Instantons and Kaluza-Klein reductions over time

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Abstract

In this master thesis various aspects of instantons are explored. To begin with the $SU(2)$ Yang-Mills instanton, followed by the extremal D-instanton of type IIB string theory, both obtained via a Wick rotation. The D-instantons give rise to wormholes, which are further investigated, especially for the Schwarzschild and Reissner-Nordstrøm black holes. To obtain instantons, Kaluza-Klein reductions over a time-like component are introduced, which can be seen as an alternative way to obtain instantons, without a Wick rotation thus. This also relates instantons to solitons. For various Lagrangians the instantons obtained in this way have been explored, most notably the Reissner-Nordstrøm and dilatonic black holes, the latter resembles the D-instanton if one of the scalar fields is turned off. This system is investigated in depth, including the presence of wormholes and how they are related to different frames used in string theory. The non-extremal solutions obtained are uplifted and compared to standard literature and a prediction is made of their effect on the effective IIB string theory action. It is conjectured that besides the $\mathcal{R}^4$ contribution, which is due to the extremal D-instanton, the non-extremal D-instantons give raise to a $\mathcal{R}^8$ contribution.

[**When using standard letter size (10 pt) it is 86 pages, including appendix, list of figures... etc.**]

Contents

List of Figures

Chapter 1

Introducing the *Instanton*

1.1 The Instanton

The name **instanton** dates back to 1976, due to G. 't Hooft [28], and he describes it as

> *My starting point is the solution of classical field equations ... in four dimensional (4D)* ***Euclidean*** *gauge-field theories ... the solution is not only localized in three-space, but also* ***instantaneous*** *in time. I shall refer to such objects as "Euclidean-gauge solitons", EGS for short.*

These "Euclidean-gauge solitons" are now called instantons and the two relevant words are written in bold. In this article 't Hooft also explains why instantons are relevant to physics

> *There is a simple heuristic argument that explains why these solutions of the Euclidean field equations are relevant for describing a tunnelling mechanism in real (Minkowsky) space-time ... Consider an ordinary quantum mechanical system with a potential barrier V larger than the available energy E, which I put equal to zero. Then the leading exponential of the tunnelling amplitude is $exp(-\int \mathrm{p}\mathrm{d}x)$ with*
>
> $$p^2/2m = V - E \tag{1.1}$$
>
> *This corresponds to the classical equations of motion, except for a sign difference. Thus the leading exponential is obtained by replacing in the equations of motion $\mathtt{t}$ by $\mathtt{it}$ and computing the action S for a path from one to the other vacuum.*

Generalizing these two quotes to modern day theoretical theories, which are often field theories, the concept of instantons play a role in calculating (parts of) the path integral

$$G = \int \mathcal{D}[\phi] e^{iS_{Min}(\phi(x,t),\dot{\phi}(x,t))} \tag{1.2}$$

where ϕ stand for the fields, S_{Min} for the action in Minkowskian space-time and the path integral $\mathcal{D}[x]$ goes over all fields ϕ that satisfy appropriate boundary conditions. Only rarely can one explicitly solve these path integrals.

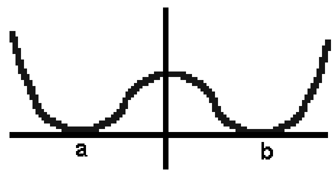

Figure 1.1: Double well potential $V(x)$, without a classical solution

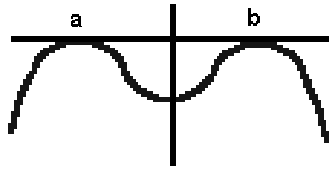

Figure 1.2: Rotated double well potential $V(x)$, with a classical solution

To explain these two quotes the instanton belonging to the double well potential, see figure 1.1, will be worked out in some detail. As a working definition of the instanton will be used

> **Definition**: An instanton is a finite action solution of the Euclidean version of the Minkowskian action.

How one obtains the Euclidean (Euc) theory from the corresponding Minkowskian (Min) action is explained in appendix A.1 [1]. Note that 't Hooft uses in his article the Euclidean version of the equations of motion, instead of the action. To calculate the tunnelling amplitude (putting $\hbar = 1$) for going from a at $\frac{-\tau}{2}$ to b at $\frac{\tau}{2}$ one can use the Feynman path integral formalism

$$< b, \frac{\tau}{2}|a, -\frac{\tau}{2} >= \int \mathcal{D}[x]e^{iS_{Min}} \tag{1.3}$$

where

$$S[x(t), \dot{x}(t)]_{Min} = \int_0^t \mathrm{d}t' \Big[\frac{1}{2}m(\dot{x}(t'))^2 - V(x(t), \dot{x}(t'))\Big] \tag{1.4}$$

and the path integral is over all paths that begin at a at $\frac{-\tau}{2}$ and end at b at $\frac{\tau}{2}$. A standard trick is to apply the stationary phase approximation (SPA) or saddle point method: one realizes that the greatest contribution to the path integral comes from a) the classical path ($\overline{x}(t)$) that extremize the action, i.e. those paths satisfying

$$\frac{\partial S[x(t)]}{\partial x(t)}\Big|_{\overline{x}(t)} = 0 \tag{1.5}$$

and b) from the fluctuations around it. Because the integrand of the sum over paths in (1.3) is an oscillating function with phase S, only those paths that have roughly the same action will interfere constructively and thus contribute to the sum. The potential in figure 1.1 however does *not* have a *classical* solution connecting the two wells at a and b. If however the action (1.4) is analytically continuated to the whole complex plane by applying a Wick rotation to the time t via

$$t = \tau e^{-i\delta} \qquad \delta \in [0, 2\pi] \qquad \tau, \delta \in \mathbb{R} \tag{1.6}$$

solutions *do* exist, as will be shown below. These are the instantons if they have finite action. This continuation is allowed if there are no singularities throughout the complex plane swept by δ, for convenience take $\delta = \frac{\pi}{2}$. This is now a Euclidean theory, since a timelike component t has been changed into a spacelike component

τ. To obtain a real action one has to multiply by $-i$, see appendix A.1 for more details. Applying rules one and four to (1.4) gives

$$S_{Euc}[x(\tau),\dot{x}(\tau)] = \int_0^\tau \mathrm{d}\tau' \Big[\frac{1}{2}m(\dot{x}(\tau'))^2 + V(x(\tau'),\dot{x}(\tau'))\Big] \tag{1.7}$$

and one achieves a sign flip before the potential, see figure 1.2. Also observe that the tunnelling amplitude becomes now

$$< b, \frac{\tau}{2}|a, -\frac{\tau}{2} >= \int \mathcal{D}[x] e^{-S_{Euc}} \tag{1.8}$$

which explains why the finite action requirement is needed. The sign change in the action (1.7) means that there is now a classical path connecting a and b, since these are now the maximum of the flipped potential, see figure 1.2. This classic path is an extreme of the Euclidean action, in general a *minimum*. It will therefore give an important contribution to the path integral (1.8), the instanton method has thus induced tunnelling between a and b and an extra (perturbative) part of the path integral has been calculated in this way. An anti-instanton solution is obtained, if one takes into account the tunnelling from b to a. For physical results a Wick rotation back to the normal time t (i.e. $\tau \to it$) must be applied to the final results. When doing the full calculation one gets the same result as one normally obtains via the WKB [34] method of quantum mechanics. This example shows that the use of instantons is (amongst other things) to obtain more information about the path integral. Later in this chapter it will be shown that instantons in string theory also give corrections to the Einstein-Hilbert (EH) action in the context of string theory.

1.2 $SU(2)$ Yang-Mills[1]

The double well example explains both what instantons are and what they are good for. Modern day theories like the Standard Model and extensions thereof require groups like $SU(N)$. Therefore a physically more interesting instanton is the $SU(2)$ Yang-Mills instanton. Let the vector-fields be A^a_μ, $a = 1,2,3$ and to make the notation simpler introduce

$$A_\mu(x) \equiv \sum_a g \frac{\sigma^a}{2i} A^a_\mu(x) \tag{1.9}$$

where x stands for (x_1, x_2, x_3, x_4), g is a coupling constant and σ^a are the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.10}$$

The $\frac{\sigma^a}{2}$ form the three generators of the two-dimensional representation of the $SU(2)$-group[1]. The field tensor is defined analogously

$$G_{\mu\nu} \equiv \sum_a g \frac{\sigma^a}{2i} G^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{1.11}$$

[1]Which satisfies the Lie algebra $[\frac{\sigma^a}{2}, \frac{\sigma^a}{2}] = i\epsilon^{abc}\frac{\sigma^c}{2}$.

where

$$G^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu \tag{1.12}$$

and ϵ^{abc} is the Levi-Civita symbol. The $SU(2)$ gauge transformation U acts as

$$\begin{aligned} A_\mu &\to U A_\mu U^{-1} + U\partial_\mu U^{-1} \\ G_{\mu\nu} &\to U G_{\mu\nu} U^{-1} \end{aligned} \tag{1.13}$$

and the Minkowskian and the corresponding Euclidean action are

$$S_{Min} = \frac{1}{2g^2} \int \mathrm{d}^4x \mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] \tag{1.14}$$

$$S_{Euc} = -\frac{1}{2g^2} \int \mathrm{d}^4x \mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] \tag{1.15}$$

Note that in Euclidean space there is no difference between co- and contravariance. A standard approach to obtain the Yang-Mills instantons is via the *Bogomol'nyi bound*. Begin with the identity (Euclidean signature)

$$-\int \mathrm{d}^4x \mathrm{Tr}[(G_{\mu\nu} \pm \tilde{G}_{\mu\nu})^2] \geq 0 \Rightarrow -\int \mathrm{d}^4x \mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] \geq \mp \int \mathrm{d}^4x \mathrm{Tr}[\tilde{G}_{\mu\nu}G^{\mu\nu}] \tag{1.16}$$

where $\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$ is the dual of $G_{\mu\nu}$[2] and using (1.15)

$$S \geq \frac{8\pi^2}{g^2}|Q| \tag{1.17}$$

where $|Q|$ is the Pontryagin index (see below)

$$Q \equiv \int Q(x)\mathrm{d}^4x = -\frac{1}{16\pi^2} \int \mathrm{d}^4x \mathrm{Tr}[\tilde{G}_{\mu\nu}G^{\mu\nu}] \tag{1.18}$$

The minimum action is according to (1.16) obtained when

$$\tilde{G}_{\mu\nu} = \mp G_{\mu\nu} \tag{1.19}$$

Thus, self-dual (+) and anti-self-dual (−) configurations extremize the action for this system[3]. Using as an ansatz

$$A_\mu(x) = i\overline{\sum}_{\mu\nu} \partial_\nu \log[\phi(x)] \tag{1.20}$$

where

$$\overline{\sum}_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 & -\sigma_2 & -\sigma_1 \\ -\sigma_3 & 0 & \sigma_1 & -\sigma_2 \\ \sigma_2 & -\sigma_1 & 0 & -\sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 & 0 \end{pmatrix} = \overline{\eta}^{i\mu\nu}\frac{\sigma^i}{2} \tag{1.21}$$

and

$$\overline{\eta}^{i\mu\nu} = -\overline{\eta}^{i\nu\mu} = \begin{cases} \epsilon^{i\mu\nu} & \text{for } \mu,\, \nu = 1,2,3 \\ -\delta^{i\mu} & \text{for } \nu = 4 \end{cases} \tag{1.22}$$

[2] $\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] = \mathrm{Tr}[\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu}]$

[3] This of course need not be it's only extreme, when however the system has supersymmetry extra conditions follow from that and fix the extreme uniquely.

which means that $\overline{\sum}_{\mu\nu}$ is both antisymmetric and anti-self dual in its indices. Substituting this into (1.19) and using (1.20) leads to the N-instanton solution

$$\phi(x) = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{|x_\mu - a_{i\mu}|^2} \Rightarrow A_{\mu(x)} = i\overline{\sum}_{\mu\nu} \partial_\nu \Big(\log \Big[1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{|y_i|^2} \Big] \Big) \tag{1.23}$$

where $y_\mu \equiv (x-a)_\mu$ and $y^2 \equiv y_\mu y^\mu$. The special case $N = 1$ gives

$$A_{\mu(x)} = -2i\lambda_1^2 \overline{\sum}_{\mu\nu} \frac{y_\nu}{y^2(y^2+\lambda_1^2)} \tag{1.24}$$

This instanton is clearly singular at $y = 0$. This can be removed by applying the gauge transformation

$$U_1(x) = \frac{(x_4 + ix_j\sigma_j)}{|x|} = \sum_\mu \hat{x}_\mu s_\mu \tag{1.25}$$

leading to

$$\boxed{A_\mu(x)' = -2i \sum_{\mu\nu} \frac{y_\nu}{y^2+\lambda_1^2} = -2i \sum_{\mu\nu} \frac{(x-a_1)_\nu}{|(x-a_1)|^2 + \lambda_1^2}} \tag{1.26}$$

where

$$\sum_{\mu\nu} = \eta^{i\mu\nu} \frac{\sigma^i}{2} \tag{1.27}$$

$$\eta^{i\mu\nu} = -\eta^{i\nu\mu} = \begin{cases} \epsilon^{i\mu\nu} & \text{for } \mu, \quad \nu = 1,2,3 \\ \delta^{i\mu} & \text{for } \nu = 4 \end{cases} \tag{1.28}$$

This is the one instanton solution[4] with the corresponding action

$$\boxed{S_{Euc} = -\frac{1}{2g^2} \int \mathrm{d}^4x \mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] = \frac{8\pi^2}{g^2}} \tag{1.29}$$

Note that $\sum_{\mu\nu}$ is self-dual and hence it solves (1.19) with the plus sign. The corresponding Pontryagin index is via (1.17) $Q = 1$. Now that an instanton solution has been obtained, a tunnelling interpretation will be described. First analyze the classical vacuum configurations of the Minkowskian action (1.14): $S_{Min} = 0 \longleftrightarrow G_{\mu\nu} = 0$. However according to (1.13) this is a gauge invariant statement and thus the vacuum is described by the *pure gauges*

$$A_\mu(\mathbf{x}) = U(\mathbf{x})\partial_\mu \Big(U^{-1}(\mathbf{x}) \Big) \tag{1.30}$$

When working in the $A_0(\mathbf{x}, t) = 0$, gauge the only gauge transformations still permitted are the time-independent ones described by the time-independent matrices $\Lambda(\mathbf{x}, t) = \alpha(\mathbf{x})$, i.e.

$$A_i(\mathbf{x}) = e^{-\alpha(\mathbf{x})} \nabla_i e^{\alpha(\mathbf{x})} \tag{1.31}$$

[4]The anti-instanton solution can be obtained by replacing $\sum_{\mu\nu}$ by $\overline{\sum}_{\mu\nu}$.

Restricting to those $\alpha(\mathbf{x})$ which satisfy $e^{\alpha(\mathbf{x})} = 1$ at spatial infinity (all points $|\mathbf{x}| = \infty$) implies that three-dimensional space is compacted into the surface S^3_{phy}, where the subscript *phy* stands for physical space. The topology of the $SU(2)$ group is the three-dimensional surface of a unit sphere in four dimensions, S^3_{int}, where the subscript *int* refers to the fact that this is the "group space" or internal space. The $e^{\alpha(\mathbf{x})}$ therefore map S^3_{phy} into S^3_{int} and such mappings can be classified into homotopy sectors which form the homotopy group $\pi_3(SU(2)) = \mathbb{Z}$. Each sector is characterized by an integer, the Pontryagin index N

$$N = \frac{1}{24\pi^2}\int \mathrm{d}^3x \mathrm{Tr}[(e^{-\alpha(\mathbf{x})}\nabla_i e^{\alpha(\mathbf{x})})(e^{-\alpha(\mathbf{x})}\nabla_j e^{\alpha(\mathbf{x})})(e^{-\alpha(\mathbf{x})}\nabla_k e^{\alpha(\mathbf{x})})]\epsilon_{ijk} \quad (1.32)$$

Thus the classical vacua described by (1.31) can be divided into sectors, labelled by the index N. As an example look at

$$A_i^{(1)}(\mathbf{x}) = e^{-\alpha_1}\nabla_i e^{\alpha_1} \quad (1.33)$$

with

$$\alpha_1(\mathbf{x}) = \frac{i\pi\sigma\cdot\mathbf{x}}{(\mathbf{x}^2 + a^2)^{1/2}} + i\pi\sigma_3 \quad (1.34)$$

which belongs to the $N = 1$ sector and at spatial infinity $e^{\alpha_1(\mathbf{x})} \to 1$. The gauge transformation

$$g_1(\mathbf{x}) = e^{-\alpha_1(\mathbf{x})} \quad (1.35)$$

takes the $N = 0$ configurations $A_i = 0$ into the $N = 1$ configuration $A_i^{(1)}(x)$. Around each homotopy class N, of classical vacua, one can construct a topological vacuum state $|N >$, these states in different homotopy sectors are not gauge equivalent. To explain this look at Gauss's law

$$I(x) = \nabla E(x) - j_0(x) = 0 \quad (1.36)$$

In a quantized theory this becomes an *operator* $I(x)$ and cannot be equal to zero, for it must obey the canonical commutation relation

$$[I(x_1), A_x(x_2)]_{t=0} = i\frac{\partial}{\partial x_1}[\delta(x_1 - x_2)] \neq 0 \quad (1.37)$$

where $t = 0$ is taken because of the temporal gauge chosen. To satisfy the above two relations in a quantum theory, define the *physical* states as those states that satisfy

$$I(x)|\text{Phys} >= 0 \quad (1.38)$$

The Yang-Mills version of this is

$$I(\mathbf{x}) \equiv D_i G^{0i} = \partial_i E^i + [A_i, E^i] = 0 \quad (1.39)$$

where $E^i \equiv G^{0i}$ is the Yang-Mills 'electric' field. To explain where this generalized Gauss's law follows from, look at the equation of motion that follows from the Yang-Mills action (1.15)

$$D_\mu G_{\mu\nu} \equiv \partial_\mu G_{\mu\nu} + [A_\mu, G_{\mu\nu}] = 0 \quad (1.40)$$

In the temporal gauge this leads to (1.39) and quantizing this leads to the physical state constraint

$$D_i E^i|\text{Phys} >= 0 \quad (1.41)$$

Infinitesimal the gauge transformation (1.13) with $U = e^{-\Lambda(\mathbf{x})} \simeq 1 - \delta\Lambda(\mathbf{x})$ leads to

$$A_i \to A_i + [A_i, \delta\Lambda] + \nabla_i(\delta\Lambda) = A_i + D_i(\delta\Lambda) \tag{1.42}$$

with D_i the covariant derivative. Restrict to those $\hat{\Lambda}(\mathbf{x})$ which behave at spatial infinity as

$$\lim_{\mathbf{x}\to\infty} \hat{\Lambda}(\mathbf{x}) \to 0 \tag{1.43}$$

which are called the 'little' gauges. The canonical momentum conjugate to A_i is

$$\frac{\partial\mathcal{L}}{\partial_0 A_i} = \frac{2}{g^2} E_i \tag{1.44}$$

and thus the operator in quantum theory generating these gauge transformations $e^{-\hat{\Lambda}}$ is

$$U = e^{\frac{2i}{g^2}\int \mathrm{d}^3x \mathrm{Tr}(D_i\hat{\Lambda}E)} = e^{\frac{-2i}{g^2}\int \mathrm{d}^3x \mathrm{Tr}[\hat{\Lambda}D_i E^i]} \tag{1.45}$$

where integration by parts has been used and the fact that $\hat{\Lambda}$ is a little gauge. This operator acts on physical states due to the generalized Gauss's law as

$$U|\mathrm{Phys}>=|\mathrm{Phys}> \tag{1.46}$$

Gauge equivalence applies only to these little gauges. The example given by $g_1(x)$ (1.35) does not belong to these, since at spatial infinity e^{α} goes to one. To obtain the tunnelling interpretation a link must be made to the Euclidean action (1.15)

$$S_{Euc} = \frac{1}{2g^2}\int \mathrm{d}^4x \mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] \tag{1.47}$$

To have finite action, the fields $A_\mu(\mathbf{x}, t)$ must approach vacuum (pure gauge) configurations at the boundary of Euclidean space-time. This boundary is again a S^3_{phy}, e.g. $S^3_{\mathrm{phy}} \Rightarrow SU(2)$. Such mappings can be described by the Pontryagin index Q

$$Q = -\frac{1}{16\pi^2}\int \mathrm{d}^4x \mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}] = \frac{1}{24\pi^2}\oint_{s^3} \mathrm{d}\sigma_\mu \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[A_\nu A_\rho A_\sigma] \tag{1.48}$$

Note that in the classical vacuum analysis the index N goes only over the spatial part, i.e. $\int \mathrm{d}^3x$, due to the $A_0 = 0$ gauge chosen, whereas here the integral goes over the full Euclidean space. The relevant Euclidean path integral G is

$$G = \int \mathcal{D}[A_\mu]e^{-S_{Euc}} \tag{1.49}$$

where the path integral is over all fields A_μ that satisfy the vacuum boundary conditions at spatial infinity. The Pontryagin index is a homotopy index and hence two different values of Q *cannot* be related. Therefore (1.49) can be split into a sum over all different values Q

$$G = \sum_Q \int \mathcal{D}[A_\mu]_Q e^{-S_{Euc}} \equiv \sum_Q G_Q \tag{1.50}$$

Go again to the $A_0 = 0$ gauge and picture the boundary of space-time as shown in figure 1.3. The two flat surfaces of the cylinder stand for all three-dimensional

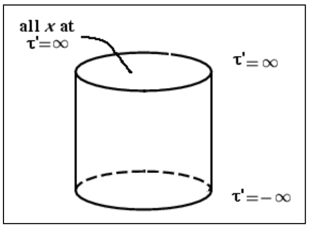

Figure 1.3: Boundary of the spacetime for the Yang-Mills system

space at $\tau' = \pm\infty$, while the curved surface stands for the boundary of space, $\mathbf{x} \to \infty$ at all τ'. In the $A_0 = 0$ gauge the index μ of (1.48) can only take the value $\mu = 0$. Therefore

$$Q = \frac{1}{24\pi^2} \int \mathrm{d}^3x \epsilon_{ijk} \mathrm{Tr}[A_i A_j A_k]_{\tau'=\pm\infty} \tag{1.51}$$

Now compare this to the classical vacuum index N (1.32)

$$Q = N_+ - N_- \tag{1.52}$$

where N_+ stands for $\tau' = +\infty$ and N_- on $\tau' = -\infty$. To use the full gauge freedom available, set $\alpha(\mathbf{x}) = 0$ at $\tau' = -\infty$. This means that $N_- = 0$ and thus $Q = N_+$. The G_Q's can then be interpreted as the Euclidean transition amplitude, connecting the classical vacuum labelled by $N = 0$ to the classical vacuum labelled by $N = Q$. Of course this interpretation is only valid within the gauges chosen. An interesting consequence of this all is that the *true* vacuum is no longer described by N but one should introduce the so called θ-vacuum

$$\boxed{|\theta >= \sum_{N=-\infty}^{+\infty} e^{iN\theta} |N >} \tag{1.53}$$

and this is the true vacuum of the $SU(2)$ Yang-Mills system. As explained the operator $U(g_1)$, (1.45)[5] which performs the gauge transformation g_1, acts on a state $|N >$ as

$$U(g_1)|N >= |N + 1 > \tag{1.54}$$

because this is not a little gauge transformation and hence can change the homotopy index N. It still has to commutate with the Hamiltonian H, $[U(g_1), H] = 0$, and since $U(g_1)$ is unitary its eigenvalues are of the form $e^{-i\theta}$. The eigenstates of the hamiltonian are $|N >$ and in combination with the unitary requirement it follows that the θ-vacuum does the job, since

$$U(g_1)|\theta >= \sum_{N=-\infty}^{+\infty} e^{iN\theta} |N + 1 >= \sum_{N=-\infty}^{+\infty} e^{i(N+1)\theta} |N + 1 >= |\theta > \tag{1.55}$$

[5]This means take for $\hat{\Lambda} = g_1$.

The action will be influenced as follows

$$
\begin{aligned}
<\theta|e^{-H\tau}|\theta>= & \sum_{N,Q} e^{-iQ\theta} <N+Q|e^{-H\tau}|N> \\
= & \; 2\pi\delta(0)\sum_{Q} e^{-iQ\theta}\int \mathcal{D}[A_\mu]_Q e^{-S_{Euc}}] \\
= & \pi\delta(0)\int \mathcal{D}[A_\mu]_{\text{all Q}} e^{\left(-S_{Euc}+\frac{i\theta}{16\pi^2}\int \text{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu}\text{d}^4x)\right)}
\end{aligned}
\tag{1.56}
$$

where the first equality follows from the definition of the θ-vacuum (1.53), the second one follows from the Feynman path integral formalism for calculating expectation values and the last one from the definition of Q, (1.48). The exponent now defines a new contribution to the action

$$
S_{total} \equiv S_{Euc} - \frac{i\theta}{16\pi^2}\int \text{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}]\text{d}^4x \tag{1.57}
$$

This results into adding to the Minkowskian Lagrangian density the extra term

$$
\boxed{\Delta\mathcal{L}_\theta = \frac{\theta}{16\pi^2}\text{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}]} \tag{1.58}
$$

This too is a gauge invariant result like the θ-vacuum and can be shown to be a total divergence, it will thus not affect the classical Yang-Mills equations. However for each θ it will yield a different quantum theory, where the corresponding θ-vacuum is the ground state of the system. Another interesting aspect of this additional Lagrangian is that it can be rewritten as

$$
\Delta\mathcal{L}_\theta = \frac{\theta}{16\pi^2}\text{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}] = \frac{\theta}{16\pi^2}\text{Tr}[4E_iB^i] \tag{1.59}
$$

if the definition of $G_{\mu\nu}$ is substituted. Now under time-reversal T the electric field is invariant, but the magnetic field changes sign, whereas the opposite happens under a parity transformation P[6]. The original Lagrangian can be shown to be equal to $\text{Tr}[E_iE^i - B_iB^i]$ and thus invariant under both parity and time reversal. Hence for all $\theta \neq 0$ the total Lagrangian becomes P and T violating. Finally observe that the original Lagrangian of this theory (1.14) does not have a clear potential as the double well system has (1.4), only by going to the temporal gauge it was possible to give a tunnelling interpretation.

1.3 D-instantons in type IIB String theory

The massless bosonic content of the chiral type IIB superstring consists of a graviton $g_{\mu\nu}$, an antisymmetric two-index tensor $B^{(2)}_{\mu\nu}$, a dilaton ϕ from the NS-NS (Neveu-Schwarz) sector, an axion (zero-form) a, a 2-form $C^{(2)}$ and a four form $C^{(4)}$ with a self-dual field strength from the R-R (Ramond) sector. These fields correspond to a total of 128 physical degrees of freedom, of which 35 are associated with the graviton, 28 with the 2-form $B^{(2)}_{\mu\nu}$, one with the dilaton, one with

[6]Under a parity transformation the time t remains unchanged and the spatial part changes sign, $(t, \mathbf{x}) \rightarrow (t, -\mathbf{x})$.

the zero-form a, 28 with the 2-form $C^{(2)}$ and 35 with the four-form $C^{(4)}$. The signature being used in this section is diag $= (-, +, ..., +)$.

A simple way to explain why the axion a has to be a pseudoscalar is the following. Type IIA string theory is defined by multiplying the Majorana-Weyl spinors[7] with opposite chirality $(8_s + 8_v) \times (8_s + 8_v)$, [27], causing a *left-right* symmetry under parity. Here $(8_c, 8_s)$ are the two fundamental eight dimensional spinor spaces, 8_s with positive chirality, 8_c with negative chirality and 8_v the massless Neveu-Schwarz representation. The particle spectrum has this mirror symmetry too. The action described below (1.61) is the massless (bosonic) sector only, for massless particles helicity and chirality agree

$$\text{chirality} = \text{helicity} = \frac{\vec{s} \cdot \vec{p}}{|\vec{p}|} \tag{1.60}$$

where $\vec{s}$ is spin and $\vec{p}$ is the momentum. Applying a parity transformation, as defined at the end of the previous section, changes the helicity to its opposite sign as is clear from the momentum. In type IIA this merely interchanges the left- and right sector and parity and thus helicity is conserved. For type IIB the two Majorana-Weyl spinors have the *same* chirality: $(8_s + 8_v) \times (8_s + 8_v)$. Under a parity transformation there is no left-right symmetry then. This should apply to *all* particles, including the scalars. A scalar is invariant under a parity transformation and thus does not agree with this. A pseudoscalar however changes sign under a parity transformation[8] and thus does the job.

The low energy effective action of the type IIB string can be written in the *string frame* (SF) as [4]

$$\begin{aligned} S^{SF}_{Min,IIB} = \int \mathrm{d}^{10}x\sqrt{g}e^{-2\phi}\mathcal{R} + \int \Big[\ & e^{-2\phi}\Big(4\mathrm{d}\phi \wedge *\mathrm{d}\phi - \frac{1}{2}H^{(3)} \wedge *H^{(3)}\Big) \\ & - \frac{1}{2}F^{(1)} \wedge *F^{(1)} - \frac{1}{2}\tilde{F}^{(3)} \wedge *\tilde{F}^{(3)} - \frac{1}{4}\tilde{F}^{(5)} \wedge *\tilde{F}^{(5)} - \\ & \frac{1}{2}C^{(4)} \wedge H^{(3)} \wedge F^{(3)}\Big] \end{aligned} \tag{1.61}$$

Where $\mathcal{R}$ stands for the Ricci scalar, $g = \mathrm{Det}[g_{\mu\nu}]$,

$$H_{(3)} = \mathrm{d}B(2), \;\; F_{(1)} = \mathrm{d}a, \;\; F_{(3)} = \mathrm{d}C(2), \;\; F_{(5)} = \mathrm{d}C(4) \tag{1.62}$$

and

$$\tilde{F}^{(3)} = F^{(3)} - a \wedge H^{(3)}, \;\; \tilde{F}^{(5)} = F^{(5)} - \frac{1}{2}C^{(2)} \wedge H^{(3)} + \frac{1}{2}B^{(2)} \wedge F^{(3)} \tag{1.63}$$

The use of the wedge product $\wedge$ implicitly already takes into account the presence of $\sqrt{|g|}$, it is therefore not written except in the Ricci scalar term. The string frame is that metric[9] for which there is no coupling between the kinetic term of the axion a and the dilaton ϕ. Solving all fields in the action (1.61) is a very difficult task. To make things simpler, truncate away all field except ϕ, a and the

[7] Weyl means that they are helicity eigenstates and Majorana means that the particle is equal to its anti-particle, i.e. that it is neutral like for example the photon.

