i} = 0 \tag{4.31}$$

4.5 Lagrange multipliers

What do the last equations mean? It turns out that N and N_i are in fact Lagrange multipliers. To show what this means here follows a brief summary of some basic properties of the Hamiltonian formalism concerning Lagrange multipliers.

Consider the following example: A pendulum has coordinates (x, y) and has the normalization condition $x^2 + y^2 = l^2$. The Lagrangian for such a pendulum is given by $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - mgy$. The variational principle leads to:

$$\delta S = \delta \int L dt = \int (\frac{\delta L}{\delta x}\delta x + \frac{\delta L}{\delta y}\delta y) dt \tag{4.32}$$

Since $x^2 + y^2 = l^2$ the constraint-term $\lambda(l^2 - x^2 - y^2)$ can be added without changing the action. If now x, y and λ are considered to be dynamical variables the following always holds:

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0 \tag{4.33}$$

The conclusion from this result is that if a constraint is hidden in a Lagrangian, or in the Hamiltonian for that matter, the conjugate momentum to the apparent dynamical variable (and thus a degree of freedom) turns out to be 0. The constraint can then be retrieved from the fact that if the conjugate momentum equals 0, then also:

$$\frac{\partial H}{\partial \lambda} = 0 \tag{4.34}$$

The conclusion from the fact that $\frac{\partial \mathcal{L}}{\partial \dot{N}} = 0$ and $\frac{\partial \mathcal{L}}{\partial \dot{N}_i} = 0$ is therefore that these apparent variables in fact are constraints on the system. They can be retrieved by setting up the Hamiltonian and take the derivate with respect to these 'variables'.

4.6 The Hamiltonian and the Hamiltonian constraints

The Hamiltonian density can now be defined in terms of the true degree of freedom h_{ij}.

$$\begin{aligned}\mathcal{H} &= \pi_{ij}\dot{h}_{ij} - \mathcal{L} \\ &= \pi^{ij}(2K_{ij}N + \mathfrak{L}_N h_{ij}) - \sqrt{h}N({}^{(3)}R + K_{ij}K^{ij} - K^2)\end{aligned} \tag{4.35}$$

For the exterior derivative introduced in section 4.4 holds:

$$D_i h_{jk} = 0 \tag{4.36}$$

This is equivalent to the covariant derivative defined as $\nabla_\mu g_{\nu\rho} = 0$.
The Lie-derivative of the spatial metric can, in analogy to the Lie derivative of the spacetime metric, be written as:

$$\mathfrak{L}_V h_{ij} = D_i V_j + D_j V_i \tag{4.37}$$

The Hamiltonian can be rewritten in terms of D^i and the momentum π^{ij}. In order to do so first the extrinsic curvature in terms of π^{ij} and the spatial metric h^{ij} has to be defined.

$$K_{ij} = h^{-1/2}(\frac{1}{2}\pi_{kl}h^{kl}h_{ij} - \pi_{ij}) \tag{4.38}$$

This definition can be checked by filling (4.38) in in (4.29). The rewritten Hamiltonian becomes:

$$\begin{aligned}\mathcal{H} &= \pi^{ij}(2K_{ij}N + D_iN_j + D_jN_i) - \sqrt{h}N({}^{(3)}R + K_{ij}K^{ij} - K^2) \\ &= \sqrt{h}N(-{}^{(3)}R - K_{ij}K^{ij} + K^2 + 2h^{-1/2}\pi^{ij}K_{ij}) \\ &\quad + \pi^{ij}(D_iN_j + D_jN_i)\end{aligned} \tag{4.39}$$

First the left hand side of (4.39) is devoloped further. To do so the following relations are needed:

$$K_{ij}K^{ij} = h^{-1}(\pi_{ij}\pi^{ij} - \pi_{kl}h^{kl}h_{ij}\pi^{ij} + \frac{3}{4}\pi^2) \tag{4.40a}$$

$$\pi^{ij}K_{ij} = h^{-1/2}(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi_{kl}h^{kl}h_{ij}\pi^{ij}) \tag{4.40b}$$

$$K^2 = \frac{1}{4}h^{-1}\pi^2 \tag{4.40c}$$

If (4.40) is inserted into (4.39) this becomes :

$$\begin{aligned}\mathcal{H} = \sqrt{h}N(-^{(3)}R + h^{-1}\pi_{ij}\pi^{ij} - \frac{1}{2}h^{-1}\pi^2) \\ - 2N_j D_i(h^{-1/2}\pi^{ij}) + 2D_i(h^{-1/2}N_j\pi^{ij})\end{aligned} \tag{4.41}$$

where the last term is only a boundary term in the integral to obtain the Hamiltonian and can therefore be ignored.

For Lagrange multipliers holds $\frac{\partial \mathcal{H}}{\partial \lambda} = 0$, so that:

$$\frac{\partial \mathcal{H}}{\partial N} = -^{(3)}R + h^{-1}\pi^{ij}\pi_{ij} - \frac{1}{2}h^{-1}\pi^2 = 0 \tag{4.42}$$

which is called the Hamiltonian constraint, and

$$\frac{\partial \mathcal{H}}{\partial N_j} = D_i(\sqrt{h}\pi^{ij}) = 0 \tag{4.43}$$

which is called the diffeomorphism constraint.

Now the constraint equations have been found, an analysis can be made of the problems they bring along. This problems can be partly solved by the introduction of new variables, Ashtekar's variables.

