Classical membranes

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With the use of the principle of least action the equation of motion of a relativistic particle can be derived. With the Nambu-Goto action, we are able to derive the equations of motion of higher dimensional objects such as strings and the classical membrane. There are several highly symmetric solutions of the classical membrane such as the string, the periodic pulsating spherical and cylindrical membrane and the pancake model.

1. Introduction

In the course "Introduction to string theory" [1] the basic aspects of string theory are discussed, such as the relativistic particle and the classical theory of the open and the closed string. First a coordinate system is set up and with the use of the principle of least action the equations of motion of the particle and the string can be derived. The Nambu-Goto action that was used to derive the equation of motion of the string can, in a more general way, also be used to derive the equation of motion of an extended object with higher dimensions such as the classical membrane.

In section 2 we discuss briefly the relativistic particle and the action principle of higher dimension objects such as the string and the membrane.
In section 3 we present the equation of motion of the classical membrane from the Nambu-Goto action which is derived in section 2.
In section 4 we show some solutions of this equation of motion of the membrane and we discuss the movement and the boundary conditions of these membranes.

2. The principle of least action

The relativistic Particle
To obtain the equation of motion for a relativistic particle we us the principle of least action. For the relativistic particle the action is given by the length of the path that the particle travels trough space-time as shown in figure 1.
The path length that is swept out by the particle is:

$$dl = (-ds^2)^{\frac{1}{2}} = (-\eta_{\mu\nu}dx^\mu dx^\nu) = (-dx^\mu dx_\mu)^{\frac{1}{2}} \quad (1)$$

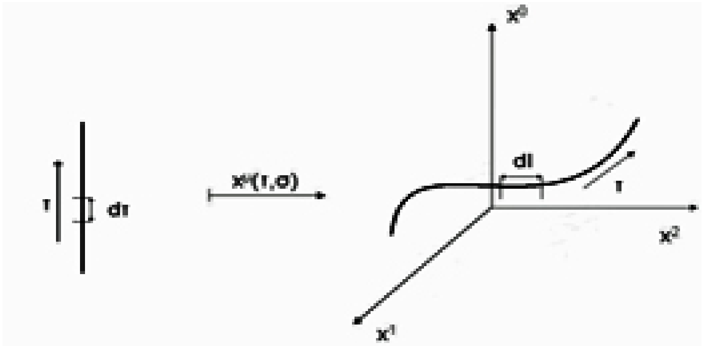

Figure 1: The embedding coordinates $x^\mu(\sigma)$ sweeps out path length on the world-sheet

Hence, the action for the particle is given by the total length of the trajectory swept out by the particle in space-time:

$$S[x] = -m \int d\tau \sqrt{-\dot{x}^2} \tag{2}$$

with

$$\dot{x}^2 = g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu, \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}$$

where we use the Einstein summation convention for μ and ν. m is a parameter with dimension of mass and $x^\mu(\tau)$ are called the embedding coordinates with the dimension of length. We choose the spacetime metric $g_{\mu\nu}$ as the flat Minkowski metric:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

After applying the principle of least action we find for a relativistic particle the following equation of motion:

$$-\frac{d}{d\tau}\left(\frac{m\, \dot{x}^\mu}{\sqrt{-\dot{x}^2}}\right) = 0 \tag{3}$$

Higher dimensional objects

The theory of the relativistic particle can be extended to a theory for higher dimensional objects such as strings and membranes.

To use the principle of least action we need to find the action for such an extended object.

First we set up a coordinate system σ^α,$\alpha = 0, ..., p$, where p is the number

of dimensions of the extended object (for the membrane $p = 2$). The embedding coordinates are given by $x^\mu(\sigma^\alpha)$, with $\mu = 0, ..., d-1$ and d the number of dimensions in which the object moves. Figure 2 shows the string trajectory embedded in the worldsheet.
The zeroth components of x and σ are the timelike directions. From this point we will use $\sigma^0 = \tau$.

