Embedding Tensor Approach to Maximal $D = 8$ Supergravity

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Chapter 1

Introduction

Einstein's Theory of General Relativity, published in 1915, achieved to unify Gravity with space-time itself. Two concepts thought to be independent since the times of Newton, were then merged into a unique, more general theory. The same way Maxwell did some decades before unifying the electric and magnetic phenomena into Electromagnetism. Unifications appear repeatedly throughout the history of physics and always represent a dramatic change in our understanding of Nature. At the time Einstein's theory was published, Quantum Mechanics were already uncovering a whole new insight on matter at very small scales that would lead, through the twentieth century, to the successful Standard Model of Particles. The next natural step was to attempt to bring together the quantum world (very small scales) with Gravity (very large scales) in a single theory. Various theories were proposed, but the one that took the lead was Superstring Theory. A theory in ten dimensions were all particles are vibration modes of fundamental one-dimensional objects, called strings. It turns out there are five different consistent formulations of Superstring Theory which share some characteristics and differ in some others. They all predict a new tower of states, where the lowest mass states correspond to massless particles and the first non-zero mass states have masses of the order of the Planck mass $\sim 10^{19}$GeV $\sim 20\mu g$. They all also suffer from fundamental pathologies (anomalies) which make them unstable. In the 70's a new idea was introduced to take care of these anomalies, Supersymmetry.

Supersymmetry is a symmetry between the fermionic degrees of freedom and the bosonic degrees of freedom, i.e. a supersymmetric theory is invariant under the interchange of fermions and bosons. It has very appealing properties and it is widely used, although it has a major disadvantage, Nature does not seem to be supersymmetric. However, a theory can still be supersymmetric even if the solutions do not seem to be supersymmetric.

Since the massive states predicted by Superstring Theory are simply out of rich for our present technology, we are mainly interested in the massless states. These states can be described at a classical level by Supergravity theories. A Supergravity theory is a classical filed theory that describes gravity taking into account supersymmetry. They can be formulated in dimensions equal or less than eleven. Many supergravities that live in space-time dimensions less than eleven can be obtained from a eleven-dimensional supergravity by writing eleven-dimensional space-time as a direct product space $X \times Y$ where X is D-dimensional and Y is a q-dimensional manifold (11=D+q), compact and wrapped in such a way we can make it of size small enough not to be observed. There are different ways to wrap these extra dimensions by means of dimensional reduction, a procedure to obtain lower-dimensional theories from higher dimensional ones. If string theory is to describe the world around us, a proper understanding of dimensional reductions is indispensable (See Appendix A), since the observable world surely is not eleven-dimensional.

Supergravity theories are given a number $\mathcal{N}$ corresponding to the amount of supersymmetry they exhibit. The purpose of this thesis is the study of a maximal case, $\mathcal{N} = 8$ in $D = 8$. Over recent years it has been understood that the structure of supergravity theories with a maximal number of supercharges is far more complex than originally expected. Maximal supergravities are obtained as deformations (gaugings) of toroidally compactified eleven-dimensional supergravity by coupling the original Abelian vector fields to charges assigned to elementary fields. Two features have been proven universal in the construction of such theories. First, when reduced to $D = 11 - q$ dimensions, the theory is organized by a global symmetry group $G = \mathrm{E}_{\mathrm{q(q)}}$. This same group determines the possible deformations. All gaugings can be parameterized in terms of a constant embedding tensor Θ. Consistency of the theory can then be encoded in a small number of constraints on Θ.

Second, the gaugings involve tensor fields of various ranks (p-forms) together with their duals ($D-p-2$-forms), (See appendix B). In order to realize the full symmetry group G, the theory must be expressed in terms of the lowest possible rank forms, i.e. dual fields must be used when being of lower rank than the ones they are dual from. The specific form of the embedding tensor in a particular gauging encodes the proper distribution of degrees of freedom among these fields. Nevertheless, as long as no particular gauging is considered, the collection of all possible deformations can be formulated in a manifestly G-covariant way. The embedding tensor keeps all the information of the theory. Although most of this formalism has been established for the global symmetry groups $G = \mathrm{E}_{\mathrm{q(q)}}$ of the maximal supergravities, the structure is not restricted to maximal supergravity and similarly underlays theories with lower number of supercharges.

In the present thesis we realize this framework for the maximal $D = 8$ case. The ungauged maximal supergravity in eight dimensions possesses a global $\mathrm{E}_{3(3)}$=SL(2)×SL(3) symmetry. This theory is formulated entirely in terms of vector, two-form and three-form tensor fields, transforming in the $(2,3)$, $(1,\overline{3})$ and $(2,1)$ representation of SL(2)×SL(3), respectively. Literature on the gaugings of the eight-dimensional theory is rather inexistent. As theories in many other dimensions have been widely investigated and worked out it appears strange to the author why the eight-dimensional case was left apart and constitutes one of the main motivations of this thesis. Together with the fact that the global symmetry group of the $D = 8$ Supergravity is a product group. Characteristic also present in more realistic theories in four dimensions. And with the major advantage that the symmetry groups involved are relatively small. So we can fully study the formalism.

This work is structured as follows. In section two we review the gauge procedure in general and make connection with the embedding tensor formalism for $D = 8$. We discuss the constraints in the embedding tensor in detail and we also present the field content of the theory. The inclusion of Stückelberg type terms in the field strengths is necessary for them to covariantly transform and we discuss the explicit constraints these leads to. In section three the structure of the solutions is depicted and some particular cases are worked out in detail. Section four is for the conclusions and acknowledgments and references will follow.

Chapter 2

Maximal $D = 8$ supergravity

In this chapter we start with a supersymmetric theory with global symmetry group G_0 and ask for the possible gaugings of this theory that are compatible with supersymmetry, i.e. we demand the deformations of the theory not to break supersymmetry. Although the answer to this question needs a case by case study, there is a general technique to parameterize the deformations via an embedding tensor Θ, which is a tensor under the global symmetry group G_0 and has to satisfy certain group theoretical constraints. Every single gauging breaks the global symmetry G_0 down to a local gauge group $G \subset G_0$, but the set of all possible gaugings can be described in a G_0 covariant way by using Θ. This embedding tensor and the constraints it has to satisfy are introduced in the following section.

2.1 Gauge Theories

Before we get started it is convenient to review the concepts of gauge theory and gauge invariance as we will be using them constantly throughout this thesis. To gauge a theory means to turn certain global symmetry the theory exhibits into a local symmetry, which translates in turning a free non-interacting theory into an interacting theory. Let's see an example. Consider the following Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - \frac{1}{4}(F_{\mu\nu})^2, \tag{2.1}$$

where the first term is the Dirac term describing fermions and the second term is the Maxwell term describing electromagnetism (photons). $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength and A_μ the electromagnetic vector potential. This Lagrangian is invariant under the global transformation $\psi(x) \to e^{i\alpha}\psi(x)$, where α is a constant. Now we turn the global transformation into a local gauge transformation, i.e.

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \qquad\qquad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x). \tag{2.2}$$

For the Lagrangian to still be invariant under such transformation we must introduce the gauge covariant derivative,

$$D_\mu \equiv \partial_\mu + ieA_\mu. \tag{2.3}$$

What we obtain after gauging this theory is the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2 = \mathcal{L}_0 - e\bar{\psi}\gamma^\mu\psi A_\mu \tag{2.4}$$

The QED Lagrangian is invariant under a U(1) local gauge transformation (local phase rotation). This is an example of an Abelian gauge theory (U(1) is Abelian). We see that gauging the theory naturally arises an interacting term between fermions and gauge fields (photons).

In order to understand the meaning of the embedding tensor it is useful to go a little further and see an example of non-Abelian gauge theories. Let's review a Yang-Mills Lagrangian. First we generalize the fermionic sector to be a triplet of Dirac fields,

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}. \tag{2.5}$$

As a side note u, d and s would correspond to the up, down and strange quarks. But this is not relevant to discuss the structure of the gauge procedure. So in general the three fermionic fields will transform under a global SU(3). The generators of the group do not commute (the group is non-Abelian), they obey the usual commutation relation $[t_a, t_b] = f_{ab}{}^c t_c$, with $f_{ab}^{\ c}$ being the structure constants of the group. The theory will contain as many vector fields as generators of the symmetry group and the free field lagrangian reads

$$\mathcal{L}_0 = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2, \tag{2.6}$$

with $a = 1, ..., 8$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$. Without going into the details we introduce the corresponding infinitesimal gauge transformations

$$\begin{aligned} \psi &\to (1 + i\alpha^a(x)t_a)\psi, \\ A^a_\mu &\to A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a(x) + f_{bc}{}^a A^b_\mu \alpha^c(x). \end{aligned} \tag{2.7}$$

In order to restore gauge invariance in the Lagrangian we use the covariant derivative and we redefine the field strength of the vector fields as follow

$$\begin{aligned} D_\mu &= \partial_\mu - igA^a_\mu t_a, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}{}^a A^b_\mu A^c_\nu. \end{aligned} \tag{2.8}$$

The interaction terms emerge in the lagrangian analogously to the QED case

$$\mathcal{L} = \mathcal{L}_0 + gA^a_\lambda \bar{\psi}\gamma^\lambda t_a \psi - gf^{bc}_{\ \ a}(\partial_\kappa A^a_\lambda)A^\kappa_b A^\lambda_c - \frac{1}{4}g^2(f_{ab}^{\ \ e}A^a_\kappa A^b_\lambda)(f^{cd}_{\ \ e}A^\kappa_c A^\lambda_d). \tag{2.9}$$

The point in this discussion is that we could be interested in gauging only a subgroup of SU(3) instead of the whole group. Then we would repeat the procedure starting from a subgroup, SU(2) for instance. The fermionic fields would be represented by a doublet

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \tag{2.10}$$

and we would only gauge three of the eight vector fields of the theory. The rest would remain as non-interacting vector fields. This would break the global SU(3) symmetry down to the gauge group symmetry. To avoid a case by case study we introduce the embedding tensor in the way described in next section. We will preserve invariance under the global symmetry. The theory will transform covariantly and the embedding tensor will act as a map from the representation of the vector fields to the global symmetry group defining all possible gaugings of subgroups of the global symmetry group.