[8] See appendix A.1

[9] Obtained by a conformal mapping via $e^{\alpha\phi}$, where α is a constant. See for example the Einstein frame below.

metric $g_{\mu\nu}$. This system allows an instanton solution, the so called D**-instanton**. The truncated version of the action (1.61) is

$$S^{SF}_{Min} = \int \mathrm{d}^{10}x\sqrt{g}\{\ e^{-2\phi}\{\mathcal{R} + 4\partial_\mu\phi\partial^\mu\phi\} - \frac{1}{2}\partial_\mu a\partial^\mu a\} \tag{1.64}$$

The coupling between $\mathcal{R}$ and ϕ makes calculating the equations of motion difficult. Removing this coupling can be achieved by going to the Einstein frame (EF) via a *conformal* mapping of the metric

$$\begin{aligned} &g^{SF}_{\mu\nu} = e^{\frac{\phi}{2}}G^{EF}_{\mu\nu} \\ &\Rightarrow \sqrt{g_{SF}}e^{-2\phi}\mathcal{R}_{SF} = \sqrt{G_{EF}}e^{-2\phi}\Big[\mathcal{R}_{EF} - 4\frac{D-1}{D-2}(\partial\phi)^2\Big] \\ &= \sqrt{G_{SF}}e^{-2\phi}\Big[\mathcal{R}_{EF} - \frac{9}{2}(\partial\phi)^2\Big] \end{aligned} \tag{1.65}$$

for D = 10. The definition of the Einstein frame is that frame in which the Einstein-Hilbert action, $\sqrt{|G|}\mathcal{R}$, is not coupled to the dilaton. Note that this implies two *different* infinitesimal line elements

$$\begin{aligned} \mathrm{d}s^2_{SF} &= G^{SF}_{\mu\nu}\mathrm{d}x_\mu\mathrm{d}x_\nu \\ \mathrm{d}s^2_{EF} &= G^{EF}_{\mu\nu}\mathrm{d}x_\mu\mathrm{d}x_\nu \\ \Longleftrightarrow \mathrm{d}s^2_{SF} &= e^{\frac{\phi}{2}}\mathrm{d}s^2_{EF} \end{aligned} \tag{1.66}$$

The word frame here has nothing to do with the word frame as used in special relativity. There a boost means viewing the event in a different frame, e.g. a co-moving or a laboratory frame, the infinitesimal line element $\mathrm{d}s^2$ is invariant under such transformations, see also below (1.79). Applying this conformal change to (1.64) gives[10]

$$S^{EF}_{Min} = \int \mathrm{d}^{10}x\sqrt{G}\{\mathcal{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu a\partial^\mu a\} \tag{1.67}$$

To obtain an instanton solution, a Wick rotation needs to be applied. As explained in appendix A.1 the pseudoscalar a picks up a factor of i. The *Euclidean* (Euc) action in the Einstein frame becomes then

$$\boxed{S^{EF}_{Euc} = \int \mathrm{d}^{10}x\sqrt{G}\{-\mathcal{R} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu a\partial^\mu a\}} \tag{1.68}$$

The field equations that follow from (1.68) are for respectively $G_{\mu\nu}$, a and ϕ

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\Big(\partial_\mu\phi\partial_\nu\phi - e^{2\phi}\partial_\mu a\partial_\nu a\Big) \tag{1.69}$$

$$\partial_\mu\Big(\sqrt{|G|}e^{2\phi}G^{\mu\nu}\partial_\nu a\Big) = 0 \tag{1.70}$$

and

$$\partial_\mu(\sqrt{|G|}G^{\mu\nu}\partial_\nu\phi) + \sqrt{|G|}e^{2\phi}G^{\mu\nu}\partial_\mu a\partial_\nu a = 0 \tag{1.71}$$

[10] $\sqrt{|g|}e^{-2\phi}\partial_\mu\phi\partial^\mu\phi = \sqrt{|g|}e^{-2\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \sqrt{|G|}e^{+\frac{5\phi}{2}}e^{-2\phi}\partial_\mu\phi\partial_\nu\phi e^{-\frac{\phi}{2}}G^{\mu\nu} = \sqrt{|G|}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu}$

Note that the first equation of motion is nothing else than the Einstein equation

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}G_{\mu\nu}\mathcal{R} = T_{\mu\nu} \tag{1.72}$$

To see this realize that to derive this equation of motion also the term

$$\frac{\delta(\sqrt{|G|})}{\delta G^{\mu\nu}} = -\frac{1}{2}\sqrt{|G|}G_{\mu\nu} \tag{1.73}$$

needs to be taken into account. To solve these equations of motion use as a metric ansatz flat space. In flat Euclidean space $R = 0$, taking the trace of (1.69) and substituting this into (1.71) gives

$$\partial_\mu\left(\sqrt{|G|}G^{\mu\nu}\partial_\nu e^\phi\right) = \Box e^\phi = 0 \tag{1.74}$$

which is the spatial D'Alembertian in spherical coordinates. This means that e^ϕ is a harmonic function and hence the spherically symmetric solution for the dilaton is

$$\boxed{e^\phi = e^{\phi_\infty} + \frac{c}{r^8}} \tag{1.75}$$

Here $e^{\phi_\infty} \equiv g$, the coupling constant and c is an integration constant, which will later be related to a conserved charge. The low energy approximation used here is only valid if g is small. From the trace of (1.69) and realizing that $e^{-\phi}\partial_\mu\phi = -\partial_\mu e^{-\phi}$ follows

$$\partial_\mu a = \pm\partial_\mu e^{-\phi} \Rightarrow \int_a^{a_\infty} \mathrm{d}^{10}x \;\partial_\mu a = \pm\int_\phi^{\phi_\infty} \mathrm{d}^{10}x \;\partial_\mu e^{-\phi} \tag{1.76}$$

And thus the axion solution is

$$\boxed{a = \pm(e^{-\phi} - e^{-\phi_\infty}) + a_\infty = \pm(e^{-\phi}) + \tilde{a}_\infty} \tag{1.77}$$

where $+$ refers to the instanton and $-$ to the anti-instanton. Harmonic functions are valid for all r, except at the origin. In general one solves this problem by adding a delta function as a source term to the action [31]. The solutions for ϕ and a still solve the equations of motion in the string frame,

$$S^{SF}_{Min} = \int \mathrm{d}^{10}x\sqrt{g}\{e^{-2\phi}\{-\mathcal{R} - 4\partial_\mu\phi\partial^\mu\phi\} - \frac{1}{2}\partial_\mu a\partial^\mu a\} \tag{1.78}$$

For example the corresponding equation of motion for a is

$$\partial_\mu\left(\frac{1}{2}\sqrt{|G|}G^{\mu\nu}\partial_\nu a\right) = 0 \tag{1.79}$$

which is still solved with (1.77). Besides this mathematical proof, there is a more physical argument why this had to be so. A metric is used to measure time and distances, by a conformal mapping of the metric one only changes the "units" in which one wishes to measure. Note that the coordinates are not changed by this mapping, only the overall factor, therefore the solutions should not change.

The action of the instanton (the triplet dilaton, axion and the metric) is clearly zero when plugged into (1.68)[11]. This seems awkward and can be resolved by realizing that adding a *total derivative* (e.g. a boundary term) to the action will not change the equations of motion. As derived in appendix A.2 the correct boundary term S_{surf} is

$$S_{surf} = \int \mathrm{d}^{10}x\partial_\mu\{\sqrt{G}G^{\mu\nu}e^{2\phi}(a\partial_\nu a)\} = \oint e^{(2\phi)}a \wedge *\mathrm{d}a \tag{1.80}$$

This will give for the instanton action (see appendix A.2 for details)

$$S_{Inst} = \frac{8cVol(S^{(9)})}{g} \tag{1.81}$$

The action above can be connected to a conserved charge. Namely one of the conserved Noether currents $J_\mu^{Noether} = e^{2\phi}\partial_\mu a$, which comes from $a \Rightarrow a + constant$. The corresponding conserved charge Q is

$$\begin{aligned} |Q_{Inst,a}| &= \left|\oint_{S^9_{r=\infty}} e^{2\phi}\partial_\mu a \mathrm{d}\Sigma^\mu\right| = \left|\oint_{S^9_{r=\infty}} e^{\phi}\partial_\mu \phi \mathrm{d}\Sigma^\mu\right| \\ &= 8cVol(S^{(9)}) \end{aligned} \tag{1.82}$$

and thus the following relation between the charge and the action holds

$$\boxed{S_{Inst} = \frac{|Q_{Inst}|}{g}} \tag{1.83}$$

Note the characteristic $S_{Int} \propto \frac{1}{g}$ for instantons. So far no tunnelling phenomenons have come into play. To observe this the flat infinitesimal line-element must be written in the string frame (1.65)

$$\boxed{ds^2 = (e^{\phi_\infty} + \frac{c}{r^8})^{\frac{1}{2}}(dr^2 + r^2 d\Omega_9^2) = \sqrt{e^{\phi_\infty} \ r^4 + \frac{c}{r^4}}\Big[(\frac{dr}{r})^2 + d\Omega_9^2\Big]} \tag{1.84}$$

where $d\Omega_9^2$ is

$$d\Omega_9^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2 + ... + \Pi_{m=1}^8 \sin^2\theta_m \mathrm{d}\theta_9^2 \tag{1.85}$$

It is clear that

$$e^{\phi_\infty} r^4 \longleftrightarrow \frac{c}{r^4} \tag{1.86}$$

is a symmetry of the metric. This symmetry interchanges the regions around $r = 0$ with those around $r = \infty$, both are asymptotically flat spaces. To see that the region near $r = 0$ is an asymptotically flat space, approximate it by

$$\mathrm{d}s^2 = \sqrt{\frac{c}{r^4}}[(\frac{\mathrm{d}r}{r})^2 + \mathrm{d}\Omega_9^2] \tag{1.87}$$

and substitute the new coordinate $\rho = \frac{c^{1/4}}{r}$ in it

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_9^2 \tag{1.88}$$

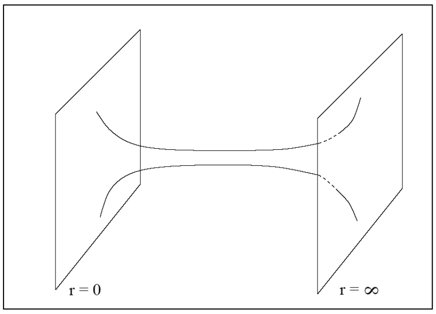

Figure 1.4: Wormhole interpretation (in the string frame)

Such a connection between two distant areas of space or a connection between two different spaces is often called a wormhole, see figure 1.4 and chapter two for more details.

It is important to realize that *only* in the string frame this wormhole interpretation holds. In a way this is similar to the $SU(2)$ Yang-Mills instanton, for there only the tunnel interpretation works in the $A_0 = 0$ gauge. In chapter four the D-instantons obtained in this section will be extended to so called non-extremal D-instantons (i.e. non-flat space ansatz) and their wormhole behavior will be investigated.

According to Feynman [5] the connection between the D-instanton and path integrals can be made via

$$< g_2, \phi_2, S_2 | g_1, \phi_1, S_1 >= \Big(\int \mathcal{D}[g,\phi] e^{iS[g,\phi]} \Big)_{Min} = \Big(\int \mathcal{D}[g,\phi] e^{-S[g,\phi]}) \Big)_{Euc} \tag{1.89}$$

where $\mathcal{D}[g,\phi]$ is a measure on the space of all field configurations g (the metric) and ϕ, $S[g,\phi]$ is the action of the fields, and the integral is taken over all fields which have the given values on S_1 and S_2.

1.3.1 The Bogomol'nyi bound

A much more straightforward derivation of the D-instanton can be achieved by going to the dual form (*dual*) of (1.68). For a full derivation of how to obtain this correctly and how this is related to the added boundary term (1.80), see appendix A.2. Also is explained there why this procedure *only* works in the dual formulation and not for the axion formulation. Ignoring the Ricci scalar and let $F^{(9)} = \mathrm{d}C^{(8)} = e^{2\phi} * \mathrm{d}a$

$$S^{EF}_{Euc,dual} = -\frac{1}{2} \int \mathrm{d}\phi \wedge *\mathrm{d}\phi + e^{-\phi} \mathrm{d}C^{(8)} \wedge *\mathrm{d}C^{(8)} \tag{1.90}$$

The idea of a Bogomol'nyi bound is to rewrite this as

$$S = -\frac{1}{2} \int (\mathrm{d}\phi \pm e^{-\phi} * \mathrm{d}C^{(8)}) \wedge *(\mathrm{d}\phi \pm e^{-\phi} * \mathrm{d}C^{(8)}) \mp \int e^{-\phi} \mathrm{d}\phi \wedge \mathrm{d}C^{(8)} \tag{1.91}$$

and realizing that the first term on the right hand side is ≥ 0, see also (1.16). Taking the equal sign gives for the instanton action

[11] Since $\mathcal{R}$ is zero in flat space and via (1.76).

$$S \geq \int_M e^{-\phi} \mathrm{d}\phi \wedge \mathrm{d}C^{(8)} = -\int_M \mathrm{d}\Big(e^{-\phi}\mathrm{d}C^{(8)}\Big) = -\oint_{\partial M} e^{-\phi} F^{(9)} \qquad (1.92)$$

Here $\partial M = \partial M_\infty + \partial M_0$ denotes the boundary of space-time which consist of the S^9 at $r = \infty$ (∂M_∞) and the S^9 at $r = 0$ (∂M_0)[12]. The minimum of (1.91) is obtained when

$$\mathrm{d}\phi = \mp e^{-\phi} * \mathrm{d}C^{(8)} \longleftrightarrow *\mathrm{d}\phi = \pm e^{-\phi} F^{(9)} \qquad (1.93)$$

Using $F^{(9)} = \mathrm{d}C^{(8)} = e^{2\phi} * \mathrm{d}a$ this can be rewritten too

$$\mathrm{d}\phi = \pm e^{\phi} \mathrm{d}a \qquad (1.94)$$

The D-instanton solutions (1.75) and (1.77) satisfy this relation and (1.92) becomes with these solutions

$$S = \left| \oint_{\partial M} *\mathrm{d}\phi \right| = \frac{8cVol(S^9)}{e^{\phi_\infty}} \qquad (1.95)$$

Two remarks can be made now. The first one deals with the fact that due to the added boundary term the action is now bounded from below in the *scalar* sector, as is evident from (1.92). Note that this is *not* the case if one ignores the boundary term (1.68). The second remark has to do with solutions that obey the Bogomol'nyi bound. A special condition in string theory is the so called BPS-condition. This BPS condition is useful because it results in the cancellation of quantum corrections to the effective action for string theory, so that precise answers can be found by simple calculations at lowest order in perturbation theory. In general it can be shown that a BPS-state preserves precisely one half of the supersymmetry, this characterizes a BPS state. So to turn it around, a BPS state can be seen as preserving half of the supersymmetry transformations. If a system satisfies the Bogomol'nyi bound, it is a BPS state, which implies that the D-instanton is a BPS-state. More about supersymmetry in section 1.5.

1.3.2 Multi-instantons solution

It is interesting to note that there are also multi-instanton solutions to this system. Looking at (1.75)

$$e^{\phi} = e^{\phi_\infty} + \frac{c}{r^8} \qquad (1.96)$$

which obeys

$$\partial^2(e^{\phi}) = 0 \qquad (1.97)$$

a more general solution is obviously

$$\boxed{e^{\phi} = e^{\phi_\infty} + \sum_i \frac{\lambda_i}{|\mathbf{x} - \mathbf{x_i}|^8}} \qquad (1.98)$$

where the λ_i are the corresponding Noether charges. This is consistent with (1.76)

$$\partial_\mu a = \pm \partial_\mu e^{-\phi} \Rightarrow a = \pm(e^{-\phi} - e^{-\phi_\infty}) + a_\infty \qquad (1.99)$$

and e^{ϕ} from (1.98).

[12]Taking both nine-spheres as a boundary is consistent with the wormhole interpretation.

1.3.3 $SL(2,\mathbb{R})$-symmetry

The action (1.67) [13] possesses an $SL(2,\mathbb{R})$ symmetry (the set of all 2×2 matrices over the real numbers with determinant 1), which becomes apparent if the complex field

$$\tau = a + ie^{-\phi} \tag{1.100}$$

and the 2×2 matrix

$$\begin{aligned}\mathcal{M} &= \frac{1}{\text{Im }\tau}\begin{pmatrix} |\tau|^2 & -\text{Re }\tau \\ -\text{Re }\tau & 1\end{pmatrix} = e^{\phi}\begin{pmatrix} |\tau|^2 & a \\ a & 1\end{pmatrix} \\ &= \begin{pmatrix} a^2+e^{-2\phi} & a \\ a & 1\end{pmatrix}\end{aligned} \tag{1.101}$$

are introduced. This changes (1.67) into

$$S^{EF}_{Min} = \int \mathrm{d}^{10}x\sqrt{G}\Big(\mathcal{R} + \frac{1}{4}\text{Tr}[\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}]\Big) \tag{1.102}$$

which is invariant under the following transformations

$$\mathcal{M} \to \mathcal{M}' = \Lambda\mathcal{M}\Lambda^T \tag{1.103}$$

$$G_{\mu\nu} \to G'_{\mu\nu} = G_{\mu\nu} \tag{1.104}$$

where T stands for transpose and

$$\Lambda = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d}\end{pmatrix} \quad \tilde{a}\tilde{d} - \tilde{b}\tilde{c} = 1 \tag{1.105}$$

Alternatively (1.103) can also be written as the rational form

$$\tau \to \tau' = \frac{\tilde{a}\tau + \tilde{b}}{\tilde{c}\tau + \tilde{d}} \tag{1.106}$$

Consider the special $SL(2,\mathbb{R})$ matrix

$$\Lambda_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \tag{1.107}$$

Then from (1.103) it is clear that

$$\mathcal{M}_{|a=0} = \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi}\end{pmatrix} \to \mathcal{M}'_{|a=0} = \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi}\end{pmatrix} \tag{1.108}$$

i.e. $\phi \to -\phi$ and thus for the coupling constant g

$$g_s = < e^{\phi} > \;\to\; < e^{-\phi} > = \frac{1}{g_s} \tag{1.109}$$

This is the weak/strong coupling duality, called S duality, which is a symmetry of the effective action of the type IIB string theory.

[13] This symmetry also applies to the full action(1.61).

Noether currents and conserved charges

$SL(2,\mathbb{R})$ has three generators and thus there should also be three corresponding conserved charges. In appendix A.4 the general derivation of Noether currents is given for Lagrangians of the type

$$\mathcal{L} = \frac{1}{4}\sqrt{|G|}\mathrm{Tr}[\partial_\mu \mathcal{M}^{-1}\partial^\mu \mathcal{M}] \tag{1.110}$$

As is shown there, these currents can be represented in terms of the 2×2 matrix (A.37)

$$\mathcal{J}_\mu = -\mathcal{M}^{-1}\partial_\mu \mathcal{M} = \begin{pmatrix} \mathcal{J}_{11_\mu} & \mathcal{J}_{12_\mu} \\ \mathcal{J}_{21_\mu} & \mathcal{J}_{22_\mu} \end{pmatrix} \tag{1.111}$$

Applying this to the $SL(2,\mathbb{R})$ matrix (1.105) gives

$$\begin{aligned} \mathcal{J}_\mu &= e^{2\phi}\begin{pmatrix} 1 & -a \\ -a & |\tau|^2 \end{pmatrix} \times \begin{pmatrix} |\tau|^2\partial_\mu\phi + \partial_\mu|\tau|^2 & a\partial_\mu\phi + \partial_\mu a \\ a\partial_\mu\phi + \partial_\mu a & \partial_\mu\phi \end{pmatrix} \rightarrow \\ \mathcal{J}_{11_\mu} &= e^{2\phi}\Big(|\tau|^2\partial_\mu\phi + \partial_\mu|\tau|^2 - a^2\partial_\mu\phi - a\partial_\mu a\Big) = -\mathcal{J}_{22_\mu} \\ \mathcal{J}_{12_\mu} &= e^{2\phi}\partial_\mu a \\ \mathcal{J}_{21_\mu} &= -a^2e^{2\phi}\partial_\mu a + \partial_\mu a + 2a\partial_\mu\phi \end{aligned} \tag{1.112}$$

This is all in terms of the matrix $\mathcal{M}$. To make connection to the symmetries of the individual fields note that besides the current already mentioned that follows from the shift in a (1.82), one other symmetry K_μ follow from the re-scaling $e^\phi \rightarrow e^\nu e^\phi,\ a \rightarrow e^{-\nu}a$ [16]

$$\begin{aligned} J_\mu &= e^{2\phi}\partial_\mu a \\ K_\mu &= -2\partial_\mu\phi + 2e^{2\phi}a\partial_\mu a \\ L_\mu &= (a^2 + e^{-2\phi})J_\mu - aK_\mu \end{aligned} \tag{1.113}$$

and this agrees with (1.112). An other interesting alternative approach is by Meessen and Ortin [15] and also by Bergshoeff *et al* [17]. They rewrite (1.102) to[14]

$$S^{EF} = \int \mathrm{d}^{10}x\sqrt{G}\Big(\mathcal{R} - \frac{1}{4}\mathrm{Tr}[(\partial\mathcal{M}\mathcal{M}^{-1})^2]\Big) \tag{1.114}$$

The corresponding $SL(2,\mathbb{R})$ currents can then nicely be written in matrix form

$$\mathcal{J}_\mu = (\partial_\mu\mathcal{M})\mathcal{M}^{-1} = \begin{pmatrix} K_\mu & L_\mu \\ J_\mu & -K_\mu \end{pmatrix} \tag{1.115}$$

Note that the off-diagonal elements are switched by this definition for $\mathcal{J}_\mu$. The advantage of working with the matrix $\mathcal{M}$ is that this allows for an easy generalization by allowing complex scalars ϕ and a, a generalized coupling $e^{b\phi}$ instead of just $b = 2$ and a complex metric $g_{\mu\nu}$. Instead of the real matrix $\mathcal{M}$ (1.101) introduce the complex matrix[15]

$$\mathcal{M}_C = e^{\frac{b\phi}{2}}\begin{pmatrix} \frac{1}{4}b^2a^2 + e^{-b\phi} & \frac{1}{2}ba \\ \frac{1}{2}ba & 1 \end{pmatrix} \tag{1.116}$$

[14]Using $\mathcal{M}\mathcal{M}^{-1} = 1$ it follows that $\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1} = \partial_\mu\mathcal{M}(\mathcal{M}^{-1}\mathcal{M})\partial^\mu\mathcal{M}^{-1} = -\partial_\mu\mathcal{M}\mathcal{M}^{-1}\partial^\mu\mathcal{M}\mathcal{M}^{-1}$

[15]The subscript C stands for complex, also a different metric sign convention is used as in the two cited articles.

which gives via the same way as in the previous section rise to the Lagrangian

$$\mathcal{L}_C = \sqrt{|G|}\Big(\mathcal{R} + \frac{1}{b^2}\mathrm{Tr}[\partial M_C \partial M_C^{-1}]\Big) = \sqrt{|G|}\Big(\mathcal{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{b\phi}\partial_\mu a\partial^\mu a\Big) \tag{1.117}$$

Obviously the symmetry of this Lagrangian is now a $SL(2,\mathbb{C})$

$$\mathcal{M} \to \mathcal{M}' = \Lambda\mathcal{M}\Lambda^T \tag{1.118}$$

$$G_{\mu\nu} \to G'_{\mu\nu} = G_{\mu\nu} \tag{1.119}$$

where

$$\Lambda = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \quad \epsilon \;\; SL(2,\mathbb{C}) \tag{1.120}$$

and corresponding currents[16]

$$\begin{aligned} J_\mu &= \frac{1}{2}be^{b\phi}\partial_\mu a \\ K_\mu &= \frac{1}{2}e^{b\phi}\partial_\mu\Big(e^{-b\phi} + \frac{1}{4}b^2a^2\Big) \\ L_\mu &= -baK_\mu + J_\mu\Big(e^{-b\phi} + \frac{1}{4}b^2a^2\Big) \end{aligned} \tag{1.121}$$

To obtain (1.101) choose all fields and $(\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ real and $b = 2$. To obtain the Euclidean case of the D-instanton, redefine $a \to ia$ and choose all fields and constants real, with the resulting matrix

$$\mathcal{M}_{Euc} = e^{\frac{b\phi}{2}}\begin{pmatrix} -\frac{1}{4}b^2a^2 + e^{-b\phi} & \frac{1}{2}iba \\ \frac{1}{2}iba & 1 \end{pmatrix} \tag{1.122}$$

The corresponding Lagrangian for $b = 2$ is

$$\mathcal{L}_{Euc} = -\sqrt{|G|}\Big(\mathcal{R} + \frac{1}{4}\mathrm{Tr}[\partial M_{Euc}\partial M_{Euc}^{-1}]\Big) = \sqrt{|G|}\Big(-\mathcal{R} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu a\partial^\mu a\Big) \tag{1.123}$$

which corresponds to (1.68). The corresponding currents are[17]

$$\begin{aligned} J_\mu &= e^{2\phi}\partial_\mu a = -e^{\phi}\partial_\mu\phi \\ K_\mu &= \frac{1}{2}e^{2\phi}\partial_\mu(e^{-2\phi} - a^2) = -\partial_\mu\phi - ae^{2\phi}\partial_\mu a \\ L_\mu &= (-a^2 + e^{-2\phi})J_\mu - 2aK_\mu \end{aligned} \tag{1.124}$$

Via the generalized Gauss's law the conserved charge matric $\mathcal{Q}$ becomes then by definition

$$\mathcal{Q} = \oint_{S^9}\mathcal{J} = \oint_{S^9}\mathcal{J}_\mu n^\mu \equiv \begin{pmatrix} q_3 & iq_2 \\ iq_1 & -q_3 \end{pmatrix} \tag{1.125}$$

In appendix A.4 (A.41) it is explained how to calculate this charge matrix

$$\mathcal{Q} = \mathrm{Vol}(S_9)r^{D-1}g_{rr}^{\frac{D-2}{2}}\mathcal{J}_r \tag{1.126}$$

Besides the already calculate charge q_1 (1.81), the other two follow from straightforward substitution of $\mathcal{J}_r$ into the above charge matrix.

[16]Note that K_μ is redefined by a factor two.

[17]Note that to have real charges, J_μ and L_μ are redefined to iJ_μ and iL_μ.

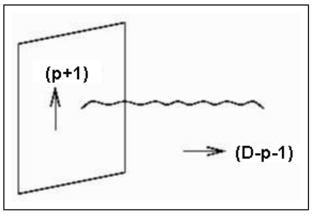

Figure 1.5: Dirichlet *Dp*-brane

1.4 *Dp*-branes

In terms of strings, D-instantons can be regarded as $p = -1$- *branes.* But what exactly is a brane? An open string in D dimensions[18] has endpoints, which have to satisfy appropriate boundary conditions. The first type of boundary conditions is that of Neumann, in short this condition allows the string endpoints to be located anywhere, it is the only choice compatible with translation invariance and Lorentz invariance in D dimensions. If some (or all) of the endpoints are constrained to lie on some definite surface, one speaks of Dirichlet boundary conditions. These are clearly incompatible with translation and Lorentz invariance, and such objects may preserve at most a part of these invariants. As an example consider Dirichlet boundary conditions applied to all endpoints of the open string, i.e. along all the $D-1$ spatial coordinates $(x_1, x_2, ..., x_{D-1})$. This constrains the endpoint of the string to lie at a *single* point in space, while the rest of the string is free to fluctuate, since only time is a Neumann condition. Then this point becomes like a point particle, which is called a *D-particle*. If one applies p Neumann and $D-p-1$ Dirichlet boundary conditions, the corresponding state is extended in p spatial directions and one time direction, and is called a Dirichlet p-brane or Dp-brane, see figure 1.5. The $p+1 = d$ dimensional volume is called the world volume and the $D-p-1$ directions are orthogonal to this. Coming back to D-instantons, it is called a $p = -1$ brane since also the time direction is "fixed". All directions are thus transverse and form a Euclidean D-dimensional space. In more mathematical terms Dp-branes can be described as

$$\mathrm{d}s^2 = e^{2A}(-\mathrm{d}t^2 + \mathrm{d}x_p^2) + e^{2B}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{D-p-2}^2) \tag{1.127}$$

where the first block is the $(p+1)$-dimensional Minkowskian world volume and the second block the transverse Euclidean space. The functions A and B depend only on r and determine the gravity in each block. If one now looks at D-instantons, which is a $p = -1$ brane, the first part disappears and the second part covers the whole space. It is thus a natural extension of Dp-branes, taking time as a Dirichlet condition too. There exist various generalizations (deformations), for example one can add a term $e^{f(r)}$ in front of the t and r components.

It is surprising that such extended objects are present in string theory, since only open and closed strings are put explicitly in the theory. At various points in this master thesis, p-form fields appear. *It turns out that a* $(p+2)$*-form field strength couples to sources that are Dp-branes.* To see this take a D-dimensional spacetime with a $(p+1)$-form vector potential A, the corresponding field strength

[18]Here D stands for one timelike and $(D-1)$ spacelike coordinates, this notation is different from what is used in the rest of the text.

F is then a $(p+2)$-form and the volume element Υ is a D-form: $\Upsilon = \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{D-1}$ The dual field strength $*F$ is a $(D-p-2)$-form and $d*F$ is a $(D-p-1)$-form. The generalized Maxwell relation

$$\mathrm{d} * F = 4\pi * \mathcal{J} \tag{1.128}$$

shows that the current (and source term) $\mathcal{J}$ is also a $(D-p-1)$-form. To make the analysis simpler go to the rest frame, where $\mathcal{J} = (\rho, \vec{0})$ and thus

$$*\mathcal{J} \propto \rho \times \Upsilon_{D-p-1} \tag{1.129}$$

where Υ_{D-p-1} is the volume element of a $(D-1-p)$-dimensional subspace of the D-dimensional space. This subspace needs to be taken since $*\mathcal{J}$ is a $(D-p-1)$-form. Charges are defined in the standard way via Gauss's law, i.e.

$$\mathcal{Q} = \int_{\Upsilon_{D-p-1}} *\mathcal{J} = \int_{S^{D-p-2}} *\mathcal{J} \tag{1.130}$$

Theses charges act as sources and are objects with $(D-1)-(D-p-1) = p$ dimensions, which are exactly the Dp-branes.

For instantons $p = -1$ and the corresponding field strength is a one form or a zero form potential, i.e. indeed the kinetic term of the scalar a. The dilaton ϕ cannot be taken as a source, since it is also present as $e^{\alpha\phi}$.

1.5 Effect of D-instantons: $\mathcal{R}^4$ and $\mathcal{R}^8$ contributions

In the case of the double well potential, the reason why one should look at instantons is that they allow for a tunnelling calculation, which gives the same result as the WKB approximation of quantum mechanics. But what does the D-instanton contribute? A clear tunnelling interpretation cannot be given, since there is no potential present, only a wormhole picture in the string frame. To answer this question one needs to know a bit about supersymmetry (SUSY) and fermionic zero modes.

SUSY and fermionic zero modes

String theory is a supersymmetric theory, which means that at each mass level as many bosons as fermions degrees of freedom are present. This is a topic in itself and therefore only some basic facts will be mentioned.

As mentioned at the beginning of section three, type IIB is defined as $(8_s + 8_v) \times (8_s + 8_v)$. Group theory says that the bosonic Neveu-Neveu sector $8_v \times 8_v$ is a 64 dimensional representation, consisting of the graviton (35_v), the B-field (a 28-dimensional antisymmetric tensor) and the dilaton. The bosonic $8_s \times 8_s$ Ramond-Ramond sector consists of representations of also 35, 28 and 1 dimensions. The cross terms $8_s \times 8_v$ and $8_v \times 8_s$ are identical for type IIB theory (i.e. same chirality) and consists out of the gravitino ψ_μ (56) and the dilatino λ (8).

A zero mode is an eigenfunction of an operator $\hat{O}$ with zero eigenvalue(s) [31]. As an example look at the *bosonic* Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - U(\phi)$ and the path integral

$$G = \int \mathcal{D}[\phi(x,t)] e^{iS[\phi]}\Big|_{Min} = \int \mathcal{D}[\phi(x,\tau)] e^{-S[\phi]}\Big|_{Euc} \tag{1.131}$$

Applying the stationary phase approximation to the Euclidean action S around the classical path ϕ_{cl} gives

$$S_{Euc}[\phi(x,\tau)] = V[\phi_{cl}]\tau + \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}\tau'[y(x,\tau')\hat{O}_{Euc}(x,\tau')y(x,\tau')] + O(y^3) \tag{1.132}$$

where $\tau = -it$ as before and

$$\begin{aligned} \hat{O}_{Euc}(x,\tau') &\equiv -\frac{\partial^2}{\partial\tau'^2} - \frac{\partial^2}{\partial x^2} + \Big(\frac{\partial^2 U}{\partial\phi^2}\Big)_{\phi_{cl}} \\ y(x,\tau') &\equiv \phi(x,\tau') - \phi_{cl} \end{aligned} \tag{1.133}$$

Via standard mathematical tricks [1] this can be rewritten as

$$G \propto e^{-V[\phi_{cl}]\tau}\{\mathrm{Det}[\hat{O}_{Euc}(x,\tau')]^{-1/2}\} \tag{1.134}$$

As is evident from the last expression, a zero mode causes a divergence in G and must therefore be extracted, see for example [31]. Naively one would expect that the same problem occurs for fermions, however fermions obey the anti-commutator or Grassmann algebra. The path integral G becomes then

$$G \propto \mathrm{Det}[\hat{O}] \tag{1.135}$$

and no divergence take place for a zero mode of $\hat{O}$. Type IIB theory has two gravitino's ($N = 2$) and therefore there are two Majorana spinors Q_i^A, $i = 1, 2$ and $A = 1, \ldots, 16$ giving in total 32 supersymmetries. The gravitino and dilatino supersymmetry (SUSY) rules are for flat Euclidean space in the Einstein frame [33]

$$\begin{aligned} \delta\psi_\mu^\pm &\propto (\partial_\mu \mp \frac{1}{4}e^\phi\partial_\mu a)\epsilon^\pm \\ \delta\lambda^\pm &\propto \epsilon^\mp\frac{1}{4}(\partial_\mu\phi \pm e^\phi\partial_\mu a) \end{aligned} \tag{1.136}$$

where ϵ are the 16 components of the Majorana spinor[19]. Substituting the anti-instanton solution (1.77) leads to

$$\epsilon^+ = 0, \quad \epsilon^- = e^{\frac{\phi}{4}}\epsilon_0^- \tag{1.137}$$

where ϵ_0^- is a constant spinor and because $\epsilon^+ = 0$ these 16 symmetries are broken.