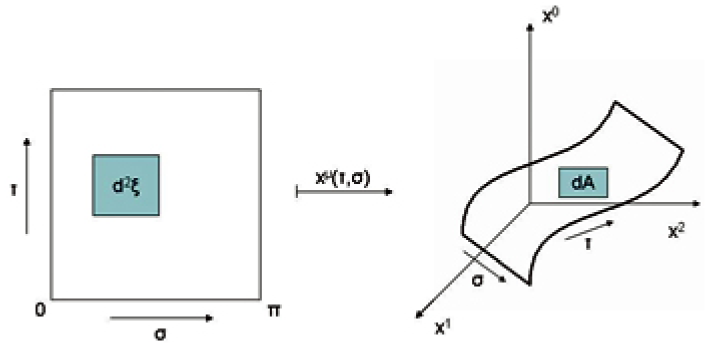

Figure 2: The embedding coordinates $x^\mu(\sigma)$ sweeps out the shape on the worldsheet

The square of the distance between the points with coordinates σ and $\sigma + d\sigma$ is given by:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}d\sigma^\alpha d\sigma^\beta \equiv -g_{\alpha\beta}d\sigma^\alpha d\sigma^\beta$$

Now the action can be written in terms of the induced metric $g_{\alpha\beta}$:

$$S = -T\int d^{p+1}\sigma\sqrt{-Det(g_{\alpha\beta})} \tag{4}$$

This action is called the Nambu-Goto form of the action. The parameter T has a dimension of $mass$ x $(length)^{-p}$. The Nambu-Goto action is invariant under coordinate transformations on the world-sheet:

$$g'_{\alpha\beta}(\sigma') = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta}g_{\gamma\delta}(\sigma)$$

3. The equation of motion of the classical membrane

As we mentioned in the second paragraph, the Nambu-Goto action of an extended object can be written as:

$$S = -T\int d^{p+1}\sigma\;\sqrt{-g} \tag{5}$$

where g is the determinant of the induce metric $g_{\alpha\beta}$. To derive the equation of motion of the classical membrane we minimize the difference in action and choose $p = 2$.

$$0 = \delta S = -T \int d^3\sigma \, \frac{1}{2\sqrt{-g}}(-\delta g)$$

where

$$\begin{aligned} \delta g &= Det(g_{\alpha\beta} + \delta g_{\alpha\beta}) - Det(g_{\alpha\beta}) \\ &= g.Det(1 + g^{\alpha\beta}\delta g_{\alpha\beta}) - g \\ &= g.Tr(g^{\alpha\beta}\delta g_{\alpha\beta}) \\ &= g.g^{\alpha\beta}\delta g_{\alpha\beta} \end{aligned}$$

with

$$\delta g_{\alpha\beta} = g_{\mu\nu}\left(\frac{\partial\delta x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta} + \frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial\delta x^\nu}{\partial\sigma^\beta}\right)$$

When we combine this we get:

$$\delta S = -T \int d^3\sigma\sqrt{-g}\; g^{\alpha\beta} g_{\mu\nu}\left(\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial\delta x^\nu}{\partial\sigma^\beta}\right)$$

After partial integration and by using the boundary condition:

$$\frac{\partial x^\mu}{\partial\sigma^\alpha} = 0 \tag{6}$$

at the surface of the membrane, we obtain

$$\delta S = T \int d^3\sigma \frac{\partial}{\partial\sigma^\beta}\left(\sqrt{-g}\; g^{\alpha\beta} g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^\alpha}\right)\delta x^\nu = 0$$

Which gives us the equation of motion for the classical membrane:

$$\frac{\partial}{\partial\sigma^\beta}\left(\sqrt{-g}\; g^{\alpha\beta} g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^\alpha}\right) = 0 \tag{7}$$

For calculating the different solutions in section 4 we use $g_{\mu\nu} = \eta_{\mu\nu}$, the flat Minkowski metric.

4. Solutions of the equation of motion

The string solution

One of the solutions of the equation of motion of the membrane should be the string solution. The string is described by:

$$x^0 = r\tau$$

$$x^1 = rcos(\sigma)cos(\tau)$$

$$x^2 = rcos(\sigma)sin(\tau)$$

where r is a parameter which describes the length of the string. The second spatial coordinate ρ is not included in the parametrization because the string model is independent of ρ.