2.2 The Embedding tensor formalism

2.2.1 General structure

We start from an ungauged supersymmetric theory with global symmetry group G_0. The symmetry generators of the corresponding algebra $\mathfrak{g}_0$ are denoted t_α, α=1,...,dim($\mathfrak{g}_0$). They obey the usual commutation relation

$$[t_\alpha, t_\beta] = f_{\alpha\beta}^{\ \ \gamma} t_\gamma, \tag{2.11}$$

where $f_{\alpha\beta}^{\ \ \gamma}$ are the structure constants of G_0. In this case gauging the theory means to turn part of this global symmetry into a local one. To preserve gauge invariance one needs to introduce minimal couplings of vector gauge fields. i.e. one replaces the derivatives ∂_μ by covariant derivatives D_μ. The theory to start with contains vector fields A^M_μ that transform in some representation V (indicated by the index M) of the global symmetry group G_0. These vector fields are U(1) gauge fields, so they will transform both under $\mathfrak{g}_0$-transformations L^α and under local gauge transformations $\Lambda^M(x)$:

$$\delta_L A^M_\mu = -L^\alpha t_{\alpha N}^{\ \ M} A^N_\mu, \qquad \delta_\Lambda A^M_\mu = \partial_\mu \Lambda^M. \tag{2.12}$$

In the covariant derivative of the gauged theory these vector fields A^M_μ need to be coupled to the G_0 symmetry generators t_α,

$$D_\mu = \partial_\mu - gA^M_\mu \Theta_M^{\ \alpha} t_\alpha, \tag{2.13}$$

where $\Theta_M^{\ \alpha}$ is the so-called embedding tensor and $g \in \mathbb{R}$ is the gauge coupling constant, which could also be absorbed into $\Theta_M^{\ \alpha}$. The embedding tensor $\Theta_M^{\ \alpha}$ has to be real and appears in (2.13) as a map $\Theta : V \to \mathfrak{g}_0$. The image of this map defines the gauge group G and the possible gauge transformations are parameterized by $\Lambda^M(x)$. For example, a field B_M in the dual representation $\overline{V}$ of the vector gauge fields transforms under G as

$$\delta B_M = g\Lambda^N \Theta_N^{\ \alpha} t_{\alpha M}^{\ \ \ P} B_P = g\Lambda^N X_{NM}^{\ \ \ \ P} B_P. \tag{2.14}$$

Here we introduce the gauge group generators $X_M = \Theta_M^{\ \alpha} t_\alpha$, which in the vector field representation $X_{NM}^{\ \ \ \ P} = \Theta_M^{\ \alpha} t_{\alpha N}^{\ \ \ P}$ take the role of generalized structure constants for the gauge group G. Note that $X_{MN}^{\ \ \ \ P}$ contains the whole information on Θ and although the embedding tensor is not invariant under the global symmetry group G_0 we can demand Θ to be gauge invariant, i.e. $\delta\Theta = \Lambda^M \delta_M \Theta{=}0$, to ensure the closure of the gauge group and a manifestly covariant transformation of the construction. Explicitly

$$\delta_M \Theta_N^{\ \alpha} = g\Theta_M^{\ \beta} \left(t_{\beta N}^{\ \ \ P} \Theta_P^{\ \alpha} - f_{\beta\gamma}^{\ \ \ \alpha} \Theta_N^{\ \gamma} \right) = 0. \tag{2.15}$$

Equivalently one can demand the generators $X_{MN}^{\ \ \ \ P}$ to be gauge invariant, and the equation $\delta_M X_{NP}^{\ \ \ \ Q} = 0$ can be written as

$$[X_M, X_N] = -X_{MN}^{\ \ \ \ P} X_P. \tag{2.16}$$

This equation guarantees the closure of the gauge group and it is the generalized Jacobi identity when evaluated in the vector field representation. The two equivalent relations (2.15) and (2.16) represent a quadratic constraint on Θ. The embedding tensor has to satisfy this constraint in order to describe a valid gauging.

In addition to this quadratic constraint a linear constraint on Θ is also needed. Eventually, it is supersymmetry which demands this linear constraint. The embedding tensor transforms in the representation $\overline{V} \otimes \mathfrak{g}_0 = \theta_1 \oplus \theta_2 \oplus ... \oplus \theta_n$, where θ_i, $i = 1, ..., n$, are the irreducible components of the tensor product. The linear constraint needs to be G_0 invariant. Thus, each irreducible component θ_i is either completely forbidden by the linear constraint or not restricted at all, as θ_i are either invariant under G_0 or they are not. This constraint can be written as a projector equation $\mathbb{P}_1\Theta = 0$, where $\mathbb{P}_1$ projects onto those representations in Θ that are forbidden. Similarly, the quadratic constraint can be written as $\mathbb{P}_2 (\Theta \otimes \Theta) = 0$, where $\mathbb{P}_2$ projects on the appropriate representation in the symmetric tensor product of $\overline{V} \otimes \mathfrak{g}_0$. One could also imagine higher order constraints like $\mathbb{P}_3 (\Theta \otimes \Theta \otimes \Theta) = 0$, but it turns out that the linear and quadratic constraint are sufficient for the construction of the gauged theory.

We summarize this section. When describing the general gauging of a supersymmetric theory, the embedding tensor Θ can be used to parameterize the gauging. Any Θ that satisfies the appropriate linear constraint and quadratic constraint (2.15) describes a valid gauging and the construction of the gauged theory only requires these constraints for consistency. When Θ is treated as a spurionic object, i.e it transforms under global symmetry group G_0, one does formally preserve the G_0 symmetry in the gauged theory. This reflects the fact that the set of all possible gaugings is G_0 invariant. But as soon as a particular gauging is considered, the embedding tensor that describes this gauging breaks the G_0 invariance down to the gauge group $G \subset G_0$.

2.2.2 Maximal $D = 8$ Supergravity linear and quadratic constraints

The maximal $D = 8$ supergravity comes from a toroidal dimensional reduction from the unique 11-dimensional supergravity. The global symmetry group of the ungauged theory after dimensional reduction is $G_0 = \mathrm{E}_{\mathrm{q(q)}} = SL(2) \times SL(3)$. The theory contains one graviton, seven scalars organized in a coset structure (See appendix A), six vector fields transforming in the $(2, 3)$ representation of G_0, three two-forms $(1, \bar{3})$ and a three-form $(2, 1)$ which is self-dual (that is why it transforms in the (2,1) instead of (1,1)). The embedding tensor is the result of the inner product of the global symmetry group G_0 and the dual vector representation of the vector fields. And it can be decomposed in irreducible representations. After applying the linear constrain only some of these are allowed [1].

$$\left.\begin{array}{l} G_0 = SL(2) \times SL(3) \\ A_\mu^{\alpha M} \rightarrow V = (2,3) \end{array}\right\} \quad \Theta \rightarrow \mathfrak{g}_0 \otimes \overline{V} = (2, \bar{3}) \oplus (2, 6) \oplus ...$$

The two irreducible representations above are the only ones allowed by the linear constraint.

For SL(2) we have 3 generators t_β^α with $\alpha, \beta = 1, 2$ and $t_\alpha^\alpha = 0$. Analogously, for SL(3) we have eight generators t_M^N with $M, N = 1, 2, 3$ and $t_N^N = 0$.

$$\begin{array}{l} SL(2)\left\{\begin{array}{l} (t_\alpha^\beta)_\gamma^\delta = \delta_\gamma^\beta \delta_\alpha^\delta - \frac{1}{2}\delta_\alpha^\beta \delta_\gamma^\delta \\ f_{\alpha\gamma\rho}^{\beta\delta\sigma} = \delta_\alpha^\delta \delta_\gamma^\sigma \delta_\rho^\beta - \delta_\gamma^\beta \delta_\rho^\delta \delta_\alpha^\sigma \end{array}\right. \\ \\ SL(3)\left\{\begin{array}{l} (t_M^N)_P^Q = \delta_P^N \delta_M^Q - \frac{1}{3}\delta_M^N \delta_P^Q \\ f_{BDF}^{ACE} = \delta_B^C \delta_D^E \delta_F^A - \delta_D^A \delta_F^C \delta_B^E \end{array}\right. \end{array} \tag{2.17}$$

We can split the embedding tensor into two terms. One for the SL(2) part and one for the SL(3) part. Now the covariant derivative has two terms (besides the partial derivative), the SL(2) piece of the embedding tensor contracted with the adjoint representation of SL(2) will generate the gauging of SL(2) and the SL(3) part contracted with the adjoint representation of SL(3) will generate the gauging of the SL(3).

$$D_\mu = \partial_\mu - g A_\mu^{\alpha M} \Theta_{\alpha M,\gamma}{}^{\beta} t_\beta^\gamma - g A_\mu^{\alpha M} \Theta_{\alpha M,R}{}^{N} t_N^R \tag{2.18}$$

Let's write explicitly the quadratic constrain (2.15) for the two terms of the embedding tensor

$$\delta\Theta_{\alpha M,\gamma}{}^{\beta} = \Lambda^{\delta N}\left[\Theta_{\delta N,\sigma}{}^{\rho}\left((t^{\sigma}_{\rho})^{\tau}_{\alpha}\Theta^{\beta}_{\tau M,\gamma} + f^{\sigma\kappa\beta}_{\rho\pi\gamma}\Theta_{\alpha M,\kappa}{}^{\pi}\right) + \Theta_{\delta N,P}{}^{Q}(t^{P}_{Q})^{R}_{M}\Theta_{\alpha R,\gamma}{}^{\beta}\right] = 0$$
$$\delta\Theta_{\alpha M,D}{}^{A} = \Lambda^{\delta N}\left[\Theta_{\delta N,E}{}^{B}\left((t^{E}_{B})^{R}_{M}\Theta_{\alpha R,D}{}^{A} + f^{EFA}_{BCD}\Theta_{\alpha M,F}{}^{C}\right) + \Theta_{\delta N,\rho}{}^{\gamma}(t^{\rho}_{\gamma})^{\sigma}_{\alpha}\Theta_{\sigma M,D}{}^{A}\right] = 0 \tag{2.19}$$

or, if we substitute the explicit expressions for the generators and structure constants, the more compact form

$$\Theta_{\delta N,\alpha}{}^{\tau}\Theta_{\tau M,\gamma}{}^{\beta} + \Theta_{\delta N,\gamma}{}^{\kappa}\Theta_{\alpha M,\kappa}{}^{\beta} - \Theta_{\delta N,\pi}{}^{\beta}\Theta_{\alpha M,\gamma}{}^{\pi} + \Theta_{\delta N,M}{}^{B}\Theta_{\alpha B,\gamma}{}^{\beta} = 0$$
$$\Theta_{\delta N,M}{}^{C}\Theta_{\alpha C,D}{}^{A} + \Theta_{\delta N,D}{}^{B}\Theta_{\alpha M,B}{}^{A} - \Theta_{\delta N,E}{}^{A}\Theta_{\alpha M,D}{}^{E} + \Theta_{\delta N,\alpha}{}^{\beta}\Theta_{\beta M,D}{}^{A} = 0 \tag{2.20}$$