To understand the consequences of this one needs to know a bit about string scattering, look at figure 1.6. The s broken spacetime SUSY generators have the form of open string fermion vertex operators and have a natural interpretation as generating fermionic zero modes in the instanton background [32]. In this figure the circle represents the D-instanton, which has Dirichlet boundary conditions in all space-time directions, attached to this is a state labelled by Ψ, which can be for example the graviton. Each broken SUSY leads to the attachment of an open string, in the D-instanton case $s = 16$ thus. In other words an instanton carrying some zero modes corresponds, at lowest order, to a disk world-sheet[20] with open-string states attached to the boundary, see figure 1.6.

[19] In 2D dimensions the dimension d of a spinor is $d = 2^{D/2}$, but the Majorana condition reduces the number of dimension by 2.

[20] Since it is a $p = -1$-brane, the word sheet is a point.

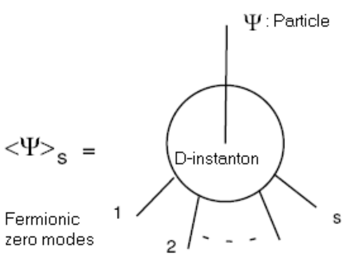

Figure 1.6: Tadpole: Scattering picture

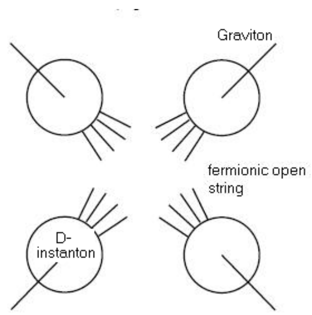

Figure 1.7: Scattering of four gravitons in a D-instanton background, the 16 fermionic open strings are due to the broken SUSY

Various types of scattering can of course happen in the presence of the D-instanton, a very interesting one is shown in figure 1.7. It represents the scattering of four gravitons with each four fermionic zero modes attached to the D-instanton disk. This scattering diagram has an interesting effect for general relativity. The Einstein vacuum equation

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 0 \tag{1.138}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor and $\mathcal{R} = \mathcal{R}_{\mu\nu} g^{\mu\nu}$ the Ricci scalar can be obtained, when as a starting point is used the Einstein-Hilbert Lagrangian

$$\mathcal{L} = \sqrt{|g|}\mathcal{R} \tag{1.139}$$

One of the reasons why string theory is so popular is that it unifies all force, including gravity. Therefore it is expected that at least (1.139) is obtained in the low energy domain. Doing however an exact calculation as done for the one-loop case by [23] and at tree level by [24] gives an additional factor proportional to $\mathcal{R}^4$. Taking into account the tadpole of figure 1.7 gives also a $\mathcal{R}^4$ contribution

$$\begin{aligned} A \propto & \int \mathrm{d}^{10} y \mathrm{d}^{16} \epsilon_0 < h^1 >< h^2 >< h^3 >< h^4 > \\ \propto & \int \mathrm{d}^{10} y \mathcal{R}^{m_1 n_1}_{i_1 j_1} \mathcal{R}^{m_2 n_2}_{i_2 j_2} \mathcal{R}^{m_3 n_1 3}_{i_3 j_3} \mathcal{R}^{m_4 n_4}_{i_4 j 4} \end{aligned} \tag{1.140}$$

where $< h^i >$ stands for the contribution for each graviton with four open string attached to it. Adding up all three contributions have led to conclusion that the second order correction to the Einstein-Hilbert action in the Einstein frame is

$$\Delta S_{IIB} \propto \int \mathrm{d}^{10} x \sqrt{-g_{EF}} f(\tau, \bar{\tau}) \mathcal{R}^4 \tag{1.141}$$

where

$$\begin{aligned} \tau &= \tau_1 + i\tau_2 = a + i e^{-\phi} \\ f(\tau, \bar{\tau}) &= \sum_{(p,n) \neq (0,0)} \frac{\tau_2^{3/2}}{|p + n\tau|^3|} \end{aligned} \tag{1.142}$$

It can be shown that expanding this around large τ_2 gives back the terms corresponding to the tree level, one loop and D-instanton contributions.

A simpler reason why such a contribution was to be expected is by realizing that *new* D-instantons are generated due to the broken SUSY's, since applying a broken SUSY transformations means that one longer stays "within" the system but gets "outside", i.e. new solutions. Therefore to include all *different* D-instantons one should integrate over all the 16 degrees of freedom of the spinor ϵ^+. Furthermore one can show that when comparing units, for example in three dimensions, the product of four fermions ($[F] = 3/2$) has the same unit as the Ricci scalar ($[\mathcal{R}] = 2$) or to be precise

$$[\frac{1}{\kappa^4}(\text{four fermions})] = [\mathcal{R}], \quad \kappa \propto \sqrt{G_N} \propto 1/M_{Planck} \tag{1.143}$$

where $\sqrt{G_N}$ is the gravitational constant in four dimensions. Since fermions obey the Grassmann algebra, integration gives only a non-zero contribution if

$$\begin{aligned} \int \mathrm{d}a_1 \int \mathrm{d}a_2 &= \int \mathrm{d}a_1 a_1 \int \mathrm{d}a_2 = 0 \\ \int \mathrm{d}a_1 \int \mathrm{d}a_2 a_1 a_2 &= 1 \end{aligned} \tag{1.144}$$

where a_i are fermions. Generalizing this to the case at hand here, one can have a non-zero contribution only if for each integrated fermion, one also has a fermion to integrate over, meaning in total 16 fermions are needed. But the unit of these 16 fermions is equal to $\mathcal{R}^4$, therefore it was to be expected that the effect would be $\propto \mathcal{R}^4$.

In chapter four the non-extremal (non-flat space) version of the D-instanton will be discussed, which breaks all 32 SUSY's. It is to be expected that these non-extremal solutions give a correction to the effective low energy action of IIB string theory of the form

$$\Delta S_{IIB} \propto \int \mathcal{R}^8 \tag{1.145}$$

since now one has to integrated over all 32 broken SUSY's.

So far in three different system the instantons have been investigated. Any theory which has in its Euclidean version a solution with finite (Euclidean) action, has by definition an instanton solution, therefore many other instantons exist. Most notably is the instanton in QCD, for this may solve the problem of quark confinement, although it is not yet clear if it will really solve this important issue, but this will not be discussed any further.

1.6 Solitons

The instanton related to the Lagrangian in a D-dimensional Minkowskian space-time is a solution of the corresponding Euclidean theory. At the classical level, instantons are not very different from static solutions of the corresponding Minkowskian equations of motion, since the static solutions involve only spatial coordinates, i.e the Euclidean subspace of the full Minkowskian space-time. This subspace is linked to the domain of solitary waves and solitons. Rajaraman defines a *solitary wave* as:

Definition: A solitary wave is the localized non-singular solution of any non-linear field equation (or coupled equations, when several fields are involved) whose energy density ϵ, as well as being localized, has a space-time dependence of the form

$$\epsilon(\vec{x},t) = \epsilon(\vec{x} - \vec{u}t) \tag{1.146}$$

where $\vec{u}$ is some velocity vector.

In other words, the energy density should move undistorted and with constant velocity. Note that any time-independent ($\vec{u} = 0$) localized solutions is automatically a solitary waves. *Solitons* are a special kind of solitary waves, namely those solitary waves whose energy density profiles are asymptotically restored to their original shapes and velocities in the limit $t \to \infty$. The distinction between solitons and solitary waves is often ignored and most people just talk about solitons or solitonic interpretations.

In some systems, instantons in a D-dimensional system can be seen as static solitons. As an illustration look at the Klein-Gordon system

$$S_{Min} = \int \mathrm{d}t \int \mathrm{d}\vec{x}\Big[\frac{1}{2}\Big(\frac{\partial\phi}{\partial t}\Big)^2 - \frac{1}{2}(\nabla\phi)^2 - m^2\phi^2\Big] \tag{1.147}$$

The Euclidean version of this is

$$S_{Euc} = \int \mathrm{d}x_4 \int \mathrm{d}\vec{x}\Big[\frac{1}{2}\Big(\frac{\partial\phi}{\partial x_4}\Big)^2 + \frac{1}{2}(\nabla\phi)^2 + m^2\phi^2\Big] \tag{1.148}$$

via $t = -ix_4$. This Euclidean action has the same structure as the energy of a *static* field. Finiteness of energy is replaced by finiteness of the action for instantons. That this is not a mere coincidence, look at the energy E

$$E = \int \mathrm{d}^{D+1}x\mathcal{H} = \int \mathrm{d}t\mathrm{d}^D x[p\dot{x} - \mathcal{L}] \longrightarrow \text{"}E\text{"} = -\int \mathrm{d}^D x\mathcal{L} \equiv -S \tag{1.149}$$

where $\mathcal{H}$ is the hamiltonian, $\mathcal{L}$ the Lagrangian, p the momentum and after the $\longrightarrow$ one ignores the time.

To obtain instanton solutions one can look at a system and search for a way to turn "off" the time, a successful way, besides the Wick rotation, will be via *Kaluza-Klein reductions* (KK) over the timelike coordinate, as will be explained in later chapters in some detail. This is an alternative to the Wick rotation method, to see this realize that after a Wick rotation one has the same number of dimensions, whereas after a KK reductions one loses a (timelike) direction.

One can of course turn it around. Given an instanton solution, can it be considered a soliton, i.e. can it be linked to a one (timelike) dimensional higher system? To answer this question one should do the inverse Kaluza-Klein reduction, which is known by the name *uplifting*. In chapter five this will be done for the non-extremal D-instanton. This way an instanton is considered as a (static) soliton of the corresponding higher dimensional theory.

Chapter 2

Wormholes

In the string frame of type IIB string theory a wormhole interpretation is found for the metric. So far the concept of a wormhole has not been specified. It is "something" that connects two asymptotically flat spaces via a symmetry of the metric, $r \longleftrightarrow \frac{1}{r}$, see (1.86).

2.1 What is a wormhole?

According to Matt Visser [25] a definition for a wormhole is

> **Definition**: A wormhole is any compact region of spacetime with a topologically simple boundary but a topologically nontrivial interior.

To understand this look at (1.86). At $r = 0$ and $r = \infty$ there are asymptotically flat spaces (=boundary) but at $r^8 = \frac{c}{e^\infty}$ it has a nontrivial interior, namely an Einstein-Rosen bridge, see below. Notice that this is *only* a symmetry of the metric, not that of any fields also present in the Lagrangian, whereas instantons are the whole set of solutions to the Euclidean Lagrangian. So the above definition of Visser can best be casted in the following working definition

> **Working definition**: A wormhole is a symmetry of the metric of the form $r \longleftrightarrow \frac{\text{constant}}{r}$, which connects two asymptotically flat space(times).

Wormholes are one of the most fascinating objects appearing in the General Theory of Relativity (GRT)[1]. As early as 1916 the first 'wormhole' physics took place by Flamm and the famous *Einstein-Rosen bridge* was looked into in 1935 by A. Einstein and N. Rosen [26]. Wormholes can be classified into two classes

- Lorentzian wormholes
- Euclidean wormholes

The difference is merely the metric used, respectively Minkowskian and Euclidean. Note that the D-instanton belongs to the later class and is also an *inter*-universe wormhole for it connects two different universes, see figure 1.4. A wormhole that connects to distant regions in one universe is called an *intra*-universe wormhole. Note however that GRT only talks about local physics and not fixes the global

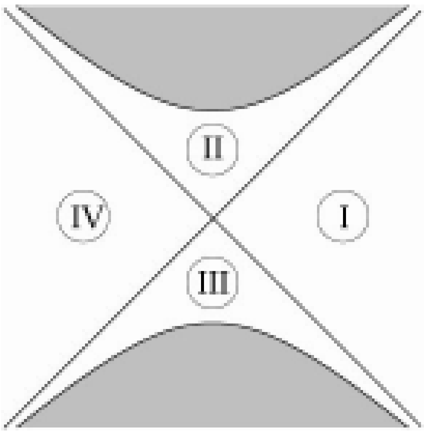

Figure 2.1: Sectors in Kruskal-Szekeres spacetime

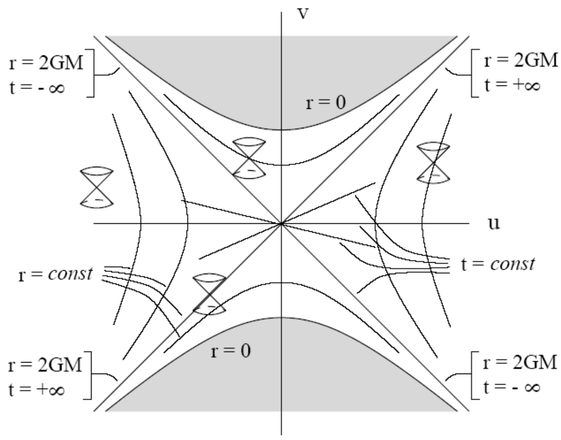

Figure 2.2: Kruskal-Szekeres spacetime diagram

topology. To describe what an Einstein-Rosen bridge is, look at the Schwarzschild metric

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega_2^2 \tag{2.1}$$

which describes the spacetime outside a heavy object ($r > 2m$) and has a (coordinate) singularity at $r = 2m$. The maximally extended version is obtained by introducing Kruskal-Szekeres coordinates [2]

$$ds^2 = \frac{32m^3 e^{-\frac{r}{2m}}}{r}\left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.2}$$

The relations between the coordinates (t, r) of the Schwarzschild metric and the Kruskal-Szekeres coordinates (T, X) are given by

$$\left(\frac{r}{2m} - 1\right)e^{\frac{r}{2m}} = X^2 - T^2 \tag{2.3}$$

$$\frac{t}{2m} = \log\left(\frac{T + X}{X - T}\right) = 2\tanh^{-1}(\frac{T}{X}) \tag{2.4}$$

This metric can be split into four sectors, see figure 2.1. Sector I is the ordinary Schwarzschild metric solution for $r > 2m$ (2.1), sector II and III are respectively the black hole and white hole and sector IV is like sector I an asymptotically flat spacetime but note that because the lightcones are always at 45 degrees, figure 2.2, two observers in sectors I and IV cannot communicate with each other, for light always ends up at the black hole sector II. In figure 2.2 the mass m used in the text can be related to the mass M of the object via $m = GM$, where G is the gravitational constant and the speed of light $c = 1$. So in Kruskal-Szekeres coordinates there are four sectors and in Schwarzschild coordinates there is only one sector, I. To see the Einstein-Rosen bridge introduce the new coordinate $u^2 = r - 2m$ with $u \, \epsilon \, (-\infty, \infty)$. This changes the Schwarzschild metric into

$$ds^2 = -\frac{u^2}{u^2 + 2m}dt^2 + 4(u^2 + 2m)du^2 + (u^2 + 2m)^2 d\Omega_2^2 \tag{2.5}$$

[1] Quantum effects will not be discussed in relation to wormholes.

By the definition of u the coordinate singularity at $r = 2m$ has been excluded. In the Kruskal-Szekeres picture, figure 2.1, this means only sectors I and IV remain, i.e. in these coordinates two asymptotically flat regions are covered instead of just one. The Einstein-Rosen bridge is the 'bridge' at $u = 0 \longleftrightarrow r = 2m$. This bridge connects the asymptotically flat regions near $u = \pm\infty$ (I and IV). *Thus* the Einstein-Rosen bridge (or Schwarzschild 'wormhole') is identical to a part of the maximally extended Schwarzschild geometry.

2.2 Morris and Thorne wormholes

The previous wormhole has been generalized by the pioneering work of Morris and Thorne [12]. They describe a whole class of static Lorentzian wormholes[2] by taking as an ansatz for the metric

$$\mathrm{d}s^2 = e^{2\phi(r)}\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{b(r)}{r}} - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{2.6}$$

where the functions $\phi(r)$ and $b(r)$ are referred to as redshift function[3] and shape function respectively. Cataldo *et al* state four general constraints for these functions

1. The no-horizon condition, i.e. $e^{2\phi(r)}$ is finite and nonzero throughout spacetime, so that there are no horizons nor singularities present.

2. Minimum value of the r-coordinate, i.e. at the throat of the wormhole $r = b(r) = b_0$, b_0 being the minimum value of r. This is the generalized Einstein-Rosen bridge, which also defines the position of the wormhole.

3. Finiteness of the proper radial distance $\frac{b(r)}{r} \leq 1$ for $r \geq b_0$ throughout the spacetime. This is required in order to ensure the finiteness of the proper radial distance $l(r)$ defined by

$$l(r) = \pm \int_{b_0}^{r} \frac{\mathrm{d}r}{\sqrt{1 - \frac{b(r)}{r}}} \tag{2.7}$$

   where the $\pm$ refers to the two asymptotically flat regions which are connected by the wormhole. The equality sign in $\frac{b(r)}{r} \leq 1$ holds only at the throat.

4. Asymptotic flatness condition, i.e. when $r \to \infty$ flat spacetime must be obtained, $\frac{b(r)}{r} \to 0$.

To see that the metric (2.6) implies a wormhole, it is useful to visualize the geometry of curved four dimensional space at a fixed moment of time t. Since the ansatz is spherically symmetric, without loss of information take $\theta = \frac{\pi}{2}$. This leads to

$$\mathrm{d}s^2 = -\frac{\mathrm{d}r^2}{1 - \frac{b(r)}{r}} - r^2\mathrm{d}\phi^2 \tag{2.8}$$

[2] A more recent review is by Cataldo *et al* [11].

[3] The redshift of light is related to the g_{00} component of the metric via $\frac{\Delta\lambda}{\lambda} = (g_{00})^{-\frac{1}{2}} - 1$.

The goal is now to construct, in flat three dimensional Euclidean space, a two-dimensional surface with the same geometry as this slice[4]. Take for the embedding coordinates $(z(r), r, \psi)$ and the Euclidean metric of the embedding space becomes

$$-\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\psi^2 + \mathrm{d}z^2 = \left[1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right]\mathrm{d}r^2 + r^2\mathrm{d}\psi^2 \tag{2.9}$$

The last equality holds since the embedded surface will be axially symmetric. This line element will be the same as that of (2.8) if ϕ is identified with ψ and if the following holds

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \pm\left(\frac{r}{b(r)} - 1\right)^{-\frac{1}{2}} \tag{2.10}$$

The shape function $b(r)$ clearly determines the shape of the wormhole. The third condition is needed since a wormhole has a minimum radius $b(r) = b_0$ at which the embedded surface becomes vertical, i.e. $\frac{\mathrm{d}z}{\mathrm{d}r} \to \infty$ when $r \to b_0$. To overcome this difficulty near the throat use the proper radial distance l defined by (2.7). This quantity is well behaved everywhere, if $b(r)/r \leq 1$ is required throughout spacetime. The above two formulas imply

$$\frac{\mathrm{d}z}{\mathrm{d}l} = \pm\sqrt{\frac{b(r)}{r}} \tag{2.11}$$

In the following sections various examples will be shown, especially the Schwarzschild and Reissner-Nordstrøm wormholes will be investigated thoroughly.

2.3 Schwarzschild metric

Begin with the well-known spherically symmetric Schwarzschild metric at constant time t and in the equatorial plane $\theta = \frac{\pi}{2}$

$$-\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} + r^2\mathrm{d}\phi^2 \tag{2.12}$$

Embed this into a flat three-dimensional Euclidean space $\mathbb{R}^3$ (ρ, z, ψ) as explained in (2.9)

$$\mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 + \mathrm{d}z^2 = \left[1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right]\mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 \equiv \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} + r^2\mathrm{d}\phi^2 \tag{2.13}$$

Comparing and concluding that $\phi = \psi$, $r = \rho$ and solving the differential equation leads to

$$\boxed{z(r) = \pm\sqrt{8m(r - 2m)} \Rightarrow r(z) = \frac{1}{8m}z^2 + 2m} \tag{2.14}$$

which is non-singular for *all* z, including $z = 0 \longleftrightarrow r = 2m$[5]. $r(z)$ is called Flamm's parabola, which is the analytically continuated solution to the Schwarzschild metric. Plotting $z(r)$ with both signs then gives the wormhole, see figure 2.3, where the plus sign of $z(r)$ is the upper half of the figure, the minus sign the lower half.

[4]This is like picturing a sphere in flat three dimensional space, instead of looking at it two dimensionally.

[5](2.14) is only valid for $r > 2m$, i.e. outside the Schwarzschild radius.

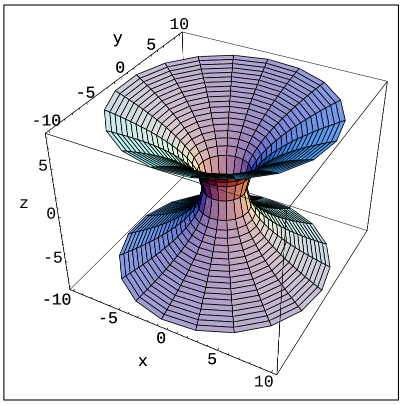

Figure 2.3: The Schwarzschild wormhole, here $x = r\cos\theta$, $y = r\sin\theta$ and $z = z(r)$.

It is important to realize that *only* the drawn surface resembles any physics in figure 2.3. The way this visualization has been set up is that the curvature of the *two* dimensional (truncated) metric (2.12) has been made visible in flat *three* dimensional Euclidean space (2.13). Drawing a spaceship "falling" into the throat, free from the surface is thus un-physical.

The position of the wormhole throat is at $r = 2m$, which is the Schwarzschild radius, *this* is the Einstein-Rosen bridge. Clearly $\frac{\mathrm{d}z}{\mathrm{d}r} \to \infty$ when $r \to 2m$, this is merely a coordinate deficit, caused by the fact that $z(r)$ only describes the curvature outside the Schwarzschild radius.

Looking back it is perhaps surprising that this wormhole, figure 2.3, is found since the Schwarzschild metric describes one spacetime outside a heavy object. To explain the origin of this second spacetime look at the Kruskal-Szekeres coordinates system (T, X) and the relations with the coordinates (t, r) of the Schwarzschild metric (2.3, 2.4). Take $t = 0 \longleftrightarrow T = 0$ and $\theta = \frac{\pi}{2}$ and from (2.3) it follows that

$$\mathrm{d}u = \pm \frac{r\sqrt{\frac{e^{\frac{r}{2m}}(r-2m)}{m}}}{4\sqrt{2}(r-2m)} \Rightarrow -\mathrm{d}s^2 = \frac{r}{r-2m}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \qquad (2.15)$$

which is indeed the Schwarzschild metric of the previous section. Now a good interpretation is possible. For this look at the figures 2.1 and 2.2, the two asymptotically sectors I and IV are now connected via this wormhole. This leads also to the conclusion that this is an inter-wormhole. So the method of embedding diagrams "analytically" extends the Schwarzschild metric here, similarly as (2.5).

2.3.1 Embedding diagrams versus symmetry of the metric

Remember that in the D-instanton case the wormhole was not described via an embedding diagram, but via a symmetry of the metric, see (1.86). To make the connection to this wormhole interpretation, which should of course agree with the embedding diagram approach, introduce the coordinate $\tilde{\rho}$

$$r = \tilde{\rho}\Big(1 + \frac{m}{2\tilde{\rho}}\Big)^2 \longleftrightarrow \tilde{\rho}(r) = \frac{1}{2}\Big(r - m \pm \sqrt{r^2 - 2mr}\Big) \qquad (2.16)$$

and substitute this into (2.1). Then the metric in *isotropic coordinates*[6] becomes

$$\mathrm{d}s^2 = \Big(\frac{1-\frac{m}{2\tilde{\rho}}}{1+\frac{m}{2\tilde{\rho}}}\Big)^2\mathrm{d}t^2 - \Big(1+\frac{m}{2\tilde{\rho}}\Big)^4\Big[\mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2(d\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\Big] \tag{2.17}$$

This metric has as a symmetry

$$\tilde{\rho} \longleftrightarrow \frac{m^2}{4\tilde{\rho}} \tag{2.18}$$

similar to the D-instanton symmetry. Defining the position of the wormhole by that point which get mapped into itself gives

$$\tilde{\rho} = \frac{m}{2} \longleftrightarrow r = 2m \tag{2.19}$$

which confirms the Schwarzschild case. The region near $\tilde{\rho} = 0$ is a second asymptotically flat region, to show this look at the asymptotic geometry near $\tilde{\rho} = 0$

$$\begin{aligned} \mathrm{d}s^2_{\tilde{\rho}\to 0} =&\ \mathrm{d}t^2 - \Big(\frac{m}{2\tilde{\rho}}\Big)^4\Big[\mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2\mathrm{d}\Omega_2^2\Big] \\ \equiv&\ \mathrm{d}t^2 - \mathrm{d}\tilde{\tilde{\rho}}^2 - \tilde{\tilde{\rho}}^2\mathrm{d}\Omega_2^2, \quad \text{where} \quad \tilde{\tilde{\rho}} = \frac{m^2}{4\tilde{\rho}} \end{aligned} \tag{2.20}$$

An important observation can be made now. For what happens when this metric is embedded in flat three dimensional Euclidian space? One would guess that since the same information is present as in the original Schwarzschild metric, the same embedding diagram should appear. Embedding (2.17) in flat three dimensional Euclidean space as before $(\tilde{\rho}, \psi, z)$ gives

$$\Big[1 + \Big(\frac{\mathrm{d}z}{\mathrm{d}\tilde{\rho}}\Big)^2\Big]\mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2\mathrm{d}\psi^2 \equiv \Big(1+\frac{m}{2\tilde{\rho}}\Big)^4\Big[\mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2 d\phi^2\Big] \tag{2.21}$$

but this last equality can never happen. Introducing therefore (ρ, ψ, z) and rewrite the above to

$$\begin{aligned} -\mathrm{d}s^2 =&\ \mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 + \mathrm{d}z^2 = \Big[\Big(\frac{\mathrm{d}\rho}{\mathrm{d}\tilde{\rho}}\Big)^2 + \Big(\frac{\mathrm{d}z}{\mathrm{d}\tilde{\rho}}\Big)^2\Big]\mathrm{d}\tilde{\rho}^2 + \rho^2(\tilde{\rho})\mathrm{d}\psi^2 \\ \equiv&\ \Big(1+\frac{m}{2\tilde{\rho}}\Big)^4\Big[\mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2 d\phi^2\Big] \end{aligned} \tag{2.22}$$

Upon identifying $\phi = \psi$ and comparing both sides leads to

$$\begin{aligned} \rho(\tilde{\rho}) &= \tilde{\rho}\Big(1+\frac{m}{2\tilde{\rho}}\Big)^2, \quad \rho \geq 2m \\ \tilde{\rho}(\rho)_\pm &= \frac{1}{2}\Big(-m + \rho \pm \sqrt{-2m\rho + \rho^2}\Big), \quad \tilde{\rho}(2m)_\pm = \frac{m}{2} \\ z(\tilde{\rho}) &= \pm\frac{\sqrt{2m}(m-2\tilde{\rho})}{\sqrt{\tilde{\rho}}} \end{aligned} \tag{2.23}$$

[6]The name isotropic comes from the fact that in these coordinates the spatial part of the metric is written as a function of r times the flat three dimensional Euclidean metric (in spherical coordinates), and this Euclidean metric does not pick out any preferred direction.

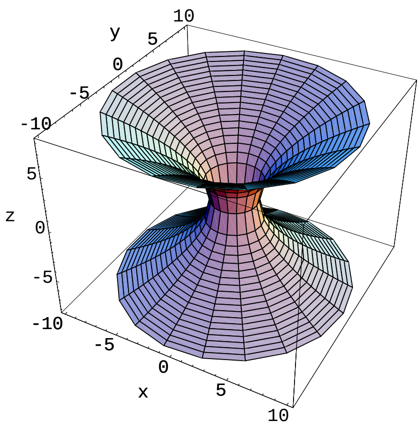

Figure 2.4: Wormhole of the isotropic version of the Schwarzschild metric

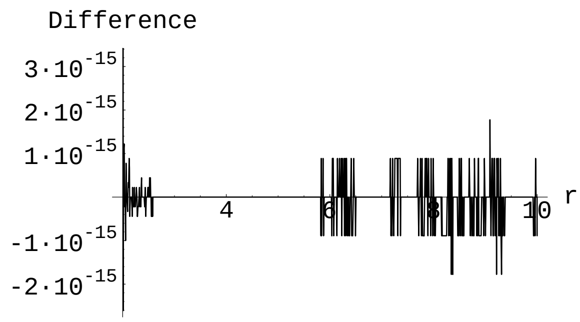

Figure 2.5: Difference between the embedding functions (2.24) and (2.14).

The minus sign in $\tilde{\rho}(\rho)_\pm$ is applicable to the interval $0 < \tilde{\rho} \leq \frac{m}{2}$, the plus sign in $\tilde{\rho} \geq \frac{m}{2}$. Since the event horizon is at $\tilde{\rho} = \frac{m}{2}$, the plus sign is the correct one to use then. The final result in terms of ρ is then

$$z(\rho) = \pm \frac{2\sqrt{m}\left(-2m + \rho + \sqrt{-2m\rho + \rho^2}\right)}{\sqrt{\left(-m + \rho + \sqrt{-2m\rho + \rho^2}\right)}} \tag{2.24}$$

The embedding diagram of this solution is shown in figure 2.4. An important question that needs some attention is why the embedding function $z(\rho)$ is not *exactly* the same as in the normal Schwarzschild case $z(r) = \sqrt{8m(r - 2m)}$? This is what one would expect since the same information is present, changing coordinates should not make a wormhole disappear or be altered[7]. The answer lies in the difference of these two embedding functions, see figure 2.5, which is (almost) equal to zero and realizing that the horizon is at the correct position, see (2.19) and (2.23).

2.4 Reissner-Nordstrøm metric

Adding charge ϵ to the Schwarzschild metric leads to the Reissner-Nordstrøm metric (A.24)

$$\mathrm{d}s^2_{NR} = \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)} - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{2.25}$$

and embedding this in flat Euclidean space leads to the differential equation

$$\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 = \frac{1}{\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)} - 1 \tag{2.26}$$

[7]Unless of course one analytically continuates or discontinues the metric.