To know if this string is a solution of the membrane we have to check if (7) holds.
For the string yields:

$$\frac{\partial x^\mu}{\partial \sigma^\alpha} = \begin{pmatrix} r & -rcos(\sigma)sin(\tau) & rcos(\sigma)cos(\tau) \\ 0 & -rsin(\sigma)cos(\tau) & -rsin(\sigma)sin(\tau) \end{pmatrix}$$

$$\frac{\partial x^\nu}{\partial \sigma^\beta} = \begin{pmatrix} r & 0 \\ -rcos(\sigma)cos(\tau) & -rsin(\sigma)cos(\tau) \\ rcos(\sigma)cos(\tau) & -rsin(\sigma)sin(\tau) \end{pmatrix}$$

With the flat Minkowsky metric $\eta_{\mu\nu}$ we find:

$$g^{\alpha\beta} = (g_{\alpha\beta})^{-1} = \left(\eta_{\mu\nu}\frac{\partial x^\mu}{\partial \sigma^\alpha}\frac{\partial x^\nu}{\partial \sigma^\beta}\right)^{-1} = \begin{pmatrix} \frac{1}{-r^2(1+cos^2(\sigma))} & 0 \\ 0 & \frac{1}{r^2 sin^2(\sigma)} \end{pmatrix}$$

and

$$g = Det(g_{\alpha\beta}) = r^4 sin^2(\sigma)(1 + cos^2(\sigma))$$

So the following three equations have to hold:

$$\frac{\partial}{\partial \tau}\left(\frac{rsin(\sigma)}{\sqrt{1 + cos^2(\sigma)}}\right) \stackrel{?}{=} 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{-rcos(\sigma)\sin(\sigma)\sin(\tau)}{\sqrt{1 + cos^2(\sigma)}}\right) + \frac{\partial}{\partial \sigma}\left(-rcos(\tau)\sqrt{1 + cos^2(\sigma)}\right) \stackrel{?}{=} 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{rcos(\sigma)sin(\sigma)cos(\tau)}{\sqrt{1 + cos^2(\sigma)}}\right) + \frac{\partial}{\partial \sigma}\left(-rsin(\tau)\sqrt{1 + cos^2(\sigma)}\right) \stackrel{?}{=} 0$$

The first equation is trivial because the argument is independent of τ. In the second and third equation the derivative of τ and σ cancel each other, so the tree equations hold. This means that,as expected, the string is a solution of the equation of motion of the membrane.
This solution represents a line which is rotating around the origin (figure 3).

We see that:

$$\dot{x}^\mu = (r, -rcos(\sigma)sin(\tau), rcos(\sigma)cos(\tau)) \quad \rightarrow \quad \dot{x}^2 = r^2(-1 + cos^2(\sigma))$$

This vanishes at the endpoints ($\dot{x}^2 = -1 + \vec{v}^2$), which means that the end-points of the string move with the speed of light. The fact the the end-points move with the speed of light prevents the string from collapsing due to the string tension.

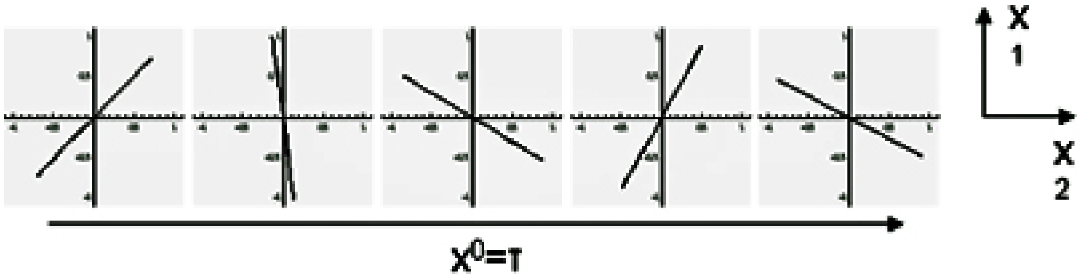

Figure 3: Sequence of five pictures of the rotating string with increasing τ and $r = 1$

The Pancake membrane

An other solution for the equation of motion for the membrane is the pancake membrane:

$$x^0 = \tau$$
$$x^1 = r(\sigma, \rho)cos(\omega_1 \tau)$$
$$x^2 = r(\sigma, \rho)sin(\omega_1 \tau)$$
$$x^3 = R(\sigma, \rho)cos(\omega_2 \tau)$$
$$x^4 = R(\sigma, \rho)sin(\omega_2 \tau)$$

This pancake membrane is only a solution for the equation of motion if the following conditions are satisfied:

$$\frac{\partial r}{\partial \sigma} = \frac{\partial R}{\partial \rho} = constant$$

and

$$\frac{\partial r}{\partial \rho} = \frac{\partial R}{\partial \sigma} = 0$$

So this means that $r(\sigma, \rho) = C_1 * \sigma$ and $R(\sigma, \rho) = C_2 * \rho$, with C_1 and C_2 arbitrary constants which we will put to 1.