It will be useful to express the embedding tensor in terms of objects that transform under the allowed irreducible representations, so the information of the linear constraint is included in the quadratic constraint.

$$\xi_{\alpha M} \to (2,\bar{3}); \qquad \Theta_{\alpha M,\gamma}{}^{\beta} = \xi_{\gamma M}\delta^{\beta}_{\alpha} - \frac{1}{n}\delta^{\beta}_{\gamma}\xi_{\alpha M}$$
$$f_{\alpha(MN)} \to (2,6); \qquad \Theta_{\alpha M,Q}{}^{P} = f^{PB}_{\alpha}\epsilon_{BMQ} + x\left(\xi_{\alpha Q}\delta^{P}_{M} - \frac{1}{N}\xi_{\alpha M}\delta^{P}_{Q}\right) \tag{2.21}$$

where n is equal to the maximal value the SL(2) indices take (two in our case) and N is equal to the maximal value of the SL(3) indices (three in our notation). This assures the embedding tensor is traceless. The inclusion of a SL(2) type term in the SL(3) part is necessary for the quadratic constraint to have non-trivial solutions and will lead to important features of the possible gaugings as we will discuss in following chapter.

If we substitute (2.21) in (2.20) we obtain two sets of equations. For the SL(2) sector we have

$$\delta^{\gamma}_{\alpha}\left[\xi_{\beta M}\xi_{\mu A} - \left(\frac{1}{n} + \frac{x}{N}\right)\xi_{\mu M}\xi_{\beta A} + x\xi_{\mu A}\xi_{\beta M} + f^{BC}_{\mu}\epsilon_{CMA}\xi_{\beta B}\right]$$
$$+\frac{\delta^{\gamma}_{\beta}}{n}\left[\left(\frac{1}{n} + \frac{x}{N}\right)\xi_{\mu M}\xi_{\alpha A} - x\xi_{\alpha M}\xi_{\mu A} - \xi_{\mu A}\xi_{\alpha M} - f^{BC}_{\mu}\epsilon_{CMA}\xi_{\alpha B}\right] = 0 \tag{2.22}$$

If we symmetrize (2.22) in M and A we obtain a constraint for x

$$\left(\delta^{\gamma}_{\alpha}\xi_{\beta(M}\xi_{\mu A)} - \frac{\delta^{\gamma}_{\beta}}{n}\xi_{\alpha(M}\xi_{\mu A)}\right)\left(1 - \frac{1}{n} - \frac{x}{N} + x\right) = 0, \tag{2.23}$$

substituting $n = 2$ and $N = 3$ the value for x is $x = -3/4$

We can also anti-symmetrize in M and A

$$\left(\delta^\gamma_\alpha \xi_{\beta[M}\xi_{\mu A]} - \frac{\delta^\gamma_\beta}{n}\xi_{\alpha[M}\xi_{\mu A]}\right)\left(1 + \frac{1}{n} + \frac{x}{N} + x\right) \\ +\delta^\gamma_\alpha f^{BC}_\mu \epsilon_{C[MA]}\xi_{\beta B} - \frac{\delta^\gamma_\beta}{n} f^{BC}_\mu \epsilon_{C[MA]}\xi_{\alpha B} = 0 \tag{2.24}$$

Already (2.24) can be simplified into two definite constraints

$$\begin{aligned} &\xi_{\beta[M}\xi_{\mu A]} = 0, \text{ which implies the two vectors must be parallel,} \\ &f^{BC}_\mu \xi_{\beta B} = 0, \text{ which means } \vec{\xi}_\beta \text{ are eigenvectors of the matrices } f_{\mu(BC)}. \end{aligned} \tag{2.25}$$

For the SL(3) part we obtain a more complicated equation

$$\begin{aligned} f^{LP}_\mu f^{DQ}_\alpha \epsilon_{MAP}\epsilon_{LCQ} - f^{LP}_\mu f^{DQ}_\alpha \epsilon_{MCP}\epsilon_{LAQ} - f^{DP}_\mu f^{LQ}_\alpha \epsilon_{MLP}\epsilon_{ACQ} \\ +x f^{DP}_\alpha \xi_{\mu A}\epsilon_{MCP} - x f^{DQ}_\alpha \xi_{\mu C}\epsilon_{MAQ} + f^{DP}_\mu \xi_{\alpha M}\epsilon_{ACP} - \left(\frac{x}{N} + \frac{1}{n}\right) f^{DP}_\alpha \xi_{\mu M}\epsilon_{ACP} \\ +x\delta^D_A f^{LP}_\mu \xi_{\alpha L}\epsilon_{MCP} - x\delta^D_M f^{LP}_\alpha \xi_{\mu L}\epsilon_{ACP} - \frac{x}{N}\delta^D_C f^{LP}_\mu \xi_{\alpha L}\epsilon_{MAP} \\ -\left(\frac{x^2}{N} + \frac{x}{N}\right)\xi_{\mu A}\xi_{\alpha M}\delta^D_C + x(1+x)\xi_{\mu C}\xi_{\alpha M}\delta^D_A \\ +\frac{x}{N}\left(\frac{x}{N} + \frac{1}{n}\right)\xi_{\mu M}\xi_{\alpha A}\delta^D_C - \left(\frac{x^2}{N} + \frac{x}{n}\right)\xi_{\mu M}\xi_{\alpha C}\delta^D_A = 0 \\ . \end{aligned} \tag{2.26}$$

However, using (2.25) this equation reduces to

$$\epsilon^{\alpha\mu}[f^{DP}_\alpha f^{LR}_\mu \epsilon_{MLP} - f^{DR}_\alpha \xi_{\mu M}] = 0. \tag{2.27}$$

To summarize, valid gaugings are parameterized by appropriate tensors $\xi_{\alpha M}$ and $f_{\alpha(MN)}$ which transform under the allowed representations of G_0 dictated by the linear constraint. The quadratic constraints in terms of such tensor reduce to three simple constraints, (2.25) and (2.27). For any particular gauging the entries of $\xi_{\alpha M}$ and $f_{\mu(MA)}$ are fixed numbers and the global symmetry group G_0 is broken down to the gauge group G. but as long as the general construction is kept, the G_0 invariance is formally retained. This is possible because constraints (2.22) and (2.26) are G_0 invariant, i.e. a G_0 transformation of any of their solutions yields to another solution. Different solutions that are related in this way describe equivalent gauged theories.

2.3 Field content of the theory

In this section we present the field content of the $D = 8$ Supergravity and how the fields involved transform under gauge transformations. We will not give the explicit form of the Lagrangian but we will point out which p-form fields are needed. We will also show that $(p+1)$-forms are always necessary in order to make the fields strengths of the p-forms gauge invariant. Finally, how to truncate this tower of gauge fields to a finite subset without loosing gauge invariance will be explained.

2.3.1 Gauge transformations and covariant field strengths

First we want to introduce the covariant field strengths for the p-form gauge fields that appear in the gauged supergravity theory. The construction is general for any supergravity but we will use the representations of the $D = 8$ case. With indices defined as in section 2.1.2. Then we will see how they transform under gauge transformations.

In the ungauged $D = 8$ theory we have vector gauge fields $A_\mu^{\alpha M}$, two-form gauge fields $B_{\mu\nu I}$, three-from gauge fields $S^\beta_{\mu\nu\rho}$ and four-form gauge fields $T^U_{\mu\nu\rho\lambda}$. Each transforming on the appropriate representation of G_0. The Abelian field strengths of these tensor gauge fields take the form

$$\begin{aligned}
\mathcal{F}^{0,\alpha M}_{\mu\nu} &= 2\partial_{[\mu}A^{\alpha M}_{\nu]}, \\
\mathcal{F}^{0}_{\mu\nu\rho I} &= 3\partial_{[\mu}B_{\nu\rho]I} + 6d_{I\alpha M\beta N}A^{\alpha M}_{[\mu}\partial_\nu A^{\beta N}_{\rho]}, \\
\mathcal{F}^{0,\tau}_{\mu\nu\rho\lambda} &= 4\partial_{[\mu}S^\tau_{\nu\rho\lambda]} - c^{\tau J}_{\beta M}(12B_{[\mu\nu J}\partial_\rho A^{\beta M}_{\lambda]} + 8d_{J\epsilon K\delta L}A^{\beta M}_{[\mu}A^{\epsilon K}_{\nu}\partial_\rho A^{\delta L}_{\lambda]}).
\end{aligned} \tag{2.28}$$

Terms of the type $A\partial A$ come from dimensional reduction and are always present in the reduced theory. The objects $d_{I\alpha M\beta N}$ and $c^{\tau J}_{\beta M}$ are some appropriate G_0-invariant tensors. To ensure invariance under Abelian gauge transformations these tensors have to satisfy

$$d_{I[\alpha M\beta N]} = 0, \qquad\qquad d_{I(\alpha M\beta N}c^{\tau I}_{\delta L)}, \tag{2.29}$$

and they must be defined by covariant tensors, i.e. in terms of δ's and ϵ's. For example

$$\begin{aligned}
d_{I\alpha M\beta N} &\propto \epsilon_{\alpha\beta}\epsilon_{IMN}, \\
c^{\tau I}_{\delta L} &\propto \delta^\tau_\delta \delta^L_I.
\end{aligned} \tag{2.30}$$

We have the freedom to add numerical factors or modify the structure of (2.29) in order to ensure the right covariant transformation of the field strengths. This will become clear later in this section.

