The anharmonic oscillator in quantum mechanics

Summary

In this article the anharmonic oscillator in quantum mechanics is discussed. First propagators of less complex systems are found by using the path integral method.
With perturbation theory an expression, up to any order, has been found for the vacuum propagator of the anharmonic oscillator. In this article only the first and second order were calculated and by looking at the n-pointfunctions the Feynman diagrams of these orders were found. It appears that the anharmonic term up to first order causes a change in the frequency of the system.
Looking at second order perturbation theory gives five diagrams. The first four of these diagrams are simple diagrams and again these diagrams alter the frequency of the system. The change in the frequency looks a lot like a series and it is possible to 'guess' a general expression up to any order for all simple diagrams. This expression seems to fit qualitatively for higher orders however the prefactors were not checked nor is there a proof formulated.
The fifth diagrams is a complex diagram and this diagram has not been solved in a satisfying way.
Finally some recommendations for additional research are given.

Jordy de Vries, 1381156

Introduction

In this article we discuss the quantum anharmonic oscillator. This system is of great importance in many branches of physics and it is interesting to look at it with different methods. In quantum field theory it the anharmonic oscillator has been studied in great detail. In that area of physics it is possible to find Feynman diagrams and Feynman rules which determine the vacuum propagator of the system. In this article these diagrams and rules are searched for in standard quantum mechanics. It appears that there is a great analogy between these two branches of physics.
This article starts out with basic propagator theory and path integral methods. Then the propagators of several physical systems are calculated and with those results an expression is found for the vacuum propagator of the anharmonic oscillator.
This propagator is then calculated and the influence of the anharmonic term is determined up to second order.

Propagator in quantum mechanics

Let's look at the Schrodinger equation of a particle in one dimension

$$i\hbar \frac{\partial}{\partial t}\Psi(q,t) = H(p,q)\Psi(q,t) \tag{1}$$

We are only considering Hamiltonians of the form

$$H(p,q) = \frac{p^2}{2m} + V(q) \qquad p = -i\hbar\frac{\partial}{\partial q}. \tag{2}$$

If the wavefunction is known at a certain time t_i the Schrodinger equation determines the wavefunction at later times and it is possible to write

$$\Psi(q,t) = \int dq' K(q,t;q',t_i)\Psi(q',t_i). \tag{3}$$

In the Heisenberg (time-independent states and time-dependant operators) picture we can write

$$\Psi(q,t) =< q,t \mid \Psi >_H, \tag{4}$$

and since the states $\mid q,t >$ form a complete set at every time

$$\Psi(q,t) = \int dq' < q,t \mid q',t_i >< q',t_i \mid \Psi >_H . \tag{5}$$

Compare this with (3) and it is easily seen that

$$K(q,t;q',t_i) =< q,t \mid q',t_i >=< q \mid e^{\frac{-i}{\hbar}(t-t_i)H} \mid q' > . \tag{6}$$

If the eigenstates of the system are known the propagator can be written in a more simple way. If $H \mid n >= E_n \mid n >$,

$$K(q,t;q',t_i) = \sum_n < q \mid n >< n \mid q' > e^{\frac{-i}{\hbar}E_n(t-t_i)} =$$

$$\sum_n \Psi_n(q)\Psi_n^*(q')e^{\frac{-i}{\hbar}E_n(t-t_i)}. \tag{7}$$

It is useful to look at an important property of the propagator. If we introduce a time $t_1,\ t_i < t_1 < t_f$,

$$\Psi(q'',t_1) = \int dq' K(q'',t_1;q',t_i)\Psi(q',t_i)$$

$$\Psi(q,t_f) = \int dq'' K(q,t_f;q'',t_1)\Psi(q'',t_1) =$$

$$\int dq'' K(q,t_f;q'',t_1) \int dq' K(q'',t_1;q',t_i)\Psi(q',t_i). \tag{8}$$

So together with (3) this gives,

$$K(q,t_f;q',t_i) = \int dq'' K(q,t_f;q'',t_1)K(q'',t_1;q',t_i). \tag{9}$$

Now it is possible with (7) to calculate the propagator of some systems, however it useful to look at different method of calculation; the path-integral method.

The path-integral method

With property (9) it is possible to find $K(q,t_f;q',t_i)$ in a different way. (9) can be written as:

$$< q,t_f \mid q',t_i >= \int dq'' < q,t_f \mid q'',t_1 >< q'',t_1 \mid q',t_i >, \tag{10}$$

however we can introduce a whole set of times $t_i < t_1 < \cdots < t_n < t_f$ so that:

$$< q,t_f \mid q',t_i >=$$

$$\int dq_1 dq_2 \cdots dq_n < q,t_f \mid q_n,t_n >< q_n,t_n \mid \cdots \mid q_1,t_1 >< q_1,t_1 \mid q',t_i > . \tag{11}$$

So in order to find $K(q,t_f;q',t_i)$ we are integrating over every possible path which begins at q' and ends at q. Now we can find an explicit expression for $K(q,t_f;q',t_i)$. First it is necessary to look at the small steps between q_j and q_{j+1}. If we take the limit of n going to infinity and assume all time-intervals are equal and given by $\tau \equiv t_{j+1} - t_j$ then τ is infinitely small.

$$< q_{j+1},t_{j+1} \mid q_j,t_j >=< q_{j+1} \mid e^{\frac{-i}{\hbar}H\tau} \mid q_j >=$$

$$< q_{j+1}, q_j > -\frac{i}{\hbar}\tau < q_{j+1} \mid H \mid q_j > +O(\tau^2). \qquad (12)$$

In the last step a Taylor expansion is used. Because H is of the form (2) the second term of the righthand side of (12) can be written as two terms; the kinetic term and the potential. First let's look at the kinetic term.

$$< q_{j+1} \mid \frac{p^2}{2m} \mid q_j >= \int dp' dp'' < q_{j+1} \mid p' >< p' \mid \frac{p^2}{2m} \mid p'' >< p'' \mid q_j >=$$

$$\frac{1}{2\pi\hbar}\int dp' dp'' e^{\frac{i}{\hbar}(p' q_{j+1} - p'' q_j)} \frac{p''^2}{2m}\delta(p' - p'') = \frac{1}{2\pi\hbar}\int dp' e^{\frac{i}{\hbar}p'(q_{j+1}-q_j)}\frac{p'^2}{2m}. \quad (13)$$

Now for the potential term

$$< q_{j+1} \mid V(q) \mid q_j >= \frac{V(\bar{q}_j)}{2\pi\hbar}\int dp' e^{\frac{i}{\hbar}p'(q_{j+1}-q_j)}, \quad \bar{q}_j = \frac{q_{j+1} - q_j}{2}. \qquad (14)$$

Filling this in (12) gives

$$< q_{j+1}, t_{j+1} \mid q_j, t_j >= \frac{1}{2\pi\hbar}\int dp_j e^{\frac{i}{\hbar}p_j(q_{j+1}-q_j)}(1 - \frac{i}{\hbar}\tau(\frac{p_j^2}{2m} + V(\bar{q}_j)) + O(\tau^2) =$$

$$\frac{1}{2\pi\hbar}\int dp_j e^{\frac{i}{\hbar}p_j(q_{j+1}-q_j)} e^{\frac{-i}{\hbar}\tau H(p_j, \bar{q}_j)}. \qquad (15)$$

If we write $q_{n+1} = q$ and $q_0 = q'$ we see

$$< q, t_f \mid q', t_i >= \lim_{n\to\infty}\int \prod_{j=1}^{n} dq_j \prod_{j=0}^{n}\frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar}\sum_{j=0}^{n}(p_j(q_{j+1}-q_j) - \tau H(p_j, \bar{q}_j))} \qquad (16)$$

If we only use differentiable paths we can write $q_{j+1} - q_j = \tau\dot{q}(t)$ and write the sum over τ as an integral over time. In that case

$$< q, t_f \mid q', t_i >= \int \frac{DqDp}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'(p\dot{q} - H(p,q))} \qquad (17)$$

Since we are considering Hamiltonians of the form (2) we can write (17) in a more convenient form by actually doing the integration over the impulses.

$$\int \prod_{j=0}^{n}\frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar}\sum_{j=0}^{n}(p_j(q_{j+1}-q_j) - \tau\frac{p_j^2}{2m})} \qquad (18)$$

In order to do this integral we to have to use the following standard integral:

$$\int dx\, e^{-a(b-x)^2} e^{cx} = e^{bc + \frac{c^2}{4a}}\sqrt{\frac{\pi}{a}} \qquad (19)$$

Which can be proven quite easily by changing variables and doing a Gauss integral. In this case we have $x = p_j$, $a = \frac{i\tau}{2m\hbar}$, $b = 0$, $c = \frac{(iq_{j+1} - q_j)}{\hbar}$ so performing the integral gives:

$$\prod_{j=0}^{n} \frac{1}{2\pi\hbar} \sqrt{\frac{2m\pi\hbar}{i\tau}} e^{\frac{-(q_{j+1}-q_j)^2 2m\hbar}{4i\tau\hbar^2}} = (\frac{m}{2\pi\hbar i\tau})^{\frac{n+1}{2}} e^{\sum_{j=0}^{n} \frac{im\tau}{2\hbar}(\frac{q_{j+1}-q_j}{\tau})^2} \tag{20}$$

Filling this in (18) gives:

$$< q, t_f \mid q', t_i >= \lim_{n \to \infty} \int \prod_{j=1}^{n} dq_j (\frac{m}{2\pi\hbar i\tau})^{\frac{n+1}{2}} e^{\frac{i}{\hbar}\tau \sum_{j=0}^{n} (\frac{m}{2}(\frac{q_{j+1}-q_j}{\tau})^2 - V(\bar{q}_j))} \tag{21}$$

And finally, by taking the limit we get:

$$< q, t_f \mid q', t_i >= N \int Dq\, e^{\frac{i}{\hbar}\int dt\, (\frac{1}{2}m\dot{q}^2 - V(q))} \tag{22}$$

We see that the term in the exponent is nothing else then $\frac{i}{\hbar}$ times the classical action. So in quantum mechanics every possible path gives a contribution to the propagator. This is completely different from classical mechanics in which only one path contributes to the propagator; the path which minimizes the action. If we take the limit from quantum mechanics to classical mechanics ($\hbar \to 0$), we get that result back.