The pancake solution represents a disc spinning in the $x^1 - x^2$ plane with frequency ω_1 and the $x^3 - x^4$ plane with frequency ω_2. We see that

$$\dot{x}^\mu = (1, -\sigma\omega_1 sin(\omega_1 \tau), \sigma\omega_1 cos(\omega_1 \tau), -\rho\omega_2 sin(\omega_2 \tau), \rho\omega_2 cos(\omega_2 \tau))$$

$$\rightarrow \dot{x}^2 = -1 + \sigma^2\omega_1^2 + \rho^2\omega_2^2$$

Again $\dot{x}^2$ should vanish, because the membrane must move with the speed of light to prevent collapsing. This means that we have found the boundary condition for the pancake membrane:

$$\sigma^2\omega_1^2 + \rho^2\omega_2^2 = 1 \tag{8}$$

Because the pancake membrane moves in a four dimensional world-sheet it is not possible to draw this membrane.

Periodic pulsating cylindrical membrane

The cylindrical membrane is given by:

$$x^0 = \tau$$

$$x^1 = r(\tau)cos(\sigma)$$
$$x^2 = r(\tau)sin(\sigma)$$
$$x^3 = \rho$$

This cylindrical membrane is only a solution of the equation of motion if the following condition on the radius $r(\tau)$ is satisfied:

$$\ddot{r}(\tau) = \frac{2(1+\dot{r}^2)}{r} \tag{9}$$

This equation can be solved in terms of Jacobi elliptic functions. The function $cn(x,k)$ solves the following equations:

$$\left(\frac{dy}{dx}\right)^2 = (1-y^2)(1-k^2+k^2y^2) \tag{10}$$

and

$$\frac{d^2y}{dx^2} + (1-2k^2)y + 2k^2y^3 = 0 \tag{11}$$

When we choose $k = \frac{1}{r_0} = \sqrt{\frac{1}{2}}$, $y = r(\tau)$ and $x = \tau$ and we use both Jacobi functions we find:

$$r(\tau) = cn\left(\tau, \sqrt{\frac{1}{2}}\right) \tag{12}$$

The function cn is a periodic pulsating function, which means that the cylindrical membrane will pulsate with a radius between 1 and 0 as shown in figure 4.

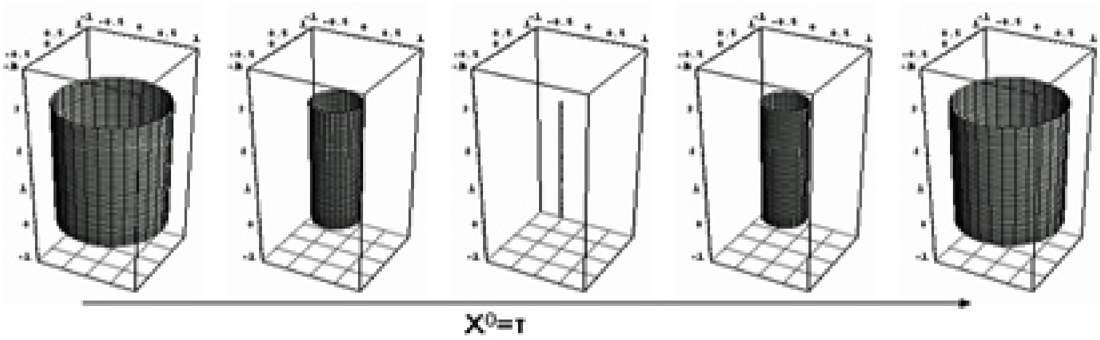

Figure 4: Sequence of five pictures of the pulsating cylindrical membrane with increasing τ

Periodic pulsating spherical membrane
The spherical membrane is given by:

$$x^0 = \tau$$

$$x^1 = r(\tau)sin(\sigma)cos(\rho)$$
$$x^2 = r(\tau)sin(\sigma)sin(\rho)$$
$$x^3 = r(\tau)cos(\sigma)$$

This is just as the cylindrical membrane only a solution if the condition (7) is satisfied for the radius $r(\tau)$. So like the cylindrical membrane:

$$r(\tau) = cn\left(\tau, \sqrt{\frac{1}{2}}\right)$$

Therefore the spherical membrane pulsates just as the cylindrical membrane with a radius varying between 1 and 0 (shown in figure 5).