4.7 Problems with the constraint equations

The ADM formulation of General Relativity can be quantized with help of the quantization process introduced by Dirac. One first calculates the Poisson-brackets:

$$\{h_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l)\delta^3(x-y) \tag{4.44}$$

Now the variables are turned into operators

$$h_{ij} \rightarrow \hat{h}_{ij} \tag{4.45}$$

$$\pi^{ij} \rightarrow -i\hbar\frac{\delta}{\delta h_{ij}} \tag{4.46}$$

The constraint equations show that $\mathcal{H} = 0$ at all time, so that the Schrodinger equation reduces to: $\hat{\mathcal{H}}|\Psi\rangle = 0$. Here

$$\mathcal{H} = \sqrt{h}N\frac{\partial \mathcal{H}}{\partial N} - 2\sqrt{h}N_i\frac{\partial \mathcal{H}}{\partial N_i} = 0 \tag{4.47}$$

which implies:

$$\frac{\partial \hat{H}}{\partial N}|\Psi\rangle = 0 \tag{4.48}$$

$$\frac{\partial \hat{H}}{\partial N_i}|\Psi\rangle = 0 \tag{4.49}$$

These equations are known as the Wheeler-DeWitt equations. The first of this equations, the quantized Hamiltonian constraint, turnes out to be a highly singular functional differential equation for which up untill now no physical solutions have been found.

5 Ashtekar's variables

The problem of the unsolvable Hamiltonian constraint seemed to disappear with the introduction of a new set of variables by Abhay Ashtekar in 1986 [4]. These were named Ashtekar's variables and ensured that the Hamiltonian constraint became in fact a polynomial equation. Here I will rewrite the constraint in terms of the new variables and show some of the problems that arose after their introduction. In this process I will follow [11] in their derivation, which uses a formulation of gravity in terms of tetrads.

5.1 The formalism: connections instead of the space-time metric

The Ashtekar formalism doesn't use the ordinary metric to describe space-time. Instead it uses an object called a tetrad or a vierbein (in 3 dimensions these are called a triad or a dreibein respectively). Physically the tetrad forms a linear map from the tangent space generated by the metric g_{ab} to Minkowsky space-time. This mapping preserves the inner product and thus the following equation holds:

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\bar{e}^{\alpha}_{\mu}(x)\bar{e}^{\beta}_{\nu}(x) \tag{5.1}$$

where μ and ν represent coordinates in the tangent space and i and j represent coordinates in Minkowsky space[1]. More on this tetrad formulation can be found in Appendix B.

In three dimensions the relationship (5.1) becomes:

$$h_{ij} = \delta_{\mu\nu}\bar{e}^{\mu}_{i}\bar{e}^{\nu}_{j} \tag{5.2}$$

where h_{ij} is the spatial metric obtained in the 3+1-decomposition of space-time.

To write GRT in terms of this triad canonically the conjugate momentum to this triad, for which the following brackets hold, has to be calculated:

$$\{\bar{e}^{\mu}_{i}(x), p^{j}{}_{\nu}(y)\} = \delta^{j}_{i}\delta^{\mu}_{\nu}\delta^{(3)}(x-y) \tag{5.3}$$

[1]This equation requires the definition by the metric tensor of a inner product in Minkowsky space time. In General Relativity this requirement is always obeyed, because the low energy solution should always generate flat space-time

This conjugate momentum is related to the ordinary momentum by:

$$p^i{}_\mu = 2p^{ij}\bar{e}_{j\mu} \tag{5.4}$$

This definition changes the Poisson-brackets for the conjugate momenta into:

$$\{p^{ij}(x), p^{kl}(y)\} = \frac{1}{4}(q^{ik}J^{jl} + q^{il}J^{jk} + q^{jk}J^{il} + q^{jl}J^{ik})\delta^{(3)}(x-y) \tag{5.5}$$

where

$$J^{ij} = \frac{1}{4}(\bar{e}^{i\mu}p^j{}_\mu - \bar{e}^{j\mu}p^i{}_\mu) \tag{5.6}$$

To maintain the original Poissonbrackets it is necessary to set $J^{ij} = 0$, which can also be represented as:

$$\mathcal{J}^\rho = \epsilon^{\rho\mu\nu}p^i{}_\mu\bar{e}_{i\nu} = 0 \tag{5.7}$$

The following three constraint equations have now been derived:

$$\mathcal{H} = -{}^{(3)}R + h^{-1}\pi^{ij}\pi_{ij} - \frac{1}{2}h^{-1}\pi^2 = 0 \tag{5.8a}$$

$$\mathcal{H}^j = D_i(\sqrt{h}\pi^{ij}) = 0 \tag{5.8b}$$

$$\mathcal{J}^\rho = \epsilon^{\rho\mu\nu}p^i{}_\mu\bar{e}_{i\nu} = 0 \tag{5.8c}$$

where

$$\pi^{ij} = \frac{1}{4}(\bar{e_\mu}^i\pi^{\mu j} + \bar{e_\mu}^j\pi^{\mu i}) \tag{5.9}$$

One can now perform a 3+1 decomposition of the tetrad variables which is analogue to the 3+1 decomposition of the metric. Again a hypersurface hypersurface for which x^0 = constant and the normal to these hypersurfaces n_μ are introduced. Then the tetrad $\bar{e}_{\mu k}$ is decomposed into the following components:

$$\bar{e}_{ok} = -e_{ak}\omega^a \tag{5.10a}$$

$$\bar{e}_{ak} = e_{ak} + \frac{1}{1+\gamma}e_{bk}\omega^b\omega_a \tag{5.10b}$$

with $a = 1, 2, 3$.