Using (2.29) we can show that under arbitrary variations $\delta A^{\alpha M}_\mu$, $\delta B_{\mu\nu I}$ and $\delta S^\tau_{\mu\nu\rho}$ the field strengths vary as

$$
\begin{aligned}
\delta\mathcal{F}^{0,\alpha M}_{\mu\nu} &= 2\partial_{[\mu}(\Delta A^{\alpha M}_{\nu]}),\\
\delta\mathcal{F}^{0}_{\mu\nu\rho I} &= 3\partial_{[\mu}(\Delta B_{\nu\rho]I}) + 6d_{I\alpha M\beta N}\mathcal{F}^{0,\alpha M}_{[\mu\nu}\Delta A^{\beta N}_{\rho]},\\
\delta\mathcal{F}^{0,\tau}_{\mu\nu\rho\lambda} &= 4\partial_{[\mu}(\Delta S^{\tau}_{\nu\rho\lambda]}) - c^{\tau J}_{\beta M}\mathcal{F}^{0,\beta M}_{[\mu\nu}\Delta B_{\rho\lambda]J} + 4c^{\tau J}_{\beta M}\mathcal{F}^{0}_{[\mu\nu\rho J}\Delta A^{\beta M}_{\lambda]},
\end{aligned}
\tag{2.31}
$$

where we used the covariant variations

$$
\begin{aligned}
\Delta A^{\alpha M}_{\mu} &= \delta A^{\alpha M}_{\mu},\\
\Delta B_{\mu\nu I} &= \delta B_{\mu\nu I} - 2d_{I\alpha M\beta N}A^{\alpha M}_{[\mu}\delta A^{\beta N}_{\nu]},\\
\Delta S^{\tau}_{\mu\nu\rho} &= \delta S^{\tau}_{\mu\nu\rho} - 3c^{\tau J}_{\alpha N}B_{[\mu\nu J}\delta A^{\alpha M}_{\rho]} - 2c^{\tau I}_{\alpha M}d_{I\beta N\gamma L}A^{\alpha M}_{[\mu}A^{\beta N}_{\nu}\delta A^{\gamma L}_{\rho]}.
\end{aligned}
\tag{2.32}
$$

These covariant variations are very useful since they allow to express gauge transformations and variations of gauge invariant objects in a manifestly covariant form, i.e. without explicit appearance of gauge fields.

Let's now gauge the theory. To do so we must introduce in the fields strengths couplings to the different p-form gauge fields. The gauge generators $X_{MN}{}^{P}$ were already introduced in a previous section, and according to (2.16) they take the role of generalized structure constants. Therefore, it would be natural to define the non-Abelian field strength of the vector fields $A^{\alpha M}_{\mu}$ as

$$
\mathcal{F}^{\alpha M}_{\mu\nu} = 2\partial_{[\mu}A^{\alpha M}_{\nu]} + gX_{\beta N\gamma P}{}^{\alpha M}A^{\beta N}_{[\mu}A^{\gamma P}_{\nu]},
\tag{2.33}
$$

but under gauge transformations $\delta A^{\alpha M}_{\mu} = D_{\mu}\Lambda^{\alpha M}$ one finds this field strength to transform as

$$
\begin{aligned}
\delta\mathcal{F}^{\alpha M}_{\mu\nu} &= 2D_{[\mu}\delta A^{\alpha M}_{\nu]} = 2D_{[\mu}D_{\nu]}\Lambda^{\alpha M} = g\mathcal{F}^{\beta N}_{\mu\nu}X_{\beta N\gamma L}{}^{\alpha M}\Lambda^{\gamma L}\\
&= -g\Lambda^{\beta N}X_{\beta N\gamma L}{}^{\alpha M}\mathcal{F}^{\gamma L}_{\mu\nu} + 2g\Lambda^{\beta N}X_{(\beta N\gamma L)}{}^{\alpha M}\mathcal{F}^{\gamma L}_{\mu\nu}
\end{aligned}
\tag{2.34}
$$

where we used the Ricci identity $[D_{\mu}, D_{\nu}] = -g\mathcal{F}^{\alpha M}_{\mu\nu}X_{\alpha M}$, which is valid due to the quadratic constraint. Here and in the following we use the covariant derivative as defined in (2.18). In the second line of equation (2.34) the first term alone would describe the correct covariant transformation of the field strength, but there is an unwanted second term since $X_{\beta N\gamma L}{}^{\alpha M}$ are typically not antisymmetric in lower indices. Thus the field strength $\mathcal{F}^{\alpha M}_{\mu\nu}$ does not transform covariantly under gauge transformations.

This problem comes from the fact that the dimension of the gauge group G can be smaller than the number of Abelian vector fields $A^{\alpha M}_{\mu}$. It is therefore possible that not all vector fields are necessary as gauge fields. To ensure no vector fields are left "unused" and the formulation is still valid for any allowed embedding tensor, we introduce Stückelberg type couplings to the two-form gauge fields. A mechanism by which the reminding vector fields are absorbed into massive two-forms and the unwanted term in the covariant transformation is canceled. The new field strength will read

$$\mathcal{H}_{\mu\nu}^{\alpha M} = 2\partial_{[\mu} A_{\nu]}^{\alpha M} + gX_{\beta N\gamma P}{}^{\alpha M} A_{[\mu}^{\beta N} A_{\nu]}^{\gamma P} + gZ^{\alpha M,I} B_{\mu\nu I}. \tag{2.35}$$

The tensor $Z^{\alpha M,I}$ should be such that the unwanted terms in (2.34) can be absorbed into an appropriate gauge transformation of the two-form gauge fields. Explicitly, $\delta B_{\mu\nu I}$ should include a term $(-2d_{I\beta N\gamma L}\Lambda^{\beta N}\mathcal{F}_{\mu\nu}^{\gamma L})$ and we need

$$X_{(\beta N\gamma L)}{}^{\alpha M} = d_{I\beta N\gamma L} Z^{\alpha M,I}. \tag{2.36}$$

Using this equation one can replace the field strength $\mathcal{F}_{\mu\nu}^{\alpha M}$ in the Ricci identity by the covariant derivative, i.e. we have

$$[D_\mu, D_\nu] = -g\mathcal{H}_{\mu\nu}^{\alpha M} X_{\alpha M} \tag{2.37}$$

To see how the Stückelberg mechanism leads to the absorption of the reminding vector fields into massive two forms is enough to note the Lagrangian will now include the following term

$$\mathcal{L} \propto \mathcal{H}_{\mu\nu}^{\alpha M}\mathcal{H}^{\mu\nu,\beta N} + ... = -g^2 Z^{\alpha M,I} Z^{\beta N,J} B_{\mu\nu,I} B_J^{\mu\nu}, \tag{2.38}$$

which is a mass term for two-form fields. It is through the tensors Z's that the reminding vector fields are projected into the massive two-forms.

Continuing the analysis to higher rank gauge fields one finds that, analogous to (2.35), one needs a Stückelberg type coupling to the three-forms in the field strength of the two-forms, and so on. We will give know the explicit form of this couplings and the conditions of the type (2.36) needed. Recall we had certain freedom in the choice (2.30) which will extend to the rest of tensors of the same kind will find in the final set of conditions.

The covariant field strengths including the Stückelberg terms read

$$\begin{aligned}
\mathcal{H}_{\mu\nu}^{\alpha M} &= 2\partial_{[\mu} A_{\nu]}^{\alpha M} + gX_{\beta N,\gamma L}{}^{\alpha M} A_{[\mu}^{\beta N} A_{\nu]}^{\gamma L} + gZ^{\alpha M,I} B_{\mu\nu,I}, \\
\mathcal{H}_{\mu\nu\rho,I} &= 3D_{[\mu} B_{\nu\rho],I} + 6d_{I\beta N\gamma L} A_{[\mu}^{\beta N} (\partial_\nu A_{\rho]}^{\gamma L} + \frac{1}{3} gX_{[\lambda P,\kappa Q]}{}^{\gamma L} A_\nu^{\lambda P} A_{\rho]}^{\kappa Q}) + gY_{I\beta} S_{\mu\nu\rho}^{\beta}, \\
\mathcal{H}_{\mu\nu\rho\lambda}^{\tau} &= 4D_{[\mu} S_{\nu\rho\lambda]}^{\tau} - c_{\beta N}^{\tau I} \Big(6B_{[\mu\nu,I}\mathcal{H}_{\rho\lambda]}^{\beta N} - 3gZ^{\beta NJ} B_{[\mu\nu,I} B_{\rho\lambda],J} + 8d_{I\kappa Q\epsilon P} A_{[\mu}^{\beta N} A_\nu^{\kappa Q} \partial_\rho A_{\lambda]}^{\epsilon Q} \\
&\quad + 2gd_{I\kappa Q\epsilon P} X_{\sigma S\pi T}{}^{\epsilon P} A_{[\mu}^{\beta N} A_\nu^{\kappa Q} A_\rho^{\sigma S} A_{\lambda]}^{\pi T} \Big) + gW_U^{\tau} T_{\mu\nu\rho\lambda}^{U}.
\end{aligned} \tag{2.39}$$

In the beginning of this section we made the statement that the inclusion of $p+1$-form fields in the p-form field strengths is always necessary in order to keep gauge invariance. It becomes now clear this is due to the fact that Stückelberg terms are necessary to guarantee the appropriate covariant transformation.