In general there are different three sectors

- $r > r_+$
- $r_- < r < r_+ \quad$ where $r_\pm = m \pm \sqrt{m^2 - \epsilon^2}$
- $r < r_-$

To begin with the first sector: $r > r_+$. The solution to the differential equation (2.26) is

$$z(r) = \pm \int_{r_+}^{r} \mathrm{d}r \sqrt{\frac{2mr - \epsilon^2}{r^2 - 2mr + \epsilon^2}} \tag{2.27}$$

The embedding diagram that follows from this function is similar to the Schwarzschild wormhole of the previous section, figure (2.4). In the second region the r coordinate becomes *timelike*. Embedding must therefore take place in $(2+1)$-Minkowskian spacetime, with the resulting embedding equation

$$\mathrm{d}s^2 = \frac{-1}{\frac{2m}{r} - 1 - \frac{\epsilon^2}{r^2}} \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2 = -\mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2 \tag{2.28}$$

and the corresponding solutions

$$\begin{aligned} \rho &= r \\ z(r) &= \pm \int_{r_-}^{r_+} \mathrm{d}r \sqrt{\frac{2mr - \epsilon^2}{2mr - r^2 - \epsilon^2}} \end{aligned} \tag{2.29}$$

The third section is the same as the first one, except that now $r < r_-$. There is however a special distance related to this solution, since the solution (2.27) becomes complex at $r = r_{sp}$ where

$$r_{sp} = \frac{\epsilon^2}{2m} \tag{2.30}$$

The last two sectors are in the *interior* of the black hole. The second sector is behind the first horizon which is located at r_+, the last sector is even behind both horizons. Although an embedding diagram merely reflects the curvature, a possible interpretation to these sectors will be given for the extremal case, $m = \epsilon$. The three sectors above now collapse to two sectors

- $r > r_\pm$
- $r < r_\pm, \quad$ where $r_\pm = m$
- $r_{sp} = \frac{m}{2}$

The embedding differential equation that needs to be solved is

$$z(r) = \pm \int \mathrm{d}r \sqrt{\frac{2rm - m^2}{(m-r)^2}} \tag{2.31}$$

Mathematica gives as an exact solution

$$z(r) = \pm \Big\{ \frac{2(m-r)\sqrt{-m^2 + 2mr}}{\sqrt{m^2 - 2mr + r^2}} - \frac{2m(m-r)\mathrm{arctanh}[\frac{\sqrt{-m^2+2mr}}{m}]}{\sqrt{m^2 - 2mr + r^2}} \Big\} \tag{2.32}$$

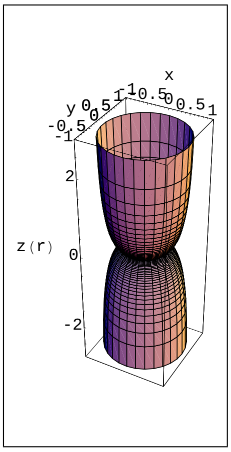

Figure 2.6: Wormhole of the internal Reissner Nordstrøm with $m = \epsilon = 1$.

For general m this function is valid only in the region $r \ \epsilon \ [0.5m, m)$ due to the arctanh. This is all within the second sector, i.e. the *interior* region of the extremal Reissner-Nordstrøm black hole. Also note that indeed the embedding fails for $r < r_{sp} = \frac{m}{2}$.

Two interpretations can be done here for the extremal case. First consider it a failure, so embedding fails for the *exterior* wormhole, this is however to be expected since in the extremal case spacetime is flat. To understand this, realize that gravity is attractive and opposite charges repel, having the same amount of both means no net force. This is similar to D-instanton case before, only in the string frame was a wormhole present, not in the Einstein frame. The presence of a wormhole is rather special, not all frames obtained by a conformal re-scaling of the metric have such a $r \longleftrightarrow \frac{1}{r}$ symmetry. In the next chapters the Kaluza-Klein versions (over the time) of these systems are investigated and more remarks about in what frame a wormhole is present are made.

The alternative interpretation is that the wormhole is now behind the horizon, it is a wormhole connecting two of such spaces. This seems a valid interpretation, although what it means physically is open for debate, this possible *interior* wormhole is shown in figure 2.6.

2.5 Type IIB String Theory

So far all wormholes have been of the Lorentzian type. Now let's take a look at the proposed *Euclidean* wormhole of the type IIB string theory metric (1.84)

$$ds^2 = (e^{\phi_\infty} + \frac{c}{r^8})^{\frac{1}{2}}(dr^2 + r^2 d\Omega_9^2) \tag{2.33}$$

Due to the $SO(10)$ symmetry take the angles $\theta_1 = ... = \theta_8 = \frac{\pi}{2}$ and θ_9 variable. The effective line element which needs to be embedded is then

$$(e^{\phi_\infty} + \frac{c}{r^8})^{\frac{1}{2}}\Big(dr^2 + r^2 d\theta_9\Big) \longleftrightarrow r^8 = \frac{c}{e^{\phi_\infty}} \tag{2.34}$$

Embedding this in flat Euclidean space (ρ, ψ, z) results in

$$\rho(r) = (e^{\phi_\infty} r^4 + \frac{c}{r^4})^{\frac{1}{4}} \geq 2^{\frac{1}{4}} e^{\frac{1}{8}\phi_\infty} c^{\frac{1}{8}} \tag{2.35}$$

Note that the minimum of ρ is indeed at $r^8 = \frac{c}{e^{\phi_\infty}}$. The embedding function z should be a function of ρ, since this is the radial parameter defined in the flat Euclidean three space. Inverting $\rho(r)$ gives a total of eight functions and taking only the physical ones

$$\begin{aligned} r_1(\rho) &= \Big(\frac{\rho^4}{2e^{\phi_\infty}} + \sqrt{\frac{\rho^8 - 4e^{\phi_\infty} c}{2e^{\phi_\infty}}}\Big)^{\frac{1}{4}}, \quad r_1^8 \geq \frac{c}{e^{\phi_\infty}} \\ r_2(\rho) &= \Big(\frac{\rho^4}{2e^{\phi_\infty}} - \sqrt{\frac{\rho^8 - 4e^{\phi_\infty} c}{2e^{\phi_\infty}}}\Big)^{\frac{1}{4}}, \quad 0 < r_1^8 \leq \frac{c}{e^{\phi_\infty}} \end{aligned} \tag{2.36}$$

Mathematica does unfortunately not give an analytical solution to the embedding differential equation for $z(r)$, drawing a wormhole picture is therefore not possible. But based on the experience so far it seems that a figure like 1.4 will appear if one could solve the differential equation.

There can be said a lot more about wormholes, for example about the exotic matter that is needed to keep the wormhole open and many energy conditions that exist. A good overview of the material can be found in [25], but for the analysis of instantons in the coming chapters this is all the wormhole physics needed.

Chapter 3

Introducing Kaluza-Klein reductions

3.1 Introduction

In chapter one various examples of instantons were given. As said an instanton is a Euclidean version of the original Minkowskian system with finite action. Effectively one could say that it is a system without any time coordinate. An alternative way to obtain systems without time is by considering a Kaluza-Klein reduction over a timelike coordinate, normally a Kaluza-Klein reduction takes place over one of the spatial coordinates. In the case of a static metric, it seems likely that the wormhole in the full theory and the wormhole corresponding to the instanton metric obtained via a time compactification have similarities, since a wormhole embedding diagram is obtained by taking slices of constant t.

Begin with the four dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ [1] and write it as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \hat{g}_{00} & \hat{g}_{0\nu} \\ \hat{g}_{\mu 0} & g_{\mu\nu} \end{pmatrix} \tag{3.1}$$

The idea is to consider $g_{\mu\nu}$ as the corresponding metric in 3 spatial dimensions and the time coordinate to have a circle symmetry S^1, i.e. compactifications of $\mathcal{M}_{3+1} = \mathcal{M}_3 \times S^1$. Since the time-direction has an assumed S^1 manifold with radius R_t, in the limit $R_t \to \infty$ one gets back the normal behavior of time. The boundary condition that needs to be satisfied is

$$\hat{\Phi}(x^\mu, t + 2\pi R_t) = \hat{\Phi}(x^\mu, t) \tag{3.2}$$

and the Fourier-expansions in terms of the eigenfunctions of the circle become

$$\hat{\Phi}(x^\mu, t) = \Sigma_n \Phi_n(x^\mu) e^{\frac{int}{R_t}} \tag{3.3}$$

Note that $\Phi_n(x^\mu)$ is assumed to be independent of t. The equation of motion for massless scalar fields is the Klein-Gordon equation gives

$$\hat{\partial}_{\hat{\mu}} \hat{\partial}^{\hat{\mu}} \hat{\Phi} = 0 \tag{3.4}$$

[1] Hatted (unhatted) objects refer to the uncompactified (compactified) case.

and inserting $\hat{\Phi}(x^\mu, t)$ (3.3) in this equation

$$\begin{aligned} \hat{\partial}_{\hat{\mu}}\hat{\partial}^{\hat{\mu}}\hat{\Phi} = & \Big(\partial_\mu\partial^\mu + \partial_t\partial^t\Big)\hat{\Phi} = \Sigma_n\Big[e^{\frac{int}{R_t}}\partial_\mu\partial^\mu\Phi_n(x^\mu) - (\frac{n}{R_t})^2\Phi_n(x^\mu)e^{\frac{int}{R_t}}\Big] \\ \Rightarrow & \Big[\partial_\mu\partial^\mu\Phi_n(x^\mu) - (\frac{n}{R_t})^2\Big]\Phi_n(x^\mu) = 0 \end{aligned} \tag{3.5}$$

which is the *massive* Klein-Gordon equation. In the limit $R_t \to 0$ the mass $m \propto \frac{1}{R_t}$ decouples in the low energy limit and only the massless state $n = 0$ needs to be taken into account. The results is that, after dimensional reduction over a circle of infinitesimal radius, a massless $(D+1)$ scalar field is effectively described by a massless scalar field in D Euclidean dimensions. Whereas taking the limit $R_t \to \infty$ makes this mass spectrum continuously again and the uncompactified $(D+1)$-dimensional theory is regained[2]. The best way to interpreted this physically is by comparing it to a movie. In the limit $R_t \to \infty$ the normal time situation is obtained and the limit $R_t \to 0$ can best be seen as "pause" or as taking a picture.

A relevant question is whether or not it is consistent to ignore these massive fields. For the $(D+1)$-dimensional example above, it is consistent, since both the reduced equation and the starting point (3.4) are massless Klein-Gordon equations. For $n \neq 0$ the mass terms are infinite heavy in the limit $R_t \to 0$ and decouple from the theory, consistency is therefore guaranteed.

Looking back at the D-instanton of the truncated type IIB string theory of chapter one it is clear that there are, from daily life experience, six spacelike dimensions too many. In light of the example above one can imagine that a similar reduction can take place over these six dimensions, to obtain a four dimensional description. Difficulties arise now from choosing which way to reduce and possibly inconsistencies as a result of that, but this will not be discussed here. For a good introduction on Kaluza-Klein reductions, see for example [7].

Finally a word on the physics of all this. If string theory is the correct way to describe the world, an (future) accelerator should be able to see these extra dimensions. To understand this one has to realize that the particles have to match the symmetry of these extra dimensions, i.e. a circle symmetry in the scalar field example above. The "wavelength" of this field has to match the radius R of this circle dimension, for example the wavelength is equal to the circumference, similar to the behavior of the strings of a guitar. These are the states $n \neq 0$, which are ignored in (3.5), i.e. the higher Fourier modes. If one has a strong enough accelerator, then from a certain energy point extra excited states of this scalar field will be observed. The mass spectrum of these excited fields will be such that it matches the wavelength pattern, since mass and distances are inversely proportional, via analyzing this mass spectrum one can estimate the radii of extra dimensions.

The (time) compactification together with keeping only the $n = 0$ term (massless fields) in (3.5), is called Kaluza-Klein reduction. One way of justifying the ansatz (3.1) is by requiring that the $\hat{g}_{0\nu}$ and $\hat{g}_{00}$ transform as vectors A_μ and scalars ϕ should do in three dimensions. To be specific consider an infinitesimal transformation in four dimensions

$$x^{\hat{\mu}} \to x^{\hat{\mu}} + \epsilon\xi^{\hat{\mu}}(x^\mu) \tag{3.6}$$

[2]As an example here has been taken a timelike reduction, but as is clear from the derivation it works equally well for a spacelike reduction.

where the transformation is independent of the zeroth coordinate, i.e. $\hat{\mu}$ runs from $(0, 1, 2, 3)$ and μ from $(1, 2, 3)$. The transformation (3.6) implies the use of the Lie-derivative. For example

$$\delta\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{g}_{\hat{\mu}\hat{\rho}}(\partial_{\hat{\nu}}\xi^{\hat{\rho}}) + \hat{g}_{\hat{\rho}\hat{\nu}}(\partial_{\hat{\mu}}\xi^{\hat{\rho}}) + \xi^{\hat{\rho}}(\delta_{\hat{\rho}}\delta\hat{g}_{\hat{\mu}\hat{\nu}}) \tag{3.7}$$

For example the transformation of $\hat{g}_{00}$

$$\begin{aligned} \delta\hat{g}_{00} &= \delta\hat{g}_{\hat{\mu}\hat{\nu}}|_{\hat{\mu}=\hat{\nu}=0} = \hat{g}_{\hat{0}\hat{\rho}}(\partial_{\hat{0}}\xi^{\hat{\rho}}) + \hat{g}_{\hat{\rho}\hat{0}}(\partial_{\hat{0}}\xi^{\hat{\rho}}) + \xi^{\hat{\rho}}(\delta_{\hat{\rho}}\delta\hat{g}_{\hat{0}\hat{0}}) \\ &= \hat{g}_{00}\partial_0\xi^0 + \hat{g}_{0\mu}\partial_0\xi^\mu + \hat{g}_{00}\partial_0\xi^0 + \hat{g}_{0\mu}\partial_0\xi^\mu + \xi^0\partial_0\hat{g}_{00} + \xi^\mu\partial_\mu\hat{g}_{00} \\ &= \xi^\mu\partial_\mu\hat{g}_{00} \end{aligned} \tag{3.8}$$

since $\xi^{\hat{\mu}}$ does not depend on the zeroth coordinate x^0 by construction (3.6). This is the transformation rule for a *scalar* and thus identifying $\hat{g}_{00} = \varphi = e^{\frac{\chi}{2}}$, ensuring that it is a time-like component ($\varphi > 0$). Similarly for $\left(\frac{\hat{g}_{0\nu}}{g_{00}}\right)$

$$\delta\left(\frac{\hat{g}_{0\nu}}{g_{00}}\right) = \left(\frac{\hat{g}_{0\nu}}{g_{00}}\right)\partial_\mu\xi^\mu + \xi^\mu\partial_\nu\left(\frac{\hat{g}_{0\mu}}{g_{00}}\right) + \partial_\nu\xi^0 \tag{3.9}$$

By rewriting $\xi^{\hat{\mu}} = (\Lambda, \xi^\mu)$ and$\left(\frac{\hat{g}_{0\mu}}{g_{00}}\right) = A_\mu$ one sees then that A_μ has the right transformation property in three dimensions for a vector and note that the last term in (3.9) is a $U(1)$-gauge term. Thus (3.1) can be rewritten as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \phi & \phi A_\nu \\ \phi A_\mu & g_{\mu\nu} + \phi A_\mu A_\nu \end{pmatrix} = \begin{pmatrix} e^{\frac{\chi}{2}} & e^{\frac{\chi}{2}} A_\nu \\ e^{\frac{\chi}{2}} A_\mu & g_{\mu\nu} + e^{\frac{\chi}{2}} A_\mu A_\nu \end{pmatrix} \tag{3.10}$$

To obtain the action in pure three dimensions $\hat{g} = \mathrm{Det}[\hat{g}_{\hat{\mu}\hat{\nu}}]$ is needed. Because of the nice form of (3.10) this turns out to be

$$\hat{g} = \mathrm{Det}[g_{\mu\nu}]\phi = g\sqrt{\phi} = ge^{\frac{\chi}{4}} \tag{3.11}$$

From the relation $\hat{g}_{\hat{\mu}\hat{\rho}}\hat{g}^{\hat{\rho}\hat{\nu}} = \delta^{\hat{\nu}}_{\hat{\mu}}$ follows the inverse metric

$$\hat{g}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \frac{1}{\phi} + A^2 & -A^\nu \\ -A^\mu & g^{\mu\nu} \end{pmatrix} \tag{3.12}$$

As an explicit example the four dimensional Einstein-Hilbert action will be reduced over a timelike coordinate [9][3]

$$\hat{S}_4 = -\int \mathrm{d}^4x\sqrt{|\hat{g}|}\hat{\mathcal{R}}(\hat{g}) \tag{3.13}$$

The metric ansatz (3.10) has a Kaluza-Klein vector A_μ. If one is interested in a theory with scalars only (besides the metric), then this vector should be set equal to zero. It can be shown that this is consistent to do, but setting the scalar field equal to zero cannot be done in general [7]. Setting $A_\mu = 0$ turns (3.10) into

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix} \tag{3.14}$$

[3]The minus sign is due to the chosen metric convention $g_{\mu\nu}$ =diag$(1, -1, -1, -1)$. To understand this better realize that $\Gamma^c_{ab} \propto g^2$, $\mathcal{R}^a_{bcd} \propto \Gamma^c_{ab} \to \mathcal{R}_{ab} \propto g^2$ and finally $\mathcal{R} = g^{ab}\mathcal{R}_{ab} \to \mathcal{R} \propto g_{ab}$ and thus clearly influenced by the choice of the metric.

The four dimensional Christoffel symbol

$$\hat{\Gamma}^{\hat{\rho}}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\hat{g}^{\hat{\rho}\hat{\sigma}}\left(\partial_{\hat{\mu}}\hat{g}_{\hat{\nu}\hat{\sigma}} + \partial_{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\sigma}} - \partial_{\hat{\sigma}}\hat{g}_{\hat{\mu}\hat{\nu}}\right) \tag{3.15}$$

with the non-zero entries[4]

$$\hat{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu},\ \hat{\Gamma}^{0}_{\mu 0} = \frac{1}{2}g^{00}\partial_\mu g_{00} = \frac{1}{4}\partial_\mu\chi,\ \hat{\Gamma}^{\rho}_{00} = -\frac{1}{2}g^{\mu\rho}\partial_\mu g_{00} = -\frac{1}{4}e^{\frac{\chi}{2}}\partial^\rho\chi \tag{3.16}$$

A very useful formula is

$$\begin{aligned}\sqrt{|g|}e^{\alpha\phi}\mathcal{R} = & \sqrt{|g|}g^{\mu\rho}e^{\alpha\phi}\left(\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\rho} - \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho}\right)\\ & + \alpha\sqrt{|g|}e^{\alpha\phi}\partial_\mu\phi g^{\mu\rho}\Gamma^{\sigma}_{\sigma\rho} - \alpha\sqrt{|g|}e^{\alpha\phi}\partial_\sigma\phi g^{\mu\rho}\Gamma^{\sigma}_{\mu\rho}\end{aligned} \tag{3.17}$$

It is important to note that this is only valid under an integral sign, for the straightforward (but a bit lengthy) proof makes use of partial integration:

$$\begin{aligned}&\int \mathrm{d}^4x\sqrt{|g|}e^{\alpha\phi}\mathcal{R} = \int \mathrm{d}^4x\sqrt{|g|}e^{\alpha\phi}g^{\mu\rho}\left(\partial_\sigma\Gamma^{\sigma}_{\mu\rho} - \partial_\mu\Gamma^{\sigma}_{\sigma\rho} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\rho}\right)\\ &= -\int \mathrm{d}^4x\partial_\sigma\left(\sqrt{|g|}g^{\mu\rho}e^{\alpha\phi}\right)\Gamma^{\sigma}_{\mu\rho} + \int \mathrm{d}^4x\partial_\sigma\left(\sqrt{|g|}g^{\mu\rho}e^{\alpha\phi}\right)\Gamma^{\sigma}_{\sigma\rho}\\ &+ \int \mathrm{d}^4x\sqrt{|g|}e^{\alpha\phi}g^{\mu\rho}\left(\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\rho}\right) = \int \mathrm{d}^4x\sqrt{|g|}g^{\mu\rho}e^{\alpha\phi}\left(\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\rho} - \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho}\right)\\ &+ \alpha\sqrt{|g|}e^{\alpha\phi}\partial_\mu\phi g^{\mu\rho}\Gamma^{\sigma}_{\sigma\rho} - \alpha\sqrt{|g|}e^{\alpha\phi}\partial_\sigma\phi g^{\mu\rho}\Gamma^{\sigma}_{\mu\rho}\Big)\end{aligned} \tag{3.18}$$

where in the second step a total derivative has been ignored and in the third equality the following standard rules are used

$$\begin{aligned}&\partial_\mu g = g g^{\rho\eta}\partial_\mu g_{\rho\eta}\\ &\nabla_\mu g_{\rho\eta} = 0 \Rightarrow \partial_\mu g_{\rho\eta} = \Gamma^{\nu}_{\rho\mu}g_{\nu\eta} + \Gamma^{\nu}_{\eta\mu}g_{\rho\nu}\\ &\partial_\mu g^{\rho\eta} = -\Gamma^{\rho}_{\nu\mu}g^{\nu\eta} - \Gamma^{\eta}_{\nu\mu}g^{\rho\nu}\end{aligned} \tag{3.19}$$

Let's apply this to the action (3.13) with $\alpha = 0$ in (3.17)

$$\begin{aligned}\hat{\mathcal{L}}_4 = & -\sqrt{|\hat{g}|}\hat{g}^{\hat{\mu}\hat{\rho}}\left(\hat{\Gamma}^{\hat{\sigma}}_{\hat{\mu}\hat{\lambda}}\hat{\Gamma}^{\hat{\lambda}}_{\hat{\sigma}\hat{\rho}} - \hat{\Gamma}^{\hat{\lambda}}_{\hat{\mu}\hat{\rho}}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\lambda}\hat{\sigma}}\right)\\ = & -e^{\frac{\chi}{4}}\sqrt{|g|}g^{\mu\rho}\left(\hat{\Gamma}^{\hat{\sigma}}_{\mu\hat{\lambda}}\hat{\Gamma}^{\hat{\lambda}}_{\hat{\sigma}\rho} - \hat{\Gamma}^{\hat{\lambda}}_{\mu\rho}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\lambda}\hat{\sigma}}\right) - e^{-\frac{\chi}{4}}\sqrt{|g|}\left(\hat{\Gamma}^{\hat{\sigma}}_{0\hat{\lambda}}\hat{\Gamma}^{\hat{\lambda}}_{\hat{\sigma}0} - \hat{\Gamma}^{\hat{\lambda}}_{00}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\lambda}\hat{\sigma}}\right)\end{aligned} \tag{3.20}$$

since $\hat{g}^{\mu 0} = 0$ due to (3.14). Upon substituting (3.16) this becomes

$$\mathcal{L}_3 = -\sqrt{|g|}e^{\frac{\chi}{4}}\mathcal{R}(g) \tag{3.21}$$

At first sight it seems strange that a kinetic term like $\partial_\mu\chi\partial^\mu\chi$ is absent. This can be understood however if one realizes that this Lagrangian is not written in the standard Einstein-Hilbert canonical form, i.e. there is a scalar - metric coupling (no EF). To remove this perform a conformal re-scaling of the metric

$$g^{\mu\nu} = e^{\frac{\chi}{2}}G^{\mu\nu}_{EF} \tag{3.22}$$

[4]The minus sign in front of $\hat{\Gamma}^{\rho}_{00}$ is due to the time-compactification instead of the more familiar space-compactification.

Because there are now two metrics involved, it can be confusing which metric to raise and lower indices. To prevent this, all raising of indices will be avoided by keeping them covariant, so $g^{\mu\nu}x_\nu$ will be used instead of x^μ. The conformal change of the metric will influence the Christoffel symbols in the following way

$$\begin{aligned}\Gamma^{\mu}_{\nu\rho}(g) =& \frac{1}{2}g^{\mu\eta}\Big(\partial_\nu g_{\rho\eta} + \partial_\rho g_{\nu\eta} - \partial_\eta g_{\nu\rho}\Big) \\ =& \ \Gamma^{\mu}_{\nu\rho}(G) + \frac{1}{4}\Big\{-\delta^{\mu}_{\rho}\partial_\nu\chi - \delta^{\mu}_{\nu}\partial_\rho\chi + G_{\nu\rho}G^{\mu\eta}\partial_\eta\chi\Big\}\end{aligned} \tag{3.23}$$

where $\Gamma^{\mu}_{\mu\rho}(G)$ is the Christoffel symbol for the metric $G_{\mu\nu}$. The easiest way to calculate the effect of this conformal mapping is by using (3.17)

$$\begin{aligned}\sqrt{|g|}e^{\frac{\chi}{4}}R(g) =& \ \sqrt{|g|}g^{\mu\rho}e^{\frac{\chi}{4}}\Big(\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\rho} - \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho}\Big) \\ +& \ \frac{1}{4}\sqrt{|g|}e^{\frac{\chi}{4}}(\partial_\mu\phi)g^{\mu\rho}\Gamma^{\sigma}_{\sigma\rho} - \frac{1}{4}\sqrt{|g|}e^{\frac{\chi}{4}}(\partial_\sigma\phi)g^{\mu\rho}\Gamma^{\sigma}_{\mu\rho}\end{aligned} \tag{3.24}$$

and substituting (3.23). After some lengthy algebra one finds

$$\sqrt{|g|}e^{\frac{\chi}{4}}R(g) = \sqrt{|G|}\Big(\Gamma^{a}_{bc}(G)\Gamma^{c}_{af}(G) - \Gamma^{a}_{bf}(G)\Gamma^{d}_{da}(G)\Big)G^{bf} - \sqrt{|G|}\frac{1}{8}\partial_\mu\chi\partial_\nu\chi G^{\mu\nu} \tag{3.25}$$

Realizing that the first term on the right hand side is (3.17) with $\alpha = 0$ gives for the dimensionally reduced Einstein-Hilbert Lagrangian

$$\mathcal{L}(G) = -\sqrt{|G|}\mathcal{R} + \sqrt{|G|}\frac{1}{8}\partial_\mu\chi\partial_\nu\chi G^{\mu\nu} \tag{3.26}$$

The factor $\frac{1}{8}$ is due to the ansatz for $\hat{g}_{tt}$, redefining therefore $\chi = 2\phi$ gives the standard form

$$\boxed{\mathcal{L}(G,\phi) = -\sqrt{|G|}\mathcal{R} + \sqrt{|G|}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu}} \tag{3.27}$$

which is called the *canonical* Einstein-Hilbert action (in the Einstein frame). The equations of motion that follow from (3.26) are[5] for $G_{\mu\nu}$ and χ

$$\mathcal{R}_{\mu\nu} = \frac{1}{8}\partial_\mu\chi\partial_\nu\chi \tag{3.28}$$

$$\frac{1}{4}\partial_\mu\Big(\sqrt{|G|}\partial_\nu\chi G^{\mu\nu}\Big) = 0 \tag{3.29}$$

The question that needs attention is whether or not these two equations are with respect to a Cartesian coordinate (Car) system[6] or valid for *all* coordinate systems. To find the answer to this question [22] first observe that the presence of the determinant of the metric $\sqrt{|g|}$ implies an invariance under a coordinate transformation. As an explicit example rewrite the action belonging to the Lagrangian (3.27) in spherical coordinates (sph)

$$\begin{aligned}S_{Sch} =& \int \mathrm{d}^3x\sqrt{|G_{Car}|}\Big(-\mathcal{R}_{\mu\nu}G^{\mu\nu}_{Car} + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu}_{Car}\Big|_{Car} \\ =& \int \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\sqrt{|G_{sph}|}\Big(-\mathcal{R}_{\mu\nu}G^{\mu\nu}_{sph} + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu}_{sph}\Big)\Big|_{sph}\end{aligned} \tag{3.30}$$

[5]The variation of $\mathcal{R}_{\mu\nu}$ can be ignored in this case, for further information see Wald, appendix E and this is *only* true in the Einstein frame.

[6]The three axes of three-dimensional Cartesian coordinates, conventionally denoted the $x-$, $y-$, and z-axes are chosen to be linear and mutually perpendicular. In three dimensions, the coordinates x, y, and z may lie anywhere in the interval $(-\infty, \infty)$.

The equations of motion now obtained in spherical coordinates are the same as the previously obtained equations of motions in Cartesian coordinates, see (3.28) and (3.29). The more fundamental reason why this is so comes from the presence of the combination $\mathrm{d}^D x\sqrt{|g|}$, which is valid in all coordinate systems, the Jacobian that appears in a coordinate transformation is automatically included.

3.2 Example: Schwarzschild metric

In the previous section the four dimensional Einstein-Hilbert action has been dimensionally reduced over a timelike coordinate. The metric ansatz (3.10) requires no off-diagonal terms, since the Kaluza-Klein vector A_μ has been set equal to zero. An example of such a system is the Schwarzschild metric

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 1-\frac{2m}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{2m}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \equiv \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{-\frac{\chi}{2}} G^{EF}_{\mu\nu} \end{pmatrix} \tag{3.31}$$

Solving the above matrix equation gives as the solutions for χ and $G_{\mu\nu}$

$$\boxed{e^{\frac{\chi}{2}} = 1 - \frac{2m}{r} \longleftrightarrow \chi = 2\log(1-\frac{2m}{r})} \tag{3.32}$$

$$\boxed{G_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2mr - r^2 & 0 \\ 0 & 0 & (2mr-r^2)\sin^2\theta \end{pmatrix}} \tag{3.33}$$

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}r^2 - (r^2-2mr)\mathrm{d}\theta^2 - (r^2-2mr)\sin^2\theta\mathrm{d}\phi^2} \tag{3.34}$$

The solutions to the equations of motion belonging to the Schwarzschild system (3.28) and (3.29) are already know now. As a consistency check however one can calculate the Ricci tensor in two differen ways. First via a direct computation of the Ricci tensor via the standard formulas for Γ^c_{ab}, $\mathcal{R}^a_{bcd}$, $\mathcal{R}_{ab} = \mathcal{R}^c_{acb}$ and $\mathcal{R} = G^{ab}\mathcal{R}_{ab}$, which gives

$$\mathcal{R}_{\mu\nu}(G) = \begin{pmatrix} \frac{2m^2}{(r^2-2mr)^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathcal{R} = \frac{-2m^2}{(r^2-2mr)^2} \tag{3.35}$$

Note that it has a genuine singularity at $r = 2m$, which is at the Schwarzschild radius of the uncompactified metric (2.14), where it is merely a coordinate singularity. Further note that $\mathcal{R}_{rr}$ is non-zero, this is not the case for the full Schwarzschild metric (outside the body of mass m) and that $\mathcal{R} < 0$ and thus indeed the minus sign is needed in (3.13) to have non-negative action. The second approach is via using (3.28) and substituting the solution for χ (3.32), which gives the same answer. From the Einstein equation $\mathcal{G}_{\mu\nu}$ (1.72) follows the energy momentum tensor of this compactified system

$$T_{\mu\nu} = \frac{1}{8\pi}\begin{pmatrix} \frac{m^2}{(r^2-2mr)^2} & 0 \\ 0 & \frac{m^2}{r^2-2mr}\mathrm{d}\Omega_2^2 \end{pmatrix} \tag{3.36}$$

Thus no longer the vacuum Einstein equations are in use, instead a matter term is introduced with the resulting $T_{\mu\nu}$ from above. In section 2.3 the Schwarzschild metric was given in isotropic coordinates, the advantage of isotropic coordinates is that the possible wormhole symmetry

$$r \longleftrightarrow \frac{\text{constant}}{r} \tag{3.37}$$

can be easily found by reading it off from the conformal factor. Demanding a isotropic form leads to

$$\begin{aligned} \mathrm{d}s^2 = & -\mathrm{d}r^2 + (2mr - r^2)\mathrm{d}\theta^2 + (2mr - r^2)\sin^2\theta \mathrm{d}\phi^2 \\ = & -f^2(\rho)\mathrm{d}\rho^2 - f^2(\rho)(\rho^2 \mathrm{d}\theta^2 + \rho^2 \sin^2\theta \mathrm{d}\theta^2) \end{aligned} \tag{3.38}$$

Comparing both sides leads to the differential equation

$$\begin{aligned} \frac{\partial \rho(r)}{\partial r} = \frac{\rho(r)}{\sqrt{(r^2 - 2mr)}} \longrightarrow \rho(r) = & \ 2(r - m + \sqrt{r^2 - 2rm})C_1 \\ \longleftrightarrow r(\rho) = & \ \frac{4m^2C_1^2 + 4m\rho C_1 + \rho^2}{4\rho C_1} \\ \longleftrightarrow f^2(\rho) = & \ \frac{r^2 - 2mr}{\rho^2} = \frac{(4m^2C_1^2 - \rho^2)^2}{16\rho^4 C_1^2} \end{aligned} \tag{3.39}$$

where C_1 is a constant of integration. The infinitesimal line element changes to

$$\mathrm{d}s^2 = -\frac{(\frac{4m^2C_1^2}{\rho} - \rho)^2}{16C_1^2}[\Big(\frac{\mathrm{d}\rho}{\rho}\Big)^2 + \mathrm{d}\Omega_2^2] \tag{3.40}$$

The symmetry that follows from the conformal factor $(\frac{4m^2C_1^2}{\rho} - \rho)^2$ is

$$\boxed{\begin{array}{l} \rho \longleftrightarrow \pm\dfrac{4m^2C_1^2}{\rho} \\ \text{or} \quad r = 2m \end{array}} \tag{3.41}$$

which *seems* to satisfy the wormhole condition of chapter two at first sight, if one takes the plus sign, since $\rho \geqslant 0$ by definition. The position of the wormhole would then be

$$\rho^2 = 4m^2C_1^2 \tag{3.42}$$

and for $C_1^2 = 1/2$ this is at the same position as the original Schwarzschild wormhole. As is clear from the conformal factor, only due to the square is the plus sign a symmetry. In the next chapter the D-dimensional Schwarzschild metric will be dimensionally reduced, there the square is replaced by a different constant (dimensionally dependant) and the symmetry is no longer present, see (4.24). Why then should it be for three dimensions a wormhole symmetry? Therefore look at the embedding function $z(\tilde{\rho})$ corresponding to (3.34), this turns out to be

$$z(\tilde{\rho}) = C_1 \pm 2mi\log[\sqrt{\sqrt{m^2 + \tilde{\rho}^2} - m} + \sqrt{m + \sqrt{m^2 + \tilde{\rho}^2}}] \tag{3.43}$$

which is *complex*. One should therefore reject this as a genuine wormhole and take only the plus sign as a legitimate symmetry. But this symmetry exchanges positive and negative values of ρ, which cannot be considered an acceptable physical symmetry. The conclusion must be that this system does *not* have a wormhole in the *Einstein* frame, more about this in the next chapter when the $(D+1)$-dimensional Schwarzschild black hole is dimensionally reduced and since the Schwarzschild metric is a special case of the Reissner-Nordstrøm metric, a wormhole can be found via the latter.