It can now be checked whether the introduced triads e_{ak} are indeed triads on x^0 =constant. To do so first the following relations are defined $\gamma = \sqrt{1 + \omega_a\omega^a}$ and $\omega^a = n^a$.

$$\begin{aligned} h_{ij} &= \bar{e}_{\mu i}\bar{e}^\mu_j \\ &= -\bar{e}_{0i}\bar{e}^0_j + \bar{e}_{ai}\bar{e}^a_j \\ &= e_{ai}\omega^a e_{bj}\omega^b - (e_{ai} + \frac{1}{\gamma+1}e_{bi}\omega^b\omega_a)(e^a{}_j + \frac{1}{\gamma+1}e^c{}_j\omega_c\omega^a) \end{aligned} \tag{5.11}$$

Because the metric tensor is symmetric in i, j this is the same as:

$$\begin{aligned} h_{ij} &= -e_{ai}e_{bj}\omega^a\omega^b(\frac{\gamma-1}{\gamma+1}) + (\frac{\gamma-1}{\gamma+1})(e_{bi}e^c{}_j\omega^b\omega_c) + e_{ai}e^a{}_j \\ &= e_{ai}e^a{}_j \end{aligned} \tag{5.12}$$

since a, c are only dummy variables. So spacetime is decomposed in 3+1 dimensions and a new set of variables is introduced: e_{ai} and ω_a. To complete the transformation to these new variables the conjugate momenta to these new variables have to be found. The transformation must be canonical, so that:

$$\pi^{bl}d\bar{e}_{bl} = p^{bl}de_{bl} \tag{5.13}$$

with d indicating the extorior derivative. In this case this simplifies to:

$$p^{cj} = \pi^{bl}\frac{\partial\bar{e}_{bl}}{\partial e_{cj}} \tag{5.14}$$

From this the following expression for the canonical momenta conjugate to respectively e_{ai} and ω_a is obtained:

$$p^{ak} = \pi^{ak} - \pi^{0k}\omega^a + \frac{1}{\gamma+1}\pi^{bk}\omega_b\omega^a \tag{5.15a}$$

$$\pi^a = -\pi^{0k}e^a{}_k + \frac{1}{\gamma+1}\omega_b(e^a{}_k\pi^{bk} + \pi^{ak}e^b{}_k) \tag{5.15b}$$

If analogue to J^{ab} the spatial rotation generators are defined as:

$$j^{ab} = (p^{ak}e^b{}_k - p^{bk}e^a{}_k) \tag{5.16}$$

the following relations can be calculated

$$J^{0b} = -\pi^b + \frac{1}{\gamma+1}\omega_c j^{cb} \tag{5.17a}$$

$$J^{ab} = j^{ab} - \omega^a\pi^b + \omega^b\pi^c \tag{5.17b}$$

$$j^{ab} = J^{ab} - \omega^a J^{0b} + \omega^b J^{0a} + \frac{1}{\gamma+1}\omega_c(\omega^a J^{cb} - \omega^b J^{ca}) \tag{5.17c}$$

$$\pi^a = \frac{1}{\gamma+1}\omega_b J^{ba} + (-\delta^a_b + \frac{\omega_c\omega^a}{\gamma(\gamma+1)})J^{0c} \tag{5.17d}$$

Therefore it follows that the constraint $J^{ab} = 0$ is equivalent to $\pi^a = 0, j^{ab} = 0$.

Another canonical transformation is applied to find that the variables $(\tilde{e}^{ak}, 2K_{ak})$ also form a canonical pair. Here $\tilde{e}^{ak} = h^{1/2}h^{ik}e^a{}_i$ and

$$K_{ak} = e_a{}^iK_{ik} + \frac{1}{4}h^{-1/2}j_{ab}e^b{}_k \tag{5.18}$$

where K_{ik} is the extrinsic curvature, here defined in terms of the triad momenta as

$$K_{ik} = \frac{1}{4} h^{-1/2} p^{aj} (e_{aj} h_{ik} - e_{ai} h_{kj} - e_{ak} h_{ij}) \tag{5.19}$$

That the new variables form a canonical pair can be checked by calculating as before:

$$2K_{aj} d\tilde{e}^{aj} = p^{aj} de_{aj} \tag{5.20}$$

To do so the following formula is needed:

$$\frac{\delta \tilde{e}^{aj}}{\delta e_{bi}} = h^{1/2} (e^{aj} e^{bi} - e^{ai} e^{bj}) \tag{5.21}$$

with δ indicating the functional derivative. Now one can check relation (5.20).

$$\begin{aligned} 2K_{aj} \frac{d\tilde{e}^{aj}}{de_{aj}} &= 2K_{aj} \frac{\delta \tilde{e}^{aj}}{\delta e_{bi}} \\ &= (2{e_a}^i K_{ij} + \frac{1}{2} h^{-1/2} j_{ab} {e^b}_j) h^{1/2} (e^{aj} e^{bi} - e^{ai} e^{bj}) \\ &= (2{e_a}^i (\frac{1}{4} h^{-1/2} p^{cl} (e_{cl} h_{ij} - e_{ci} h_{jl} - e_{cj} h_{il})) \\ &\quad + \frac{1}{2} h^{-1/2} (p_{ak} {e_b}^k - p_{bk} {e_a}^k) {e^b}_j) h^{1/2} (e^{aj} e^{bi} - e^{ai} e^{bj}) \\ &= 0 \\ &\quad + \frac{1}{2} h^{-1/2} (p_{ak} {e_b}^k - p_{bk} {e_a}^k) {e^b}_j (h^{1/2} (e^{aj} e^{bi} - e^{ai} e^{bj}) \\ &= p^{bi} \end{aligned} \tag{5.22}$$

And with setting $b = a, i = j$ (5.20) is obeyed.

The following step is to perform another canonical transformation which will bring us to the Ashtekar variables which are $\tilde{e}$ and

$$A_{ak} = 2K_{ak} + \frac{i}{2} \epsilon_{abc} \omega_k^{bc} \tag{5.23}$$

These also form a canonical pair, because the second term on the right hand side of (5.23) is just a canonical phase transormation. The factor i in front of the second term on the right hand side is the already mentioned Barbero-Immirzi parameter. The variables A_{ak} are connections and one can define their field strength F_{aij}, which is defined on the constraint surface $j_{ab} = 0$ as:

$$F_{aij} = \frac{i}{4} \epsilon^{klm} ({}^{(3)}R_{klij} + 2K_{ki} K_{lj}) e_{am} + \frac{1}{2} (K_{kj|i} - K_{ki|j}) {e_a}^k \tag{5.24}$$