For Hamiltonians with a potential which is no more then a quadratic polynomial of q, thus $V(q) = V_0 + V_1 q + V_2 q^2$ it is possible to find a real simple expression for the propagator. Since the free particle, the harmonic oscillator with and without an external linear force all have this property it is useful to find this expression. This will greatly simplify calculating the propagators for these systems.

For Hamiltonians of the form (2) the equation of motion is:

$$m\ddot{q}(t) + \frac{\partial V(q(t))}{\partial q} = 0, \;\; q(t_i) = q', \;\; q(t_f) = q. \tag{23}$$

Now let's introduce the classical path $q_{cl}(t)$ and the difference between the classical and 'real' path $x(t)$ so that:

$$q(t) = q_{cl}(t) + x(t), \;\; x(t_i) = x(t_f) = 0. \tag{24}$$

Then we can write the Lagrangian as, by Taylor expanding the potential around the classical path,

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q) =$$

$$L(q_{cl}, \dot{q_{cl}}) + \frac{1}{2}m\dot{x}^2 + m\dot{q_{cl}}\dot{x} - x\frac{\partial V(q)}{\partial q}|_{q=q_{cl}} - \frac{1}{2}x^2\frac{\partial^2 V(q)}{\partial^2 q}|_{q=q_{cl}}. \tag{25}$$

Two of this terms will cancel which can be seen by integrating these terms over time:

$$\int dt \ (m\dot{q_{cl}}\dot{x} - x\frac{\partial V(q)}{\partial q}|_{q=q_{cl}}) =$$

$$m\dot{q_{cl}}x|_{t_i}^{t_f} - \int dt \, x(m\ddot{q_{cl}} + \frac{\partial V(q)}{\partial q}|_{q=q_{cl}}) = 0. \tag{26}$$

This gives

$$< q, t_f \mid q', t_i >= N \int Dq \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} \, dt \, (\frac{1}{2}m\dot{q}^2 - V(q))} =$$

$$e^{\frac{i}{\hbar}S_{cl}} \int Dx \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} \, dt \, (\frac{1}{2}m\dot{x}^2 - \frac{1}{2}x^2 \frac{\partial^2 V(q)}{\partial^2 q}|_{q=q_{cl}})}. \tag{27}$$

But since the potential is no more then a quadratic polynomial of q, the second partial derivative will be a constant. This means that we can write:

$$< q, t_f \mid q', t_i >= N e^{\frac{i}{\hbar}S_{cl}} \tag{28}$$

Propagator of a free particle

First let us look at the free particle case,

$$S = \int dt \, L(q, \dot{q}) = \int dt \, \frac{1}{2}m\dot{q}^2. \tag{29}$$

Minimizing the action gives naturally $\ddot{q} = 0$ and this means

$$\dot{q} = b, \ \ q(t) = bt + c \tag{30}$$

With the boundary conditions $q(t_i) = q', \ q(t_f) = q$ this can be easily solved for c and b:

$$c = \frac{q - q'}{t_f - t_i}, \ b = \frac{q' t_f - q t_i}{t_f - t_i}, \tag{31}$$

and this gives

$$q(t) = \frac{1}{t_f - t_i}[(t_f - t)q' + (t - t_i)q] \tag{32}$$

Differentiating this expression and filling this in the action gives

$$S = \int_{t_i}^{t_f} dt \, \frac{1}{2}m\frac{(q - q')^2}{(t_f - t_i)^2} = \frac{1}{2}m\frac{(q - q')^2}{(t_f - t_i)}, \tag{33}$$

and thus

$$K(q,t_f;q',t_i) =< q,t_f \mid q',t_i >= Ne^{\frac{i}{\hbar}\frac{1}{2}m\frac{(q-q')^2}{(t_f-t_i)}} \tag{34}$$

Of course we don't know N but in most applications N isn't of any importance.

Propagator of the harmonic oscillator with path integral method

Let's now calculate the propagator for the harmonic oscillator. Again we write down the action:

$$S = \int dt\, L(q,\dot{q}) = \int dt\, (\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2). \tag{35}$$

The equation of motion simply is:

$$\ddot{q} + \omega^2 q = 0, \;\; q(t_i) = q', \; q(t_f) = q. \tag{36}$$

Which has the solution

$$q(t) = A\sin(\omega(t-t_i)) + B\cos(\omega(t-t_i)). \tag{37}$$

Filling in the boundary conditions gives

$$A = \frac{q - q'\cos(\omega(t_f-t_i))}{sin(\omega(t_f-t_i))}, \; B = q'. \tag{38}$$

For brevity let's write $c = \cos(\omega(t_f - t_i)),\; s = \sin(\omega(t_f - t_i))$ so that

$$q(t) = \frac{q - q'c}{s}\sin(\omega(t-t_i)) + q'\cos(\omega(t-t_i)) \tag{39}$$

$$\dot{q}(t) = \omega\frac{q - q'c}{s}\cos(\omega(t-t_i)) - \omega q'\sin(\omega(t-t_i)). \tag{40}$$

With these results and a bit of algebra we can write

$$L(q,\dot{q}) = \frac{1}{2}m\omega^2[\frac{(q-q'c)^2}{s^2}\cos(2\omega(t-t_i)) -$$

$$2q'\frac{q-q'c}{s}\sin(2\omega(t-t_i)) - q'^2\cos(2\omega(t-t_i))]. \tag{41}$$

In order to find the action we now have to integrate this expression over time. A simple but tedious task. Using

$$\int_{t_i}^{t_f} dt\, \cos(2\omega(t-t_i)) = \frac{cs}{\omega}, \;\; \int_{t_i}^{t_f} dt\, \sin(2\omega(t-t_i)) = \frac{s^2}{\omega}, \tag{42}$$

we can write

$$S = \frac{1}{2}m\omega^2[\frac{(q-q'c)^2}{s^2}\frac{cs}{\omega} - 2q'\frac{q-q'c}{s}\frac{s^2}{\omega} - q'^2\frac{cs}{\omega}] =$$

$$\frac{m\omega}{2\sin(\omega(t_f-t_i))}[(q^2+q'^2)\cos(\omega(t_f-t_i))-2qq']. \tag{43}$$

With this we have found the propagator:

$$K(q,t_f;q',t_i)=N\exp(\frac{im\omega}{2\hbar\sin(\omega(t_f-t_i))}[(q^2+q'^2)\cos(\omega(t_f-t_i))-2qq']). \tag{44}$$

Propagator of the harmonic oscillator with traditional methods

In order to see the usefulness of the path integral method we will also calculate the propagator of the harmonic oscillator with (7). An advantage of this method is that we will find an expression for N.