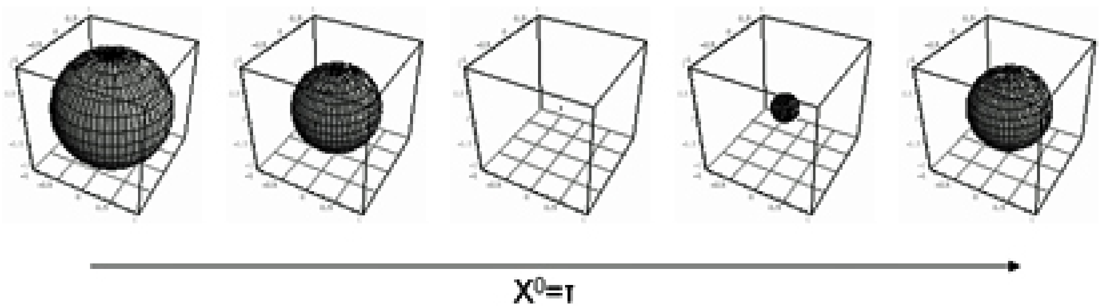

Figure 5: Sequence of five pictures of the pulsating spherical membrane with increasing τ

Periodic pulsating disk membrane

The last solution of the equation of motion of the membrane is the periodic pulsating disk membrane. This membrane is given by:

$$x^0 = \tau$$
$$x^1 = r(\tau)sin(\sigma)cos(\rho)$$
$$x^2 = r(\tau)sin(\sigma)sin(\rho)$$

By calculation of the equation of motion we find no condition for the radius $r(\tau)$. This is surprising because this means that we could choose $r(\tau)$ as a constant, which means that we have a static membrane. This is impossible because a static membrane isn't stable due to the tension in the membrane. Dirac [4] stated that a static membrane is only possible if the membrane carries an electric charge where the charge balances the surface tension.

In their paper about properties of the eleven-dimensional supermembrane theory [5] Bergshoeff, Sezgin and Townsend give a solution for this membrane. In this solution the radius $r(\tau)$ should satisfy the following condition:

$$r(\tau) = \frac{1}{r_0}\sqrt{r_0^4 - r^4} \tag{13}$$

This equation can again be solved in terms of the Jacobi Elliptic function (11), and as a result we find $cn(\tau, \sqrt{\frac{1}{2}})$. Hence, the membrane is a periodic pulsating disk which collapses to a point and expands to a disk.

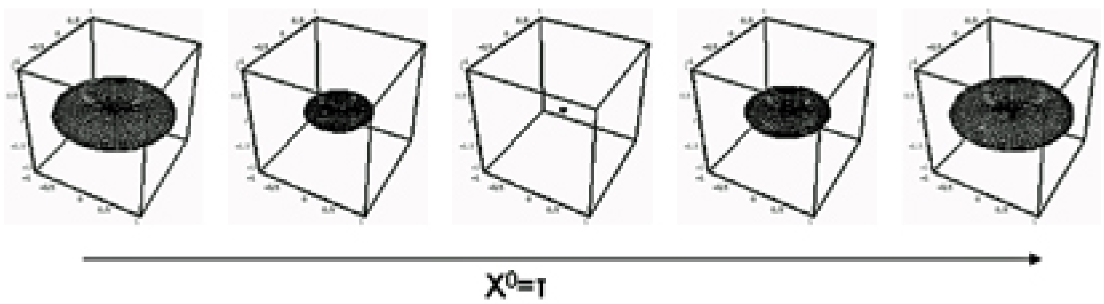

Figure 6: Sequence of five pictures of the periodic pulsating membrane with increasing τ

For a following research about the classical membrane one may look at the equation of motion with the Dirichlet boundary conditions in stead of the Neumann boundary conditions (6). Also the disk membrane can be investigated further to find the answer why this static membrane is a solution of the equation of motion.

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