In terms of the Ashtekar variables this is of the form

$$F_{aij} \approx \partial_m A_{na} - \partial_n A_{ma} + \epsilon_{abc} A_{mb} A_{nc} \tag{5.25}$$

The diffeomorphism constraint is written in these new variables as:

$$\tilde{e}^{ai} F_{aij} = 0 \tag{5.26}$$

The Hamiltonian constraint is given by:

$$\tilde{e}^i_a \tilde{e}^j_b \epsilon^{abc} F_{cij} = 0 \tag{5.27}$$

The diffeomorphism constrained can be verified in two steps. First calculate $\tilde{e}^{ai}$ acting on the second term of F_{aij}

$$\begin{aligned} 0 &= \tilde{e}^{ai} \frac{1}{2}(K_{kj|i} - K_{ki|j}) e_a{}^k = h^{1/2} h^{ni} e^a{}_n \\ &= \frac{1}{2}([\frac{1}{2}\pi_{rs} h^{rs} h^{ni} h_{ni} - h^{ni}\pi_{ni}]_{|j} \\ &\quad - [\frac{1}{2}\pi_{tv} h^{tv} h^{ni} h_{nj} - h^{ni}\pi_{nj}]_{|i} \\ &= \frac{1}{2}\pi^i{}_{j|i} \end{aligned} \tag{5.28}$$

which is the diffeomorphism constraint. Now calculate the result of $\tilde{e}^{ai}$ acting on the first term of F_{aij}:

$$\begin{aligned} &\frac{i}{4} h^{1/2} h^{ni} e^a{}_n \epsilon^{klm} ({}^{(3)}R_{klij} + 2K_{ki}K_{lj}) e_{am} \\ &= \frac{i}{4} h^{1/2} \delta^i_m \epsilon^{klm} ({}^{(3)}R_{klij} + 2K_{ki}K_{lj}) \\ &= \frac{i}{4} h^{1/2} \epsilon^{klm} ({}^{(3)}R_{klmj} + 2K_{km}K_{lj}) \end{aligned} \tag{5.29}$$

K_{ij} is invariant under the interchange of its indices and therefore that term will vanish upon multiplication with ϵ^{klm}. The values for which ϵ^{klm} is nonzero will only give components of ${}^{(3)}R_{klmj}$ which either are zero (6 components) or cancel (12 components) against eachother , so this term will vanish as well. To see this in more detail you can write out the entire equation with the nonzero values of ϵ^{klm}. Therefore (5.26) indeed yields the diffeomorphism constraint (4.43).
The second equation should yield the Hamiltonian constraint:

$$\begin{aligned} \tilde{e}^i_a \tilde{e}^j_b \epsilon^{abc} F_{cij} &= h h^{ni} h^{rj} e_{an} e_{br} \epsilon^{abc} \\ &\quad (\frac{i}{4}\epsilon^{klm} ({}^{(3)}R_{klij} + 2K_{ki}K_{lj}) e_{cm} + \frac{1}{2}(K_{kj|i} - K_{ki|j}) e_c{}^k) \end{aligned} \tag{5.30}$$

The key is to use the three dimensional identity from [12] that:

$$e_{an}e_{br}\epsilon^{abc} = h^{1/2}\epsilon_{nrt}e^{ct} \tag{5.31}$$

and the fact that:

$$\begin{aligned}\epsilon_{nrt}\epsilon_{klm}\delta^t_m &= \delta_{nk}(\delta_{rl}\delta_{tm} - \delta_{rm}\delta_{tl})\delta^t_m \\ &\delta_{nl}(\delta_{rk}\delta_{tm} - \delta_{rm}\delta_{mk})\delta^t_m \\ &\delta_{nm}(\delta_{rk}\delta_{ml} - \delta_{rl}\delta_{mk})\delta^t_m \\ &= \delta_{nk}\delta_{rl} - \delta_{nl}\delta_{rk}\end{aligned} \tag{5.32}$$

where δ^t_m comes from the fact that $e^{ct}e_{cm} = \delta^t_m$. First look at the first term of (5.30).

$$\begin{aligned}&\frac{ih^{3/2}}{4}h^{ni}h^{rj}(\delta_{nk}\delta_{rl} - \delta_{nl}\delta_{rk})({}^{(3)}R_{klij} + 2K_{ki}K_{lj}) = \\ &= \frac{ih^{3/2}}{2}({}^{(3)}R + K^2 - K_{kl}K^{kl})\end{aligned} \tag{5.33}$$

This equation is already the Hamiltonian constraint multiplied by some scalars, which can be divided out as soon is established that the second term of (5.30) yields zero.

$$\begin{aligned}&\frac{1}{2}h^{3/2}h^{ni}h^{rj}\epsilon_{nrt}(K_{kj|i} - K_{ki|j})e^{ct}e_c{}^k \\ &= \frac{1}{2}h^{3/2}h^{ni}h^{rj}\epsilon_{nrt}(K_{kj|i} - K_{ki|j})h^{tk} \\ &= \frac{1}{2}h^{3/2}\epsilon_{nrt}(K^{tr|n} - K^{tn|r}) = 0\end{aligned} \tag{5.34}$$

So it has been proven that $\tilde{e}^i_a\tilde{e}^j_b\epsilon^{abc}F_{cij}$ does indeed yield the Hamiltonian constraint.
The constraint equations have now been rewritten in such a manner that they are polynomial in e, A and derivatives of A.

5.2 Troubles with the connection representation

Unfortunately there was one choice that was made in obtaining the polynomial constraint equations that shows to be the end of the connection representation. The Barbero-Immirziparameter γ, which was chosen equal to i, should be a real number. If it is kept a complex number the phase space of general relativity is now in the complex plane. This imposes the challenge of finding reality conditions after quantizing the theorem. Finding suitable reality conditions for a complex theory of quantum gravity proved to be impossible. Therefore the complex value of γ was dropped and with that also the polynomial form of the hamiltonian constraint.