Note by the way that nothing changes when one does the more conventional space compactification. For the same Lagrangian

$$\mathcal{L}_4 = -\sqrt{|g|}e^{\frac{\chi}{4}}\mathcal{R} \tag{3.44}$$

is obtained. What is different of course is the ansatz for $\hat{g}_{\hat{\mu}\hat{\nu}}$

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \hat{g}_{zz} \end{pmatrix} = \begin{pmatrix} e^{-\frac{\chi}{2}}G_{\mu\nu} & 0 \\ 0 & -e^{\frac{\chi}{2}} \end{pmatrix} \tag{3.45}$$

but the same conformal mapping is needed to remove the scalar-metric coupling. A direct comparison between the Schwarzschild metric and this ansatz is not possible for if one compactifies in the $\hat{z}$-direction the spherically symmetric Schwarzschild metric must first be rewritten in Cartesian coordinates first. The reason that this is not needed in the time compactification case is that the time t in the Schwarzschild metric is used in an Cartesian way, i.e. $t \,\epsilon\, (-\infty, \infty)$.

Chapter 4

Various Kaluza-Klein reductions

In the previous chapter Kaluza-Klein reductions were introduced via the four dimensional Schwarzschild metric. To make connection to the D-instanton of chapter one, two scalar fields are needed. As a first step to obtain this, the (extremal) Reissner-Nordstrøm black hole is explored.

4.1 General Reissner-Nordstrøm black hole

By adding charge to the Schwarzschild black hole, one obtains the Reissner-Nordstrøm black hole. An interesting conclusion of the previous section was that the wormhole of the uncompactified theory had disappeared after the reduction (in the Einstein frame). Since the Reissner-Nordstrøm black hole is a generalization of the Schwarzschild case, new insights may appear why this wormhole disappears.

From $(3+1)$ dimensions to $(D+1)$ dimensions

In the previous chapter the compactification was done from $(3+1)$ to $(3+0)$ dimensions, generalizing to $(D+1)$ dimensions is the logical next step to do. Taking again only scalars into account by setting all vector fields that appear equal to zero, will change the metric ansatz for $\hat{g}_{\hat{\mu}\hat{\nu}}$ to

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix} \tag{4.1}$$

where $g_{\mu\nu}$ is now a $(D \times D)$ matrix. To begin with the general Reissner-Nordstrøm metric in $(D+1)$ dimensions, i.e.

$$\begin{aligned} \hat{\mathcal{L}}_{RN} &= \hat{\mathcal{L}}_{\mathcal{R}} + \hat{\mathcal{L}}_{MW} \\ &= -\sqrt{|\hat{g}|}\hat{\mathcal{R}} - \frac{1}{4}\sqrt{|\hat{g}|}\hat{g}^{\hat{\mu}\hat{\rho}}\hat{g}^{\hat{\nu}\hat{\eta}}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}_{\hat{\rho}\hat{\eta}} \end{aligned} \tag{4.2}$$

where a hat now refers to a $(D+1)$-dimensional object and the field strength is expressed through the vector potential $\hat{F}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\hat{A}_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{A}_{\hat{\mu}}$, which is the standard

Maxwell (MW) field strength to describe electromagnetism. To obtain the conformal mapping that results in the Einstein frame. the first part of (4.2) needs to be re-examined. The $(D+1)$-dimensional Christoffel symbol

$$\hat{\Gamma}^{\hat{\rho}}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\hat{g}^{\hat{\rho}\hat{\sigma}}\left(\partial_{\hat{\mu}}\hat{g}_{\hat{\nu}\hat{\sigma}} + \partial_{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\sigma}} - \partial_{\hat{\sigma}}\hat{g}_{\hat{\mu}\hat{\nu}}\right) \tag{4.3}$$

has as non-zero entries

$$\hat{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu},\ \hat{\Gamma}^{0}_{\mu 0} = \frac{1}{2}g^{00}\partial_\mu g_{00} = \frac{1}{4}\partial_\mu\chi,\ \hat{\Gamma}^{\rho}_{00} = -\frac{1}{2}g^{\rho\, u}\partial_{\, u}g_{00} = -\frac{1}{4}e^{\frac{\chi}{2}}\partial^\rho\chi \tag{4.4}$$

Using (3.17), the first part of the action (4.2) can be rewritten as (with $\alpha = 0$)

$$\begin{aligned}\hat{\mathcal{L}}_{D+1} =& -\sqrt{|\hat{g}|}\hat{g}^{\hat{\mu}\hat{\rho}}\left(\hat{\Gamma}^{\hat{\sigma}}_{\hat{\mu}\hat{\lambda}}\hat{\Gamma}^{\hat{\lambda}}_{\hat{\sigma}\hat{\rho}} - \hat{\Gamma}^{\hat{\lambda}}_{\hat{\mu}\hat{\rho}}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\lambda}\hat{\sigma}}\right)\\ =& -e^{\frac{\chi}{4}}\sqrt{|g|}g^{\mu\rho}\left(\hat{\Gamma}^{\hat{\sigma}}_{\mu\hat{\lambda}}\hat{\Gamma}^{\hat{\lambda}}_{\hat{\sigma}\rho} - \hat{\Gamma}^{\hat{\lambda}}_{\mu\rho}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\lambda}\hat{\sigma}}\right) - e^{-\frac{\chi}{4}}\sqrt{|g|}\left(\hat{\Gamma}^{\hat{\sigma}}_{0\hat{\lambda}}\hat{\Gamma}^{\hat{\lambda}}_{\hat{\sigma}0} - \hat{\Gamma}^{\hat{\lambda}}_{00}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\lambda}\hat{\sigma}}\right)\end{aligned} \tag{4.5}$$

since $\hat{g}^{\mu 0} = 0$ due to (4.1). Upon substituting (4.4) this becomes

$$\mathcal{L}_D = -\sqrt{|g|}e^{\frac{\chi}{4}}\mathcal{R} \tag{4.6}$$

So far the dimension D did not explicitly come into play, the reason for this is that the metric ansatz for $\hat{g}_{\hat{\mu}\hat{\nu}}$ (4.1) has the same form for all dimensions. But to go to the Einstein frame remember that the determinant $\sqrt{|g|}$ and the Ricci scalar $\mathcal{R}$ changes in D dimensions as

$$\begin{aligned} g_{\mu\nu} &\Rightarrow G_{\mu\nu}e^{\alpha\chi}\\ \mathcal{R}_g &\Rightarrow \mathcal{R}_G e^{-\alpha\chi}\end{aligned} \tag{4.7}$$

and thus to obtain the Einstein frame demand

$$\begin{aligned} e^{\frac{\chi}{4}}\sqrt{|g|}\mathcal{R}_g =&\ e^{\frac{\chi}{4}}\sqrt{|G|}e^{\frac{D\alpha}{2}\chi}\mathcal{R}_G e^{-\alpha\chi} \equiv \sqrt{|G|}\mathcal{R}\\ \Rightarrow&\ \frac{D\alpha}{2} + \frac{1}{4} - \alpha = 0 \Rightarrow \alpha = \frac{1}{4-2D}\end{aligned} \tag{4.8}$$

The conformal mapping of the metric will influence the Christoffel symbols in the following way

$$\begin{aligned}\Gamma^{\mu}_{\nu\rho}(g) =& \frac{1}{2}g^{\mu\eta}\left(\partial_\nu g_{\eta\rho} + \partial_\rho g_{\nu\eta} - \partial_\eta g_{\nu\rho}\right)\\ =&\ \Gamma^{\mu}_{\nu\rho}(G) - \frac{1}{8-4D}\left\{-\delta^{\mu}_{\rho}\partial_\nu b\chi - \delta^{\mu}_{\nu}\partial_\rho\chi + G_{\nu\rho}G^{\mu\eta}\partial_\eta\chi\right\}\end{aligned} \tag{4.9}$$

where $\Gamma^{\rho}_{\mu\nu}(G)$ is the Christoffel symbol for the metric $G_{\mu\nu}$. The easiest way to calculate the effect of this conformal mapping is by using (3.17)

$$\begin{aligned}\sqrt{|g|}e^{\frac{\chi}{4}}R(g) =&\ \sqrt{|g|}g^{\mu\rho}e^{\frac{\chi}{4}}\left(\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\rho} - \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho}\right)\\ &+ \frac{1}{4}\sqrt{|g|}e^{\frac{\chi}{4}}\partial_\mu\phi g^{\mu\rho}\Gamma^{\sigma}_{\sigma\rho} - \frac{1}{4}\sqrt{|g|}e^{\frac{\chi}{4}}\partial_\sigma\phi g^{\mu\rho}\Gamma^{\sigma}_{\mu\rho}\end{aligned} \tag{4.10}$$

and substituting (4.9). Note however that the factor $\frac{1}{4}$ in this formula comes purely from the $e^{\frac{\chi}{4}}$ and is *not* influenced by generalizing to D dimensions. After some lengthy algebra one obtains

$$\sqrt{|g|}e^{\frac{\chi}{4}}\mathcal{R}(g) = \sqrt{|G|}\mathcal{R}(G) + \Big[\frac{D-1}{16(2-D)}\Big]\partial_\mu\chi\partial_\nu\chi G^{\mu\nu} \tag{4.11}$$

The second part of (4.2) is straightforward

$$\begin{aligned}
\hat{\mathcal{L}}_{MW} &= -\frac{\sqrt{|\hat{g}|}}{4}\hat{g}^{\hat{\mu}\hat{\rho}}\hat{g}^{\hat{\nu}\hat{\eta}}F_{\hat{\mu}\hat{\nu}}F_{\hat{\rho}\hat{\eta}} \\
&= -\frac{\sqrt{|g|}}{4}e^{\frac{\chi}{4}}\Big(\hat{g}^{\mu\rho}\hat{g}^{\nu d}F_{\mu\nu}F_{\rho\eta} + \hat{g}^{00}\hat{g}^{\nu\eta}F_{0\nu}F_{0\eta} + \hat{g}^{\mu\rho}\hat{g}^{00}F_{\mu 0}F_{\rho 0}\Big) \\
&= -\frac{\sqrt{|g|}}{4}e^{\frac{\chi}{4}}\Big(g^{\mu\rho}g^{\nu\eta}F_{\mu\nu}F_{\rho\eta} + e^{\frac{-\chi}{2}}g^{\nu\eta}F_{0\nu}F_{0\eta} + g^{\mu\rho}e^{\frac{-\chi}{2}}F_{\mu 0}F_{\rho 0}\Big) \\
&= -\frac{\sqrt{|g|}}{4}e^{\frac{\chi}{4}}\Big(g^{\mu\rho}g^{\nu\eta}F_{\mu\nu}F_{\rho\eta} + 2e^{\frac{-\chi}{2}}g^{\nu\eta}F_{0\nu}F_{0\eta}\Big)
\end{aligned} \tag{4.12}$$

Setting the vector part $F_{\mu\nu}$ equal to zero and labelling $\hat{A}_0 = \ell$

$$\begin{aligned}
\mathcal{L}_{MW} &= -\frac{\sqrt{|g|}}{4}e^{\frac{\chi}{4}}\Big(2e^{\frac{-\chi}{2}}g^{\nu\eta}F_{0\nu}F_{0\eta}\Big) = -\frac{\sqrt{|g|}}{2}e^{-\frac{\chi}{4}}\Big(g^{\nu\eta}(\partial_0A_\nu - \partial_\nu A_0)(\partial_0A_\eta - \partial_\eta A_0)\Big) \\
&= -\frac{\sqrt{|g|}}{2}e^{\frac{-\chi}{4}}\Big(g^{\nu\eta}\partial_\nu A_0\partial_\eta A_0\Big) = -\frac{1}{2}\sqrt{|g|}e^{\frac{-\chi}{4}}g^{\nu\eta}\partial_\nu\ell\partial_\eta\ell
\end{aligned} \tag{4.13}$$

To go to the Einstein frame apply the conformal mapping with α from (4.8)

$$\mathcal{L}^{EF}_{MW} = -\frac{1}{2}\sqrt{|G|}G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell e^{\frac{-\chi}{2}} \tag{4.14}$$

which is independent of the dimension used. The time-compactified Einstein Reissner-Nordstrøm Lagrangian (CRN) in D dimensions becomes thus

$$\boxed{\mathcal{L}^{EF}_{CRN} = -\sqrt{|G|}\mathcal{R}(G) - \sqrt{|G|}\Big[\frac{D-1}{16(2-D)}\Big]\partial_\mu\chi\partial_\nu\chi G^{\mu\nu} - \frac{1}{2}\sqrt{|G|}\Big(G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell\Big)e^{\frac{-\chi}{2}}} \tag{4.15}$$

The equations of motion that follow from this Lagrangian are for respectively $G_{\mu\nu}$, ℓ and χ

$$\begin{aligned}
&-\mathcal{R}_{\mu\nu} - \Big[\frac{D-1}{16(2-D)}\Big]\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}\Big(\partial_\mu\ell\partial_\nu\ell\Big)e^{\frac{-\chi}{2}} = 0 \\
&\partial_\mu\Big(\sqrt{|G|}G^{\mu\nu}e^{\frac{-\chi}{2}}\partial_\nu\ell\Big) = 0 \\
&\partial_\mu\Big(\sqrt{|G|}\Big[\frac{D-1}{16(2-D)}\Big]G^{\mu\nu}\partial_\nu\chi\Big) + \frac{\sqrt{|G|}}{8}e^{\frac{-\chi}{2}}G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell = 0
\end{aligned} \tag{4.16}$$

The field ℓ is a pseudoscalar, to understand this go for example to four dimensions and compare in appendix A.1 rule one and three. A pseudoscalar transforms the same as time does under a Wick rotation and since ℓ is the *time component* of a four vector $\hat{A}_{\hat{\mu}}$, rule number two shows that it is a pseudoscalar.

In [20] the $(D+1)$-dimensional Schwarzschild and Reissner-Nordstrøm metric are given

$$\mathrm{d}s^2_{SCH} = \Big(1 - \frac{\alpha}{r^{D-2}}\Big)\mathrm{d}t^2 - \Big(1 - \frac{\alpha}{r^{D-2}}\Big)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2_{D-1} \tag{4.17}$$

$$\mathrm{d}s^2_{RN} = \Big(1 - \frac{\alpha}{r^{D-2}} + \frac{\beta^2}{r^{2D-4}}\Big)\mathrm{d}t^2 - \Big(1 - \frac{\alpha}{r^{D-2}} + \frac{\beta^2}{r^{2D-4}}\Big)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2_{D-1} \tag{4.18}$$

where α and β are the generalized mass and charge and $\mathrm{d}\Omega^2_{D-1}$ represents the angular parts of a S^{D-1} sphere

$$\mathrm{d}\Omega^2_{D-1} = \mathrm{d}\theta_1^2 + \sin^2\theta_2\mathrm{d}\theta_2^2 + ... + \prod_{m=1}^{D-2}\sin^2\theta_m\mathrm{d}\theta^2_{D-1} \tag{4.19}$$

Note that the Schwarzschild metric has a coordinate singularity at $r = \alpha^{\frac{1}{D-2}}$ and the general Reissner-Nordstrøm metric at $\beta^2 r^4 - \alpha r^{D+2} + r^{2D} = 0$.

Schwarzschild metric in $(D+1)$ dimensions

To obtain the equations of motion for the dimensionally reduced Schwarzschild metric in D dimensions put $\ell = 0$ in (4.16). The Schwarzschild line element in $(D+1)$ dimensions gives for $\hat{g}_{\hat{\mu}\hat{\nu}}$

$$\begin{aligned}\hat{g}_{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} \big(1 - \frac{\alpha}{r^{D-2}}\big) & 0 & 0 & 0 & 0 \\ 0 & -\big(1 - \frac{\alpha}{r^{D-2}}\big)^{-1} & 0 & 0 & 0 \\ 0 & & & & \\ 0 & 0 & -r^2 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & -r^2\prod_{m=1}^{D-2}\sin^2\theta_m \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix} = \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{\frac{\chi}{(4-2D)}}G^{EF}_{\mu\nu} \end{pmatrix}\end{aligned} \tag{4.20}$$

Comparing both sides of this matrix equation leads to

$$e^{\frac{\chi}{2}} = \Big(1 - \frac{\alpha}{r^{D-2}}\Big) \Rightarrow \boxed{\chi = 2\log\Big[1 - \frac{\alpha}{r^{D-2}}\Big]} \tag{4.21}$$

$$\boxed{\begin{aligned}&G_{\mu\nu} = \\ &\begin{pmatrix} -\big(1 - \frac{\alpha}{r^{D-2}}\big)^{-\frac{D-3}{D-2}} & 0 & 0 & 0 \\ 0 & \big(r^{2D-4} - \alpha r^{D-2}\big)^{\frac{1}{(D-2)}} & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \big(r^{2D-4} - \alpha r^{D-2}\big)^{\frac{1}{D-2}}\prod_{m=1}^{D-2}\sin^2\theta_m \end{pmatrix}\end{aligned}} \tag{4.22}$$

A problem with this reduction is that for $\alpha > 0$ (which corresponds to masses larger then zero) G_{rr} becomes imaginary, this happens for $r^{D-2} < \alpha$. Note that for $D = 3$ this problem does not arise (3.33). Demanding a conformal flat metric leads to

$$\boxed{\begin{aligned}&\rho(r) = (r^{(D-2)/2} + \sqrt{r^{D-2} - \alpha})^{2/d-2}C_1 \\ &f(\rho)^2 = \Big(\frac{(\rho^{2(D-2)} - \alpha^2 C_1^{2(D-2)})^{\frac{1}{D-2}}}{2^{\frac{2}{D-2}}\rho^2 C_1^2}\Big)^2\end{aligned}} \tag{4.23}$$

The symmetry of the conformal factor is

$$\rho^{D-2} \longleftrightarrow \underbrace{-}\frac{\alpha^2 C_1^{2(D-2)}}{\rho^{D-2}} \tag{4.24}$$

Note that *only* for $D = 3$ does (4.23) imply a (possible) wormhole symmetry, since then $1/(D-2) = 1$ and a plus sign can be chosen in (4.24) instead of the minus sign. But as was noted in the last section of the previous chapter, this gives an imaginary embedding functions, see (3.43), which was therefore discarded as a genuine wormhole. The conclusion is thus that for the $(D+1)$-dimensional Schwarzschild metric, the Kaluza-Klein reduced system does not have a wormhole interpretation in any dimension in the Einstein frame and that the metric becomes imaginary at a certain point. This last issue is due to the fact that the Schwarzschild metric is valid only for r larger than the Schwarzschild radius, which is in $(D+1)$ dimensions determined by $r^{D-2} = \alpha$.

Also not that G_{rr} is not just -1, this only happens for $D = 3$. As a consistency check compare the trace of the Ricci scalar via two different paths, similar as to the three dimensional example (3.35). The first approach leads to the $D \times D$ Ricci tensor

$$\mathcal{R}_{\mu\nu} = \begin{pmatrix} (D-1)\frac{\alpha^2(-2+D)}{4(-\alpha r + r^{D-1})^2} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.25}$$

and thus the trace of the Ricci scalar is $(D-1)\frac{\alpha^2(-2+D)r^2}{4(-\alpha r^2 + r^D)^2}$ which agrees with the first equation of motion of (4.16)

$$\begin{aligned} \mathcal{R}_{\mu\nu} = & \Big[\frac{D-1}{16(D-2)}\Big]\partial_\mu\chi\partial_\nu\chi \Rightarrow \mathrm{Tr}[\mathcal{R}_{\mu\nu}] = \Big[\frac{D-1}{16(D-2)}\Big]\partial_r\chi\partial_r\chi \\ =&(D-1)\frac{\alpha^2(D-2)r^2}{4(-\alpha r^2 + r^D)^2} \end{aligned} \tag{4.26}$$

Note that the Ricci scalar has a singularity at $r = \alpha^{\frac{1}{D-2}}$, which was merely a coordinate singularity in the uncompactified case, see below (4.19).

Reissner-Nordstrøm metric

The equations of motion for the D-dimensional time reduced Reissner-Nordstrøm metric are given by (4.16). The Reissner-Nordstrøm line element (4.18) in $(D+1)$ dimensions gives with the same metric ansatz as (4.20)

$$\boxed{\begin{aligned} & e^{\frac{\chi}{2}} = \Big(1 - \frac{\alpha}{r^{D-2}} + \frac{\beta^2}{r^{2D-4}}\Big) \Rightarrow \chi = 2\log\Big[1 - \frac{\alpha}{r^{D-2}} + \frac{\beta^2}{r^{2D-4}}\Big] \\ & G_{\mu\nu} = \\ & \begin{pmatrix} -\Big(1 - \frac{\alpha}{r^{D-2}} + \frac{D^2}{r^{2D-4}}\Big)^{-\frac{D-3}{D-2}} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & -\Big(r^{2(D-2)} - \alpha r^{D-2} + \beta^2\Big)^{\frac{1}{(D-2)}} \prod_{m=1}^{D-2}\sin^2\theta_m \end{pmatrix} \end{aligned}} \tag{4.27}$$

and the corresponding Ricci tensor is

$$\mathcal{R}_{\mu\nu} = \begin{pmatrix} \frac{(D-1)(D-2)(\alpha^2-4\beta^2)r^{2+2D}}{4(\beta^2 r^4 + r^{2D} - \alpha r^{2+D})^2} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.28}$$

Note that it has a genuine singularity at $\beta^2 r^4 - \alpha r^{D+2} + r^{2D} = 0$, which was the condition for merely a coordinate singularity in the uncompactified case, see below (4.19).

To find the radial solution for the field ℓ integrate out the second equation of (4.16) with the help of the solution for χ (4.27)

$$\boxed{\ell(r) = \frac{\beta\sqrt{2(D-1)}}{r^{D-2}\sqrt{D-2}} + \mu_2} \tag{4.29}$$

This gives for $D = 3$, $\beta = \epsilon$ and $\mu_2 = 0$ the result

$$\ell(r) = \frac{2\epsilon}{r} \tag{4.30}$$

which is nothing but the potential of a point charge and see also (A.23). Demanding a conformal metric to investigate the wormhole symmetry leads to

$$f(\rho)^2 = \frac{(\rho^{2(d-2)} - (\alpha^2 - 4\beta^2)C_1^{2(d-2)})^{2/(d-2)}}{2^{4/(d-2)}\rho^4 C_1^2} \tag{4.31}$$

This agrees with the three dimensional Schwarzschild case (3.39) if $\beta = 0$ and $\alpha = 2m$. The symmetry of the conformal factor is

$$\rho^{D-2} \longleftrightarrow -\frac{(\alpha^2 - 4\beta^2)C_1^{2D-4}}{\rho^{D-2}} \tag{4.32}$$

Only in the special case $D = 3$ can one also take the plus sign due to $2/(D-2)$ in (4.31). It is a genuine wormhole symmetry for the case $\alpha^2 - 4\beta^2 < 0$ which means a naked singularity. This condition is called naked singularity, because the uncompactified metric does not have a horizon. So one can travel to the singularity at $r = 0$ and still be able to go back, according to the so called Cosmic censorship hypothesis these objects do not appear in nature. To have a wormhole for the other situation go to the *dual frame*, see appendix A.5. From the general case (A.51) one gets with $b = -1/2$ and $\phi = \chi$

$$G_{\mu\nu}^{DF} = e^{\frac{-\chi}{2D-4}} g_{\mu\nu}^{EF} \tag{4.33}$$

Comparing this with (4.20) shows that the dual frame mapping *cancels* the initial Kaluza-Klein reduction, which explains why the wormhole obtained in the dual frame is the same one as in the **original** metric of the $(D+1)$-dimensional uncompactified system, this compactified system is simply the radial part of the full system. Further note that in the limit $\beta \to 0$ this still works and for the wormhole of the D-dimensional Schwarzschild metric one also find that it is in the dual frame and is equal to the wormhole of the $(D+1)$-dimensional Schwarzschild metric. In the special case $D = 3$ this means that in the dual frame the wormhole is at the same position as the uncompactified case, which was discussed in section 2.3 and answers the question of the last section of chapter three about where the wormhole has gone to.

Extremal Reissner-Nordstrøm metric

All that remains to investigate is the extremal Reissner-Nordstrøm metric. Based on the knowledge of the extremal D-instanton, it is expected that a wormhole appears in the string frame.

To obtain the extremal case, rewrite the Reissner-Nordstrøm metric in terms of $\tilde{\alpha} = \frac{\alpha}{2}$ and set $\tilde{\alpha} = \beta$. The infinitesimal line element becomes

$$\mathrm{d}s^2_{RN} = \Big(1 - \frac{\tilde{\alpha}}{r^{D-2}}\Big)^2 \mathrm{d}t^2 - \Big(1 - \frac{\tilde{\alpha}}{r^{D-2}}\Big)^{-2} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2_{D-1} \tag{4.34}$$

and the solutions

$$\boxed{\begin{array}{l} e^{\frac{\chi}{2}} = \Big(1 - \frac{\tilde{\alpha}}{r^{D-2}}\Big)^2 \Rightarrow \chi = 4 \log\Big[1 - \frac{\tilde{\alpha}}{r^{D-2}}\Big] \\ G_{\mu\nu} = \begin{pmatrix} -\Big(1 - \frac{\tilde{\alpha}}{r^{D-2}}\Big)^{-\frac{2(D-3)}{D-2}} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & -\Big(r^{D-2} - \tilde{\alpha}\Big)^{\frac{2}{D-2}} \prod_{m=1}^{D-2} \sin^2\theta_m \end{pmatrix} \end{array}} \tag{4.35}$$

The Ricci tensor becomes identical to zero, which of course confirms the fact that this is the *extremal* solution, see also (4.37). The solution for the field $\ell(r)$ is

$$\ell(r) = \frac{\tilde{\alpha}\sqrt{2(D-1)}}{r^{D-2}\sqrt{D-2}} \tag{4.36}$$

This system in the Einstein frame is nothing else then flat Euclidean space, to see this note that the corresponding conformal factor $f(r)$ of the isotropic version is

$$f(r) = \frac{1}{C_1} \tag{4.37}$$

The reason why this is flat space, is the same as discussed in chapter two, see below (2.32). A wormhole may be present in the string frame and to get there it follows from (4.15) that the mapping should be

$$G^{EF}_{\mu\nu} = e^{\frac{\chi}{D-2}} g^{SF}_{\mu\nu} \tag{4.38}$$

which gives for the string frame metric

$$g^{SF}_{\mu\nu} = \begin{pmatrix} \Big(1 - \frac{\tilde{\alpha}}{r^{D-2}}\Big)^{-\frac{2D-2}{D-2}} & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & -r^2\Big(1 - \frac{\tilde{\alpha}}{r^{D-2}}\Big)^{\frac{-2}{D-2}} \prod_{m=1}^{D-2} \sin^2\theta_m \end{pmatrix} \tag{4.39}$$

Demanding isotropic coordinates gives for the conformal factor

$$f(\rho) = \Big[\frac{(\rho^{D-2} + \tilde{\alpha}C_1^{D-2})^2}{\rho^{2(D-2)}C_1^{D-2}}\Big]^{\frac{1}{D-2}} \tag{4.40}$$

and the symmetry that follows is

$$\rho^{\frac{D-2}{2}} \longleftrightarrow \frac{\tilde{\alpha}C_1^{D-2}}{\rho^{\frac{D-2}{2}}} \quad \text{or} \quad r = (2m)^{\frac{1}{D-2}} \tag{4.41}$$

For the three dimensional case this extremal wormhole can be embedded in flat Euclidean space (taking one angle constant), with the resulting figure 4.1. In

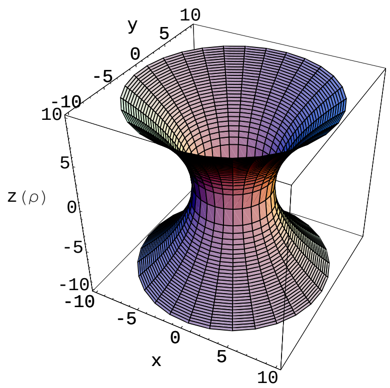

Figure 4.1: Wormhole of the compactified Extremal Reissner-Nordstrøm black hole

Figure 4.2: "Semi-wormhole"

section 2.4 the uncompactified extremal Reissner-Nordstrøm metric did not have a good wormhole interpretation, only a wormhole connecting the interior part of two black holes, see figure 2.6. In the compactified case here, there is a genuine wormhole in the string frame, so the compactification has generated an extra wormhole. This was also the case for the naked singularity, see (4.32).

The analysis of the previous cases lead to the following **conclusions**

- The wormholes in the Einstein frame of the uncompactified metrics are found back in the dual frame after the Kaluza-Klein reduction. This can be understood if one realizes that the mapping to the dual frame cancels the mapping needed to get in the Einstein frame after the Kaluza-Klein reduction, see for example (4.33).
- Extra wormholes are generated by the Kaluza-Klein reduction of the original metrics in the Einstein and string frame. Theses are for example the extremal wormholes in the string frame (4.41) or the naked singularity wormholes (4.32) in the Einstein frame.
- The coordinate singularities in the uncompactified case, become genuine singularities of the Ricci scalar in the compactified case, see for example below (4.28).