The eigenfunctions of the harmonic oscillator are

$$\Psi_n=\frac{1}{\sqrt{2^n n!}}(\frac{m\omega}{\pi\hbar})^{\frac{1}{4}}e^{-\frac{m\omega q^2}{2\hbar}}H_n(\sqrt{\frac{m\omega}{\hbar}}q) \tag{45}$$

Where H_n are the Hermite Polynomials. For the harmonic oscillator the energies are given by $E_n=\hbar\omega(n+\frac{1}{2})$ so the expression for the propagator is

$$K(q,t_f;q',t_i)=\sum_n\Psi_n(q)\Psi_n^*(q')e^{\frac{-i}{\hbar}E_n(t_f-t_i)}=$$

$$\sum_n\frac{1}{2^n n!}(\frac{m\omega}{\pi\hbar})^{\frac{1}{2}}e^{-\frac{m\omega}{2\hbar}(q^2+q'^2)}H_n(\sqrt{\frac{m\omega}{\hbar}}q)H_n(\sqrt{\frac{m\omega}{\hbar}}q')e^{-i\omega(n+\frac{1}{2})(t_f-t_i)}. \tag{46}$$

Let's define $y\equiv\sqrt{\frac{m\omega}{\hbar}}q$ and $T\equiv e^{-i\omega(t_f-t_i)}$ so we can look at

$$k=\sum_n\frac{1}{2^n n!}\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}y^2}H_n(y)e^{-\frac{1}{2}y'^2)}H_n(y')T^n. \tag{47}$$

If we find k, we can easily find K by the relation

$$K=\sqrt{T}\sqrt{\frac{m\omega}{\hbar}}k \tag{48}$$

In order to find k we'll have to use the following expression for the Hermite polynomials

$$H_n(y)=\frac{2^n(-i)^n e^{y^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty}du\,u^n e^{2iyu-u^2}. \tag{49}$$

Filling this in the expression for k gives

$$k=\pi^{-\frac{3}{2}}e^{\frac{y^2+y'^2}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dudv\sum_n\frac{1}{n!}(-2uvT)^n e^{-u^2-v^2+2iyu+2iy'v}. \tag{50}$$

Using $\sum_n \frac{1}{n!}(-2uvT)^n = e^{-2uvT}$ we get

$$k = \pi^{-\frac{3}{2}} e^{\frac{y^2+y'^2}{2}} \int_{-\infty}^{\infty} dv\, e^{-v^2+2iy'v} \int_{-\infty}^{\infty} du\, e^{-u^2+2iyu-2uvT} \tag{51}$$

Now we have two Gaussian integrals which can be easily done by using (19). Doing the integral gives

$$k = \pi^{-\frac{3}{2}} e^{\frac{y^2+y'^2}{2}} \int_{-\infty}^{\infty} dv\, e^{-v^2+2iy'v} \sqrt{\pi} e^{v^2T^2-2iyvT-y^2} =$$

$$\pi^{-1} e^{\frac{y^2+y'^2}{2}} \int_{-\infty}^{\infty} dv\, e^{-v^2(1-T^2)+v(2iy'-2iyT)} e^{-y^2} =$$

$$\pi^{-1} e^{\frac{-y^2+y'^2}{2}} \sqrt{\frac{\pi}{1-T^2}} e^{\frac{2yy'T-y'^2-y^2T^2}{1-T^2}} =$$

$$\frac{1}{\sqrt{\pi(1-T^2)}} e^{-\frac{1}{2}(y^2+y'^2)\frac{1+T^2}{1-T^2}+\frac{2yy'T}{1-T^2}}. \tag{52}$$

Using the definition of T we can write

$$\frac{1+T^2}{1-T^2} = \frac{-i\cos(\omega(t_f-t_i))}{\sin(\omega(t_f-t_i))}, \quad \frac{T}{1-T^2} = \frac{-i}{2\sin(\omega(t_f-t_i))}, \tag{53}$$

and this gives

$$k = \frac{1}{\sqrt{\pi(1-T^2)}} e^{\frac{i}{2\sin(\omega(t_f-t_i))}[(y^2+y'^2)\cos(\omega(t_f-t_i))-yy']}. \tag{54}$$

Finally using (48) we get

$$K = \sqrt{\frac{m\omega}{2i\pi\hbar\sin(\omega(t_f-t_i))}} e^{\frac{im\omega}{2\hbar\sin(\omega(t_f-t_i))}[(q^2+q'^2)\cos(\omega(t_f-t_i))-2qq']}. \tag{55}$$

This is the same result as the path integral method. However the path integral method is much more easier and straightforward then the traditional method. One can imagine that for more advanced systems the propagators are extremely hard or even impossible to calculate without path integrals.

Now we are going to calculate the propagator for the forced harmonic oscillator. This case turns out to be really important in the things we are going to do later.

Propagator of the forced harmonic oscillator

We need to find the classical action of the system with the Lagrangian

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 + \gamma(t)q \tag{56}$$

Using Euler-Lagrange equations we see that we have to solve

$$(\frac{d^2}{dt^2} + \omega^2)q(t) = \frac{\gamma(t)}{m}. \tag{57}$$

The solution of this equation has a homogeneous part and an inhomogeneous part with the homogeneous part written as

$$q_H(t) = Ae^{i\omega t} + Be^{-i\omega t}. \tag{58}$$

In order to solve for the inhomogeneous part we are going to define a Greens functions with the property

$$(\frac{d^2}{dt^2} + \omega^2)G(t - t') = -\delta(t - t') \tag{59}$$

$$q_I(t) = -\int_{t_i}^{t_f} dt' G(t - t') \frac{\gamma(t')}{m}. \tag{60}$$

By Fourier transformations it's easy to see that the expression for the Greens function becomes

$$G(t - t') = \int dk \frac{1}{\sqrt{2\pi}} exp(-ik(t - t'))G(k) = \int dk \frac{1}{2\pi} \frac{exp(-ik(t - t'))}{(k^2 - \omega^2)}. \tag{61}$$

Although this solves the system it appears that it is convenient to define the Feynman Greens function in the following manner

$$G_F(t - t') = \lim_{\epsilon \to 0} \int dk \frac{1}{2\pi} \frac{exp(-ik(t - t'))}{(k^2 - \omega^2 + i\epsilon)}. \tag{62}$$

This is convenient because now we can do a contour integration to find

$$G_F(t - t') = \theta(t - t') \frac{exp(-i\omega(t - t'))}{2i\omega} + \theta(t' - t) \frac{exp(i\omega(t - t'))}{2i\omega}. \tag{63}$$

With this we can find q_I since

$$q_I(t) = -\int_{t_i}^{t_f} dt' G_F(t - t') \frac{\gamma(t')}{m} =$$

$$-\frac{1}{2im\omega}(\int_{t_i}^{t} dt'\, exp(-i\omega(t - t')\gamma(t') + \int_{t}^{t_f} dt'\, exp(i\omega(t - t')\gamma(t')). \tag{64}$$

With $q(t) = q_H(t) + q_I(t)$ and the boundary conditions $q(t_f) = q$ and $q(t_i) = q'$ we can now solve for A and B. By defining $T = t_f - t_i$ we find the following lengthy expression for the classical action

$$S(q) = \frac{m\omega}{2\sin(\omega T)}[(q^2 + q'^2)\cos(\omega T) - 2qq']$$

$$+\frac{q'}{\sin(\omega T)}\int_{t_i}^{t_f} dt\,\gamma(t)\sin(\omega(t_f-t)+\frac{q}{\sin(\omega T)}\int_{t_i}^{t_f} dt\,\gamma(t)\sin(\omega(t-t_i)-$$

$$\frac{1}{m\omega\sin(\omega T)}\int_{t_i}^{t'_f} dt\int_{t_i}^{t'_f} dt'\,\gamma(t)\sin(\omega(t_f-t))\sin(\omega(t'-t_i))\gamma(t'). \tag{65}$$

With finding the action we have also determined the propagator of the system. We see that if we let γ vanish we get the action of the harmonic oscillator back. However we also see that this expression is not very insightful and it's also complicated to use. That's why in the next section we are going to look only at vacuum-vacuum transitions.

Vacuum to vacuum transitions

Vacuum-vacuum transitions are transitions where a particle begins in the ground state at $t_i = -\infty$ and at $t_f = \infty$ it is still in the ground state. In order to find an expression for the vacuum-propagator we have to do some additional things. We know

$$< q, t_f \mid q', t_i >= N\int Dq\, e^{\frac{i}{\hbar}S}. \tag{66}$$

Now let's look at the matrix-element of the form $< q, t_f \mid Q(t_1)Q(t_2) \mid q, t_i >$ where $Q(t) \mid q, t >= q \mid q, t >$ and $t_i < t_1 < t_2 < t_f$.

$$< q, t_f \mid Q(t_1)Q(t_2) \mid q', t_i >=$$

$$\int dq_1 dq_2\; < q, t_f \mid Q(t_1) \mid q_1, t_1 >< q_1, t_1 \mid Q(t_2) \mid q_2, t_2 >< q_2, t_2 \mid q', t_i >=$$

$$\int dq_1 dq_2 q_1 q_2 < q, t_f \mid q_1, t_1 >< q_1, t_1 \mid q_2, t_2 >< q_2, t_2 \mid q', t_i > . \tag{67}$$

The inner products can be written as a path integral and this gives

$$< q, t_f \mid Q(t_1)Q(t_2) \mid q', t_i >= N\int Dq\, q(t_1)q(t_2)e^{\frac{i}{\hbar}S}. \tag{68}$$

This was the case for $t_1 < t_2$, however we can also write the same thing down for $t_1 > t_2$. In that case

$$< q, t_f \mid Q(t_2)Q(t_1) \mid q', t_i >=$$

$$\int dq_1 dq_2\; < q, t_f \mid Q(t_2) \mid q_2, t_2 >< q_2, t_2 \mid Q(t_1) \mid q_1, t_1 >< q_1, t_1 \mid q', t_i >=$$

$$\int dq_1 dq_2 q_2 q_1 < q, t_f \mid q_1, t_1 >< q_1, t_1 \mid q_2, t_2 >< q_2, t_2 \mid q', t_i > .$$

$$< q, t_f \mid Q(t_2)Q(t_1) \mid q', t_i >= N\int Dq\, q(t_1)q(t_2)e^{\frac{i}{\hbar}S}. \tag{69}$$