5.3 Loops

With the introduction of the formulation of the triads we came across a new constraint. This is the SO(3)-invariance, or rotational invariance. There is a large class of functionals in terms of the Ashtekar's variables that is already SO(3)-invariant. These are the Wilson loops, the trace of the holonomy of the Ashtekar variable.

$$W_\gamma(A) = Tr(P \exp \oint dy^a A_a) \tag{5.35}$$

These loops, form in fact a basis for all SO(3)-invariant functionals. This loops can be taken as our basic variables. This is called the loop representation. Because these Gauge-invariant functionals are the new basic variables one can forget about the SO(3) constraint, since it will always vanish. It turns out that the Wilson loops in fact form an overcomplete basis. Therefore they themselves have to satisfy constraint equations, the Mandelstam identities. These identities play a very complicating role in the rest of the theory, as we shall see.

So what happens to the hamiltonian and diffeomorphism constraint equations with the Wilson loops as our new canonical variables? First the diffeomorphism constraint seemed to be solved naturally. This diffeomorpism constraint acts on the wavefunctions by shifting the loop infinitesimally. So if one considers loops that are invariant under such deformations the constraint is satisfied. These type of wavefunctions are called knot-invariants and were studied for a long time by mathematicians. So there was a large class of wavefunctions satisfying the diffeomorpism constraints. It also seemed that for smooth loops the hamiltonian constraint was satisfied. So this leads to the conclusion that if one uses knot invariants supported on smooth loops only, this would yield a class of solutions to all constraint equations. Unfortunately this is not possible, because (1) knot invariants support on smooth loops do not satisfy the Mandelstam identities and (2) smooth loops seem to simple to carry relevant physics. One needs loop intersections to build up a volume operator for example. Solving these problems takes the reader into a deeper, more technical explanation of the current state of LQG, including spin networks and area/volume operators, which lies beyond the scope of this article.

6 Present state of Loop Quantum Gravity

There are many open questions in LQG, some of which are very fundamental and prevent many physicists from taking the theory as serious as, for example, string theory. The following is a list of the most important ones.

Classical limit and physical interpretation

Every fundamental theory in physics should have a classical limit in which the physics of our everyday experience re-appears (i.e. Minkowsky spacetime and Newton's laws). For example one can obtain Newton's laws from GR by taking some limits in which quantum effects and velocities near the speed of light are eliminated. As mentioned in the short overview of the formalism, so far LQG lacks such a limit. The key problem here is that the space in which LQG mathematics is defined is a not a regular, seperable Hilbert space which is generally required to make physical predictions. LQG takes place in a Hilbert space which is non-seperable. This means that each continuous function in regular space time is mapped into an uncountable number of states in the LQG Hilbert space. The problem is therefore how to construct a continuous space from this discontinuous one. The fact however that LQG physics takes place in a non-seperable space is actually also the main reason why the theory gives hope to predict solutions to the constraint equations.
This unconventional Hilbert space also leads to problems when one tries to interpret the solutions to the Hamiltonian constraint. These solutions are analyzed in the mathematical branch of knot theory. So far there a link has not been established to physical reality.

Spacetime covariance

In the regular GRT formulation as well as in the canonical description the theory is fully spacetime covariant. This spacetime covariance is not necessarily maintained after quantization. There is yet no proof that LQG is fully spacetime covariant or will at least has this covariance in it's GRT limit. There is however a very interesting experiment suggested in [13], that will be able to test this and in fact will be able to test the structure of spacetime predicted by LQG. This experiment can therefore be seen as a *crucial* test for the theory. In short the experiment says the following. LQG predicts that light is scattered of the discrete structure of space, which has a very small effect on the speed of the light. This effect is larger for higher energetic photons, e.g. the speed of light is dependent on the energy of the photon. Therefore if a high and a low energy photon were emitted by the gamma ray burster at the same time, there will be a time delay in the arrival at earth. For normal gamma rays coming from our own galaxy this effect would not be measurble. There is however evidence (due to the measuring of red shifts of gamma rays incident on detectors in satellite's) of gamma ray bursters that are on the scale of cosmological distances away. It is shown in [13] that with these gamma rays the prediction by LQG can in fact be measured. There is yet no conclusive experimental data to show whether or not there is an energy dependence. If the experiment does indicate this dependence this would mean that one of the foundations of modern physics, the principle of relativity, would have to be revised.

The divergent perturbation expansion

An important question is what happens to the divergence that emerges when general relativity is expanded in a perturbation series? There should be a good explanation about the difference between the perturbative approach and the LQG approach such that it is logical that the divergence vanishes. This problem is connected to that of finding semi-classical states. These states should be able to reproduce the infinities found in the perturbative approach.

Matter coupling

It seems that LQG sets no limits on the types of mass it applies to. One can just add all sorts of different types of matter to the theory of pure gravity. This is very different from other theories such as superstring theory in which matter is necessary to take care of the inconsistencies that emerge when doing perturbative quantum gravity. Also there has not yet been a calculation that relates the predictions of LQG to a physical observable such as the scattering amplitude.

Of course there are also various results of LQG. Some of them are quite controversal though and therefore they will not be named here, foremost because it lies not in the scope of this article to check these results. If you are interested in these results you can for example read Lee Smolins article: An invitation to Loop Quantum Gravity. A very interesting result of LQG is for example the discreteness of space at the smallest scales, due to the discrete area and volume operators.

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A Appendix A: 3-dimensional gravity

A.1 The difference with 4-dimensional gravity

An example for which LQG is a bit further with developing physical solutions is 3-dimensional gravity. Investigating 3-dimensional gravity in the LQG-formalism is not a simple exercise. In this appendix loops are shown to be appropiate variables to describe 2+1 dimensional gravity. A more extended view on this subject, which included some examples can be found in [10].