The general variations of the field strengths are now

$$\begin{aligned} \delta\mathcal{H}_{\mu\nu}^{\alpha M} &= 2D_{[\mu}(\Delta A_{\nu]}^{\alpha M}) + gZ^{\alpha M,I}\Delta B_{\mu\nu,I}, \\ \delta\mathcal{H}_{\mu\nu\rho,I} &= 3D_{[\mu}(\Delta B_{\nu\rho,I]}) + 6d_{I\beta N\gamma L}\mathcal{H}_{[\mu\nu}^{\beta N}\Delta A_{\rho]}^{\gamma L} + gY_{I\beta}\Delta S_{\mu\nu\rho}^{\beta}, \\ \delta\mathcal{H}_{\mu\nu\rho\lambda}^{\tau} &= 4D_{[\mu}(\Delta S_{\nu\rho\lambda]}^{\tau}) - 6c_{\alpha M}^{\tau I}\mathcal{H}_{[\mu\nu}^{\alpha M}\Delta B_{\rho\lambda],I} + 4c_{\alpha M}^{\tau I}\mathcal{H}_{[\mu\nu\rho,I}\Delta A_{\lambda]}^{\alpha M} \\ &\quad + gW_{U}^{\tau}\Delta T_{\mu\nu\rho\lambda}^{U}. \end{aligned} \tag{2.40}$$

The unwanted terms in the variation of each field strength are canceled by the new terms in the covariant variations. Or equivalently, the unwanted terms are absorbed in the covariant variations such that

$$\begin{aligned} \Delta A_{\mu}^{\alpha M} &= D_{\mu}\Lambda^{\alpha M} - gZ^{\alpha M,I}\Sigma_{\mu,I}, \\ \Delta B_{\mu\nu,I} &= 2D_{[\mu}\Sigma_{\nu],I} - 2d_{I\alpha M\beta N}\Lambda^{\alpha M}\mathcal{H}_{\mu\nu}^{\beta N} - gY_{I\tau}\Phi_{\mu\nu}^{\tau}, \\ \Delta S_{\mu\nu\rho}^{\tau} &= 3D_{[\mu}\Phi_{\nu\rho]}^{\tau} + 3c_{\alpha M}^{\tau I}\mathcal{H}_{[\mu\nu}^{\alpha M}\Sigma_{\rho],I} + c_{\alpha M}^{\tau I}\Lambda^{\alpha M}\mathcal{H}_{\mu\nu\rho,I} - gW_{U}^{\tau}\Omega_{\mu\nu\rho}^{U}, \\ \Delta T_{\mu\nu\rho\lambda}^{U} &= 4D_{[\mu}\Omega_{\nu\rho\lambda}^{U} - 12e_{\alpha M\beta}^{U}\mathcal{H}_{[\mu\nu}^{\alpha M}\Phi_{\rho\lambda]}^{\beta} \pm 4f^{IJU}\mathcal{H}_{[\mu\nu\rho,I}\Sigma_{\lambda],J} \\ &\quad - 2e_{\alpha M\beta}^{U}\mathcal{H}_{\mu\nu\rho\lambda}^{\beta}\Lambda^{\alpha M} - gR_{\alpha M}^{U}\Xi_{\mu\nu\rho\lambda}^{\alpha M}, \end{aligned} \tag{2.41}$$

with $\Lambda^{\alpha M}(x)$, $\Sigma_{\mu,I}(x)$, $\Phi_{\mu\nu}^{\tau}(x)$, $\Omega_{\mu\nu\rho}^{U}(x)$ and $\Xi_{\mu\nu\rho\lambda}^{\alpha M}(x)$ being the gauge parameters. Finally plugging these covariant variations into (2.40) one finds that the field strengths transform covariantly, i.e. that is

$$\begin{gathered} \delta\mathcal{H}_{\mu\nu}^{\alpha M} = -g\Lambda^{\beta N}X_{\beta N\gamma L}{}^{\alpha M}\mathcal{H}_{\mu\nu}^{\gamma L}; \quad \delta\mathcal{H}_{\mu\nu\rho,I} = g\Lambda^{\beta N}X_{\beta NI}{}^{J}\mathcal{H}_{\mu\nu\rho,J}; \\ \delta\mathcal{H}_{\mu\nu\rho\lambda}^{\tau} = -g\Lambda^{\beta N}X_{\beta N\gamma}{}^{\tau}\mathcal{H}_{\mu\nu\rho\lambda}^{\gamma}. \end{gathered} \tag{2.42}$$

as long as the following set of conditions is satisfied.

- $X_{(\mu M\beta N)}{}^{\alpha L} = d_{I\mu M\beta N}Z^{\alpha L,I}$
- $X_{\mu MI}{}^{J} + 2d_{I\mu M\beta N}Z^{\beta N,J} = Y_{I\lambda}c_{\mu M}^{\lambda J}$
- $X_{\mu M\beta}{}^{\gamma} + c_{\mu M}^{\gamma I}Y_{I\beta} = 2e_{\beta\mu M}^{P}W_{P}^{\gamma}$
- $d_{I(\alpha M\beta N}c_{\rho L)}^{\lambda J} = 0$
- $W_{P}^{\alpha}f^{IJP} + c_{\beta P}^{\alpha J}Z^{\beta P,I} - c_{\beta P}^{\alpha I}Z^{\beta P,J} = 0$
- $Z^{\alpha M,I}Y_{I\beta} = 0$
- $Y_{I\tau}W_{U}^{\tau} = 0$
- $W_{L}^{\alpha}R_{\beta M}^{L} = 0$

When trying to prove (2.40) it is very convenient to use the following modified Bianchi identities for the covariant field strengths

$$\begin{aligned} D_{[\mu}\mathcal{H}^{\alpha M}_{\nu\rho]} &= \frac{1}{3} g Z^{\alpha M, I} \mathcal{H}_{\mu\nu\rho I}, \\ D_{[\mu}\mathcal{H}_{\nu\rho\lambda]I} &= \frac{3}{2} d_{I\alpha M\beta N}\mathcal{H}^{\alpha M}_{[\mu\nu}\mathcal{H}^{\beta N}_{\rho\lambda]} + \frac{1}{4} g Y_{I\tau}\mathcal{H}^{\tau}_{\mu\nu\rho\lambda}. \end{aligned} \tag{2.43}$$

2.3.2 Truncation of the tower of p-form gauge fields

In the last section we found that the inclusion of $(p+1)$-forms in the field strength of the p-forms is necessary to guarantee gauge invariance. We have an infinite tower of p-form gauge fields but we want to truncate it to a finite subset without loosing gauge invariance. There are different truncation schemes depending on the dimension of the supergravity considered. We will describe the scheme that applies to the $D = 8$ case. Details on the other cases can be found in [1].

To find a finite subset of gauge fields that close on themselves under gauge transformations we must note that the four-form $T^{U}_{\mu\nu\rho\lambda}$ only enters the three-form field strength ($\mathcal{H}^{\tau}_{\mu\nu\rho\lambda}$) projected with W^{τ}_{U}. Due to the conditions found in the end of last section the gauge transformation corresponding to $\Xi^{\alpha M}_{\mu\nu\rho\lambda}$ is therefore not needed to ensure gauge invariance of the three-form field strength. We find $\{A^{\alpha M}_{\mu}, B_{\mu\nu I}, S^{\tau}_{\mu\nu\rho}, W^{\tau}_{U}T^{U}_{\mu\nu\rho\lambda}\}$ to be a set of gauge fields that close under the corresponding set of gauge transformations $\{\Lambda^{\alpha M}, \Sigma_{\mu,I}, \Phi^{\tau}_{\mu\nu}, W^{\tau}_{U}\Omega^{U}_{\mu\nu\rho}\}$.

Formally, the consistency condition one can use as a check is

$$W^{\tau}_{R}(2e^{R}_{\gamma\alpha M}W^{\gamma}_{U} - X_{\alpha M,U}{}^{R}) = 0. \tag{2.44}$$

To prove (2.44) one starts with the gauge invariance of W^{τ}_{U}, i.e. $\delta_{\alpha M}W^{\tau}_{U}$ and apply the conditions to ensure gauge invariance found in last section.

Chapter 3

Solutions to the quadratic constraint

In this chapter we will explicitly solve the quadratic constraints on the embedding tensor in terms of a number of parameters. We will explore the structure of the different possible gaugings encoded in the embedding tensor and we will see that even though at first we find a big number of independent solutions (87), many of them are equivalent in terms of the gaugings. Finally we give some particular examples to show the most general gaugings allowed in the $D = 8$ theory.

3.1 Generalities of the solutions

Although the quadratic constraints (2.22) and (2.26) are rather complex, we can solve them in terms of 16 parameters (2.21). We obtain 87 independent solutions. We will see that some of those are equivalent. There is certain multiplicity in the solutions in the sense that with the appropriate redefinitions only a part of the solutions is truly independent.

As we introduced in last chapter, we obtain the solutions in terms of two vectors, which read

$$\left.\begin{array}{l} \xi_{1M} = (V_1, V_2, V_3) \\ \xi_{2M} = (W_1, W_2, W_3) \end{array}\right\} \quad \text{this are 6 parameters} \tag{3.1}$$

and two symmetric matrices

$$f_{1(MN)} = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12} & Z_{22} & Z_{23} \\ Z_{13} & Z_{23} & Z_{33} \end{pmatrix}; \quad f_{2(MN)} = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12} & Y_{22} & Y_{23} \\ Y_{13} & Y_{23} & Y_{33} \end{pmatrix} \tag{3.2}$$

which are another 12 parameters. One of the constraints (2.25) tells us that the two vectors must be parallel. Therefore we are left with one vector and a proportionality constant (λ), thus 4 parameters.

However, what are really interested in the generators for the gauge group (2.16) which will tell us what groups we can gauge. To obtain them we use the following

$$X_{\alpha M \beta N}{}^{\gamma L} = X_{\alpha M \beta}{}^{\gamma}\delta_N^L + X_{\alpha M N}{}^{L}\delta_\beta^\gamma = \Theta_{\alpha M,\rho}{}^{\tau}(t_\tau^\rho)_\beta^\gamma\delta_N^L + \Theta_{\alpha M,S}{}^{T}(t_T^S)_N^L\delta_\beta^\gamma \tag{3.3}$$

The first term in (3.3) corresponds to the generators for the gauging of the SL(2) part and the second term generates the SL(3) part of the gauging. We find that each solution comes in six 2×2 matrices for SL(2) and six 3×3 matrices for SL(3), i.e. the generators are labelled by a SL(2) index ($\alpha = 1,2$) and a SL(3) index (M$= 1,2,3$). But the six generators are not independent, for the SL(2) the vectors are a doublet so we will not find in the solutions more than two independent generators for the gauge group. In the same way, for the SL(3) there are only three vectors so we will find at most three independent generators since we can not gauge more vectors.