The analysis of the general Reissner-Nordstrøm black hole has given rather interesting (and surprising) results. For the well known case of three dimensions the mass β is m and the charge α is ϵ, see also (2.25). For the extremal case one finds a wormhole in the string frame, for the (un-physical) $m^2 - \epsilon^2 < 0$ case in the Einstein frame and for $m^2 - \epsilon^2 > 0$ in the dual frame (and thus also for the Schwarzschild case). The special behavior of the extremal case $m = \epsilon$ can be better understood if one looks at the proper radial distance [14]. Introduce $r_\pm = m \pm \sqrt{m^2 - \epsilon^2}$ and rewrite the metric of the Reissner-Nordstrøm black hole as

$$\mathrm{d}s^2 = -\frac{\Delta}{r^2}\mathrm{d}t^2 + \frac{r^2}{\Delta}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{4.42}$$

where

$$\Delta = (r - r_+)(r - r_-) \tag{4.43}$$

The distance to the horizon at $r = r_+$ along a curve of constant $(t,\ \theta,\ \phi)$ from $r = R$ is

$$s = \int_{r_+}^{R} \frac{\mathrm{d}r}{\sqrt{(1-\frac{r_+}{r})(1-\frac{r_-}{r})}} \tag{4.44}$$

which goes to ∞ when $(r_+ - r_-) \to 0$, i.e. as $m - \epsilon \to 0$. So the wormhole has an *infinitely long throat* when the charge becomes equal to the mass in the Einstein frame, see figure 4.2.

4.2 Dilatonic black hole

In the previous section of this chapter the dilaton χ came from the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ and the axion ℓ came from the vector $\hat{A}_{\hat{a}}$. In ten dimensional type IIB string theory there are however already from the start (i.e. before any compactification) both a dilaton and an axion present. A more general and thus interesting generalization is possible [18]

$$\mathcal{L} = \mathcal{L}_{\hat{\mathcal{R}}} + \tilde{\mathcal{L}}_{\hat{F}_{\hat{\mu}\hat{\nu}}} + \mathcal{L}_{\hat{\phi}} \tag{4.45}$$

where $\mathcal{L}_{\hat{\mathcal{R}}}$ is the same as in the previous sections and $\tilde{\mathcal{L}}_{\hat{F}_{\hat{\mu}\hat{\nu}}}$ has now an extra coupling with the initial dilaton field $\hat{\phi}$

$$\begin{aligned} \tilde{\mathcal{L}}_{\hat{F}_{\hat{\mu}\hat{\nu}}} &= -\frac{1}{2} e^{a\hat{\phi}} * \hat{F}_{\hat{\mu}\hat{\nu}} \wedge \hat{F}_{\hat{\mu}\hat{\nu}} \\ \mathcal{L}_{\hat{\phi}} &= -\frac{1}{2} * \mathrm{d}\hat{\phi} \wedge \mathrm{d}\hat{\phi} \end{aligned} \tag{4.46}$$

so that he full Lagrangian of this system becomes

$$\hat{\mathcal{L}} = \sqrt{|\hat{g}|}\hat{\mathcal{R}} - \frac{1}{4}\sqrt{|\hat{g}|} e^{a\hat{\phi}} \hat{g}^{\hat{\mu}\hat{\rho}} \hat{g}^{\hat{\nu}\hat{\eta}} \hat{F}_{\hat{\mu}\hat{\nu}} \hat{F}_{\hat{\rho}\hat{\eta}} - \frac{1}{2}\sqrt{|\hat{g}|}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi}\hat{g}^{\hat{\mu}\hat{\nu}} \tag{4.47}$$

where the metric is diag $= (-1, 1, \ldots, 1)$. To obtain the Lagrangian with the same metric convention as in the previous sections observe that both the Ricci scalar $\hat{\mathcal{R}}$ and the kinetic part $\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi}\hat{g}^{\hat{\mu}\hat{\nu}}$ are sensitive to such a change

$$\boxed{\hat{\mathcal{L}}_{DB} = -\sqrt{|\hat{g}|}\hat{\mathcal{R}} - \frac{1}{4}\sqrt{|\hat{g}|} e^{a\hat{\phi}} \hat{g}^{\hat{\mu}\hat{\rho}} \hat{g}^{\hat{\nu}\hat{\eta}} \hat{F}_{\hat{\mu}\hat{\nu}} \hat{F}_{\hat{\rho}\hat{\eta}} + \frac{1}{2}\sqrt{|\hat{g}|}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi}\hat{g}^{\hat{\mu}\hat{\nu}}} \tag{4.48}$$

where DB stands for dilatonic black hole, it is a charged black hole and an extra scalar field ϕ, the dilaton, is taken into account. Note however that in this case there is only a Lagrangian present and not an explicit metric, see for example (4.20). Will the extra dilaton field $\hat{\phi}$ destroy the metric ansatz, i.e. does still hold

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} u & n & k & n \\ o & w & n & - \\ u & n & k & n \\ o & w & n & - \end{pmatrix} = \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix} \ ? \tag{4.49}$$

Taking in the ansatz $\hat{g}_{0\mu} \neq 0$ means that an extra vector field will appear, see (3.10). So although the exact metric form of the full $\hat{g}_{\hat{\mu}\hat{\nu}}$ is unknown, it seems likely that the same ansatz will still work. The extra dilaton field $\hat{\phi}$ will only show up when the Lagrangian becomes time compactified, but it is not to be expected to influence the metric ansatz as given by (4.49). Also note that in the previous section, knowing the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ from the start, meant that only the field ℓ was unknown. In the D-instanton case however it was the other way around, by assuming a flat Minkowskian space in the Einstein frame the D-instanton was solved.

The Einstein-Hilbert and Maxwell terms with extra dilaton coupling $e^{\alpha\hat{\phi}}$ in $(D+1)$ dimensions are

$$\hat{\mathcal{L}}_{RN} = -\sqrt{|\hat{g}|}\hat{\mathcal{R}} - \frac{1}{4}\hat{g}^{\hat{\mu}\hat{\rho}}\hat{g}^{\hat{\nu}\hat{\eta}}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}_{\hat{\rho}\hat{\eta}}e^{\alpha\hat{\phi}} \tag{4.50}$$

and reduces to

$$\begin{aligned}\mathcal{L}_{RN} =& -\sqrt{|G|}\mathcal{R}(G) - \sqrt{|G|}\Big[\frac{D-1}{16(2-D)}\Big]\partial_\mu\chi\partial_\nu\chi G^{\mu\nu} \\ &- \frac{\sqrt{|G|}}{2}\Big(G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell\Big)e^{\frac{-\chi}{2}}e^{\alpha\phi}\end{aligned} \tag{4.51}$$

in the Einstein frame. The extra dilaton field in $(D+1)$ dimensions

$$\hat{\mathcal{L}}_{\hat{\phi}} = \frac{1}{2}\sqrt{|\hat{g}|}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi}\hat{g}^{\mu\nu} \tag{4.52}$$

becomes under the time compactification

$$\begin{aligned}&\mathcal{L}_\phi = \frac{1}{2}\sqrt{|g|}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}e^{\frac{\chi}{4}} \Rightarrow \mathcal{L}_\phi^{EF} = \frac{1}{2}\sqrt{|G|}e^{\frac{D\chi}{2(4-2D)}}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu}e^{\frac{-\chi}{4-2D}}e^{\frac{\chi}{4}} \\ &= \frac{1}{2}\sqrt{|G|}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu}\end{aligned} \tag{4.53}$$

Thus the D-dimensional Lagrangian for the compactified dilatonic black hole (CDB) becomes in the Einstein frame

$$\begin{aligned}\mathcal{L}_{CDB}^{EF} =& -\sqrt{|G|}\mathcal{R}(G) + \sqrt{|G|}\Big[\frac{D-1}{16(D-2)}\Big]\partial_\mu\chi\partial_\nu\chi G^{\mu\nu} \\ &- \frac{\sqrt{|G|}}{2}G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell e^{\frac{-\chi}{2}}e^{\alpha\phi} + \frac{1}{2}\sqrt{|G|}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu}\end{aligned} \tag{4.54}$$

The two fields χ and ϕ are both scalars, whereas ℓ is a pseudoscalar, so a more democratic treatment must be possible for the scalars. To help achieve this, compare to the work of Cremmer *et al* [21]

$$\hat{\mathcal{L}} = \sqrt{|\hat{g}|}\hat{\mathcal{R}} - \frac{1}{2}\sqrt{|\hat{g}|}(\partial\hat{\phi})^2 - \frac{1}{2n!}\sqrt{|\hat{g}|}e^{\hat{a}\hat{\phi}}\hat{F}_n^2 \tag{4.55}$$

Here $\hat{F_n}$ is an n-index antisymmetric tensor in $(D+1)$ dimensions. Start with the Kaluza-Klein reduction over a timelike component t, i.e. $x^{\hat{M}} = (x^M, t)$. The metric ansatz of Cremmer *et al* is

$$\begin{aligned}&\mathrm{d}\hat{s}^2 = e^{-2\alpha\varphi}\mathrm{d}s^2 - e^{2(D-2)\alpha\varphi}(\mathrm{d}t + \mathcal{A}_{(1)}^2)^2 \\ &\hat{A}_{(n-1)}(x,t) = A_{(n-1)}(x) + A_{(n-2)}(x)\wedge\mathrm{d}t \\ &\hat{\phi}(x,t) = \phi(x)\end{aligned} \tag{4.56}$$

where $\alpha = (2(D-1)(D-2))^{-1/2}$. The time-compactified metric becomes then

$$\begin{aligned} \mathcal{L} = & \sqrt{|g}\mathcal{R} - \frac{1}{2}\sqrt{|g|}(\partial\phi)^2 - \frac{1}{2}\sqrt{|g|}(\partial\varphi)^2 + \frac{1}{4}\sqrt{|g|}e^{2(D-1)\alpha\varphi}\mathcal{F}^2_{(2)} \\ - & \frac{\sqrt{|g|}}{2n!}e^{2(n-1)\alpha\phi+\hat{a}\phi}F^2_{(n)} + \frac{\sqrt{|g|}}{2(n-1)!}e^{-2(D-n)\alpha\varphi+\hat{a}\phi}F^2_{(n-1)} \end{aligned} \tag{4.57}$$

where

$$\begin{aligned} F_{(n)} &= \mathrm{d}A_{(n-1)} - \mathrm{d}A_{(n-2)} \wedge \mathcal{A}_{(1)} \\ F_{(n-1)} &= \mathrm{d}A_{(n-2)} \end{aligned} \tag{4.58}$$

To make connection to (4.54) observe first the following [7]. Reducing the $(D+1)$-dimensional Einstein-Hilbert Lagrangian with the Kaluza-Klein vector *not* turned off leads to

$$\mathcal{L} = \sqrt{|\hat{g}|}\hat{\mathcal{R}} = \sqrt{|G|}\Big(\mathcal{R} - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{4}e^{-2(D-1)\beta\varphi}\mathcal{F}^2\Big) \tag{4.59}$$

where β is a constant here. Thus the fourth term on the right hand side of (4.57) is the vector part of the metric ansatz, which was set equal to zero in (4.54), this term must thus be discarded. To make connection to the D-instanton, only the scalars must be kept, this means that $n = 2$. The field $F_{(n)} = F_{(2)}$ must be turned off, since else there has to be a $A_\mu \neq 0$. Further $F^2_{(n-1)} = F^2_{(1)} = \partial_\mu\ell\partial_\nu\ell G^{\mu\nu}$ and the non-zero terms of the Lagrangian become

$$\begin{aligned} \mathcal{L} = & \sqrt{|G|}\mathcal{R} - \frac{1}{2}\sqrt{|G|}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu} - \frac{1}{2}\sqrt{|G|}\partial_\mu\varphi\partial_\nu\varphi G^{\mu\nu} \\ + & \frac{1}{2}e^{-2(D-2)\alpha\varphi+\hat{a}\phi}\partial_\mu\chi\partial_\nu\chi G^{\mu\nu} \end{aligned} \tag{4.60}$$

Note that the last term on the right hand side depends on D. To explain this look at the metric ansatz Cremmer *et al* use (4.56), there is an explicit D in the exponent. They then rewrite the non-zero parts of this Lagrangian to

$$\mathcal{L} = \sqrt{|G|}\mathcal{R} - \frac{1}{2}\sqrt{|G|}(\partial\phi_1)^2 - \frac{1}{2}\sqrt{|G|}(\partial\phi_2)^2 + \sqrt{|G|}e^{2\phi_1}(\partial\chi)^2 \tag{4.61}$$

where

$$\phi_1 = \frac{1}{2}\hat{a}\phi - (D-2)\alpha\varphi, \quad \phi_2 = (D-2)\alpha\phi + \frac{1}{2}\hat{a}\varphi \tag{4.62}$$

How do they get this last equality? First of all there seems to be a $\frac{1}{2}$ missing in front of the last term. Secondly this equality is good if and only if $a^2 \equiv (\frac{\hat{a}^2}{4} + (D-2)^2\alpha^2) = 1$. When correcting for this, the good normalized Lagrangian becomes

$$\begin{aligned} \mathcal{L} = & \sqrt{|G|}\mathcal{R} - \frac{1}{\frac{\hat{a}^2}{4} + (D-2)^2\alpha^2}\Big[\frac{1}{2}\sqrt{|G|}(\partial\phi_2)^2 + \frac{1}{2}\sqrt{|G|}(\partial\phi_1)^2\Big] \\ + & \frac{1}{2}\sqrt{|G|}e^{2\phi_1}(\partial\chi)^2 \end{aligned} \tag{4.63}$$

where both modifications are taken into account. The solutions for the fields are in the case $a^2 = 1$

$$\boxed{\begin{aligned} &\mathrm{d}s^2 = e^{\frac{2f}{D-2}}(e^{-2f}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2) \\ &\phi_1 = -f + \log[H], \quad \phi_2 = f\sqrt{\frac{D}{D-2}} \\ &\chi = H^{-1}\coth(\mu) \\ &H = 1 + \frac{k\sinh^2\mu}{r^{\tilde{d}}}, \quad e^{2f} = 1 - \frac{k}{r^{\tilde{d}}} \end{aligned}} \tag{4.64}$$

where $\tilde{d} = D - 2$ and k and μ are constants related to the mass and charge. Note that there are three scalar fields present, to link this to the D-instanton case one of the fields must be set equal to zero. For consistency ϕ_2 must be turned off, since in the Lagrangian (4.61) this is the only field not coupled to any of the other fields. The solution for ϕ_2 means that setting this field equal to zero, also $f = 0$, i.e. it is not possible to obtain solutions with non-flat space, the so called non-extremal D-instantons. This is not interesting, because it does not allow for a generalization of the extremal D-instanton of chapter one.

But nonetheless, this same procedure can also be applied to (4.54) and this may give a more fruitful solution. To write this in terms of fields ϕ_1 and ϕ_2 first introduce $\chi = \sqrt{(8(D-2))/(D-1)}\tilde{\chi}$ which turns the Lagrangian into

$$\begin{aligned} \mathcal{L}_D = & -\sqrt{|G|}\mathcal{R}(G) + \sqrt{|G|}\frac{1}{2}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi}G^{\mu\nu} \\ & + \frac{1}{2}\sqrt{|G|}\partial_\mu\phi\partial_\nu\phi G^{\mu\nu} - \frac{1}{2}\sqrt{|G|}G^{\mu\nu}\partial_\mu\ell\partial_\nu^{\frac{-\sqrt{\frac{8(D-2)}{D-1}}\tilde{\chi}}{2}} e^{\alpha\phi} \end{aligned} \tag{4.65}$$

Introduce now $\gamma = \frac{\sqrt{\frac{8(D-2)}{D-1}}}{4}$, $\alpha = 2\tilde{\alpha}$ and

$$\begin{aligned} \phi_1 &= -\gamma\tilde{\chi} + \tilde{\alpha}\phi \\ \phi_2 &= \tilde{\alpha}\tilde{\chi} + \gamma\phi \end{aligned} \tag{4.66}$$

which yields for the Lagrangian

$$\begin{aligned} \mathcal{L}_D = & -\sqrt{|G|}\mathcal{R}(G) + \frac{1}{\tilde{\alpha}^2 + \gamma^2}\Big(\frac{1}{2}\sqrt{|G|}\partial_\mu\phi_1\partial_\nu\phi_1 G^{\mu\nu} \\ & + \frac{1}{2}\sqrt{|G|}\partial_\mu\phi_2\partial_\nu\phi_2 G^{\mu\nu}\Big) - \frac{1}{2}\sqrt{|G|}G^{\mu\nu}\partial_\mu\ell\partial_\nu^{2\phi_1} \end{aligned} \tag{4.67}$$

Further introduce

$$\tilde{b}^2 = \tilde{\alpha}^2 + \gamma^2, \quad \tilde{b}\varphi_i = \phi_i, \quad b = 2\tilde{b} \tag{4.68}$$

which turns the Lagrangian into

$$\boxed{\begin{aligned} \mathcal{L}_D = & -\sqrt{|G|}\mathcal{R}(G) + \frac{1}{2}\sqrt{|G|}\partial_\mu\varphi_1\partial_\nu\varphi_1 G^{\mu\nu} \\ & + \frac{1}{2}\sqrt{|G|}\partial_\mu\varphi_2\partial_\nu\varphi_2 G^{\mu\nu} - \frac{1}{2}\sqrt{|G|}G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell e^{b\varphi_1} \end{aligned}} \tag{4.69}$$

and the corresponding equations of motion

$$\boxed{\begin{aligned}&-\mathcal{R}_{\mu\nu}+\frac{1}{2}\partial_\mu\varphi_1\partial_\nu\varphi_1+\frac{1}{2}\partial_\mu\varphi_2\partial_\nu\varphi_2-\frac{1}{2}e^{b\varphi_1}\partial_\mu\ell\partial_\nu\ell=0\\&-\frac{b}{2}\sqrt{|G|}e^{b\varphi_1}G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell-\partial_\mu\Big(\sqrt{|G|}G^{\mu\nu}\partial_\nu\varphi_1\Big)=0\\&\partial_\mu\Big(\sqrt{|G|}G^{\mu\nu}\partial_\nu\varphi_2\Big)=0\\&\partial_\mu\Big(\sqrt{|G|}e^{b\varphi_1}G^{\mu\nu}\partial_\nu\ell\Big)=0\end{aligned}} \tag{4.70}$$

Note that the difference with Cremmer *et al* is the general coupling b and that there is one scalar field too many here too, but that it is consistent to set $\varphi_2 = 0$ since it does not interact with any of the other fields as is evident from the Lagrangian and equations of motion above.

4.3 Non-extremal D-instanton

To help solving these equations of motion, a metric ansatz is needed. The extremal D-instanton has a $SO(10)$ symmetry (1.85), it makes therefore sense to consider in this case a metric with $SO(D)$ symmetry. Furthermore having flat space at infinitely is customary, i.e. demand thus a conformally flat metric

$$\mathrm{d}s^2=-e^{2B(r)}\Big(\mathrm{d}r^2+r^2\mathrm{d}\Omega^2_{D-1}\Big) \tag{4.71}$$

To have a $SO(D)$ symmetry, only radial field solutions are acceptable, the angular components of the Ricci scalar must vanish, see the Einstein equation (4.70). This leads to a differential equation for $B(r)$, with as a solution

$$B(r)=C_2+\log\Big[\Big(1+\frac{\kappa^2}{r^{2(D-2)}}\Big)^{\frac{1}{D-2}}\Big] \tag{4.72}$$

where κ^2 and C_2 are constants of integration. Note that C_2 is related to the following symmetry of the equations of motion

$$g_{\mu\nu}\to e^{\lambda}g_{\mu\nu} \tag{4.73}$$

where λ is a constant, it thus also determines the behavior at spatial infinity, only for $C_2 = 0$ does the metric approaches pure flat space. This symmetry is called the Weyl re-scaling of the metric, it is an extra symmetry besides the $SL(2,\mathbb{R})$-symmetry[1]. The line element becomes with this choice for $B(r)$

$$\boxed{\mathrm{d}s^2=-e^{2C_2}\Big(1+\frac{\kappa^2}{r^{2(D-2)}}\Big)^{\frac{2}{D-2}}\Big(\mathrm{d}r^2+r^2\mathrm{d}\Omega^2_{D-1}\Big)} \tag{4.74}$$

The difference with the case of Cremmer *et al* is that they do not take a manifestly $SO(D)$ symmetric line element, see (4.64). The metric above has the following symmetry (for the case $\kappa^2 > 0$ *only*)

$$r^{D-2}\longleftrightarrow\frac{\kappa^2}{r^{D-2}} \tag{4.75}$$

[1] Of the equations of motion only, not the action.

at the position

$$r = \kappa^{\frac{1}{D-2}} \tag{4.76}$$

To see that it indeed connects two asymptotically flat spaces first observe that for large r the infinitesimal line element becomes

$$\mathrm{d}s^2 = -e^{2C_2}\Big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-1}\Big) \tag{4.77}$$

and for small values of r

$$\mathrm{d}s^2 = -e^{2C_2}\Big(\frac{\kappa^2}{r^{(D-2)}}\Big)^{\frac{2}{D-2}}\Big[\Big(\frac{\mathrm{d}r}{r}\Big)^2 + \mathrm{d}\Omega^2_{D-1}\Big] \equiv -\mathrm{d}\rho^2 - \rho^2\mathrm{d}\Omega^2_{D-1} \tag{4.78}$$

where

$$\rho = e^{C_2}\frac{\kappa^{\frac{2}{D-2}}}{r} \tag{4.79}$$

This clearly satisfies the working definition of a wormhole as defined in section 2.1, for it connects two asymptotically flat spaces. Now the following interesting observation follows. For the extremal D-instanton there is a wormhole interpretation possible only in the string frame, whereas for the non-extremal D-instanton (to be obtained below), a wormhole is present in the Einstein frame (4.75) (if $\kappa^2 > 0$). This should be compared to the instantons of for example the three dimensional compactified Reissner-Nordstrøm black holes. There was concluded that the extremal case $m^2 = \epsilon^2$ ($\approx \kappa^2 = 0$) has a wormhole in the string frame (4.41), whereas for the non-extremal case $m^2 - \epsilon^2 < 0$ ($\approx \kappa^2 > 0$) in the Einstein frame (4.32) and finally the wormhole of the physical situation $m^2 - \epsilon^2 > 0$ ($\approx \kappa^2 < 0$) was obtained in the dual frame (4.70). Also observe that taking the combined limit $\kappa \to 0$ and $C_2 \to 0$ implies that the metric (4.74) becomes that of flat space. The wormhole symmetry disappears however, see (4.75).

The Ricci tensor belonging to this conformally flat ansatz is for all values of κ^2

$$\mathcal{R}_{\mu\nu} = \begin{pmatrix} \frac{\alpha\kappa^2 r^{2(D-3)}}{(\kappa^2 + r^{2(D-2)})^2} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.80}$$

which is independent of C_2 and where

$$\alpha = 2[(2D-2)(D-2)] \tag{4.81}$$

For the case $\kappa^2 < 0$ it has a singularity at $r^{2(D-2)} = \kappa^2$. Since the field $g_{\mu\nu}$ is already know, it may seem that there is one equation of motion too many. But since the conformal factor of the metric depends only on $B(r)$ and not on any fields, all four turn out to be relevant.

First case: $\kappa^2 > 0$

To solve the coupled differential equations (4.70) begin with the third one, for it only contains the unknown function φ_2

$$\partial_r\Big(\sqrt{|G|}\partial_r\varphi_2 G^{rr}\Big) = 0 \tag{4.82}$$

The radial solution to this equation of motion is

$$\boxed{\varphi_2(r) = \frac{\arctan[\frac{r^{D-2}}{\kappa}]C_3}{(D-2)\kappa} + C_4} \tag{4.83}$$

As said at the end of the previous section, setting this field zero is consistent with the third equation of motion (4.70), since there are no source terms on the right hand side and note that this solution is valid for $0 < r < \infty$. Further it is interesting to point out that the argument of the arctan can also be inverted, i.e.

$$\varphi_2(r) = \frac{\arctan[\frac{\kappa}{r^{D-2}}]C_3}{(D-2)\kappa} + C_4 \tag{4.84}$$

where arccot[r] = arctan[$\frac{1}{r}$]. This too solves the equation of motion for $\varphi_2(r)$, because these two functions are related

$$\arctan(x) + \text{arccot}(x) = \frac{\pi}{2}, \quad x > 0 \tag{4.85}$$

For the non-extremal D-instantons this field will be set equal to zero, but for completeness the solution has been given.

To obtain the solution for the field φ_1 integrate out the last equation of motion

$$\partial_r \ell(r) = \frac{e^{-b\varphi_1}C_3}{g^{rr}\sqrt{|g|}} = \frac{e^{-b\varphi_1}r^{D-3}C_3}{\kappa^2 + r^{2(D-2)}} \tag{4.86}$$

and substitute this into the first equation of motion

$$-\frac{b}{2}\sqrt{|G|}e^{b\varphi_1}G^{\mu\nu}\Big(\frac{e^{-b\varphi_1}r^{D-3}C_3}{\kappa^2 + r^{2(D-2)}}\Big)^2 - \partial_r\Big(\sqrt{|G|}\partial_r\varphi_1 G^{rr}\Big) = 0 \tag{4.87}$$

Solving this differential equation gives

$$\boxed{e^{-\frac{b\varphi_1(r)_\pm}{2}} = b\sqrt{C_4}\csc\Big[b^2\sqrt{C_4}\Big(\frac{\pm\arctan[\frac{r^{D-2}}{\kappa}]C_3 + (D-2)\kappa C_5}{2(D-2)\kappa}\Big)\Big]} \tag{4.88}$$

where $\csc(x) = 1/\sin(x)$. This solution is valid for $0 < r < \infty$ only if

$$b^2\sqrt{C_4}\frac{[\frac{\pi}{2}C_3 + (D-2)\kappa C_5]}{2(D-2)\kappa} < \frac{\pi}{2} \tag{4.89}$$

since the $\arctan(x)\ \epsilon\ (-\frac{\pi}{2}, \frac{\pi}{2})$ and this condition makes sure that the $\sin(x) > 0$ so that $e^{-b\varphi_1(r)_\pm/2} > 0$, else the field φ_1 becomes imaginary. If this condition is not satisfied then define the point R as that point at which the argument of the csc takes on the value $\pi/2$. This happens then at the point where $\varphi_1(R)$ goes to zero, hence at weak coupling and thus there is no hope that string theory may come to the rescue.

To obtain the solution for the pseudoscalar use (4.86)

$$\boxed{\ell(r)_\pm = C_6 - 2\sqrt{C_4}\cot\Big[\frac{b^2\sqrt{C_4}\Big(\arctan[\frac{r^{D-2}}{\kappa}]C_3 \pm (D-2)\kappa C_5\Big)}{2(D-2)\kappa}\Big]} \tag{4.90}$$

The $\pm$ in both the solution above and in φ_1 can be absorbed in the constant of integrations. Finally the Einstein equation with $\varphi_2 \neq 0$ is only satisfied if

$$\boxed{2\alpha\kappa^2 + C_3^2(1 - b^2C_4) = 0}_{\varphi_2\neq 0} \tag{4.91}$$

and with $\varphi_2 = 0$ this condition becomes

$$\boxed{2\alpha\kappa^2 - b^2 C_3^2 C_4 = 0}_{\varphi_2=0} \tag{4.92}$$

So there are only four independent constant of integration: (κ, C_3, C_5, C_6). The constant C_2 is of course also a free parameter, but if one requires pure flat space at infinity it should be set equal to zero.

If one uses the opposite argument of the arctan the ℓ solution becomes

$$\ell(r)_\pm = C_6 + 2\sqrt{C_4}\cot\Big[\frac{b^2\sqrt{C_4}\Big(\mathrm{arccot}[\frac{r^{D-2}}{\kappa}]C_3 \pm (D-2)\kappa C_5\Big)}{2(D-2)\kappa}\Big] \tag{4.93}$$

Second case:$\kappa^2 < 0$

The second case is when $\kappa^2 < 0$, this is the same as replacing κ by $i\kappa$ in all solutions obtained above. The following relations are important

- $\arctan(ir) = i\mathrm{arctanh(r)}$
- $\mathrm{arccot}(ir) = -i\mathrm{arccot}(r)$

Looking back at the solutions of the previous section, this means that one can replace everywhere in these solutions the corresponding function on the left hand side of this list by the corresponding right hand side function. Reality of the fields is guaranteed, because everywhere there are compensating κ's, for example the three dimensional case

$$\boxed{\begin{aligned}
&\varphi_2(r) = \frac{\mathrm{arctanh}[\underbrace{\tfrac{r}{\kappa}}]C_3}{\underbrace{\kappa}} + C_4 \\
&e^{-\frac{b\varphi_{1\pm}}{2}} = b\sqrt{C_4}\mathrm{csc}\Big[b^2\sqrt{C_4}\Big(\frac{\pm\mathrm{arctanh}[\underbrace{\tfrac{r}{\kappa}}]C_3 + \underbrace{\kappa}C_5}{2\underbrace{\kappa}}\Big)\Big] \\
&\ell(r)_\pm = C_6 + 2\sqrt{C_4}\cot\Big[b^2\sqrt{C_4}\Big(\frac{\mathrm{arctanh}[\frac{r}{\kappa}]C_3 \pm \kappa C_5}{2\kappa}\Big)\Big] \\
&16\kappa^2 + C_3^2(1 - b^2C_4) = 0_{|\varphi_2\neq 0}, \quad 16\kappa^2 - b^2C_3^2C_4 = 0_{|\varphi_2=0}
\end{aligned}} \tag{4.94}$$

where, as in the previous section, one can also choose the inverted argument via replacing the arctanh by arccoth. Note however that in this case C_4 is *negative* and that thus the solutions for ϕ and ℓ have csch and coth instead of csc and cot, if the relation for C_4 is substituted in these solutions. The relations between the integration constants C_i and the ones by Bergshoeff *et al* [17] are

$$\kappa = q, \quad C_3 = \sqrt{2\alpha}q_-, \quad C_5 = \frac{2C_1q_-}{b\mathbf{q}}, \quad C_6 = \frac{-2q_3}{bq_-}, \quad C_2 = 0 \tag{4.95}$$

It is important to note that these relations are obtained via the solutions with arccoth.

The infinitesimal line element becomes in this situation with the re-definition $-\kappa^2 = \tilde{\kappa}^2 > 0$

$$\mathrm{d}s^2 = -e^{2C_2}\Big(1 - \frac{\tilde{\kappa}^2}{r^{2(D-2)}}\Big)^{\frac{2}{D-2}}\Big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{D-1}^2\Big) \tag{4.96}$$

Two interesting observations can be made now. First note that the metric becomes imaginary if

$$r^{2D-4} < \tilde{\kappa}^2 \tag{4.97}$$

but this happens at strong coupling g, since e^{ϕ} blows up at this point. The low energy effective theory is not sufficient in this case, higher order corrections to the type IIB effective action need to be considered to see if this really is a problem. Secondly the metric has the symmetry

$$r^{D-2} \longleftrightarrow -\frac{\tilde{\kappa}^2}{r^{D-2}} < 0 \tag{4.98}$$

which exchanges positive values of r with negative values. In this case the metric does *not* have a wormhole in the Einstein frame.

Dual frame wormhole?

For the non-extremal Reissner-Nordstrøm black hole with $m^2 - \epsilon^2 > 0$ the same situation as above happened. There the wormhole issue could be resolved by going to the dual frame, because this frame cancels the initial Kaluza-Klein reduction.