The main feature of 3-dimensional gravity is that it has no local degrees of freedom, i.e. there are no gravitational waves. This can be seen through the following calculation. From now in 3-dimensional gravity will be named 2+1-dimensional gravity, 4-dimensional gravity will be named 3+1-dimensional gravity.

The Einstein equations for 2+1-dimensional gravity are given by the same expression as in 3+1 dimensions.

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{A.1}$$

The Riemann-curvature tensor can be written in a different way [13], namely

$$\begin{aligned} R^{\mu\nu}{}_{\alpha\beta} &= \epsilon^{\mu\nu\lambda}\epsilon_{\alpha\beta\sigma}G^{\sigma}_{\lambda} \\ &= 8\pi G\epsilon^{\mu\nu\lambda}\epsilon_{\alpha\beta\sigma}T^{\sigma}_{\lambda} + \Lambda(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}) \end{aligned} \tag{A.2}$$

Where the Einstein equations have been used to obtain the last step. The curvature outside sources now is equal to:

$$R = R^{\mu\nu}{}_{\mu\nu} = 0 + \Lambda(9-3) = 6\Lambda \tag{A.3}$$

which is a constant. This proofs that in 2+1-dimensional gravity there are no gravitational waves, which simplifies the theory.

Although in 2+1 dimensions there are no local degrees of freedom the structure of 2+1 gravity in the loop representation is rather similar to that in 3+1 dimensions. The theory of LQG in 2+1 dimensions has a strong resemblance of that of Chern-Simons theory. In the following section this resemblance will be shown and loops will be introduced as appropiate basic variables.

A.2 Chern-Simons formulation of 2+1 dimensional gravity

The Chern-Simons formulation is a first order formulation of 2+1 dimensional gravity. The fundamental variables that will be used to derive the Chern-Simons action are the earlier encountered terms: the triad ${e_\mu}^a$ and the spin connection ${\omega_\mu}^{ab}$. Just as in the Palatini formulation these variables are treated independently. The action should now be defined in terms of the one forms of the triad and the spin connection. These are defined as:

$$\begin{aligned} e^a &= {e_\mu}^a dx^\mu \\ \omega^a &= \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu \end{aligned} \tag{A.4}$$

The action for a vanishing cosmological constant as derived for 4 dimensions in appendix B given by these quantities is:

$$S = 2\int [e^a \wedge (d\omega_a + \frac{1}{2}\epsilon_{abc}\omega_b \wedge \omega_c)] \tag{A.5}$$

This can be seen by using Cartan's first and second structural equation (in 3 dimensions) and fill them in in the ordinary Einstein Hilbert action. This action can be interpretated as an action in the Chern-Simons form as is argued by Edward Witten in [11]. The action is then:

$$S = \frac{k}{4\pi}\int tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \tag{A.6}$$

where A is the Gauge potential.
The field equations obtained by taking $\frac{\delta I}{\delta A}$ give:

$$F[A] = dA + A \wedge A = 0 \tag{A.7}$$

which is only obeyed if A is a flat connection. Such a flat connection is completely determined by its holonomies.
The physical observables are those functions on phase space whose Poisson brackets with the constraint equations vanish. It turns out that these observables for the Chern-Simmons theory are the Wilson loops:

$$U_\gamma = P\exp(\int_\gamma A) \tag{A.8}$$

With that loops have proven to be appropiate variables to describe 2+1 dimensional gravity and a link to the LQG-formalism is established.

B Appendix B: The tetrad formulation

The purpose of this appendix is to discuss some elementary properties that are important in the tetrad formulation of general relativity. In this discussion [16] will be followed, which is a very good and much more elaborate introduction to the subject. The goal is to derive Cartan's first and second structural equation, which can be used to write the Einstein Hilbert action in terms of a tetrad and a quantity called the spin connection. This result is also used in appendix A.

One forms

A one form α is a linear function from a vector space V into $\mathbb{R}$, i.e. $\alpha(v)$ is a real number. To write this one form into component form one needs to define the basis $\{\omega^\mu\}$ of the one form so that $\alpha = a_\mu \omega^\mu$:

$$\omega^\mu(e_\nu) = \delta^\mu_\nu \tag{B.1}$$

So now:

$$\alpha(e_\mu) = a_\nu \omega^\nu(e_\mu) = a_\mu \tag{B.2a}$$

$$\alpha(v) = \alpha(v^\mu e_\mu) = v^\mu a_\mu = i_v \alpha \tag{B.2b}$$

where the last is called the contraction or interior product of α with v.

Tensor properties

A tensor maps vectors as well as one forms onto $\mathbb{R}$. The tensor product of two covariant tensors T and S of rank m and n is defined as:

$$T \otimes S(u_1...u_m, v_1...v_n) = T(u_1...u_m)S(v_1...v_n) \tag{B.3}$$

For example, if $u \otimes v = T$, then:

$$T(\alpha, \beta) = u \otimes v(\alpha, \beta) = u(\alpha)v(\beta) = u^\mu a_\mu v^\nu b_\nu \tag{B.4}$$

The component form of a contravariant tensor of rank q in n dimensions is:

$$R = R^{\mu_1 \cdots \mu_q} e_{\mu_1} \otimes ... \otimes e_{\mu_q} \tag{B.5}$$

where $\{e_{\mu_1}...e_{\mu_q}\}$ is a maximally independent set of basis elements and

$$R^{\mu_1 \cdots \mu_q} \equiv R(\omega^{\mu_1}...\omega^{\mu_q}) \tag{B.6}$$

A covariant tensor is expressed in a similarly fashion.