In the last step we used that $q(t_1)$ and $q(t_2)$ are classical things and thus we are allowed to swap them around. So now we know what the path integral gives:

$$< q, t_f \mid T(Q(t_1)Q(t_2)) \mid q', t_i >= N \int Dq\, q(t_1)q(t_2)e^{\frac{i}{\hbar}S}, \tag{70}$$

where $T(Q(t_1)Q(t_2))$ gives the time-ordered product of A and B;

$$T(Q(t_1)Q(t_2)) = \theta(t_1 - t_2)Q(t_1)Q(t_2) + \theta(t_2 - t_1)Q(t_2)Q(t_1). \tag{71}$$

It can be easily seen that this can be extended to any set of operators:

$$< q, t_f \mid T(O_1(Q(t_1)) \dots O_n(Q(t_n))) \mid q', t_i >= N \int Dq\, O_1(Q(t_1)) \dots O_n(Q(t_n))e^{\frac{i}{\hbar}S}. \tag{72}$$

Now again let's look at the harmonic oscillator with an external force

$$< q, t_f \mid q', t_i >_\gamma = N \int Dq\, e^{\frac{i}{\hbar}[\int_{t_i}^{t_f} dt\,(S_{HO}) + \int_{t_i}^{t_f} dt\,\gamma(t)q(t)]}. \tag{73}$$

By using (72) we can write this as:

$$< q, t_f \mid q', t_i >_\gamma = < q, t_f \mid T(e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,\gamma(t)q(t)]}) \mid q', t_i > . \tag{74}$$

Now let's look at the probability that a particle in the infinite past from the coordinate state q' transfers to the coordinate state q in the infinite future in the presence of the external force $\gamma(t)$. So we have to take the limits $t_i \to -\infty,\ t_f \to \infty$. Now we are also going to assume that the system has a discrete set of eigenenergies with $H \mid 0 >= 0,\ H \mid n >= E_n > 0$. Of course in most quantum mechanical systems the ground state energy does not vanish. However the derivation becomes very complicated without this assumption. Now we can write

$$\lim_{t_i \to -\infty, t_f \to \infty} < q, t_f \mid q', t_i >_\gamma = \lim_{t_i \to -\infty} \lim_{t_f \to \infty} < q, t_f \mid T(e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,\gamma(t)q(t)]}) \mid q', t_i >=$$

$$\lim_{t_i \to -\infty} \lim_{t_f \to \infty} \sum_{n,m} < q, t_f \mid n >< n \mid T(e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,\gamma(t)q(t)}) \mid m >< m \mid q', t_i >=$$

$$\lim_{t_i \to -\infty} \lim_{t_f \to \infty} \sum_{n,m} < q \mid e^{-\frac{i}{\hbar}t_f H} \mid n >< n \mid T(e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,\gamma(t)q(t)}) \mid m >< m \mid\ e^{\frac{i}{\hbar}t_i H} \mid q' >=$$

$$\lim_{t_i \to -\infty} \lim_{t_f \to \infty} \sum_{n,m} e^{\frac{i}{\hbar}(E_m t_i - E_n t_f)} < q \mid n >< n \mid T(e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\,\gamma(t)q(t)]}) \mid m >< m \mid q' > .$$

If we take the limits and look at the exponent we can conclude that the only terms contributing anything are $n, m = 0$ because all other terms will oscillate out. This means

$$\lim_{t_i \to -\infty} \lim_{t_f \to \infty} < q, t_f \mid q', t_i >_\gamma =$$

$$< q \mid 0 >< 0 \mid q' >< 0 \mid T(e^{\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \, \gamma(t) q(t)]}) \mid 0 > . \quad (75)$$

Rearranging this gives

$$< 0 \mid T(e^{\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \, \gamma(t) q(t)]}) \mid 0 >= \lim_{t_i \to -\infty} \lim_{t_f \to \infty} \frac{< q, t_f \mid q', t_i >_\gamma}{< q \mid 0 >< 0 \mid q' >}. \quad (76)$$

Now we see two things, the left hand side is independent of q and q' so the right hand side should also be independent of q and q'. The other thing we notice is that the right hand side can be simply written as a path integral. Now we define Z_γ as

$$Z_\gamma =< 0 \mid T(e^{\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \, \gamma(t) q(t)]}) \mid 0 >= N \int Dq \, e^{\frac{i}{\hbar} S_\gamma}. \quad (77)$$

Now we are going to calculate Z_γ. We are going to have to use some tricks we used in calculating the propagator for the forced harmonic oscillator.

$$Z_\gamma = N \int Dq \, e^{\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \, (\frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 + \gamma q)} =$$

$$\lim_{\epsilon \to 0} N \int Dq \, e^{\frac{-im}{2\hbar} \int_{-\infty}^{\infty} dt \, (q(\frac{\partial^2}{\partial t^2} + \omega^2 - i\epsilon) q - \frac{2}{m} \gamma q)}. \quad (78)$$

Where we have used the same trick as in (62), $\omega^2 \to \omega^2 - i\epsilon$. Remember from (59) that

$$\lim_{\epsilon \to 0} (\frac{\partial^2}{\partial t^2} + \omega^2 - i\epsilon) G_F(t - t') = -\delta(t - t'). \quad (79)$$

Using this to define

$$\tilde{q}(t) = q(t) + \frac{1}{m} \int_{-\infty}^{\infty} dt' \, G_F(t - t') \gamma(t'), \quad (80)$$

we can write

$$Z_\gamma = \lim_{\epsilon \to 0} N \int D\tilde{q} \, e^{\frac{-im}{2\hbar} \int_{-\infty}^{\infty} dt \, \tilde{q} (\frac{\partial^2}{\partial t^2} + \omega^2 - i\epsilon) \tilde{q}} \times e^{\frac{-i}{2\hbar m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' \, \gamma(t) G_F(t - t') \gamma(t')} =$$

$$Z_{HO} e^{\frac{-i}{2\hbar m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' \, \gamma(t) G_F(t - t') \gamma(t')}. \quad (81)$$

This result is very important in the upcoming calculations.

Perturbation theory

In fact we are not interested in the harmonic oscillator with an external force but we are interested in the anharmonic oscillator. However it appears we can find an expression for Z of this system by using Z_γ. Let's write down the Lagrangian for the anharmonic oscillator with an external force,

$$L = \frac{1}{2}m\dot{q}(t)^2 - \frac{1}{2}m\omega^2 q(t)^2 + \lambda q(t)^4 + \gamma(t)q(t) = L_\gamma + \lambda q(t)^4. \tag{82}$$

By definition

$$Z_\lambda = N\int Dq\, e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, \lambda q^4} e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, L_\gamma} \tag{83}$$

Functional derivatives

Now we are going to do a functional derivative. A functional derivative is just like a normal derivative but instead of taking a derivative with respect to a variable we take the derivative with respect to a function. The definition is

$$\frac{\delta F(f)}{\delta f(y)} = \lim_{\epsilon\to 0}\frac{F[f(x)+\epsilon\delta(x-y)] - F[f(x)]}{\epsilon} \tag{84}$$

The chain- and productrule for normal derivatives also hold for functional derivatives. A few important functional derivatives are:

$$\frac{\delta f(x)}{\delta f(y)} = \frac{1}{\epsilon}[f(x)+\epsilon\delta(x-y) - f(x)] = \delta(x-y). \tag{85}$$

If

$$F_x(f) = \int dz\, G(x,z)f(z) \tag{86}$$

then

$$\frac{\delta F_x(f)}{\delta f(y)} = G(x,y). \tag{87}$$

Now let's take a functional derivative of $N\int Dq\, e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, L_\gamma}$,

$$\frac{\delta}{\delta\gamma(t)} N\int Dq\, e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, \frac{1}{2}m\dot{q}(t)^2 - \frac{1}{2}m\omega^2 q(t)^2 + \gamma(t)q(t)} =$$

$$N\frac{i}{\hbar}\int Dq\, q(t) e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, L_\gamma}. \tag{88}$$

It is easy to see that

$$\frac{\delta^n}{\delta\gamma(t)^n} N\int Dq\, e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, L_\gamma} = N(\frac{i}{\hbar})^n \int Dq\, q(t)^n e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, L_\gamma}. \tag{89}$$

Now if we have a function $f(q)$ which can be written as a Taylor series then we can say

$$N\int Dq\, f(q) e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, L_\gamma} =$$

$$f(\frac{h}{i}\frac{\delta}{\delta\gamma(t)}) N\int e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, L_\gamma} = Nf(\frac{h}{i}\frac{\delta}{\delta\gamma(t)})Z_\gamma \tag{90}$$

But if we look at (83) we have

$$f(q) = e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, \lambda q^4}. \tag{91}$$

So we can write

$$Z_\lambda = Ne^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, \lambda (\frac{\hbar}{i})^4 (\frac{\delta}{\delta\gamma(t)})^4} Z_\gamma. \tag{92}$$

If we write the exponent as a Taylor series we get

$$Z_\lambda = N(1 + i\hbar^3\lambda \int dt\, [\frac{\delta^4}{\delta\gamma(t)^4}] - \frac{1}{2}\hbar^6\lambda^2 \int dt dt'\, [\frac{\delta^4}{\delta\gamma(t)^4}\frac{\delta^4}{\delta\gamma(t')^4}] + \ldots)Z_\gamma. \tag{93}$$

This is perturbation theory in the path integral approach. Since we know Z_γ (see (81)) we can know determine the influence of the λ-term on the system up to any order.