3.1.1 The SL(2) part

Although each solution comes in a set of six generators, we only have two independent ones for any gauging we can think of. To show this statement let's see how a SL(2) field transforms under such gauge transformations

$$\begin{aligned}
\delta B^\alpha &= -g\Lambda^{\beta N}X_{\beta N\gamma}{}^{\alpha}B^\gamma \\
&= -g\left[(\Lambda^{11}X_{11} + \Lambda^{12}X_{12} + \Lambda^{13}X_{13})_\gamma^\alpha B^\gamma + (\Lambda^{21}X_{21} + \Lambda^{22}X_{22} + \Lambda^{23}X_{23})_\gamma^\alpha B^\gamma\right] \\
&= -g\left[(\Lambda^{11} + \Lambda^{12}\lambda_1 + \Lambda^{13}\lambda_2)X_{11\gamma}{}^{\alpha} + (\Lambda^{21} + \Lambda^{22}\lambda_1 + \Lambda^{33}\lambda_2)X_{21\gamma}{}^{\alpha}\right]B^\gamma \\
&= -g\left(\Lambda'^1 X'_{1\gamma}{}^{\alpha} + \Lambda'^2 X'_{2\gamma}{}^{\alpha}\right)B^\gamma,
\end{aligned} \tag{3.4}$$

where we have redefined the gauge parameters and generators in the last line. It is easy to check that the proportionality $X_{11} \propto X_{12} \propto X_{13}$ is due to the fact that they are all linear combinations of the original SL(2) generators and also the two vectors $\xi_{\alpha M}$ are proportional.

$$\begin{aligned}
X_{\alpha M,\gamma}{}^{\beta} &= \Theta_{\alpha M,\rho}{}^{\sigma}(t_\sigma^\rho)_\gamma^\beta = \xi_{\rho M}(t_\alpha^\rho)_\gamma^\beta \\
X_{\alpha M} &= \xi_{1M}t_\alpha^1 + \xi_{2M}t_\alpha^2 = \xi_{1M}(t_\alpha^1 + \lambda t_\alpha^2)
\end{aligned} \tag{3.5}$$

To sum up, each solution for the SL(2) sector has two independent generators which are linear combinations of the original SL(2).

3.1.2 The SL(3) part

Analogously we can do the same for the SL(3) sector. A SL(3) field will transform under a general gauge transformations as

$$\begin{aligned}\delta B^M &= -g\Lambda^{\beta N} X_{\beta N S}{}^{M} B^S \\ &= -g\left(\Lambda^{11}X_{11} + \Lambda^{12}X_{12} + \Lambda^{13}X_{13} + \Lambda^{21}X_{21} + \Lambda^{22}X_{22} + \Lambda^{23}X_{23}\right)_S{}^M B^S \\ &= -g\left[(\Lambda^{11} + \Lambda^{21}\kappa_1)X_{11} + (\Lambda^{12} + \Lambda^{22}\kappa_2)X_{12} + (\Lambda^{13} + \Lambda^{23}\kappa_3)X_{13}\right]_S{}^M B^S \\ &= -g\left(\Lambda'^1 X'_1 + \Lambda'^2 X'_2 + \Lambda'^3 X'_3\right)_S{}^M B^S\end{aligned} \tag{3.6}$$

This is the general case for the SL(3) part. We are always allowed to reexpress the generators so $X_{2M} = \kappa_M X_{1M}$. The dimension of the gauge group will be smaller or equal to three.

Even though the previous are the correct transformations for the fields it might occur that not all generators are really generating the gauging. The see which generators form the gauging we project the transformation with the generators. In general

$$X_{M\alpha}\left(\delta B^{M\alpha} = -g\Lambda^{\beta N} X_{N\beta P\gamma}{}^{M\alpha} B^{P\gamma}\right) \tag{3.7}$$

After which only the antisymmetric part of the generators $X_{[N\beta P\gamma]}{}^{M\alpha}$ is left and guarantees the closure of the group. This is due to the fact that in general $X_{N\beta P\gamma}{}^{M\alpha}$ have no definite symmetry as we explained in Chapter 2.

3.2 SL(3) Solutions

Let's explicitly see some particular gaugings. We will set the vectors $\xi_{\alpha M}$ to zero, i.e. we will gauge only the SL(3) sector of the theory. Now we obtain seven independent solutions, the most general of them being

$$\begin{aligned}X_{11}^{(3)} &= \begin{pmatrix} 0 & 0 & 0 \\ Z_{13} & Z_{23} & Z_{33} \\ -Z_{12} & -Z_{22} & -Z_{23} \end{pmatrix}, \; X_{12}^{(3)} = \begin{pmatrix} -Z_{13} & -Z_{23} & -Z_{33} \\ 0 & 0 & 0 \\ Z_{11} & Z_{12} & Z_{13} \end{pmatrix}, \\ X_{13}^{(3)} &= \begin{pmatrix} Z_{12} & Z_{22} & Z_{23} \\ -Z_{11} & -Z_{12} & -Z_{13} \\ 0 & 0 & 0 \end{pmatrix}, \; X_{21}^{(3)} = \frac{Y_{23}}{Z_{23}} \begin{pmatrix} 0 & 0 & 0 \\ Z_{13} & Z_{23} & Z_{33} \\ -Z_{12} & -Z_{22} & -Z_{23} \end{pmatrix}, \\ X_{22}^{(3)} &= \frac{Y_{23}}{Z_{23}} \begin{pmatrix} -Z_{13} & -Z_{23} & -Z_{33} \\ 0 & 0 & 0 \\ Z_{11} & Z_{12} & Z_{13} \end{pmatrix}, \; X_{23}^{(3)} = \frac{Y_{23}}{Z_{23}} \begin{pmatrix} Z_{12} & Z_{22} & Z_{23} \\ -Z_{11} & -Z_{12} & -Z_{13} \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned} \tag{3.8}$$

These are two proportional sets of three independent generators. The superscript indicates they come from the SL(3) sector, we will keep this notation in the following sections. To see what groups these generators produce we follow the analysis described in [3]. The solutions for f corresponding to the generators above read

$$f_1 = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12} & Z_{22} & Z_{23} \\ Z_{13} & Z_{23} & Z_{33} \end{pmatrix}, \quad f_2 = \frac{Y_{23}}{Z_{23}} \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12} & Z_{22} & Z_{23} \\ Z_{13} & Z_{23} & Z_{33} \end{pmatrix}. \tag{3.9}$$

And the generators in terms of the f's

$$X_{\alpha M,R}{}^{P} = \Theta_{\alpha M,Q}{}^{T} (t_T^Q)_R^P = f_\alpha^{PB} \epsilon_{BMR}, \tag{3.10}$$

which is analogous to the definition of the structure constants in [3]. In this analysis the distinction between generators and structure constants is not relevant.

The Lie algebras are we are looking for are only defined up to changes of basis generated by SL(3) transformations. We can always find a transformation on the generators that turns f into a diagonal form. The different three-dimensional Lie algebras are obtained by taking different signatures of f. They correspond to contractions of a SO(3) subgroup of SL(3) and they are classified by the Bianchi classification. So f being of rank three, i.e. no zero eigenvalues, generates a $so(3)$ or a $so(2,1)$ algebra (type VIII and IX in Bianchi classification). Which means there is no contraction. f having one zero eigenvalue generates either an $iso(1,1)$ or an $iso(2)$ algebra (types VI_0 and VII_0). If it has two zero eigenvalues, f generates a $heis_3$ Heisenberg algebra (type II). And finally f having the three eigenvalues equal to zero, corresponds to a $u(1)^3$ algebra, i.e. we are not gauging anything, as the vector fields are already U(1) invariant (2.12).

3.3 Example of SL(2)×SL(3) solution

In this section we will see an example of SL(2)×SL(3) solution. For simplicity we now take $f_{\alpha(MN)} = 0$ and we are left only with $\xi_{\alpha M}$. For the quadratic constraints to have non-trivial solutions we can not gauge only the SL(2) sector. We will always be gauging at the same time a SL(2) subsector of the SL(3) part (2.17). A particular solution for this case reads

$$\begin{aligned} X_{11}^{(2)} &= \begin{pmatrix} V_3/2 & 0 \\ W_3 & -V_3/2 \end{pmatrix}, \; X_{21}^{(2)} = \begin{pmatrix} -W_3/2 & V_3 \\ 0 & W_3/2 \end{pmatrix}, \\ X_{12}^{(2)} &= \frac{W_2}{W_3} \begin{pmatrix} V_3/2 & 0 \\ W_3 & -V_3/2 \end{pmatrix}, \; X_{22}^{(2)} = \frac{W_2}{W_3} \begin{pmatrix} -W_3/2 & V_3 \\ 0 & W_3/2 \end{pmatrix}, \\ X_{13}^{(2)} &= \frac{W_1}{W_3} \begin{pmatrix} V_3/2 & 0 \\ W_3 & -V_3/2 \end{pmatrix}, \; X_{23}^{(2)} = \frac{W_1}{W_3} \begin{pmatrix} -W_3/2 & V_3 \\ 0 & W_3/2 \end{pmatrix}, \end{aligned} \tag{3.11}$$

for the SL(2) sector. And

$$X_{11}^{(3)} = \begin{pmatrix} -W_1/2 & 0 & 0 \\ -3W_2/4 & W_1/4 & 0 \\ -3W_3/4 & 0 & W_1/4 \end{pmatrix},\; X_{12}^{(3)} = \begin{pmatrix} W_2/4 & -3W_1/4 & 0 \\ 0 & -W_2/2 & 0 \\ 0 & -3W_3/4 & W_2/4 \end{pmatrix},$$
$$X_{13}^{(3)} = \begin{pmatrix} W_3/4 & 0 & -3W_1/4 \\ 0 & W_3/4 & -3W_2/4 \\ 0 & 0 & -W_3/2 \end{pmatrix},\; X_{21}^{(3)} = \frac{V_3}{W_3}\begin{pmatrix} -W_1/2 & 0 & 0 \\ -3W_2/4 & W_1/4 & 0 \\ -3W_3/4 & 0 & W_1/4 \end{pmatrix},$$
$$X_{22}^{(3)} = \frac{V_3}{W_3}\begin{pmatrix} W_2/4 & -3W_1/4 & 0 \\ 0 & -W_2/2 & 0 \\ 0 & -3W_3/4 & W_2/4 \end{pmatrix},\; X_{23}^{(3)} = \frac{V_3}{W_3}\begin{pmatrix} W_3/4 & 0 & -3W_1/4 \\ 0 & W_3/4 & -3W_2/4 \\ 0 & 0 & -W_3/2 \end{pmatrix}, \tag{3.12}$$

for the SL(3) sector.