For the dilatonic black hole the same metric ansatz is used in the KK reduction as for the D-dimensional Reissner-Nordstrøm black hole

$$G^{EF}_{\mu\nu} = g_{\mu\nu} e^{\frac{\chi}{2D-4}} \tag{4.99}$$

and, as explained in appendix A.5, the mapping to the dual frame is achieved by

$$g^{DF}_{\mu\nu} = e^{\frac{b\varphi_1}{D-2}} G^{EF}_{\mu\nu} \tag{4.100}$$

The net effect of these two mapping is via using amongst others (4.66) (and see also (5.22))

$$g^{DF}_{\mu\nu} = e^{(b^2 - \frac{2D-4}{D-1})\varphi_1} g_{\mu\nu} \tag{4.101}$$

where $g_{\mu\nu}$ is the spatial part of the uncompactified metric $\hat{g}_{\hat{\mu}\hat{\nu}}$. This becomes the identity if

$$b = \pm\sqrt{\frac{2D-4}{D-1}} \tag{4.102}$$

What system is obtained under this identity mapping? First note that the wormhole pattern described in this section, was also found for the Schwarzschild and Reissner-Nordstrøm black holes in sections 3.2 and 4.1. Can this be understood in the light of the conclusion above? In other words, can these systems be seen as a subclass of the compactified dilatonic black hole for a proper choice of b? As is evident from a comparison between the dilatonic black hole Lagrangian (4.48) and the Lagrangian of the Reissner-Nordstrøm black hole (4.2), this is not immediately clear. *However* to obtain a useful compactified dilatonic system, various redefinition are used and φ_2 is set equal to zero. It is therefore better to compare the compactified dilatonic black hole with $\varphi_2 = 0$ (4.69)

$$\mathcal{L}_D = -\sqrt{|G|}\mathcal{R}(G) + \frac{1}{2}\sqrt{|G|}\partial_\mu\varphi_1\partial_\nu\varphi_1 G^{\mu\nu} - \frac{1}{2}\sqrt{|G|}G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell e^{b\varphi_1} \tag{4.103}$$

with the compactified Reissner-Nordstrøm black hole

$$\mathcal{L}_{RN} = -\sqrt{|G|}\mathcal{R}(G) - \sqrt{|G|}\Big[\frac{D-1}{16(2-D)}\Big]\partial_\mu\chi\partial_\nu\chi G^{\mu\nu} - \frac{\sqrt{|G|}}{2}G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell e^{\frac{-\chi}{2}} \tag{4.104}$$

and introduce $\chi = \tilde{\chi}\sqrt{\frac{8(D-2)}{D-1}}$, which results in

$$\mathcal{L}_{RN} = -\sqrt{|G|}\mathcal{R}(G) + \frac{1}{2}\sqrt{|G|}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi}G^{\mu\nu} - \frac{\sqrt{|G|}}{2}G^{\mu\nu}\partial_\mu\ell\partial_\nu\ell e^{\frac{-\tilde{\chi}\sqrt{\frac{8(D-2)}{D-1}}}{2}} \tag{4.105}$$

A quick look shows that if

$$\varphi_1 = \tilde{\chi} \quad \text{and} \quad b = -\frac{\sqrt{\frac{8(D-2)}{D-1}}}{2} = -\sqrt{\frac{2D-4}{D-1}} \tag{4.106}$$

indeed the compactified Reissner-Nordstrøm black hole can be seen as a subclass of the compactified dilatonic black hole. Comparing to (4.102) shows that this happens under the identity mapping. So the system obtained under this mapping is precisely the compactified Reissner-Nordstrøm black hole, but from section 4.1 it is known that it has a wormhole in the dual frame (4.33). So there is a wormhole present in the dual frame for this special choice of b, which is identical to the radial part of the uncompactified Reissner-Nordstrøm black hole.

It is interesting to observe that this (almost) matches the claim by Bergshoeff *et al*, for they say that if they take in their article $bc = 2$, the solution with[2] $\mathbf{q}^2 \geq 0$ can be related to the radial part of the Reissner-Nordstrøm black hole. To be precise

$$bc = 2 \longleftrightarrow b = \sqrt{\frac{2D-4}{D-1}} \tag{4.107}$$

which agrees with (4.106) up to a sign. A second thing they claim is that for $bc = 2$ their non-extremal D-instanton lifts up to a Reissner-Nordstrøm black hole. But this can be understood now, for the interesting thing is that **both** (4.106) and (4.107) agree with the identity map (4.102) and thus under this condition the compactified dilatonic black hole is identical to the compactified Reissner-Nordstrøm black hole. This implies that the D-instanton of $bc = \pm 2$ in the dual frame indeed corresponds to the radial part of the Reissner-Nordstrøm black hole, since this dual map brings the compactified metric back to the spatial part of the original metric (4.102). Or to put it in other words, the $bc = \pm 2$ cases are special situations of the radial part of the $p = 0$-brane of Pope *et al* [29], namely those that have $\mu = 0$, as will be shown in the next chapter.

Third case: $\kappa^2 = 0$

Looking at $B(r)$ (4.72) this should agree with flat Euclidean space if $\kappa = 0$. Taking the limit

$$\kappa \to 0 \tag{4.108}$$

in the solutions for φ_1 and ℓ lead to

$$e^{\frac{b\varphi_1}{2}} = \frac{b}{2}\Big(\frac{r^{2-D}C_3}{D-2} + C_5\Big), \quad \ell = \frac{2}{b}e^{-b\varphi_1/2} + C_6 \tag{4.109}$$

[2]The case $\mathbf{q}^2 \geq 0$ is the same as $\kappa^2 < 0$.

To see if this is correct, look at the generalization of the extremal D-instanton

$$S^{EF}_{Euc} = \int \mathrm{d}^D x \sqrt{|g|}\{\mathcal{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{b\phi}\partial_\mu a\partial^\mu a\} \tag{4.110}$$

with the corresponding solutions

$$e^{b\phi/2} = \frac{b\alpha}{2(D-2)r^{D-2}} + \frac{b}{2}e^{\phi_\infty}, \quad a = a_\infty \pm \frac{2}{b}e^{-b\phi/2} \tag{4.111}$$

where α can be related to a conserved charge and e^{ϕ_∞} is the coupling constant in D dimensions. Evidently these solutions correspond to the solutions obtained in the limit $\kappa \to 0$ above (4.109) if

$$\ell = a, \quad \varphi_1 = \phi, \quad C_3 = \alpha, \quad C_5 = e^{\phi_\infty}, \quad C_6 = a_\infty \tag{4.112}$$

4.4 Noether currents, Bogomol'nyi bound and $\mathcal{R}^8$

In chapter one the charge matrix $\mathcal{Q}$ has been introduced and in appendix A.4 a general expression for $\mathcal{Q}$ (A.41) has been given. Substituting the non-extremal solutions in this expression give

$$\mathcal{Q} = \begin{pmatrix} -\frac{1}{4}b^2C_3C_6 & \gamma \\ \frac{bC_3}{2} & \frac{1}{4}b^2C_3C_6 \end{pmatrix} \tag{4.113}$$

where γ is defined by

$$\mathrm{Det}[\mathcal{Q}] = \kappa^2 b\sqrt{\frac{\alpha}{2}} \tag{4.114}$$

As said in chapter one (above (1.113)), the equations of motion are invariant under the shift of the pseudoscalar ℓ by a constant, which gave rise to the conserved current J_μ. This is the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$ and it is evident from ℓ (4.90) that C_6 is that constant. The $SO(1,1)$ symmetry related to K_μ has the effect $e^\phi \to e^\nu e^\phi,\ \ell \to e^{-\nu}\ell$. Looking at the solutions for ϕ (4.88) and ℓ (4.90) shows that C_4 takes on this role, in combination with C_5 to counterbalance the effect in the argument of csc and cot.

Bogomol'nyi bound

The extremal D-instanton satisfied the Bogomol'nyi bound, which meant it broke half the SUSY's as was shown in chapter one. For the non-extremal case, with general dilaton coupling b in D dimensions, the Bogomol'nyi bound can be obtained in a similar way as explained in section 1.3.1 and also for the same reasons as explained there does one need to introduce a boundary term. The only difference is now a general dilaton coupling b, general dimension D and that the dual of a one form is now a $(D-1)$ form. Take the Euclidean version of (A.5)

$$S_{Euc}(\varphi_1, \mathrm{d}C^{(D-2)}) = -\frac{1}{2}\int(\mathrm{d}\varphi_1 \wedge *\mathrm{d}\varphi_1 + e^{-b\varphi_1}\mathrm{d}C^{(D-2)} \wedge *\mathrm{d}C^{(D-2)}) \tag{4.115}$$

where $\mathrm{d}C^{(D-2)} = e^{b\varphi_1} * \mathrm{d}\ell$ and use the fact that in Euclidean space $* * \mathrm{d}C^{(D-2)} = (-)^{D-1}\mathrm{d}C^{(D-2)}$ which means that it can be rewritten too[3]

$$\begin{aligned} S_{Euc}(\varphi_1, \mathrm{d}C^{(D-2)}) = & -\frac{1}{2}\int \left[(\mathrm{d}\varphi_1 \pm e^{-b\varphi_1/2} * \mathrm{d}C^{(D-2)}) \wedge *(\mathrm{d}\varphi_1 \pm e^{-b\varphi_1/2} * \mathrm{d}C^{(D-2)})\right] \\ & \mp (-1)^D \frac{2}{b}\int \mathrm{d}(e^{-b\varphi_1/2}\mathrm{d}C^{(D-2)}) \end{aligned} \tag{4.116}$$

Clearly this action is now bound from below if

$$\boxed{\mathrm{d}\varphi_1 \pm e^{-b\varphi_1/2} * \mathrm{d}C^{(D-2)} = 0 \longleftrightarrow \mathrm{d}\varphi_1 = \pm(-1)^D e^{b\varphi_1/2}\mathrm{d}\ell} \tag{4.117}$$

which agrees with the extremal D-instanton case of chapter one (1.93). Substituting the general solution for the non-extremal cases $\kappa^2 < 0$ and $\kappa^2 > 0$ in this lead to the conclusion that they are *not* BPS-states since the above condition is not obeyed.

$\mathcal{R}^8$ contribution

The question that remains than is how many SUSY parameters are left unbroken. To answer this question one should substitute the non-extremal D-instanton solutions in the (Euclidean) supersymmetry transformations belonging to a gravitational background. Sticking to ten dimensions these are for the dilaton and axion[8]

$$\begin{aligned} \delta\psi_\mu^\pm = & \left(\partial_\mu - \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab} \mp \frac{1}{4}e^{\varphi_1}\partial_\mu\ell\right)\epsilon^\pm = 0 \\ \delta\lambda^\pm = & \frac{1}{4}\Gamma^\mu\epsilon^\mp\left(\partial_\mu\varphi_1 \pm e^{\varphi_1}\partial_\mu\ell\right) = 0 \end{aligned} \tag{4.118}$$

The difference with the SUSY transformations of chapter one (1.136) is the presence of the spin connection term ω_μ^{ab}, this is zero in the flat space ansatz. The Γ^μ's are the Dirac gamma matrices obeying the Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. Substituting the non-extremal D-instanton solutions in these two equations lead to the conclusions that

$$\epsilon^+ = \epsilon^- = 0 \tag{4.119}$$

and thus all 32 SUSY's are broken. The same arguments can now be used as in section 1.5. Over all 32 broken spinors need to be integrated and comparing units shows that the product of 32 fermions is the same as a $\mathcal{R}^8$ term. But this should of course be confirmed by an exact D-instanton calculation, but this is beyond the scope of this master thesis.

[3]The general rule is $* * A_p = (-1)^{(D-1)p}A_p$.

Chapter 5

Uplifting instanton systems

In the previous section various instanton solution were obtained corresponding to Kaluza-Klein reduced Lagrangians. These instanton solutions live in a D-dimensional Euclidean space. To get back to a $(D+1)$-dimensional Minkowskian spacetime one must undo the Kaluza-Klein reduction, i.e. uplift the systems obtained in the previous chapters. Formally this can be achieved by realizing that the right hand side of (4.49) is known $(G_{\mu\nu}, \chi)$ and solving for the left hand side gives $\hat{g}_{\hat{\mu}\hat{\nu}}$, the required uplifted version of the metric $G_{\mu\nu}$

$$\begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{-\frac{\chi}{2}} G_{\mu\nu} \end{pmatrix} = \hat{g}_{\hat{\mu}\hat{\nu}} \tag{5.1}$$

This is only half of the story, for the Lagrangian must be uplifted too. This is however not difficult since the Lagrangian, before the Kaluza-Klein reduction is know, only the D-dimensional solutions need to be uplifted and substituted in this.

As an illustration these steps will be applied to the general Reissner-Nordstrøm black hole in four dimensions, after that it will be applied to the non-extremal D-instantons obtained in the previous chapter. General Dp-branes are investigated by Pope *et al* [29], the uplifted solutions should correspond to a subsector of this system with $p = 0$.

The compactified four dimensional Reissner-Nordstrøm black hole can be obtained by taking $D = 3$ in the $(D+1)$-dimensional case as discussed in section 4.1. Uplifting the metric is simple in this case, since in that chapter the opposite path was followed, i.e. the $(D+1)$-dimensional general Reissner-Nordstrøm black hole was given explicitly, (4.18), together with the uncompactified Lagrangian, (4.2) and then the compactified system was solved, see (4.27). One can thus simply substitute these solutions in the equations of motion that follow from the uncompactified case and note that this indeed is a solution of the system, see below (5.5).

However in general one does *not* know the uncompactified metric, for example in the non-extremal D-instanton case the compactified metric was "obtained" via the ansatz (4.71), but did not follow from a higher dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, however the system can still be uplifted. As an illustration of this, assume that only the metric of the compactified system $g_{\mu\nu}$ in three dimensions is known (4.27) and the corresponding solution for ℓ (4.29), i.e.

$$\boxed{\begin{aligned}&\chi = 2\log(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2})\\ &G_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & -(r^2-2mr+\epsilon^2)\mathrm{d}\Omega_2^2\end{pmatrix}\\ &A_0 = \ell(r) = \frac{2\epsilon}{r}+\mu_2, \quad A_i = 0\\ &\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{-\frac{\chi}{2}}G_{\mu\nu}\end{pmatrix}\end{aligned}} \tag{5.2}$$

Combining the above information ($\hat{g}_{\hat{\mu}\hat{\nu}}$, $G_{\mu\nu}$ and χ) gives for the uncompactified four dimensional Reissner-Nordstrøm metric

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} (1-\frac{2m}{r}+\frac{\epsilon^2}{r^2}) & 0 & 0 & 0 \\ 0 & \frac{-1}{(1-\frac{2m}{r}+\frac{\epsilon^2}{r^2})} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{5.3}$$

which agrees with the $(D+1)$-dimensional case (4.18) if $\alpha = 2m$ and $\beta = \epsilon$. The four dimensional Lagrangian is given by

$$\hat{\mathcal{L}}_{\hat{G}} = -\sqrt{|\hat{g}|}\hat{\mathcal{R}} - \frac{1}{4}\sqrt{|\hat{g}|}\hat{g}^{\hat{\mu}\hat{\rho}}\hat{g}^{\hat{\nu}\hat{\eta}}F_{\hat{\mu}\hat{\nu}}F_{\hat{\rho}\hat{\eta}} \tag{5.4}$$

The solutions obtained in three dimensions should also solve the equations of motion that follow from this Lagrangian, if the reduction went consistent. The equations of motion that follow from the above Lagrangian are

$$\begin{aligned}&-\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\Big(F^2_{\hat{\mu}\hat{\nu}} - \frac{1}{4}F^2 g_{\hat{\mu}\hat{\nu}}\Big) = 0\\ &-\partial_{\hat{\mu}}\Big(\sqrt{|\hat{g}|}g^{\hat{\alpha}\hat{\mu}}g^{\hat{\rho}\hat{\nu}}F_{\hat{\alpha}\hat{\rho}}\Big) = 0\end{aligned} \tag{5.5}$$

Looking at appendix A.3 it is clear that the first equation of motion is nothing else then the Einstein equation, see (A.21). The solution for $\ell = A_0$ is then (A.20) or

$$A_0(r) = \frac{2\epsilon}{r} \tag{5.6}$$

which is indeed also the solution found (5.2).

Having explained the general concept of uplifting, it is now interesting to uplift the non-extremal instanton, since this uplifted system is a $p=0$ or black hole solution, it should agree with the work as presented in an article by Pope *et al* [29], which will be discussed next.

5.1 General Dp-branes

The non-extremal Dp-branes[1] of Pope *et al* are solution of the Lagrangian

$$\mathcal{L} = \sqrt{|g|}\Big[\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(n)!}e^{a\phi}F_n^2\Big] \tag{5.7}$$

[1]In this subsection the metric is diag $= (-1,\ldots,1)$.

where n can be related to the p of a general Dp-brane via $n = p + 2$. The metric ansatz Pope *et al* use is

$$ds^2 = e^{2A} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B} dy^m dy^m = e^{2A} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B}\left(dr^2 + r^2 d\Omega^2_{D-d-1}\right) \tag{5.8}$$

where x^μ are the coordinates of the d-dimensional world volume of the Dp-brane, y^m are the coordinates of the $(D-d)$ dimensional transverse space and A and B are functions of $r = \sqrt{y^m y^m}$. It turns out that the following definitions are useful

$$a^2 = \Delta - \frac{2d\tilde{d}}{D-2}, \quad \tilde{d} = D - d - 2 \tag{5.9}$$

Pope *et al* solve this system in the most general way, i.e. no special requirements about retaining (a fraction) of the supersymmetry. The field strength F_n can be defined as

$$F_{m\mu_1\ldots\mu_{n-1}} = \epsilon_{\mu_1\ldots\mu_{n-1}} \partial_m e^C \tag{5.10}$$

where C is a function of r only. This can be done since Pope *et al* are only interested in radial solutions, i.e. taking one of the n-indices equal to r, say $m = r$, and the remaining $n-1$ indices are governed by the $n-1$ rank anti-symmetric tensor $\epsilon_{\mu_1\ldots\mu_{n-1}}$. The equation of motion [30] that follows for ϕ is

$$\boxed{\Box\phi = -\frac{a}{2n!} e^{-a\phi} F^2} \tag{5.11}$$

For $g_{\mu\nu}$ one has two contributions, to begin with the contribution from $g_{\mu\nu}$ "directly". Realizing that $F^2 = F_{\mu_1\ldots\mu_n} F_{\nu_1\ldots\nu_n} g^{\mu_1\nu_1} \ldots g^{\mu_n\nu_n}$ this gives

$$\sqrt{|g|}\mathcal{R}_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2(n-1)!}\sqrt{|g|} e^{-a\phi} F^2_{\mu\nu} \tag{5.12}$$

where

$$F^2_{\mu\nu} \equiv F^2_{\mu_1\nu_1} = F_{\mu_1\mu_2\ldots\mu_n} F_{\nu_1\nu_2\ldots\nu_n} g^{\mu_2\nu_2} \ldots g^{\mu_n\nu_n} \tag{5.13}$$

For the second contributions use (1.73) and this gives

$$-\frac{1}{2}\sqrt{g}\mathcal{R}_{\eta\rho} g^{\eta\rho} g_{\mu\nu} + \frac{1}{4}\sqrt{|g|}\partial_\eta\phi\partial_\rho\phi g^{\eta\rho} g_{\mu\nu} + \frac{1}{4n!}\sqrt{|g|} e^{-a\phi} F^2 g_{\mu\nu} \tag{5.14}$$

Adding these two up gives the following Einstein equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}\sqrt{|g|}\partial_\eta\phi\partial_\rho\phi g^{\eta\rho} g_{\mu\nu} + \frac{1}{2(n-1)!}\sqrt{|g|} e^{-a\phi} F^2_{\mu\nu} - \frac{1}{4n!}\sqrt{|g|} e^{-a\phi} F^2 g_{\mu\nu} \tag{5.15}$$

Multiplying by $g^{\mu\nu}$ gives the Ricci scalar and substituting this back leads to

$$\boxed{\begin{aligned} \mathcal{R}_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + S_{\mu\nu} \\ S_{\mu\nu} &= \frac{1}{2(n-1)!} e^{-a\phi}\left(F^2_{\mu\nu} - \frac{n-1}{n(D-2)} F^2 g_{\mu\nu}\right) \end{aligned}} \tag{5.16}$$

The last equation of motion for the anti-symmetric tensor is obtained if one realizes than any n-form can be written as $F_{M\mu_1\ldots\mu_{n-1}} = \partial_{[M}A_{\mu_1\ldots\mu_{n-1}]}$

$$\boxed{\partial_M\left(\sqrt{|g|}e^{-a\phi}F^{M\mu_1\ldots\mu_{N-1}}\right) = 0} \tag{5.17}$$

Doing the substitutions

$$\begin{aligned} &X \equiv dA + \tilde{d}B, \quad Y \equiv A + \frac{\epsilon\tilde{d}}{a(D-2)}\phi, \quad \Phi \equiv \epsilon a\phi - 2dA, \quad \rho = r^{-\tilde{d}}, \\ &k\rho = \tanh(k\xi) \end{aligned} \tag{5.18}$$

are useful to find the solutions, which turn out to be[2]

$$\boxed{\begin{aligned} e^{-\frac{(D-2)\Delta}{2d}A} &= \frac{\lambda\sqrt{\Delta}}{2\tilde{d}\beta}\sinh(\beta\xi+\alpha)e^{\frac{a^2(D-2)\mu\xi}{2\tilde{d}}} \\ e^{\frac{(D-2)\Delta}{2d}B} &= \frac{\lambda\sqrt{\Delta}}{2\tilde{d}\beta}\sinh(\beta\xi+\alpha)e^{\frac{a^2(D-2)\mu\xi}{2\tilde{d}}}(\cosh(k\xi))^{-\frac{(D-2)\Delta}{d\tilde{d}}} \\ e^{\frac{\epsilon\Delta}{2a}\phi} &= \frac{\lambda}{2\tilde{d}\beta}\sinh(\beta\xi+\alpha)e^{-d\mu\xi} \end{aligned}} \tag{5.19}$$

where μ, k and α are constants, λ is an integration constant following from the equation of motion for C and β is defined via

$$\tilde{d}\beta^2 = 2(\tilde{d}+1)\Delta k^2 - \frac{1}{2}a^2 d(D-2)\mu^2 \tag{5.20}$$

Demanding that A, B and ϕ go to zero at $r = \infty$ lead to the conclusion that $\sinh(\alpha) = \frac{2\tilde{d}\beta}{\lambda\sqrt{\Delta}}$, which implies for the metric that it approaches a D-dimensional Minkowskian spacetime. The definition for ξ implies that at $\xi = \infty \longleftrightarrow k\rho = 1 \longleftrightarrow r^{\tilde{d}} = k$ there is a horizon and demanding ϕ to be finite at this position leads to

$$\boxed{\beta = \mu d \longleftrightarrow \mu = \sqrt{(\tilde{d}+1)/(d(D-2))}} \tag{5.21}$$

To compare this work to the non-extremal D-instanton, it is important to realize the *different* meaning of the dimension D in both cases. In this section D stands for both space and time, so to avoid confusion in the next section it will be referred to as $D_{\text{this section}} \equiv D_{Pope} = \tilde{D}+1$, where $\tilde{D}$ are the number of spatial directions Pope *et al* use, i.e. $\tilde{D} = D_{Pope} - 1$

5.2 Uplifting the non-extremal D-instanton

The special case $p = 0$ of the general Dp-branes obtained in the previous section, should agree with the solutions obtained if one uplifts the D-instanton of the dilatonic black hole obtained in chapter four. Clearly it will depend on the sign chosen for κ^2 and looking at the work of Pope *et al* it seems that the negative sign is the correct one to take and also the arccoth (4.93) is needed. Although arctanh

[2] As will be explained later, the formula for ϕ is not correct, a factor $\sqrt{\Delta}$ has been forgotten.

will of course work too, but to match it to the work of Pope *et al* at the end, the same transformation is needed.

Via the relations in section 4.3, amongst others (4.66), one finds

$$\boxed{\phi = \frac{a\varphi_1}{b}, \quad \chi = \frac{-4(D-2)\varphi_1}{b(D-1)}, \quad b^2 = a^2 + \frac{2(D-2)}{D-1}} \tag{5.22}$$

The D-dimensional solutions are with $\varphi_2 = 0$

$$\boxed{\begin{aligned} &\mathrm{d}s^2 = -e^{2C_2}\Big(1 - \frac{\kappa^2}{r^{2(D-2)}}\Big)^{\frac{2}{D-2}}\Big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-1}\Big) \\ &e^{-\frac{b\varphi_1(r)_\pm}{2}} = b\sqrt{C_4}\csc\Big[b^2\sqrt{C_4}\Big(\frac{\pm\mathrm{arctanh}[\frac{\kappa}{r^{D-2}}]C_3 + (D-2)\kappa C_5}{2(D-2)\kappa}\Big)\Big] \\ &\ell(r)_\pm = C_6 + 2\sqrt{C_4}\cot\Big[\frac{b^2\sqrt{C_4}\Big(\mathrm{arctanh}[\frac{\kappa}{r^{D-2}}]C_3 \pm (D-2)C_5\Big)}{2(D-2)\kappa}\Big] \\ &2\alpha\kappa^2 + b^2C_3^2C_4 = 0 \end{aligned}} \tag{5.23}$$

and note that C_4 is negative according to the last relation. The Einstein frame mapping for the D-dimensional case was derived in chapter four, see (4.8). This gives for the higher dimensional metric

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{\frac{\chi}{4-2D}}G^{EF}_{\mu\nu} \end{pmatrix} \tag{5.24}$$

with $G^{EF}_{\mu\nu}$ from the infinitesimal line element (5.23).

To connect it to Pope *et al* first observe that their time component is[3]

$$g^{Pope}_{tt} = e^{2A} \tag{5.25}$$

The solution for ϕ follows from the first formulae of (5.22), $\hat{A}_{\hat{\mu}} = (\ell, \vec{0})$ and the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ follows finally from (5.24). This gives in terms of the solutions of the fields

$$\boxed{\begin{aligned} &g_{tt} = e^{-\frac{b\varphi_1(r)_\pm}{2}\frac{4(D-2)}{b^2(D-1)}}, \\ &g_{rr} = -e^{-\frac{b\varphi_1(r)_\pm}{2}-\frac{4}{b^2(D-1)}}e^{2C_2}\Big(1 - \frac{\kappa^2}{r^{2(D-2)}}\Big)^{\frac{2}{D-2}} \\ &g_{\theta\theta} = g_{rr}r^2, \quad g_{\phi\phi} = g_{rr}r^2\sin^2\theta \end{aligned}} \tag{5.26}$$

This agrees with to the general solutions of Pope *et al* for $\mu = 0$, $d = 1$ and $D_{Pope} = \tilde{D} + 1$:

$$\boxed{\begin{aligned} &g^{Pope}_{tt} = e^{2A} = \Big(\frac{\lambda\sqrt{\Delta}}{2\tilde{d}\beta}\sinh(\beta\xi + \alpha)\Big)^{\frac{-4(\tilde{D}-2)}{(\tilde{D}-1)\Delta}} \\ &g^{Pope}_{rr} = e^{2B} = e^{\frac{(\tilde{D}-2)\Delta}{2\tilde{d}}B} = \Big(\frac{\lambda\sqrt{\Delta}}{2\tilde{d}\beta}\sinh(\beta\xi + \alpha)\Big)^{\frac{4}{(\tilde{D}-1)\Delta}}(1 - \frac{k^2}{r^{2(\tilde{D}-2)}})^{\frac{2}{\tilde{D}-2}} \\ &e^{\phi} = \Big(\frac{\lambda}{2(\tilde{D}-2)\beta}\sinh(\beta\xi + \alpha)\Big)^{\frac{2a}{\epsilon\Delta}} \end{aligned}} \tag{5.27}$$

[3]Pope uses as a sign convention: diag $= (-1, 1, \ldots, 1)$.

if the following relations are used

$$\boxed{k=\mu,\quad C_3=\frac{\lambda\sqrt{2\alpha}k\sqrt{\Delta}}{2(\tilde{D}-2)\beta},\quad b=\frac{2(\tilde{D}-2)\beta}{k\sqrt{2\alpha}},\quad C_2=0,\quad C_5=\frac{\alpha\lambda\sqrt{\Delta}}{(\tilde{D}-2)\beta b}} \tag{5.28}$$

A remark must be made. When comparing the expressions for ϕ, agreement can *only* be made if the one by Pope *et al* is

$$e^{\frac{\epsilon\Delta}{2a}\phi}=\frac{\lambda\overbrace{\sqrt{\Delta}}}{2\tilde{d}\beta}\sinh(\beta\xi+\alpha)e^{-d\mu\xi} \tag{5.29}$$

In other words an extra factor $\sqrt{\Delta}$ is needed, this turns out to be also the case if one calculates ϕ via the relations as given in the article of Pope *et al.*

As a final check the uncompactified Lagrangian in $(D+1)$ dimensions is (4.47)

$$\hat{\mathcal{L}}=-\sqrt{|\hat{g}|}\hat{\mathcal{R}}-\frac{1}{4}\sqrt{|\hat{g}|}e^{a\hat{\phi}}\hat{g}^{\hat{\mu}\hat{\rho}}\hat{g}^{\hat{\nu}\hat{\eta}}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}_{\hat{\rho}\hat{\eta}}+\frac{1}{2}\sqrt{|\hat{g}|}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi}\hat{g}^{\hat{\mu}\hat{\nu}} \tag{5.30}$$

where $\hat{\mathcal{R}}$ follows now from (5.26), the field $\hat{\phi}=\phi$ from (5.22) and $\hat{A}_{\hat{\mu}}=(\ell,\vec{0})$. The equations of motion that need to be satisfied are

$$\begin{aligned}&\partial_r\Big(\sqrt{|\hat{g}|}\hat{g}^{rr}\partial_r\hat{\phi}\Big)+\frac{a}{4}\sqrt{|\hat{g}|}e^{a\hat{\phi}}\hat{F}_2^2=0\\&\hat{\mathcal{R}}_{\hat{\mu}\hat{\nu}}+\frac{1}{2}e^{a\hat{\phi}}\big(\hat{F}^2_{\hat{\mu}\hat{\nu}}-\frac{1}{4}\hat{F}^2\hat{g}_{\hat{\mu}\hat{\nu}}\Big)-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi=0\\&\partial_r\Big(\sqrt{|\hat{g}|}e^{a\hat{\phi}}\hat{F}^{r\hat{\mu}}\Big)0=0\end{aligned} \tag{5.31}$$

By a direct substitution of the metric and fields just obtained, it follows that indeed these are solutions. For this realize that

$$\hat{F}_2^2=\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}}=2\hat{F}_{0r}\hat{F}^{0r}=2(\partial_r\hat{A}_0)^2\hat{g}^{00}\hat{g}^{rr} \tag{5.32}$$

and

$$\hat{F}^2_{\hat{\mu}\hat{\nu}}\equiv\hat{F}^2_{\hat{\mu}_1\hat{\nu}_1}=\hat{F}_{\hat{\mu}_1\hat{\mu}_2}\hat{F}_{\hat{\nu}_1\hat{\nu}_2}\hat{g}^{\hat{\mu_2}\hat{\nu_2}} \tag{5.33}$$

In this section the general dilatonic black hole or $p=0$-brane has been solved, via first determining the instantons belonging to the compactified version of the Lagrangian (5.30). Besides the rather complicated solutions, an interesting conclusion is that the uplifting works *only* if $b^2=a^2+\frac{2(D-2)}{D-1}$, see (5.22), which puts for a given dimension D, a minimum on b

$$b\geq\sqrt{\frac{2(D-2)}{D-1}} \tag{5.34}$$

This concludes the investigation of non-extremal dilatonic black holes.

Chapter 6

Summary, conclusions and remarks

Summary and conclusions

This thesis started with the introduction of instantons, most notably the D-instanton. Various aspects were being investigated and the wormhole (1.86) found in the string frame, made clear that a firmer introduction to these special objects was needed. This was done in chapter two, especially for the Schwarzschild and Reissner-Nordstrøm metric. The solitonic interpretation via embedding diagrams was explained and that these diagrams are equivalent to having a metric with a symmetry of the form

$$r \longleftrightarrow \frac{\text{constant}}{r} \tag{6.1}$$

see section 2.3.1. To combine instantons and wormholes and ultimately generalize the extremal D-instantons, Kaluza-Klein reductions over the time were introduced for the Schwarzschild and (extremal) Reissner-Nordstrøm black holes. An interesting observation was that for the extremal case a wormhole was present in the string frame (figure 4.1), whereas for the non-extremal cases the wormhole was found in the Einstein frame if $\alpha^2 - 4\beta^2 < 0$ and for the opposite sign in the dual frame, see below (4.41).