So in order to find Z_λ we have to find functional derivatives of the vacuum propagator of the forced harmonic oscillator.

$$\frac{\delta Z_\gamma}{\delta\gamma(t)} = Z_\gamma \times \frac{-i}{m\hbar}\int dt'\, G_F(t-t')\gamma(t') = \frac{-i}{m\hbar} g(t) Z_\gamma \tag{94}$$

$$g(t) = \int dt'\, G_F(t-t')\gamma(t'). \tag{95}$$

The second functional gives:

$$\frac{\delta^2 Z_\gamma}{\delta\gamma^2(t)} = \frac{-i}{m\hbar}[g(t)\frac{\delta Z_\gamma}{\delta\gamma(t)} + Z_\gamma \frac{\delta g(t)}{\delta\gamma(t)}]. \tag{96}$$

Looking at $\frac{\delta g(t)}{\delta\gamma(t)}$ gives

$$\frac{\delta g(t_a)}{\delta\gamma(t_b)} = G_F(t_a - t_b) \tag{97}$$

and this gives for the second derivative of the Z_γ

$$\frac{\delta^2 Z_\gamma}{\delta\gamma^2(t)} = \frac{-i}{m\hbar}[g(t)\times\frac{-i}{\hbar m} g(t) Z_\gamma + Z_\gamma G_F(0)] =$$

$$Z_\gamma[(\frac{i}{\hbar m})^2 g^2(t) - \frac{i}{\hbar m} G_F(0)]. \tag{98}$$

If we continue this process we find the following expression for first order perturbation theory:

$$Z_\lambda = N\{1 + i\hbar^3 \times$$

$$\lambda \int dt\,[3(\frac{i}{\hbar m})^2 G_F^2(0) - 6(\frac{i}{\hbar m})^3 G_F(0) g^2(t) + (\frac{i}{\hbar m})^4 g^4(t)]\} Z_\gamma. \quad (99)$$

However we have to normalize this in order to get a useful expression. We know that if there is no external force then a particle in the ground state at $t_i = -\infty$ will still be in the ground state at $t_f = \infty$. This means that $\lim_{\gamma\to 0} Z_\gamma = 1$ and that the constant N has to make sure that $\lim_{\gamma\to 0} Z_\lambda = 1$.

Since $g(t)$ vanishes when γ vanishes we get the following expression for N

$$1 = N\{1 + i\hbar^3\lambda \int dt\, 3(\frac{i}{m\hbar})^2 G_F^2(0)\}$$

$$N = \{1 + i\hbar^3\lambda \int dt\, 3(\frac{i}{m\hbar})^2 G_F^2(0)\}^{-1} \approx \{1 - i\hbar^3\lambda \int dt\, 3(\frac{i}{m\hbar})^2 G_F^2(0)\}. \quad (100)$$

Finally this gives for the first order

$$Z_\lambda = \{1 + i\hbar^3\lambda \int dt\,[-6(\frac{i}{\hbar m})^3 G_F(0) g^2(t) + (\frac{i}{\hbar m})^4 g^4(t)]\} Z_\gamma. \quad (101)$$

In order to see what the anharmonic term does to the system we have to use so called n-pointfunctions

n-pointfunctions

With expression (72) it can be seen:

$$< T(Q(t_1)\cdots Q(t_n)) >= (\frac{\hbar}{i})^n \frac{\delta^n Z_\lambda}{\delta\gamma(t_1)\cdots\delta\gamma(t_n)} |_{\gamma\to 0}= D^{(n)}. \quad (102)$$

So we see that the Z_λ generates time-ordered correlation functions. The $D^{(N)}$ are called the n-pointfunctions, and with these functions the influence of the anharmonic term can be determined.

Propagator of the anharmonic oscillator up to first order

Let us calculate the 2-pointfunction up to first order. We know Z_λ so in order to find $D^{(2)}$ we need to take some more partial derivatives:

$$\frac{\delta Z_\lambda}{\delta\gamma(t_1)} = \{-\frac{i}{m\hbar} g(t_1) + i\hbar^3\lambda \int dt\,[-12(\frac{i}{m\hbar})^3 G_F(0) G_F(t-t_1) g(t) +$$

$$6(\frac{i}{m\hbar})^4 g^2(t)g(t_1) + 4(\frac{i}{m\hbar})^4 G_F(t-t_1)g^3(t) - (\frac{i}{m\hbar})^5 g^4(t)g(t_1)]\}. \quad (103)$$

We have to take one more functional derivative and then we have to let γ vanish. But this means that all terms that have a g in them after taking the derivative will become zero. This makes the calculation a lot shorter since the only thing we have to look at is

$$\frac{\delta}{\delta\gamma(t_2)}\{-\frac{i}{m\hbar}g(t_1) + i\hbar^3\lambda \int dt\,[-12(\frac{i}{m\hbar})^3 G_F(0)G_F(t-t_1)g(t)]\}. \quad (104)$$

This gives for the 2-pointfunction:

$$D^{(2)} = (\frac{\hbar}{i})^2\{(-\frac{i}{m\hbar}G_F(t_1-t_2) + i\hbar^3\lambda\times$$

$$\int dt[-12(\frac{i}{m\hbar})^3 G_F(0)G_F(t-t_1)G_F(t-t_2)]\}. \quad (105)$$

This expression can be written as a sum of Feynman diagrams. This gives the following diagrams:

Figure 1: First order diagrams

The first diagram is just the diagram of the harmonic oscillator. So we see that the anharmonic term gives, up to first order, a perturbation to the system given by the second diagram.
Since we know the expressions for the Greens functions (62) we can write:

$$dt\,G_F(t-t_1)G_F(t-t_2) =$$

$$\int\int dt dk_1 dk_2\,(\frac{1}{2\pi})^2 \frac{exp(-ik_1(t-t_1))exp(-ik_2(t_2-t))}{(k_1^2-\omega^2+i\epsilon)(k_2^2-\omega^2+i\epsilon)}. \quad (106)$$

By doing the integral over time this becomes:

$$\int dk_1 dk_2\,\frac{1}{2\pi}\delta(k_1-k_2)\frac{exp(+ik_1t_1-ik_2t_2)}{(k_1^2-\omega^2+i\epsilon)(k_2^2-\omega^2+i\epsilon)} =$$

$$\int dk \frac{1}{2\pi}\frac{exp(-ik(t_1-t_2))}{(k^2-\omega^2+i\epsilon)^2}. \quad (107)$$

Filling this in the expression for the 2-point function gives

$$D^{(2)} = (\frac{\hbar}{i})^2 \frac{-i}{\hbar m} \int dk \frac{1}{2\pi} \frac{exp(-ik(t_1-t_2))}{(k^2-\omega^2+i\epsilon)} [1 - \frac{12i\hbar\lambda G_F(0)}{m^2} \frac{1}{(k^2-\omega^2+i\epsilon)}].$$

Since we are looking at the first order and thus are ignoring terms of λ^2 and higher we can use $1+\lambda \approx \frac{1}{1-\lambda}$ to write the two-pointfunction as

$$D^{(2)} = \frac{i\hbar}{m} \int dk \frac{1}{2\pi} \frac{exp(-ik(t_1-t_2))}{(k^2-\omega^2+\frac{12i\hbar G_F(0)}{m^2}\lambda+i\epsilon)}. \tag{108}$$

Now we clearly see what the anharmonic term, up to first order, does to the system. We see that the anharmonic term changes ω by:

$$\omega^2 \rightarrow \omega^2 - \frac{12i\hbar G_F(0)}{m^2}\lambda. \tag{109}$$

Of course the change is linear in λ since we looked only up to first order. There is also a term $G_F(0)$, this is a convergent integral given by $\int dk \frac{1}{k^2-\omega^2+i\epsilon}$. So now we can solve the system up to first order.