So we have two independent generators for SL(2) and three independent generators for SL(3). We can always absorb the dependent ones in the redefinitions of (3.4) and (3.6)

3.3.1 SL(2) gauge groups

Let's take $X_{11}^{(2)}$ and $X_{12}^{(2)}$. In this "raw" form they don't tell us much but let's work a little on them. We can always reexpress them in terms of linear combinations to obtain a more convenient form

$$\begin{aligned} X_1'^{(2)} &= X_{11}^{(2)} + \frac{V_3}{W_3}X_{12}^{(2)} = \begin{pmatrix} 0 & V_3^2/W_3 \\ W_3 & 0 \end{pmatrix}, \\ X_2'^{(2)} &= X_{11}^{(2)} - \frac{V_3}{W_3}X_{12}^{(2)} = \begin{pmatrix} V_3 & -V_3^2/W_3 \\ W_3 & -V_3 \end{pmatrix}, \end{aligned} \tag{3.13}$$

Now we diagonalize $X_1'^{(2)}$

$$X_1^{(d2)} = S \cdot X_1'^{(2)} \cdot S^{-1} = \mathrm{diag}[X_1'^{(2)}] = \begin{pmatrix} -V_3 & 0 \\ 0 & V_3 \end{pmatrix}, \tag{3.14}$$

where S is the appropriate transformation that diagonalizes the generator. We can use the same transformation on $X_2'^{(2)}$

$$X_2^{(d2)} = S \cdot X_2'^{(2)} \cdot S^{-1} = \begin{pmatrix} 0 & 0 \\ -2V_3 & 0 \end{pmatrix} \tag{3.15}$$

The structure becomes now clear. We end up with two of the original SL(2) generators. Nevertheless, the only two-dimensional subgroup of SL(2) is the Borel subgroup which would be the diagonal generator and the upper triangular one. In this case we can check that when projected as in (3.7), $X_2^{(d2)}$ drops out and does not contribute to the gauging. The gauge group is generated by $X_1^{(d2)}$ alone and corresponds to SL(2) rescalings, SO(1,1) group.

3.3.2 SL(3) gauge groups

We will play the same trick to make the structure of the SL(3) part become clear. Taking $X_{11}^{(3)}$, $X_{12}^{(3)}$ and $X_{13}^{(3)}$ we reexpress them in terms of a convenient set of linear combinations

$$X_{11}^{\prime(3)} = X_{11}^{(3)} - \frac{W_1}{W_2} X_{12}^{(3)} = \begin{pmatrix} -3W_1/4 & 3W_1^2/4W_2 & 0 \\ -3W_2/4 & 3W_1/4 & 0 \\ -3W_3/4 & 3W_1W_3/4W_2 & 0 \end{pmatrix},$$
$$X_{12}^{\prime(3)} = X_{12}^{(3)} - \frac{W_1}{W_3} X_{13}^{(3)} = \begin{pmatrix} -3W_1/4 & 0 & 3W_1^2/4W_3 \\ -3W_2/4 & 0 & 3W_1W_2/4W_3 \\ -3W_3/4 & 0 & 3W_1/4 \end{pmatrix}, \tag{3.16}$$
$$X_{13}^{\prime 3} = X_{13}^{(3)} = \begin{pmatrix} W_3/4 & 0 & -3W_1/4 \\ 0 & W_3/4 & -3W_2/4 \\ 0 & 0 & -W_3/2 \end{pmatrix}.$$

And we diagonalize $X_3^{\prime(3)}$

$$X_{13}^{(d3)} = R \cdot X_{13}^{\prime(3)} \cdot R^{-1} = \mathrm{diag}[X_3^{\prime(3)}] = \begin{pmatrix} -W_3/2 & 0 & 0 \\ 0 & W_3/4 & 0 \\ 0 & 0 & W_3/4 \end{pmatrix}, \tag{3.17}$$

and again use the transformation to find the corresponding expressions for the other generators

$$X_{12}^{(d3)} = R \cdot X_{12}^{\prime(3)} \cdot R^{-1} = \begin{pmatrix} 0 & -3W_3/4 & 3W_3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$X_{11}^{(d3)} = R \cdot X_{11}^{\prime(3)} \cdot R^{-1} = \begin{pmatrix} 0 & -3(W_1+W_2)W_3/4W_2 & 3(2W_1+W_2)W_3/4W_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.18}$$

And still a final rearrangement

$$X_{13}^{\prime(d3)} = X_{13}^{(d3)} = \begin{pmatrix} -W_3/2 & 0 & 0 \\ 0 & W_3/4 & 0 \\ 0 & 0 & W_3/4 \end{pmatrix},$$
$$X_{12}^{\prime(d3)} = X_{12}^{(d3)} - \frac{W_2}{(2W_1+W_2)} X_{11}^{(d3)} = \begin{pmatrix} 0 & -3W_3/4 + \frac{3(W_1+W_2)W_3}{4(2W_1+W_2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.19}$$
$$X_{11}^{\prime(d3)} = X_2^{(d3)} - \frac{W_2}{(W_1+W_2)} X_{11}^{(d3)} = \begin{pmatrix} 0 & 0 & 3W_3/4 - \frac{3(2W_1+W_2)W_3}{4(W_1+W_2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The structure becomes clear now. We will find this a general feature of the solutions, we can always reexpress them in an appropriate set of linear combinations and then find the right transformations to turn one of the generators into diagonal form. We must then apply the same transformation to the other generators. And we can still rearrange them to find generators of the original SL(3) group. In this particular example this structure corresponds to a three-dimensional Borel Algebra.

Summing up, in this case we are gauging a one-dimensional SO(1,1) subgroup of SL(2), rescalings. For the SL(3) part we gauge a three-dimensional subgroup of SL(3). Which consists of a diagonal generator and two upper triangular generators. This corresponds to a three-dimensional Borel Algebra with center element (or ordering) $\mathfrak{h}$ and two positive weighted subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$ where $\mathfrak{B} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{h}$

3.4 Most General Solutions

In the last two subsections we depicted a particular solution and learnt how to deal with it in order to extract useful information. We allow now the most general gauging possible in the $D = 8$ supergravity. We obtain similar results using the same analysis.

Among the solutions for the SL(2) gauge generators we find two possible gaugings. The only generators that play a role in the gauging are two of the original SL(2) generators

$$t_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \qquad t_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.20}$$

and there are solutions including both generators and solutions including only t_1. So we either gauge rescalings or rescalings plus shifts $so(1,1) \oplus \mathbb{R}$ which define the Borel subgroup of SL(2).

For the SL(3) something similar occurs and among the solutions we find that only the following generators of the original SL(3) are involved in the gauging

$$\begin{aligned} t_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad t_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ t_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.21}$$

These generators define the Borel subgroup of SL(3) (See Appendix C).

$$\mathfrak{B} = \mathfrak{h}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \tag{3.22}$$

Nevertheless the biggest gauge group embedded in the SL(3) is three-dimensional (recall we only have three vector fields in respect to SL(3)), so the four generators will not appear in the same solution. We find three possible gaugings of the SL(3)

$$\begin{aligned} \{t_1, t_2, t_3\} &\rightarrow \text{three-dimensional Borel subgroup, } B_3 \\ \{t_1, t_4\} &\rightarrow \text{two-dimensional Borel subgroup, } B_2 \\ \{t_1\} &\rightarrow \text{SL(3) rescalings, } B_1 \end{aligned} \tag{3.23}$$

As we saw in last section the generators do not appear in the solutions in the simple form just presented but in terms of the parameters of $\xi_{\alpha M}$ and $f_{\alpha(MN)}$. After all, the gauge structure of the maximal $D = 8$ Supergravity has only a small number of possibilities. But we obtain 87 independent solutions from the quadratic constraint. To understand the difference between the solutions is useful to look at the covariant derivative

$$\begin{aligned} D_\mu =& \partial_\mu \\ &- g\left[(A^{11}_\mu + \lambda_1 A^{12}_\mu + \lambda_2 A^{13}_\mu) X^{(d2)}_{11} + (A^{21}_\mu + \lambda_1 A^{22}_\mu + \lambda_2 A^{23}_\mu) X^{(d2)}_{21}\right] \\ &- g\left[(A^{11}_\mu + \kappa_1 A^{21}_\mu) X'^{(d3)}_{11} + (A^{12}_\mu + \kappa_2 A^{22}_\mu) X'^{(d3)}_{12} + (A^{13}_\mu + \kappa_3 A^{23}_\mu) X'^{(d3)}_{13}\right] \\ =& \partial_\mu - g\left[B^{11}_\mu X^{(d2)}_{11} + B^{21}_\mu X^{(d2)}_{21}\right] - g\left[C^{11}_\mu X'^{(d3)}_{11} + C^{12}_\mu X'^{(d3)}_{12} + C^{13}_\mu X'^{(d3)}_{13}\right] \end{aligned} \tag{3.24}$$

In general we are coupling the generators of the gauge group to linear combinations of the original set of six vector fields. The constants λ and κ are the ones in (3.4) and (3.6) and depend very non-trivially on the parameters of $\xi_{\alpha M}$ and $f_{\alpha(MN)}$, (3.1) and (3.2). So the different solutions correspond to the several different ways we can take linear combinations of the original vector fields to make the gauging rather than to different structures of the gauge group.

Chapter 4

Conclusions

In this thesis we reviewed the embedding tensor formalism in the particular case of $D = 8$ Supergravity. Although the constraints it leads to are fairly complicated we have at our disposal methods (Mathematica) powerful enough to deal with them. The tower of solutions obtained look at first untreatable. But after some study, structure starts to emerge. Working from bottom up, that is, we begin with the simplest cases and find out what gauge groups are being generated, and then allowing every time more general cases, we have been able to explore the gauging possibilities of the theory up to the most general case embedded in Θ.

The key features we found during this work are:

- When studying gaugings of the SL(3) sector alone we had the guide of [3]. As the embedding tensor formalism is constructed in such a way the gauge group generators (and structure constants) are traceless, we were not expecting to find among the solutions any of the B type cases of the Bianchi classification. Nevertheless, the correspondence in the case of the A type gaugings is total. All A cases can be found in the solutions of the quadratic constraint for the embedding tensor. This is a reliable test for the formalism and makes the assertions for the more general cases trustful.

- It is not possible to gauge only the SL(2) part of the global symmetry group. Any SL(2) gauging induces a gauging of a subsector of SL(3). In terms of the equations this is needed to guarantee non-trivial solutions of the quadratic constraint but the interpretation for such characteristic is yet to be understood.