To make the connection to the D-instanton, an extra scalar field was needed (the dilaton), which was coupled to the Maxwell term via the constant a. This gave rise to the dilatonic black hole (4.47) which was subsequently Kaluza-Klein reduced (4.69). Turning off one of the scalar fields resulted in a Lagrangian which described non-extremal D-instantons (i.e. non-flat space) with a generalized dilaton-axion coupling parameter b. For this reason the extra dilaton field was added to the Reissner-Nordstrøm black hole, else b would have been just a number, see (4.105).

The solutions to the corresponding equations of motion were the non-extremal D-instantons (section 4.4). These are described by a total of five independent parameters $(\kappa^2, C_2, C_3, C_5, C_6)$, of which C_2 has to be set equal to zero if asymptotically flat space has to be obtained. These parameters can be related to the parameters $(\mathbf{q}, q_-, q_3, C_1)$ in the recently published article by Bergshoeff *et al*, see (4.95).

The parameter κ^2 separates the solutions in three distinct sectors: equal, smaller or larger than zero. The sector $\kappa^2 = 0$ is the standard D-instanton (4.109),

$\kappa^2 < 0$ is valid for all r larger than a critical value(4.97), beyond this point the metric becomes imaginary, but the coupling constant, $g = e^{\phi_\infty}$, becomes infinite which implies that a low energy effective theory is no longer valid to make any claims near this point. The last sector $\kappa^2 > 0$ can be made valid everywhere if the constants are chosen properly, otherwise the field ϕ will become complex (4.89). This then happens at low coupling constant.

For the sector $\kappa^2 > 0$ a wormhole was found in the Einstein frame (4.75). To obtain a wormhole for the case $\kappa^2 < 0$, a mapping to the dual frame was needed (4.101) and demanding this to be the identity mapping gave a value for b (4.102). Comparing the Lagrangian with the minus sign for this value of b with the Reissner-Nordstrøm metric lead to the conclusion that the Reissner-Nordstrøm metric can be considered as a subclass of the general Dilatonic black hole (4.106), which was not completely surprising since the same wormhole pattern was obtained for this system already, see below figure 4.1, the plus sign has been discussed in a different way by Bergshoeff *et al.* Taking the limit $\kappa^2 \to 0$ of the wormholes obtained in the two non-extremal cases do not lead to a wormhole, to be precise the limit leads to a "semi-wormhole", see figure 4.2. In the string frame it is a genuine wormhole.

To show that the (non-)extremal D-instantons can be considered as static solitons of the corresponding one (timelike) dimensional higher theory, an uplifting was needed which was performed in chapter five. This uplifting changes a $p = -1$-brane to a $p = 0$-brane. General Dp-branes have been discussed by Pope *et al* [29] and therefore this uplifting had to agree with a subsector of their solutions. It was shown that only in the limit μ goes to zero this connection can be made and that also $\kappa^2 < 0$ had to be taken. Pope chooses μ such that there are no singularities present, the field ϕ remains finite everywhere. This value of μ is different from zero and it is therefore no wonder that the metric has a singularity for this sector. To be precise the critical value found is (4.97) and the article by Pope *et al* gives the same position also if one takes $p = -1$ and $n = 1$, which is the D-instanton requirement. Finally it was shown that the uplift works only if

$$b \geq \sqrt{\frac{2(D-2)}{D-1}} \tag{6.2}$$

for a given dimension D.

As was discussed in sections 1.5 and 4.4, one reason why people should investigate (non)-extremal D-instantons is that they give higher order corrections to the effective IIB action. The extremal D-instanton gave rise to $\mathcal{R}^4$ terms, since they break half the supersymmetry. The non-extremal D-instantons do not satisfy the Bogomol'nyi bound (4.117). To see that they break all SUSY one has to check the effect of these non-extremal solutions on the SUSY transformations in a gravitational background (4.119). *If* the non-extremal D-instantons are genuine instantons ($S < \infty$), the logical conclusion seems to be that they give rise to $\mathcal{R}^8$ contributions, since now over all 32 broken SUSY's need to be integrated and the unit of 32 fermions is the same a $\mathcal{R}^8$ term. This of course should be confirmed by an exact D-instanton calculation and should be a subject of future research.

Remarks

In the introduction it was stated that instantons are solutions with finite action. Except for the extremal D-instanton, nowhere has the action explicitly been calculated, so formally one should put quotes around the word instanton: D-"instanton". Calculating the action is far from trivial, as was already shown for the extremal case. The bulk action was zero and only by realizing that a boundary term should be included (1.80), could one find finite action. A contribution which has been neglected is the so called *Gibbons-Hawking* term (GH)

$$S_{GH} = -2 \int_{\partial M} \mathrm{Tr}[K - K_0] \tag{6.3}$$

where ∂M is the boundary of the (Euclidean) space, and $K - K_0$ is the difference between the trace of the extrinsic curvature on the boundary of M and the value it would have had if the boundary were in flat space, this is needed as a normalization, see [6]. The reason why it could be ignored is because it turns out to be equal to zero.

For the non-extremal D-instantons it is not yet fully clear what the total action is for the two sectors $\kappa^2 \neq 0$. Partly this is due to the GH-term and partly related to the fact that in the case $\kappa^2 < 0$ there is a point at which the metric becomes complex, but this happens at strong coupling. Whether or not then this should be considered a genuine boundary is unclear[1]. More research is needed in this area to determine whether or not these non-extremal cases are genuine instantons as defined in section 1.1.

Finally an interesting side remark is that even in published articles mistakes still happen. See for example the article by Cremmer *et al* [21], formulae (4.61) and also Pope *et al* [29], formula (5.19).

[1]Remember that in the D-instanton case it was a boundary term that gave the action, therefore it is to be expected that boundaries are very important in this case too.

Appendix A

A.1 Rules for analytical continuation

The whole concept of instantons is based on the analytical continuation of the time t via[1]

$$t = \tau e^{-i\delta} \qquad \delta \in [0, 2\pi]$$

(for convenience one often takes $\delta = \frac{\pi}{2}$), which is also called a Wick rotation. Important is to note that τ is *not* a physical time as t is, although it is real. The rules for a proper Euclidinazation are:

Rule 1: $t \Rightarrow -i\tau$
Rule 2: $a_\mu = (a_0, a_1, \ldots, a_D) \Rightarrow (-ia_0, a_1, \ldots, a_D)$
Rule 3: *pseudoscalar a: $a \Rightarrow -ia$*
Rule 4: $\int_0^t \mathrm{d}t \Rightarrow \int_0^\tau -i\mathrm{d}\tau$
Rule 5: $S_{Euc} = -i(S_{Min})_{\text{analytical continuated}}$ *for obtaining a real Euclidean action*
Rule 6: *Results obtained must be Wick rotated back $\tau \Rightarrow it$*

A pseudoscalar a changes sign under a parity transformation P. Therefore rewrite it in terms of the scalars S_i, $i = 0, \ldots, 9$

$$a = \epsilon^{\mu_0\mu_1\ldots\mu_9}(\partial_{\mu_0} S_0)\ldots(\partial_{\mu_9} S_9)$$

This is still a pseudoscalar, since in (9+1) dimensions it picks up an odd number of minus signs (nine) under P. Going to a its Euclidean "partner" α, it is clear that it picks up an extra factor of i due to the ∂_{μ_i} in combination with the $\epsilon^{\mu_0\mu_1\ldots\mu_9}$, i.e. there is always *one* term $\partial/\partial t$ present [35]. A less mathematical "proof" is the following. A parity transformation flips the spatial coordinates $\vec{x}$ to $-\vec{x}$, i.e. a rotation of 180 degrees. A Wick rotation is thus the square root out of a parity transformation. The pseudoscalar a changes by definition sign under a parity transformation, $a \to -a$ and thus taking the square root implies $a \to -ia$, it is $-i$ since a Wick rotation is clockwise, see figure (A.1). To compensate for the $-i$ of rule four, the fifth is necessary, so that the Euclidean action is still real.

Having said all this, the question that still needs to be answered is why extra information is extracted about the path integral when using instantons. Solving a path integral is in general difficult, only in special cases does one exactly know the

[1] This section discusses only the Wick rotation method, not the Kaluza-Klein approach.

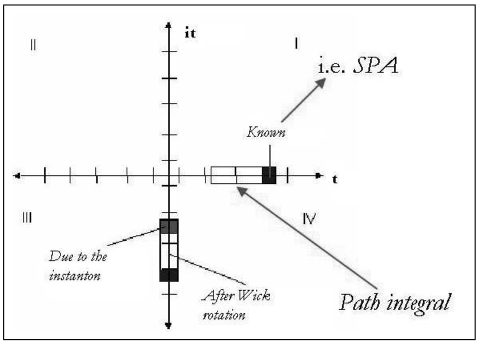

Figure A.1: Path integrals and instantons

answer. Therefore people often use tricks as the stationary phase approximation, see for example the double well of chapter one. The instanton belonging to that system introduces an extra classical path, see figure 1.2. Picture now the path integral as the rectangle shown in figure A.1. In real time t only the black part is known via for example a SPA, after a Wick rotation[2] one arrives at $\tau = it$. Here the instanton can be used as a second point to do an expansion, as indicated by the second colored box. If one then Wick rotates back via rule six, extra information about the path integral is known. This is a good way to understand why instanton can extract extra information about a path integral.

A.2 Boundary term: Giving D-instantons action

Introduce the dual of the axion a, $F^{(9)} = e^{2\phi} * \mathrm{d}a$. If as a starting point one uses the action[3]

$$S^{EF}_{Min} = \int \mathrm{d}^{10}x\sqrt{|g|}\mathcal{R} - \frac{1}{2}\int \Big[\mathrm{d}\phi \wedge *\mathrm{d}\phi + e^{2\phi}\mathrm{d}a \wedge *\mathrm{d}a\Big] \tag{A.1}$$

and treating the dual transformation $e^{-2\phi}F^{(9)} \wedge *F^{(9)} = e^{2\phi} * \mathrm{d}a \wedge * * \mathrm{d}a = -e^{2\phi}\mathrm{d}a \wedge *\mathrm{d}a$ [4] as a formal substitution, turns (A.1) into[5]

$$S^{EF}_{Min} = \int \mathrm{d}^{10}x\sqrt{|g|}\mathcal{R} - \frac{1}{2}\int \Big[\mathrm{d}\phi \wedge *\mathrm{d}\phi - e^{-2\phi}F^{(9)} \wedge *F^{(9)}\Big] \tag{A.2}$$

which has the wrong sign compared to the action used in chapter one, (1.90)[6]. This example was shown to make clear that "dualizing" a theory is something else than a substitution, see also [16]. Adding the boundary term

$$S^{EF}_{Surf} = \int d(e^{2\phi}a \wedge *\mathrm{d}a) \tag{A.3}$$

[2] It is assumed here that there are no singularities in the IV quarter.

[3] Which gives zero for the instanton solution of chapter one.

[4] Let $a^{(p)}$ be a p-form and s be the number of minuses in the metric, then $**a^{(p)} = (-1)^{p+s}a^{(P)}$ in ten dimensions.

[5] $\mathcal{R}$ is zero in flat space and can therefore be omitted.

[6] Identifying $*dC^{(8)}$ as $F^{(9)}$.

to (A.1) will not only make the action positive definite, but it will also change the sign in front of the nine-form in formula (A.2), as is shown below. The equations of motion, (1.69) - (1.71), will not be altered since (A.3) is a total derivative. A clear way of obtaining this surface term and proving the sign change is by looking at the *ansatz*

$$S_{Min}(P^{(d-1)}, \phi, a) = -\frac{1}{2}\int \Big[\mathrm{d}\phi \wedge *\mathrm{d}\phi + e^{-b\phi}P^{(D-1)} \wedge *P^{(D-1)} - 2a \wedge \mathrm{d}P^{(D-1)}\Big] \tag{A.4}$$

where $P^{(D-1)}$, ϕ and a (pseudoscalar) are the fundamental independent fields, a generalized coupling b is introduced and D dimensions are used instead of ten. Looking at the Euler-Lagrange equation for a it is clear that $\mathrm{d}P^{(D-1)} = 0$ and hence $P^{(D-1)} = \mathrm{d}C^{(D-2)}$. Substituting this back into (A.4) gives

$$S_{Min}(C^{(D-2)}, \phi) = -\frac{1}{2}\int (\mathrm{d}\phi \wedge *\mathrm{d}\phi + e^{-b\phi}\mathrm{d}C^{(D-2)} \wedge *\mathrm{d}C^{(D-2)}) \tag{A.5}$$

which is the action (1.90) for $b = 2$ and $D = 10$. To obtain the action in terms of the axion a (in the Euclidean metric) first note that the Euclidean ansatz that follows from (A.4) is

$$S_{Euc}(P^{(D-1)}, \phi, a) = \frac{1}{2}\int \Big[\mathrm{d}\phi \wedge *\mathrm{d}\phi + e^{-b\phi}(P^{(D-1)} \wedge *P^{(D-1)}) - 2ia \wedge \mathrm{d}P^{(D-1)}\Big] \tag{A.6}$$

The advantage of using the nine-form $P^{(D-1)}$ is that it is invariant under a Wick rotation. Rewriting (A.6) as

$$\begin{aligned} S_{Euc}(P^{(D-1)}, \phi, a) =& \frac{1}{2}\int \Big[\mathrm{d}\phi \wedge *\mathrm{d}\phi + e^{-b\phi}(P^{(D-1)} + ie^{b\phi} * \mathrm{d}a) \wedge *(P^{(D-1)} + ie^{b\phi} * \mathrm{d}a) \\ &+ e^{b\phi}\mathrm{d}a \wedge *\mathrm{d}a - \frac{4}{b}d(iaP^{(D-1)})\Big] \end{aligned} \tag{A.7}$$

and performing a shift of variables $P^{'(D-1)} = (P^{(D-1)} + ie^{b\phi} * \mathrm{d}a)$ and substituting the equation of motion for $P^{'(D-1)}$ leads to[7]

$$S_{Euc}(P^{(D-1)}, \phi, a) = \frac{1}{2}\int \Big[\mathrm{d}\phi \wedge *\mathrm{d}\phi + e^{b\phi}\mathrm{d}a \wedge *\mathrm{d}a) - \frac{4}{b}d(iaP^{(D-1)})\Big] \tag{A.8}$$

Realizing that the shift of variables should not lead to a complex field $P^{'(D-1)}$, $a = ia$ (a real) and the main result of this section is the Euclidean action *with* the appropriate boundary term for $b = 2$

$$\boxed{S_{Euc}(\phi, a) = \frac{1}{2}\int \Big[\mathrm{d}\phi \wedge *\mathrm{d}\phi - e^{2\phi}\mathrm{d}a \wedge *\mathrm{d}a + 2d(e^{2\phi}a \wedge *\mathrm{d}a)\Big]} \tag{A.9}$$

Via Gauss's law the last term can be written as a boundary term

$$S^{EF}_{Surf} = \oint e^{2\phi}a \wedge *\mathrm{d}a \tag{A.10}$$

[7] which is $*P^{'(D-1)} = 0$

There are thus two equivalent formulations for calculating the action of the instanton on a manifold $\mathcal{M}$

$$S_{Euc}^{EF} = \int_{\mathcal{M}} \mathrm{d}^{10}x\sqrt{|g|}(-\mathcal{R}) + \frac{1}{2}\int_{\mathcal{M}} \Big[\mathrm{d}\phi\wedge *\mathrm{d}\phi - e^{2\phi}\mathrm{d}a\wedge *\mathrm{d}a + 2d(e^{2\phi}a\wedge *\mathrm{d}a)\Big] \quad \text{(A.11)}$$

This action gives rise to the same equations of motion as (1.69) - (1.71), for it differs only a total derivative. Or

$$S_{Euc}^{EF} = \int_{\mathcal{M}} \mathrm{d}^{10}x\sqrt{|g|}(-\mathcal{R}) + \frac{1}{2}\int_{\mathcal{M}} \Big[\mathrm{d}\phi \wedge *\mathrm{d}\phi + e^{-2\phi}F^{(9)} \wedge *F^{(9)}\Big] \quad \text{(A.12)}$$

First note the different sign in front of the terms $e^{2\phi}\mathrm{d}a \wedge *\mathrm{d}a$ in (A.11) and $e^{-2\phi}F^{(9)} \wedge *F^{(9)}$ in (A.12), this is important for the Bogomol'nyi bound. For the latter action can be written in such a form (1.91) and the former not. Note secondly that in Euclidian ten dimensional space

$$e^{-2\phi}F^{(9)} \wedge *F^{(9)} = e^{2\phi} * \mathrm{d}\phi \wedge * * \mathrm{d}\phi = e^{2\phi}\mathrm{d}\phi \wedge *\mathrm{d}\phi \quad \text{(A.13)}$$

and hence (A.12) becomes

$$S_{Euc}^{EF} = \int \mathrm{d}^{10}x\sqrt{|g|}(-\mathcal{R}) + \int \mathrm{d}\phi \wedge *\mathrm{d}\phi = \int \mathrm{d}^{10}x\sqrt{|g|}\big[-\mathcal{R} + \partial_\mu\phi\partial^\mu\phi\big] \quad \text{(A.14)}$$

and thus the action of the instanton (1.75) in flat space becomes

$$S_{E,Inst}^{EF} = \int \mathrm{d}^{10}x\sqrt{|g|}\partial_\mu\phi\partial^\mu\phi = -\int \mathrm{d}^{10}x\sqrt{|g|}\partial^2\phi \quad \text{(A.15)}$$

where for the second equality (1.74) has been used. To evaluate this integral apply Gauss's law [6]

$$S_{Euc}^{EF} = -\int_{\mathbb{R}^{10}} \sqrt{|g|}\partial^2\phi = -\oint_{S^9_{r=\infty}} \sqrt{|g^{S_9}|}\partial_\mu\phi n^\mu + \oint_{S^9_{r=0}} \sqrt{|g^{S_9}|}\partial_\mu\phi n^\mu \quad \text{(A.16)}$$

where n^μ is an outward pointing unit vector, i.e. radial

$$n^\mu n^\nu g_{\mu\nu}^{S_9} = 1 \longleftrightarrow n^r = \frac{1}{\sqrt{g_{rr}}} \quad \text{(A.17)}$$

and $g_{\mu\nu}^{S_9}$ is the metric *on* the nine sphere. Upon using (1.75), $g_{rr} = 1$ and defining the unit volume of a $(D-1)$-sphere as[8]

$$\mathrm{Vol}(S_{D-1}) = \oint_{S_9} \sin^{D-2}\theta_1 \ldots \sin\theta_{D-2}\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_{D-1} \quad \text{(A.18)}$$

leads to the D-instanton action

$$\boxed{S_{Euc}^{EF} = -\mathrm{Vol}(S_9)\oint_{S^9_{r=\infty}} \sqrt{|g^{S_9}|}\partial_r\phi n^r = \frac{8c\mathrm{Vol}(S_9)}{e^{\phi_\infty}}} \quad \text{(A.19)}$$

[8] It is convention to call this a volume element, although more correctly it should be called the hypersurface area.

A.3 Reissner-Nordstrøm metric[3]

The Reissner-Nordstrøm metric is obtained via demanding spherical symmetry and a point charge at the origin of the coordinate system. The same ansatz for the metric $g_{\mu\nu}$ is used as for the Schwarzschild metric, only $T_{\mu\nu}$ is non-zero now. Let's begin with the potential field in spherical coordinates

$$F_{\mu\nu} = E(r) \times \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow F_{rt} = \partial_r A_0 = E(r) \tag{A.20}$$

and use for $T_{\mu\nu}$ and the infinitesimal line element [3]

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2}(-g^{\rho\eta}F_{\mu\rho}F_{\nu\eta} + \frac{1}{4}g_{\mu\nu}F_{\rho\eta}F^{\rho\eta}) \equiv \frac{1}{2}(-F_{\mu\nu}^2 + \frac{1}{4}g_{\mu\nu}F^2) \\ \mathrm{d}s^2 &= e^{\nu(r)}\mathrm{d}t^2 - e^{\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega_2^2 \end{aligned} \tag{A.21}$$

$T_{\mu\nu}$ can be shown to be traceless and hence the Einstein equation becomes $\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \mathcal{R}_{\mu\nu} = T_{\mu\nu}$[9]. The Maxwell's equations in source-free regions are

$$\begin{aligned} \nabla_\mu F^{\nu\mu} &= 0 \\ \partial_{[\mu}F_{\nu\rho]} &= 0 \end{aligned} \tag{A.22}$$

Substituting the metric and $F_{\mu\nu}$ ansatz in the Maxwell equations and using the Einstein equation lead to

$$E(r) = \frac{2\epsilon}{r^2} = \partial_r A_0 \longleftrightarrow A_0(r) = \frac{-2\epsilon}{r} \tag{A.23}$$

Looking at the limit $r \to \infty$, the charge of the system is ϵ. The metric becomes

$$\mathrm{d}s_{RN}^2 = \Big(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\Big)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\Big(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\Big)} - r^2\Big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\Big) \tag{A.24}$$

It contains the removable singularities

$$r_\pm = m \pm \sqrt{m^2 - \epsilon^2} \tag{A.25}$$

There are therefore three cases to consider. To begin with $m < |\epsilon|$. Clearly there are now no real roots in the case, hence there is no horizon and thus the singularity at $r = 0$ is a *naked singularity*, this is like the case $m < 0$ for the Schwarzschild metric. According to the cosmic censorship hypothesis this case cannot occur in gravitational collapse. The second case of interest is when $m > |\epsilon|$, then the two real roots are (A.25). The third case $m = |\epsilon|$ is often referred to as the extremal Reissner-Nordstrøm (*ERN*) black hole and has only one singularity

$$r = m \tag{A.26}$$

with the corresponding metric

$$\mathrm{d}s_{ERN}^2 = \Big(1 - \frac{m}{r}\Big)^2\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\Big(1 - \frac{m}{r}\Big)^2} - r^2\Big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\Big) \tag{A.27}$$

[9]The convention used in this section is: $g_{\mu\nu}$ =diag$(1, -1, -1, -1)$.

As explained many times in the text, the easiest way to see if a wormhole is present, is rewriting the metric in isotropic form. This requires the radial part of the line element (A.24) to be described by a conformal factor, giving

$$\mathrm{d}s^2 = \frac{(4\rho^2 - m^2 + \epsilon^2)^2}{(4\rho^2 + 4m\rho + m^2 - \epsilon^2)^2} - \frac{1}{\rho^2}\Big(\rho + m + \frac{m^2 - \epsilon^2}{4\rho}\Big)^2\Big(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_2^2\Big) \quad (A.28)$$

where

$$\begin{aligned}\rho(r) &= \frac{1}{2}\Big(r - m + \sqrt{\epsilon^2 - 2mr + r^2}\Big)\\ r(\rho) &= m + \rho + \frac{m^2 - \epsilon^2}{4\rho}\end{aligned} \quad (A.29)$$

The metric has only for the case $m^2 - \epsilon^2 > 0$ a wormhole symmetry

$$\rho \longleftrightarrow \frac{m^2 - \epsilon^2}{4\rho} \quad \text{or} \quad r = m + \sqrt{m^2 - \epsilon^2} \quad (A.30)$$

For general dimensions $(D + 1) > 3$ the transformation rules are

$$\begin{aligned}r(\rho) &= \frac{\Big(2\alpha\rho^{D-2} + \frac{\rho^{2(D-2)}}{C_1^{(D-2)}} + \alpha^2 C_1^{D-2} - 4\beta^2 C_1^{D-2}\Big)^{\frac{1}{D-2}}}{2^{\frac{2}{D-2}}\rho}\\ \rho(r) &= \Big(2r^{D-2} - \alpha + 2\sqrt{r^{2(D-2)} - r^{D-2}\alpha + \beta^2}\Big)^{\frac{1}{D-2}} C_1\end{aligned} \quad (A.31)$$

where α (β) is the generalized mass (charge) and the conformal factor in front of the explicit $SO(D)$ symmetry

$$f(\rho) = \frac{\Big(2\alpha\rho^{D-2} + \frac{\rho^{2(D-2)}}{C_1^{(D-2)}} + \alpha^2 C_1^{D-2} - 4\beta^2 C_1^{D-2}\Big)^{\frac{1}{D-2}}}{2^{\frac{2}{D-2}}\rho^2} \quad (A.32)$$

with the wormhole symmetry

$$\rho^{D-2} \longleftrightarrow \frac{(\alpha^2 - 4\beta^2) C_1^{2(D-2)}}{\rho^{D-2}} \quad (A.33)$$

if $\alpha^2 - 4\beta^2 > 0$. To compare to the four dimensional case above choose $C_1 = \frac{1}{4}$, $\alpha = 2m$ and $\beta = q$.

A.4 Noether currents and conserved charges[21]

In type IIB string theory Lagrangians of the form

$$\mathcal{L} = \frac{1}{4}\sqrt{|g|}\mathrm{Tr}[\partial_\mu \mathcal{M}^{-1}\partial^\mu \mathcal{M}] \quad (A.34)$$

are important. As an example take $\mathcal{M}$ with a $SL(2, \mathbb{R})$ symmetry, just like the type IIB action. Since this group has three generators, there are also three conserved currents present. To derive these observe that due to the cyclic permutation property of the trace the Lagrangian above is invariant under the global transformations

$$\mathcal{M} \to \mathcal{M}' = \Lambda^T \mathcal{M} \Lambda \tag{A.35}$$

for a $SL(2,\mathbb{R})$ matrix Λ, see for example chapter one (1.105). Infinitesimally this can be written as

$$\begin{aligned} \Lambda &= \mathbf{1} + \lambda \\ \delta\mathcal{M} &= \lambda^T \mathcal{M} + \mathcal{M}\lambda \end{aligned} \tag{A.36}$$

with λ infinitesimal. To obtain the Noether currents, the Lagrangian is varied with respect to a space-time dependent transformation and keeping only those terms where a derivative falls on the parameter λ

$$\begin{aligned} \delta\mathcal{L} &= \delta\frac{1}{4}\sqrt{|g|}\mathrm{Tr}[\partial_\mu \mathcal{M}^{-1}\partial^\mu \mathcal{M}] \\ &= -\mathrm{Tr}[\partial_\mu \lambda \mathcal{M}^{-1}\partial^\mu \mathcal{M}] \Rightarrow \\ \mathcal{J}_\mu &= -\mathcal{M}^{-1}\partial_\mu \mathcal{M} = \begin{pmatrix} \mathcal{J}_{11\mu} & \mathcal{J}_{12\mu} \\ \mathcal{J}_{21\mu} & -\mathcal{J}_{11\mu} \end{pmatrix} \quad \text{with} \\ \nabla_\mu \mathcal{J}^\mu &= 0 \end{aligned} \tag{A.37}$$

for systems with $SL(2,\mathbb{R})$ symmetry. Via the generalized Gauss's law one can define the corresponding conserved charges as

$$\mathcal{Q} = \oint_{S_9} \mathcal{J} = \oint \mathcal{J}_\mu n^\mu \tag{A.38}$$

where n^μ is an outward directed unit vector, see (A.17) and where $g^{S_9}_{\mu\nu}$ stands for the metric on the nine sphere,

$$\sqrt{|g^{S_9}|} = r^{D-1} g_{rr}^{\frac{D-1}{2}} \sin^{D-2}\theta_1 \ldots \sin\theta_{D-2} \tag{A.39}$$

To calculate this integral for the D-instanton realizes that the integrand becomes

$$\mathcal{J}_\mu n^\mu = \sqrt{|g^{S_9}|}\mathcal{J}_r n^r = r^{D-1} g_{rr}^{\frac{(D-2)}{2}} \mathcal{J}_r \sin^{D-2}\theta_1 \ldots \sin\theta_{D-2} \tag{A.40}$$

which gives for the conserved (Euclidean) charge matrix

$$\boxed{\mathcal{Q} = \mathrm{Vol}(S_9) r^{D-1} g_{rr}^{\frac{(D-2)}{2}} \mathcal{J}_r} \tag{A.41}$$

As an example look at one of the conserved currents of the extremal D-instanton

$$\mathcal{J}_{12_r} = -e^\phi \partial_r \phi \to \mathcal{Q}_{12_r} = 8c\mathrm{Vol}(S_9) \tag{A.42}$$

which gives to the constant of integration c the interpretation as a charge. Combining this with the action (A.19) gives

$$S = \frac{|Q|}{g} \tag{A.43}$$

where

$$Q = \mathcal{Q}_{12_r} \tag{A.44}$$

A.5 Another frame: The dual frame

The Einstein frame and the string frame have been used a lot in the text. The Einstein frame definition implies a canonical Einstein-Hilbert term, i.e. $\sqrt{|g_{EF}|}\mathcal{R}$, whereas the string frame has an axion a kinetic term free from dilaton ϕ coupling, $\sqrt{|G_{SF}|}(\partial a)^2$. These two frames appear in the theory naturally, one can of course choose a random conformal mapping (RM) , for example

$$g_{\mu\nu}^{EF} \equiv G_{\mu\nu}^{RM} e^{\alpha\phi} \tag{A.45}$$

for any $\alpha \in \mathbb{R}$, but what would they mean physically? It turns out that there is a frame which has a physical background, the so called *dual frame* (DF). To see this consider first the truncated type IIB low energy effective action in D dimensions with again a generalized coupling b in front of the generalized p-form F_{p+2}

$$S_{EF} = \alpha \int \mathrm{d}^D x \sqrt{|g|}\Big[\mathcal{R} - \frac{4}{D-2}(\partial\phi)^2 - \beta e^{b\phi} F_{p+2}^2\Big] \tag{A.46}$$

where α and β are constants which are needed to have proper units, but these are irrelevant for the story. Note that with $b = 2$, $p = -1$ and $D = 10$ one obtains the action for the D-instanton as discussed in chapter one. This is the so called electric formulation, taking the Hodge dual of F_{p+2} gives the magnetic formulation $\tilde{F}_{D-p-2}$

$$\tilde{F}^{\mu_1\ldots\mu_{D-p-2}} = \frac{(-1)^{D+p-1}}{\sqrt{|g|}(p+2)!} e^{b\phi} \epsilon^{\nu_1\ldots\nu_{p+2}\mu_1\ldots\mu_{D-p-2}} F_{\nu_1\ldots\nu_{p+2}} \tag{A.47}$$

Introducing $\tilde{d} = D - d - 2$ and $d = p + 1$ gives for the action

$$S = \alpha \int \mathrm{d}^D x \sqrt{|g|}\Big[\mathcal{R} - \frac{4}{D-2}(\partial\phi)^2 - \tilde{\beta} e^{-b\phi} F_{\tilde{d}+1}^2\Big] \tag{A.48}$$

where $\tilde{\beta}$ and $-b$ are a result from taking the Hodge dual (A.47). The dual frame can be obtained via demanding an overall dilaton coupling, i.e.

$$S_{DF} \propto \int \mathrm{d}^D x e^{\gamma\phi} \sqrt{|G|}\Big[\mathcal{R} - \frac{4}{D-2}(\partial\phi)^2 - \tilde{\beta} F_{\tilde{d}+1}^2\Big] \tag{A.49}$$

To determine γ observe that under (A.45) the terms change as

$$\begin{aligned} \sqrt{|g|}\mathcal{R}_g &\to \sqrt{|G^D|}\mathcal{R}_G^D e^{(\frac{D\alpha}{2}-\alpha)\phi} \\ \sqrt{|g|}e^{-b\phi} F_{\tilde{d}+1}^2 &\to \sqrt{|G^D|} F_{\tilde{d}+1}^2 e^{(\frac{D\alpha}{2}-b-(\tilde{d}+1)\alpha)\phi} \end{aligned} \tag{A.50}$$

These two changes have to be equal in the dual frame

$$\boxed{\alpha = -\frac{b}{\tilde{d}} \longrightarrow \gamma = -\frac{(D-2)b}{2\tilde{d}} = -\frac{b}{2}} \tag{A.51}$$

where the last equality holds only if $p = -1$. Note that in the electric formulation this fails, unless one takes $b = 0$. How about the physical interpretation? It can be shown that the dual frame describes a $(d+1)$-dimensional Anti-de Sitter (Ad) spacetime times a $(\tilde{d}+1)$-dimensional sphere: $AdS_{d+1} \times S^{\tilde{d}+1}$, see for example [10] for details.

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