Propagator of the anharmonic oscillator up to second order

Let us now look at second order perturbation theory, using again (93) we see that we now have to do 8 functional derivatives. However we already found the first 4. We have to calculate:

$$\int\int dtdt' \frac{\delta^4}{\delta\gamma^4(t)} \frac{\delta^4}{\delta\gamma^4(t')} Z_\gamma =$$

$$\int\int dtdt' \frac{\delta^4}{\delta\gamma^4(t')} [3(\frac{i}{\hbar m})^2 G_F^2(0) - 6(\frac{i}{\hbar m})^3 G_F(0) g^2(t) + (\frac{i}{\hbar m})^4 g^4(t)]. \tag{110}$$

We are going to do the derivatives one by one:

$$\frac{\delta}{\delta\gamma(t')} \frac{\delta^4}{\delta\gamma(t)} Z_\gamma = \{-3(\frac{i}{\hbar m})^3 G_F^2(0) g(t') -$$
$$12(\frac{i}{\hbar m})^2 G_F(0) G_F(t-t') g(t) + 6(\frac{i}{\hbar m})^4 G_F(0) g^2(t) g(t') +$$
$$4(\frac{i}{\hbar m})^4 G_F(0) g^3(t) G_F(t-t') - (\frac{i}{\hbar m})^5 g^4(t) g(t')\} Z_\gamma. \tag{111}$$

Now by simply continuing this calculation, which is a gruesome task, we arrive at the result:

$$\frac{\delta^4}{\delta\gamma^4(t)} \frac{\delta^4}{\delta\gamma^4(t')} Z_\gamma = \{9(\frac{i}{\hbar m})^4 G_F^4(0) + 72(\frac{i}{\hbar m})^4 G_F^2(0) G_F^2(t-t') +$$
$$24(\frac{i}{\hbar m})^4 G_F^4(t-t) - 18(\frac{i}{\hbar m})^5 G_F^3(0) g^2(t') -$$

$$18(\frac{i}{\hbar m})^5 G_F^{\,3}(0)g^2(t) - 72(\frac{i}{\hbar m})^5 G_F(0)G_F^{\,2}(t-t')g^2(t')-$$

$$72(\frac{i}{\hbar m})^5 G_F(0)G_F^{\,2}(t-t')g^2(t) - 144(\frac{i}{\hbar m})^5 G_F^{\,2}(0)G_F(t-t')g(t)g(t')-$$

$$96(\frac{i}{\hbar m})^5 G_F^{\,3}(t-t')g(t)g(t') + 3(\frac{i}{\hbar m})^6 G_F^{\,2}(0)g^4(t')+$$

$$3(\frac{i}{\hbar m})^6 G_F^{\,2}(0)g^4(t) + 48(\frac{i}{\hbar m})^6 G_F(0)G_F(t-t')g(t)g^3(t')$$

$$48(\frac{i}{\hbar m})^6 G_F(0)G_F(t-t')g^3(t)g(t') + 72(\frac{i}{\hbar m})^6 G_F^{\,2}(t-t')g^2(t)g^2(t')+$$

$$36(\frac{i}{\hbar m})^6 G_F^{\,2}(0)g^2(t)g^2(t') - 6(\frac{i}{\hbar m})^7 G_F(0)g^2(t)g^4(t')-$$

$$6(\frac{i}{\hbar m})^7 G_F(0)g^4(t)g^2(t') - 16(\frac{i}{\hbar m})^7 G_F(t-t')g^3(t)g^3(t')+$$

$$(\frac{i}{\hbar m})^8 g^4(t)g^4(t')\}Z_\gamma. \tag{112}$$

In order to find Z_λ we still have to normalize. If we look at (x) and let γ vanish we get the following expression for N

$$N \approx \{1 - i\hbar^3\lambda \int dt[3(\frac{i}{\hbar m})^2 G_F^{\,2}(0)]+$$

$$\frac{1}{2}\hbar^6\lambda^2 \int\int dtdt'(\frac{i}{\hbar m})^4[9G_F^{\,4}(0) + 72G_F^{\,2}(0)G_F^{\,2}(t-t') + 24G_F^{\,4}(t-t')]\}. \tag{113}$$

As you can imagine getting (112) by simply taking one derivative after the other is a very tiring task. A lot of terms appear and this means that mistakes are easily made. However this calculation can be done a lot faster by using some tricks.

A useful method for finding derivatives

If we look at $\frac{\delta^4}{\delta\gamma^4(t)}Z_\gamma$ we see that it consists of 3 terms:

$$\frac{\delta^4}{\delta\gamma^4(t)}Z_\gamma = [3(\frac{i}{\hbar m})^2 G_F^{\,2}(0) - 6(\frac{i}{\hbar m})^3 G_F(0)g^2(t) + (\frac{i}{\hbar m})^4 g^4(t)]Z_\gamma =$$

$$A + B + C. \tag{114}$$

Now let's look at the three terms A,B and C separately. We have to take 4 functional derivatives with respect to $\gamma(t')$ of all 3 terms. Let's first look at $A = 3(\frac{i}{\hbar m})^2 G_F^{\,2}(0)Z_\gamma$.

We see that the only term dependant on γ is Z_γ. So we first differentiate

Z_γ and after taking this derivative we get a $g(t')Z_\gamma$ term. In this term both $g(t')$ as Z_γ are dependant on γ. This means that for the second derivative we have two choices. The third derivative will give even more possibilities etc.
If we introduce a new notation we can quickly see what terms will appear. A row $AZZZZ$ means we differentiate the Z_γ-term in A 4 times. $AZZZg'$ means we differentiate the Z_γ-term in A 3 times and finally differentiate the g(t')-term. A Z in a row creates a $-\frac{i}{\hbar m}g(t')$-term (see (94)) and a g' changes a $g(t')$ term into a $G_F(0)$ term (see (97)). This means that a combination Zg' gives a $-\frac{i}{\hbar m}G_F(0)$ term. So by just writing down all possible combinations and then counting the number of Z, g' and Zg' in the row it's easy to see what terms will appear. Of course we don't have to write down all possible rows because a lot of them will give 0. Rows that begin with a g' and rows with more g's than Z's won't contribute anything. Same goes for rows such as $AZg'g'Z$ because the first Z creates only one $g(t')$ term and the next g' destroys it. So the following g' will give 0.
Now let's look at all possible rows:

first look at $AZZZZ$, since we have 4 Z's in this row we have to multiply A with $(\frac{i}{\hbar m})^4 g^4(t')$.

$$AZZZZ = 3(\frac{i}{\hbar m})^2 G_F^2(0)Z_\gamma \times (\frac{i}{\hbar m})^4 g^4(t') = 3(\frac{i}{\hbar m})^6 G_F^2(0)Z_\gamma. \qquad (115)$$

Now let's look at rows with one g' and 3 Z's. We know that $g'ZZZ$ gives 0 so the only rows left are $Zg'ZZ$, $ZZg'Z$ and $ZZZg'$. By simply counting the number of Z's that appear before the g' it can be seen that $AZZZg' = 3 \times AZg'ZZ$ and $AZZgZ' = 2 \times AZg'ZZ$.
$AZg'ZZ$ has a Zg' that gives a $-\frac{i}{\hbar m}G_F(0)$ term and 2 Z's which gives a $(\frac{i}{\hbar m})^2 g^2(t')$ term.

This means

$$AZZZg' + AZZg'Z + AZg'ZZ = 6 \times AZg'ZZ =$$

$$6 \times 3(\frac{i}{\hbar m})^2 G_F^2(0)Z_\gamma \times -(\frac{i}{\hbar m})^3 G_F(0) g^2(t') = -18(\frac{i}{\hbar m})^5 G_F^3(0) g^2(t') Z_\gamma. \qquad (116)$$

Finally look at terms with 2 Z's and 2 g's. The only possibilities that contribute anything are $AZZg'g'$ and $AZg'Zg'$ and with the same argument as above we see $AZZg'g' = 2 \times AZg'Zg'$. Thus

$$AZZg'g' + AZg'Zg' = 3 \times AZg'Zg' =$$

$$3 \times 3(\frac{i}{\hbar m})^2 G_F^2(0) Z_\gamma \times (\frac{i}{\hbar m})^2 G_F^2(0) = 9(\frac{i}{\hbar m})^4 G_F^4(0). \qquad (117)$$

These are all rows for A that contribute anything.

Now let's look at B, which is slightly more complicating but still very straightforward.

$$B = -6(\frac{i}{\hbar m})^3 G_F(0) g^2(t) Z_\gamma$$

So we see that this time there is an additional term dependant on γ namely the $g(t)$-term. This means that in the rows also a g appears which means we differentiate the $g(t)$ term. A g in a row changes a $g(t)$ term into a $G_F(t-t')$ term. Fortunately this does not make things a lot harder. If we look at the terms we immediately see that a gZ combination gives the same thing as a Zg combination. Also a gg' combination gives the same as a $g'g$ combination. This is clearly not the case for $g'Z$ and Zg' combinations.
Since there is a $g^2(t)$ term in B we cannot have more than two g's in a row. The third g would make the contribution 0.
Let us look at all possible rows and begin with the rows we already discussed for A:

$$BZZZZ = -6(\frac{i}{\hbar m})^3 G_F(0) g^2(t) Z_\gamma \times (\frac{i}{\hbar m})^4 g^4(t') = -6(\frac{i}{\hbar m})^7 G_F(0) g^2(t) g^4(t') Z_\gamma$$

$$6 \times BZg'ZZ = 6 \times -6(\frac{i}{\hbar m})^3 G_F(0) g^2(t) Z_\gamma \times -(\frac{i}{\hbar m})^3 G_F(0) g^2(t') = 36(\frac{i}{\hbar m})^6 G_F^2(0) g^2(t) g^2(t') Z_\gamma$$

$$3 \times BZg'Zg' = 3 \times -6(\frac{i}{\hbar m})^3 G_F(0) g^2(t) Z_\gamma \times (\frac{i}{\hbar m})^2 G_F^2(0) = -18(\frac{i}{\hbar m})^5 G_F^3(0) g^2(t) Z_\gamma.$$