For example the tensor components of $R = u \otimes v$, $S = \alpha \otimes v$ and $T = \alpha \otimes \beta$ are:

$$R^{\mu\nu} = (u \otimes v)(\omega^\mu, \omega^\nu) = u^\mu v^\nu \tag{B.7a}$$

$$S_\mu{}^\nu = (\alpha \otimes v)(e_\mu, \omega^\nu) = \alpha_\mu v^\nu \tag{B.7b}$$

$$T_{\mu\nu} = (\alpha \otimes \beta)(e_\mu, e_\nu) = \alpha_\mu \beta_\nu \tag{B.7c}$$

Forms

A p-form is defined as a antisymmetric covariant tensor of rank p. Therefore to write this in a component form one needs an antisymmtric tensor basis, which can be defined as:

$$\omega^{[\mu_1} \otimes ... \otimes \omega^{\mu_p]} = \frac{1}{p!} \sum_{i=1}^{p!} (-1)^{\pi(i)} \omega^{\mu_1} \otimes ... \otimes \omega^{\mu_p} \tag{B.8}$$

with

$$\pi(i) = \begin{cases} +1 & \text{if the permutation is even} \\ -1 & \text{if the permutation is odd} \end{cases}$$

From this it can be seen that any p-form can be written in components as:

$$\alpha = \alpha_{|\mu_1 ... \mu_p} \omega^{[\mu_1} \otimes ... \otimes \omega^{\mu_p]} \tag{B.10}$$

where the vertical bars denote that only components with increasing indices are included.

An antisymmetric tensor product is defined as:

$$\omega^{[\mu_1} \otimes ... \otimes \omega^{\mu_p]} \wedge \omega^{[\nu_1} \otimes ... \otimes \omega^{\nu_q]} = \frac{(p+q)!}{p!q!} \omega^{[\mu_1} \otimes ... \otimes \omega^{\mu_p} \otimes \omega^{\nu_1} \otimes ... \otimes \omega^{\nu_q]} \tag{B.11}$$

also called the wedge product or the exterior product. This product is linear and associative.

Equation (B.10) can now be written as:

$$\alpha = \alpha_{|\mu_1 ... \mu_p|} \omega^{\mu_1} \wedge ... \wedge \omega^{\mu_p} \tag{B.12}$$

So every p-form can be written as the exterior product of its antisymmetric basis components.

Differentiating forms

Of course one also wants to differentiate forms. This can be done with the exterior derivative, which for a zero form is defined as:

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu \tag{B.13}$$

This derivative is invariant under coordinate transformations and is thus independent of the chosen coordinate system.

The exterior derivative for a p-form is similarly:

$$d\alpha = \frac{1}{p!} \frac{\partial \alpha_{\mu_1 \dots \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{B.14}$$

which is an antisymmetric tensor with rank $p+1$

Covariant differentiation of vectors

Now let us shortly go back to the covariant differentiation of vectors, which will lead to the introduction of the well known Christoffel symbols. Consider a vector A with components A^μ in a coordinate basis $\{e_\mu\}$. If one differentiates A with respect to a parameter λ the following result is obtained:

$$\frac{dA}{d\lambda} = \frac{dA^\mu}{d\lambda} e_\mu + A^\mu \frac{de_\mu}{d\lambda} = \partial_\nu A^\mu u^\nu e_\mu + A^\mu u^\nu \partial_\nu e_\mu \tag{B.15}$$

where $u^\mu = \frac{dx^\mu}{d\lambda}$.

$$\frac{de_\mu}{d\lambda} = \Gamma^\alpha_{\mu\nu} \frac{dx^\nu}{d\lambda} e_\alpha \tag{B.16}$$

This can be put back into equation (B.15) to obtain that the covariant derivative of A^μ is:

$$A^\mu_{;\nu} = \partial_\nu A^\mu + \Gamma^\mu_{\alpha\nu} A^\alpha \tag{B.17}$$

In a graphical respresentation the covariant derivative, also denoted with ∇, can be seen in 3 For a vector now holds:

$$\nabla_u A = (e_\nu(A^\mu) u^\nu + A^\alpha \Gamma^\mu_{\alpha\nu}) e_\mu \tag{B.18}$$

Now the torsion tensor will be introduced, for which one needs the definition of structure constants. These structure constants are defined as follows. If f is a scalar function and u and v are vector fields:

$$u(f) = u^\mu e_\mu(f) \tag{B.19}$$

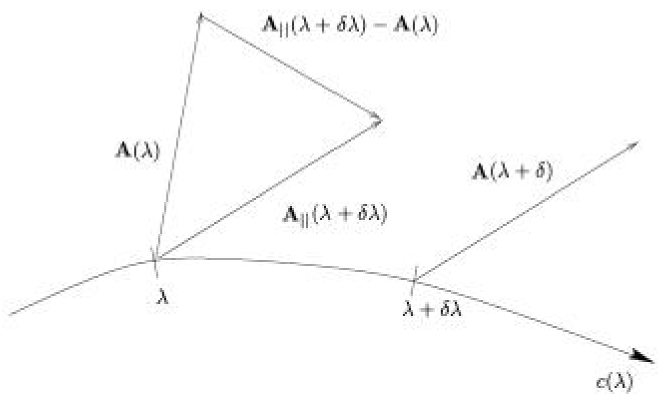

Figure 3: Graphical representation of the covariant derivative

where e_μ can be interpreted as a partial derivative. Then it can be seen that:

$$uv(f) = u^\mu e_\mu(v^\nu e_\nu(f)) = u^\mu e_\mu(v^\nu)e_\nu(f) + u^\mu u^\nu e_\mu e_\nu \tag{B.20}$$

The commutator of two vector fields is then:

$$[u, v] = (u^\mu e_\mu(v^\nu) - v^\mu e_\mu(u^\nu))e_\nu + u^\mu u^\nu [e_\mu, e_\nu] \tag{B.21}$$

The structure constants are then defined by:

$$[e_\mu, e_\nu] = c^\rho{}_{\mu\nu} e_\rho \tag{B.22}$$

With help of (B.18) and (B.21) one obtain:

$$[u, v] = \nabla_u v - \nabla_v u + (\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} + c^\rho_{\mu\nu})u^\mu u^\nu e_\rho \tag{B.23}$$