Now look at rows that have 1 g in them. The first g in a row changes the $g^2(t)$ term into a $2G_F(t-t')g(t)$ term. Since g and Z are independent, $BgZZZ = \cdots = BZZZg$ so we get:

$$4 \times BgZZZ = 48(\frac{i}{\hbar m})^6 G_F(0) G_F(t-t') g(t) g^3(t') Z_\gamma. \tag{118}$$

We know $BgZZg' = 2 \times BgZg'Z$ and by looking at the rows we see that $BgZZg' = BZgZg' = BZZgg' = BZZg'g$. Also $BgZg'Z = BZgg'Z = BZg'gZ = BZgZg'$. So this gives

$$4 \times BgZg'Z + 4 \times BgZZg' = 12 \times BgZg'Z = -144(\frac{i}{\hbar m})^5 G_F(0) G_F(t-t') g(t) g(t') Z_\gamma. \tag{119}$$

It's also easy to see that

$$\binom{4}{2} BggZZ = 6 \times BggZZ = -72(\frac{i}{\hbar m})^5 G_F(0) G_F^2(t-t') g^2(t') Z_\gamma$$

$$6 \times BggZg' = 72(\frac{i}{\hbar m})^4 G_F^2(0) G_F^2(t-t') Z_\gamma. \tag{120}$$

Finally look at $C = (\frac{i}{\hbar m})^4 g^4(t)]Z_\gamma$, using the same method as above we can now easily write down all possible rows. The difference is that this time we can have up to 4 g's in a row.

$$
\begin{aligned}
1 \times CZZZZ &= (\frac{i}{\hbar m})^8 g^4(t') g^4(t) Z_\gamma \\
6 \times CZg'ZZ &= -6(\frac{i}{\hbar m})^7 G_F(0) g^2(t') g^4(t) Z_\gamma \\
3 \times CZg'Zg' &= 3(\frac{i}{\hbar m})^6 G_F^{\,2}(0) g^4(t) Z_\gamma \\
4 \times CgZZZ &= -16(\frac{i}{\hbar m})^7 G_F(t-t') g^3(t') g^3(t) Z_\gamma \\
12 \times CgZg'Z &= 48(\frac{i}{\hbar m})^6 G_F(0) G_F(t-t') g(t') g^3(t) Z_\gamma \\
6 \times CggZZ &= 72(\frac{i}{\hbar m})^6 G_F^{\,2}(t-t') g^2(t') g^2(t) Z_\gamma \\
6 \times CggZg' &= -72(\frac{i}{\hbar m})^5 G_F(0) G_F^{\,2}(t-t') g^2(t) Z_\gamma \\
4 \times CgggZ &= -96(\frac{i}{\hbar m})^5 G_F^{\,3}(t-t') g(t') g(t) Z_\gamma \\
1 \times Cgggg &= 24(\frac{i}{\hbar m})^4 G_F^{\,4}(t-t') Z_\gamma. \qquad (121)
\end{aligned}
$$

If we now add up all rows from A,B and C we get the same expression as we get by taking the derivatives one at a time. However doing it this way is much faster and it's also a lot easier to see where terms are coming from.

2-point function up to second order

With the expression for the 2-point function

$$D^{(2)} = (\frac{\hbar}{i})^2 \frac{\delta^2 Z_\lambda}{\delta\gamma(t_1)\delta\gamma(t_2)} \mid_{\gamma\to 0}, \qquad (122)$$

we see that we have to take two more functional derivatives of Z_λ. However all terms that consists of $g^3(t)$ or $g^3(t')$ or higher orders will become zero because after differentiating there will still be g's left and they vanish when we let γ vanish.

After doing the calculation, which is really straightforward and fortunately not very long, we get the following expression

$$D^{(2)} = (\frac{\hbar}{i})^2 \{-\frac{i}{m\hbar} G_F(t_1-t_2) + i\hbar^3\lambda \int dt[-12(\frac{i}{m\hbar})^3 G_F(0) G_F(t-t_1) G_F(t-t_2)]\} -$$

$$\hbar^6\lambda^2 \int\int dtdt'[-144(\frac{i}{\hbar m})^5 G_F^{\,2}(0) G_F(t-t') G_F(t'-t_1) G_F(t-t_2) -$$

$$144(\frac{i}{\hbar m})^5 G_F(0) G_F^2(t-t') G_F(t-t_1) G_F(t-t_2) - 96(\frac{i}{\hbar m})^5 G_F^3(t-t') G_F(t'-t_1) G_F(t-t_2)]\}.$$

We can write this expression as the following diagrams.

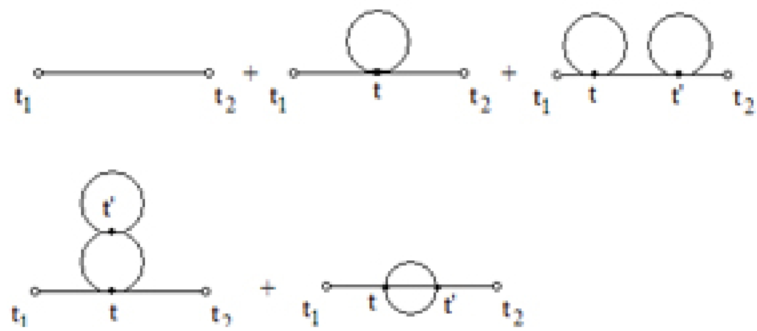

Figure 2: Second order diagrams

We see that the first two diagrams are, naturally, the same as we got in first order perturbation theory but we also get three new diagrams. We define the first four diagrams as simple diagrams because between their internal points there are no more than two internal lines and it appears that such diagrams are a lot easier to deal with. The last diagram has three internal lines between its internal points and is defined as a complex diagram

Let's look at the influence of the diagrams separately beginning with the third diagram.

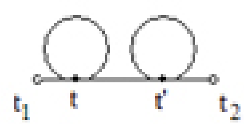

Figure 3: The first simple diagram in second order

The expression for this diagram is

$$\hbar^6 \lambda^2 \int \int dt dt' - 144(\frac{i}{\hbar m})^5 G_F^2(0) G_F(t-t') G_F(t'-t_1) G_F(t-t_2). \quad (123)$$

Let us again use the expressions for the Green functions to see what this term does to the system.

$$\int\int dtdt'\, G_F(t-t')G_F(t'-t_1)G_F(t-t_2) =$$

$$\int dtdt'dk_1dk_2dk_3(\frac{1}{2\pi})^3\frac{\exp(-ik_1(t-t_1))\exp(-ik_2(t'-t))\exp(-ik_3(t_2-t'))}{(k_1^2-\omega^2+i\epsilon)(k_2^2-\omega^2+i\epsilon)(k_3^2-\omega^2+i\epsilon)}.$$

By doing the integral over the times we get

$$\int dk_1dk_2dk_3\frac{1}{2\pi}\delta(k_1-k_2)\delta(k_2-k_3)\frac{\exp(i(k_1t_1-k_3t_2))}{(k_1^2-\omega^2+i\epsilon)(k_2^2-\omega^2+i\epsilon)(k_3^2-\omega^2+i\epsilon)} =$$

$$\int dk\frac{1}{2\pi}\frac{\exp(-ik(t_1-t_2))}{(k^2-\omega^2+i\epsilon)^3}. \tag{124}$$

Now let's use this in the expression for Z_λ and look at the 2-point function for the first three diagrams

$$D^{(2)} = (\frac{\hbar}{i})^2\frac{-i}{\hbar m}\int dk\frac{1}{2\pi}\frac{\exp(-ik(t_1-t_2))}{(k^2-\omega^2+i\epsilon)}\times$$

$$[1-\frac{12i\hbar\lambda G_F(0)}{m^2}\frac{1}{(k^2-\omega^2+i\epsilon)}-\frac{144\hbar^2\lambda^2G_F^{\,2}(0)}{m^4}\frac{1}{(k^2-\omega^2+i\epsilon)^2}]. \tag{125}$$

Since we are looking at second order we are ignoring λ^3 and higher order terms. This means we can, just like in first order, use $1+\lambda+\lambda^2 \approx \frac{1}{1-\lambda}$. So just as in first order we can write

$$D^{(2)} = \frac{i\hbar}{m}\int dk\frac{1}{2\pi}\frac{\exp(-ik(t_1-t_2))}{(k^2-\omega^2+\frac{12i\hbar G_F(0)}{m^2}\lambda+i\epsilon)}. \tag{126}$$

This means that this second order diagram does not change the system any further. It does make the approximation we used better since we are ignoring less terms. In order to make the approximation even better we have to look at diagrams in higher orders such as

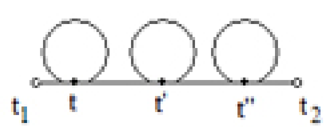

Figure 4: A simple third order diagram

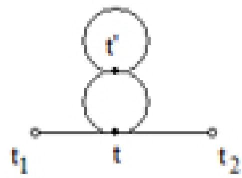

Figure 5: The second simple diagram in second order

Now look at the fourth diagram

The expression for this diagram is

$$\hbar^6\lambda^2 \int\int dtdt' - 144(\frac{i}{\hbar m})^5 G_F(0) G_F^2(t-t') G_F(t-t_1) G_F(t-t_2), \qquad (127)$$

using (62) we can write

$$\int dtdt' G_F^2(t-t') G_F(t-t_1) G_F(t-t_2) = \qquad (128)$$

$$\int dtdt' dk_1 dk_2 dk_3 dk_4 (\frac{1}{2\pi})^4 \frac{\exp(-ik_1(t-t_1))\exp(-ik_2(t'-t))\exp(-ik_3(t'-t))\exp(-ik_4(t_2-t))}{(k_1^2-\omega^2+i\epsilon)(k_2^2-\omega^2+i\epsilon)(k_3^2-\omega^2+i\epsilon)(k_4^2-\omega^2+i\epsilon)}.$$

By doing the integrals over time we get the expression

$$\int dk_1 dk_2 dk_3 dk_4 (\frac{1}{2\pi})^2 \frac{\delta(k_1-k_2-k_3-k_4)\delta(k_2+k_3)\exp(i(k_1t_1-k_4t_2))}{(k_1^2-\omega^2+i\epsilon)(k_2^2-\omega^2+i\epsilon)(k_3^2-\omega^2+i\epsilon)(k_4^2-\omega^2+i\epsilon)} =$$

$$\int dk \frac{1}{2\pi} \frac{\exp(-ik(t_1-t_2))}{(k^2-\omega^2+i\epsilon)^2} \int dk' \frac{1}{2\pi} \frac{1}{(k'^2-\omega^2+i\epsilon)^2}. \qquad (129)$$

If we look at the integral $\frac{1}{2\pi}\frac{1}{(k'^2-\omega^2+i\epsilon)^2}$ we see that this greatly resembles $G_F(0) = \frac{1}{2\pi}\frac{1}{(k'^2-\omega^2+i\epsilon)}$. Thus we define

$$G_{F2}(0) = \frac{1}{2\pi}\frac{1}{(k'^2-\omega^2+i\epsilon)^2}. \qquad (130)$$

Now let's fill all this in (129) and add this up to the previous 3 diagrams and see what this does to the system.

$$D^{(2)} = (\frac{\hbar}{i})^2 \frac{-i}{\hbar m} \int dk \frac{1}{2\pi} \frac{\exp(-ik(t_1-t_2))}{(k^2-\omega^2+i\epsilon)} \times$$

$$[1 - \frac{12i\hbar\lambda G_F(0)}{m^2(k^2-\omega^2+i\epsilon)} - \frac{144\hbar^2\lambda^2 G_F^2(0)}{m^4(k^2-\omega^2+i\epsilon)^2} - \frac{144\hbar^2\lambda^2 G_F(0) G_{F2}(0)}{m^4(k^2-\omega^2+i\epsilon)^2}]. \qquad (131)$$

Now we have to use a approximation similar to the one we already used. Look at $1 + x + x^2 + y$ with y of order x^2 and we are ignoring terms of x^3 and higher (thus also xy). Then we can write

$$1+x+x^2+y \approx \frac{1}{1-x}+y = \frac{1+y-yx}{1-x} \approx \frac{1+y}{1-x} \approx \frac{1}{(1-x)(1-y)} \approx \frac{1}{1-x-y}.$$

Using this we see

$$D^{(2)} = \frac{i\hbar}{m}\int dk \frac{1}{2\pi}\frac{\exp(-ik(t_1-t_2))}{(k^2-\omega^2+\frac{12i\hbar G_F(0)}{m^2}\lambda+\frac{144\hbar^2 G_F(0)G_{F2}(0)}{m^4}\lambda^2+i\epsilon)}. \quad (132)$$

Now we clearly see what these diagrams do to the system. Again ω has changed but this time by the amount

$$\omega^2 \to \omega^2 + (\frac{-12i\hbar\lambda}{m^2})G_F(0) + (\frac{-12i\hbar\lambda}{m^2})^2 G_F(0)G_{F2}(0). \quad (133)$$

This looks like a series and it is tempting to say that for higher orders all simple diagrams can be written in this series. If we write down the following series up to order p

$$\omega^2 \to \omega^2 + (\frac{-12i\hbar\lambda}{m^2})G_F(0) + \sum_{n=2}^{p}(\frac{-12i\hbar\lambda}{m^2})^n \sum_{m=1}^{n-1} G_F^{\,m}(0)G_{F2}^{\,n-m}(0) \quad (134)$$

we see that up to second order it is correct. In the series no terms dependant on $G_{F3}(0)$ appear because we are only looking at simple diagrams and this means there are no more than two internal lines between internal points. A $G_{F3}(0)$ term requires three internal lines.

We now need to check whether higher orders also work out.
Let us look qualitatively at third order simple diagrams. There are only three possibilities

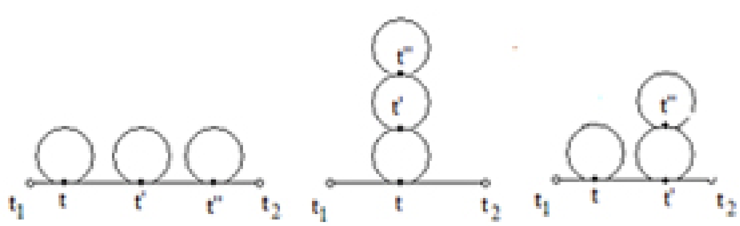

Figure 6: All simple third order diagrams

We already discussed the first of these diagrams; it is absorbed in the series $1+\lambda+\lambda^2 \approx \frac{1}{1-\lambda}$ and only makes the approximation better. The second diagram gives a term $\sim G_F(0)G_{F2}^2(0)$ and the third diagrams gives a term $\sim G_F^2(0)G_{F2}(0)$. We see that qualitatively the third order simple diagrams fit in the series however we did not check the prefactor $(\frac{-12i\hbar\lambda}{m^2})^3$.

It's looks like a series exists for all simple diagrams but whether the series we wrote down is the correct one is not clear. In this article higher orders have not been calculated nor is a proof formulated so we cannot be sure that the series is correct. It remains an interesting idea though.

Let us now look at the complex diagram

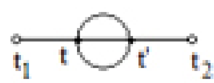

Figure 7: Second order complex diagram

The expression for this diagram is

$$\hbar^6\lambda^2 \int\int dtdt' - 96(\frac{i}{\hbar m})^5 G_F^3(t-t')G_F(t'-t_1)G_F(t-t_2)] \tag{135}$$

We can write the Green functions in the following way

$$\begin{gathered}\int dtdt' G_F^3(t-t')G_F(t'-t_1)G_F(t-t_2)] = \\ \int dk \frac{exp(-ik(t_1-t_2))}{(k^2-\omega^2+i\epsilon)^2} \int dk' \frac{1}{(k'^2-\omega^2+i\epsilon)} \times \\ \int dk'' \frac{1}{(k''^2-\omega^2+i\epsilon)} \frac{1}{((k-k'-k'')^2-\omega^2+i\epsilon)}.\end{gathered} \tag{136}$$

It appears that this integral is more complicated than the previous ones. In this project no satisfying solutions has been found for this diagram because of the final integral. The $(k-k'-k'')^2$ term in the denominator causes these problems because it makes it impossible to split the integral in independent parts.

Recommendations for new projects

For additional research on this subject there are some interesting things to do. Most of them were briefly looked after but were discarded due to a lack of time.

- It would be interesting to look at higher order perturbation theory and interpret those diagrams. In this way also the series we wrote down for the change in ω could be checked.
- Instead of only looking at 2-pointfunctions also 4-pointfunctions or even higher functions could be found. Researching those functions would surely give more insight in the physics behind this problem.
- In this project we only looked at vacuum-vacuum transitions. However since we calculated the propagator of the forced harmonic oscillator in a general case we could in principle find the general propagator of the anharmonic oscillator. With this propagator we could, for example look at transitions from the ground state to the first excited state. Also with this general propagator we should be able to derive the same things we found in this project.
- By using more advanced mathematical techniques the complex diagram perhaps can be solved in a satisfying way. This would be very interesting because than we can solve the system completely up to second order.
- In this project we assumed the ground state energy to be 0. It would be interesting to see whether the expressions change a lot if we do not assume this.

Acknowledgements

The author is grateful for the assistance of M. de Roo during this research.

References

-Ashok Das, Field theory: a path integral approach, Singapore, World Scientific Publishing, 1993

-Richard P. Feynman, edited by Laurie M. Brown, Feynman's thesis: a new approach to quantum theory, Singapore, Mainland Press, 2006

-M. de Roo, Quantumveldentheorie, Institute for theoretical physics, Rijksuniversiteit Groningen, Lecture notes, 1988

-K. Schulten, Notes on quantum mechanics, Department of Physics and Beckman institute, University of Illinois at Urbana, Lecture notes, 2000