The torsion operator is defined as:

$$T(u \wedge v) \equiv \nabla_u v - \nabla_v u - [u, v] = T^\rho_{\mu\nu} u^\mu u^\nu e_\rho \tag{B.24}$$

So that in a torsion free spacetime:

$$\Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} = c^\rho_{\mu\nu} \tag{B.25}$$

Covariant differentiation of tensors and forms

The covariant derivative on a scalar function is defined as:

$$\nabla_X f = X(f), \text{ since } \quad \nabla_X = \nabla_{X^\mu e_\mu} = X^\mu \nabla_\mu \tag{B.26}$$

The covariant derivative on a one form is defined as:

$$(\nabla_X \alpha)(A) = \nabla_X(\alpha(A) - (\nabla_X A)\alpha \tag{B.27}$$

So for basis vectors ω and e one gets:

$$(\nabla_\alpha \omega^\mu)e_\beta = -\omega^\mu(\nabla_\alpha e_\beta) = -\Gamma^\mu_{\alpha\beta} \tag{B.28}$$

And therefore:

$$\nabla_\alpha \omega^\mu = -\Gamma^\mu_{\beta\alpha}\omega^\beta \tag{B.29}$$

which is used to obtain:

$$\nabla_\lambda \alpha = [e_\lambda \alpha_\nu - \alpha_\mu \Gamma^\mu_{\nu\lambda}]\omega^\nu = \alpha_{\nu;\lambda}\omega^\nu \tag{B.30}$$

The covariant derivative acting on the tensor product is:

$$\nabla_X(A \otimes B) = (\nabla_X A) \otimes B + A \otimes (\nabla_X B) \tag{B.31}$$

From this the covariant derivative of the metric is derived as:

$$g_{\mu\nu;\alpha} = e_\alpha(g_{\mu\nu}) - g_{\beta\nu}\Gamma^\beta_{\mu\alpha} - g_{\mu\beta}\Gamma^\beta_{\nu\alpha} \tag{B.32}$$

Plugging in the standard definition of $\Gamma^\mu_{\alpha\beta}$ yields that in the coordinate basis the covariant derivative of the metric defined in this way gives zero, as one would expect.

Exterior derivative of a basis vector field

The exterior derivative of a basis vector field is given by:

$$de_\mu = \Gamma^\nu_{\mu\alpha} e_\nu \otimes \omega^\alpha \tag{B.33}$$

The exterior derivative applied to a vector field A yields

$$\begin{aligned} dA &= d(A^\mu e_\mu) \\ &= e_\mu \otimes dA^\nu + A^\mu de_\mu \\ &= [e_\lambda(A^\nu) + A^\mu \Gamma^\nu_{\mu\lambda}]e_\nu \otimes \omega^\lambda \\ &= A^\nu_{;\lambda} e_\nu \otimes \omega^\lambda \end{aligned} \tag{B.34}$$

One can then define connection forms, Ω^ν_μ by:

$$de_\mu = e_\nu \Omega^\nu_\mu \tag{B.35}$$

from which according to (A.34)it follows that:

$$\Omega^{\nu}_{\mu} = \Gamma^{\nu}_{\mu\alpha}\omega^{\alpha} \tag{B.36}$$

which is antisymmetric.

If the following four relations are calculated:

$$\alpha([u,v]) = u^{\mu}a_{\nu}v^{\nu}_{,\mu} - v^{\mu}a_{\nu}u^{\nu}_{,\mu} \tag{B.37a}$$
$$u(\alpha(v)) = u^{\mu}v^{\nu}\alpha_{\nu,\mu} + u^{\mu}\alpha_{\nu}v^{\nu}_{,\mu} \tag{B.37b}$$
$$v(\alpha(u)) = v^{\mu}u^{\nu}\alpha_{\nu,\mu} + v^{\mu}\alpha_{\nu}u^{\nu}_{,\mu} \tag{B.37c}$$
$$d\alpha(u \wedge v) = (\alpha_{\mu,\nu} - \alpha_{\nu,\mu})u^{\nu}v^{\mu} \tag{B.37d}$$

one obtains:

$$d\alpha(u \wedge v) = u(\alpha(v)) - v(\alpha(u)) - \alpha([u,v]) \tag{B.38}$$

If $\alpha = \omega^{\rho}$, $u = e_{\mu}$ and $v = e_{\nu}$ this changes into:

$$d\omega^{\rho} = -\frac{1}{2}c^{\rho}_{\mu\nu}\omega^{\mu} \wedge \omega^{\nu} \tag{B.39}$$

The torsion operator has the component form:

$$\begin{aligned} T &= \frac{1}{2}(\Gamma^{\rho}_{\mu}\nu - \Gamma^{\rho}_{\nu}\mu - c^{\rho}_{\mu}\nu)e_{\rho} \otimes \omega^{\mu} \wedge \omega^{\nu} \\ &= e_{\rho} \otimes (d\omega^{\rho} + \Omega^{\rho}_{\nu} \wedge \omega^{\nu} \\ &= e_{\rho} \otimes T^{\rho} \end{aligned} \tag{B.40}$$

In Riemannian geometry therefore holds that:

$$d\omega^{\rho} = -\Omega^{\rho}_{\nu} \wedge \omega^{\nu} \tag{B.41}$$

where ω^{ρ} is often referred to as the spin connection. (B.41) is called Cartan's first structural equation, which is needed to write our Einstein Hilbert action in terms of the variables e_{μ} and ω^{ν}. Cartan's second structural equation links the Riemann curvature tensor to a tetrad and the spin connection. Its derivation can be found in [16]. The result is just given here:

$$R^{a}_{b} = d\omega^{a}_{b} + \omega^{a}_{c} \wedge \omega^{c}_{b} \tag{B.42}$$

This can be filled in into the Einstein Hilbert action from which it follows that:

$$S = \int_{\mathcal{M}} \epsilon^{abcd} R_{ab} \wedge e_c e_d \tag{B.43}$$

where e_c denotes a tetrad.