- We showed the most general gauging embedded in Θ consists of a two-dimensional gauging of the SL(2) sector and a three-dimensional gauging of the SL(3) sector. For SL(2) the gauging is made up of two of the original $sl(2)$ generators, one of them being the center element (diagonal generator) and the other one being the upper triangular one (See appendix C). In other words we are gauging rescalings and shifts, the members of the Borel subgroup of SL(2). Gaugings of the SL(3) involve more generators and thus are richer in possibilities. Again we find the generators of the Borel subgroup, in this case of SL(3).

- Although the structure of the gauging is restricted to a small number of possibilities, we find a large set of independent solutions for the quadratic constraint. These correspond to different choices of linear combinations of the original vector fields. In general the gauge vector fields are linear combinations of the original vector fields.

- The embedding tensor technique has proven to be a very useful tool. It allows us to write the theory in a manifestly covariant way and besides it provides us with a list of all the possible deformations allowed. We can work out any aspect of the theory without loosing generality and then test it for a particular gauging we are interested in. This technique can be applied to supergravities in any dimension. Even though the quadratic constraint has been worked out for supergravities in many other dimensions, practically no information about the full tower of possible deformations can be found in the literature. We hope this work is a useful guide for whom wants to work out the detailed deformations of more relevant supergravities.

Finally, it would have been nice to work out the Lagrangian of the theory in terms of the embedding tensor, see the coset structure of the scalars, topological terms, etc. As we already found the expressions of the gauge fields involved in terms of $X_{\alpha M}$, but this would require much more time and it is out the purpose of this work. Such aspects will have to be left for further research.

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Appendix A

Dimensional Reduction

In this appendix we review how the maximal supergravities in different dimensions are obtained from the unique $D = 11$ supergravity via dimensional reduction on a torus T^q, $q = 11-D$. For simplicity we only consider the bosonic fields and we focus our attention on how the respective global symmetry groups G_0 of the lower dimensional theories emerge. Further analysis on the issue, including the half-maximal case, can be found in [1,6,7,8,9].

A.1 Torus reduction of pure gravity

To exemplify the dimensional reduction procedure we will look at the simple case of gravity alone. Conceptually, it shares the main features with more realistic models including bosonic and fermionic fields. Let us consider Einstein gravity on a 11-dimensional manifold $\mathcal{M}_{11}$ with coordinates $x^{\hat{\mu}}$, $\hat{\mu} = 0...10$. The metric $g_{\hat{\mu}\hat{\nu}}$ has Lorentzian signature (-,+,+,...,+) and its dynamic is described by the Einstein-Hilbert action

$$\mathcal{S}_{\rm EH} = \int d^{11}x \; \mathcal{L}_{\rm EH}, \qquad\qquad \mathcal{L}_{\rm EH} = \sqrt{-g}\left(R^{(11)} + \mathcal{L}_{\rm M}\right), \tag{A.1}$$

where $g = \det(g_{\hat{\mu}\hat{\nu}})$, $R^{(11)}$ is the curvature scalar of $g_{\hat{\mu}\hat{\nu}}$ and $\mathcal{L}_{\rm M}$ describes additional matter (fermions), i.e. in the case of pure gravity we have $\mathcal{L}_{\rm M} = 0$. The equations of motion are the Einstein equations

$$R^{\hat{\mu}\hat{\nu}} - \frac{1}{2}R \; g^{\hat{\mu}\hat{\nu}} \equiv G^{\hat{\mu}\hat{\nu}} = T^{\hat{\mu}\hat{\nu}} \equiv \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm M})}{\delta g_{\hat{\mu}\hat{\nu}}}, \tag{A.2}$$

where $R_{\hat{\mu}\hat{\nu}}$, $G_{\hat{\mu}\hat{\nu}}$ and $T_{\hat{\mu}\hat{\nu}}$ are the Ricci, Einstein and Energy-Momentum tensor, respectively, and as usual indices are raised and lowered using the metric $g_{\hat{\mu}\hat{\nu}}$ and the inverse metric $g^{\hat{\mu}\hat{\nu}}$.

We want to dimensionally reduce this theory on a torus down to $D = 11 - q$ spacetime dimensions, i.e. we demand the 11-dimensional manifold $\mathcal{M}_{11}$ to locally have the form $\mathcal{M}_{11} = \mathcal{M}_d \times T^q$, with $\mathcal{M}_D$ being a D-dimensional space-time manifold and T^q being the q-dimensional torus. The idea is to split the original 11-dimensional coordinates into coordinates x^μ on $\mathcal{M}_D$, $\mu = 0...D-1$, and coordinates y^a on T^q, $a = 1...q$, such that the metric on $\mathcal{M}_{11}$ can be written as

$$\begin{aligned} ds^2 &= g_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} \\ &= \tilde{g}_{\mu\nu} dx^\mu dx^\nu + \rho^{2/q} M_{ab}(dy^a + A^a_\mu dx^\mu)(dy^b + A^b_\nu dx^\nu) \end{aligned} \tag{A.3}$$

where $\tilde{g}_{\mu\nu}$, A^a_μ, ρ and M_{ab} depend on x^μ but not on y^a, i.e. we make the field content of the theory independent of the internal manifold coordinates. The metric on $\mathcal{M}_D$ is $\tilde{g}_{\mu\nu}$ and the A^a_μ are the n Kaluza-Klein vector fields. The metric on T^q has been split into the dilaton ρ and the unimodular matrix M_{ab} (i.e. $\det M = 1$). From a D-dimensional perspective these are $q(q+1)/2$ scalar fields. So dimensional reduction of gravity introduces in the reduced theory $q(q+1)/2$ scalars and q vector fields.

The reformulation of the metric (A.3) is an Ansatz. Plugging it into the Einstein-Hilbert action (A.1) and taking the limit $y^a \to 0$ yields the effective D-dimensional action

$$\begin{aligned} S_{\text{eff}} &= \int d^D x \; \mathcal{L}_{\text{eff}} \\ \mathcal{L}_{\text{eff}} &= e\rho R^{(D)} - \frac{1}{4} e\rho^{1+2/q} M_{ab} A^a_{\mu\nu} A^{b\mu\nu} - \frac{1}{4} e\rho \; \text{tr}(M^{-1}\partial_\mu M M^{-1} \partial^\mu M) \\ &+ \frac{q-1}{q} e\rho^{-1}(\partial_\mu \rho)(\partial^\mu \rho) + e\rho \mathcal{L}_M, \end{aligned} \tag{A.4}$$

where $e = \sqrt{-\det \tilde{g}_{\mu\nu}}$ and $A^a_{\mu\nu} = 2\partial_{[\mu} A^a_{\nu]}$ are the Abelian field strengths of the vector fields. To find the usual Einstein-Hilbert term in the effective action one can perform a Weyl-rescaling of the metric, namely $\tilde{g}_{\mu\nu} \to g_{\mu\nu} = \rho^\alpha \tilde{g}_{\mu\nu}$ with $\alpha = -2/(D-2)$. The Weyl-rescaled effective Lagrangian reads

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= eR^{(D)} - \frac{1}{4} e\rho^{[2/n + 2/(D-2)]} M_{ab} A^a_{\mu\nu} A^{b\mu\nu} - \frac{1}{4} e \; \text{tr}(M^{-1}\partial_\mu M M^{-1} \partial^\mu M) \\ &+ \left(\frac{n-1}{n} - \frac{D-1}{D-2} \right) e(\rho^{-1}\partial_\mu \rho)(\rho^{-1}\partial^\mu \rho) + e\rho^{-D/(D-2)} \mathcal{L}_{\text{M}}. \end{aligned} \tag{A.5}$$

In addition to the Einstein-Hilbert term we have a kinetic term for the Abelian vector fields and for the scalars.

It will be specially relevant how the global symmetry of the reduced theory emerges. So let's consider now the symmetries of the effective actions (A.4) and (A.5). From the freedom of choosing arbitrary coordinate systems on $\mathcal{M}_{11}$ we are still allowed to choose arbitrary coordinates in the space-time $\mathcal{M}_D$. On the other hand, for the internal manifold the only coordinate changes that are compatible with the torus Ansatz are arbitrary changes of the origin and global linear transformations of the internal coordinates, i.e.

$$y^a \to \lambda^{2/q}(y^b + L^b(x))\Lambda_b^a, \tag{A.6}$$

where $\lambda \in \mathbb{R}$ is a constant rescaling factor, Λ is a constant SL(q) matrix, and $L \in \mathbb{R}^n$ are x-dependent coordinate shifts. $L^a(x)$ describes the U(1)^q gauge symmetries of the vector fields, i.e. $A^a_\mu \to A^a_\mu + \partial_\mu L^a$. λ and Λ act on the D-dimensional fields as

$$A^a_\mu \to A^b_\mu \Lambda_b^a, \qquad M \to \Lambda M \Lambda^T, \qquad \rho \to \lambda\rho. \tag{A.7}$$

These are global GL(q) = $\mathbb{R}^+ \times$SL(q) transformations. The vector fields transform in the vector representation of SL(q) while the scalars form an SL(q)/SO(q) coset. The arrangement of the scalars in a coset structure is a common feature of dimensional reduction. To make this coset structure more transparent it is convenient to introduce group valued representatives $\mathcal{V} \in$ SL(q) via

$$M = \mathcal{V}\mathcal{V}^T. \tag{A.8}$$

For given $M(x)$ the last equation only specifies $\mathcal{V}(x)$ to arbitrary local SO(q) transformations from the right. The global SL(q) transformations act linearly on $\mathcal{V}$ from the left, i.e. $\mathcal{V}$ transforms as

$$\mathcal{V} \to \lambda\ \mathcal{V}\ h(x), \qquad \lambda \in \mathrm{SL}(q), \qquad h(x) \in \mathrm{SO}(q). \tag{A.9}$$

In order to express the kinetic term in the Lagrangian in terms of $\mathcal{V}$ one introduces the scalar currents

$$P_\mu + Q_\mu = \mathcal{V}^{-1}\partial_\mu \mathcal{V}, \qquad P_\mu^T = P_\mu, \qquad Q_\mu^T = -Q_\mu. \tag{A.10}$$

Note that Q_μ is $\mathfrak{so}(q)$ valued, i.e. it takes the values in the compact part of $\mathfrak{sl}(q)$, while P_μ takes values in the non-compact directions of $\mathfrak{sl}(q)$. Using the currents the kinetic term for M can be written as

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}e\ \mathrm{tr}(M^{-1}\partial_\mu M M^{-1}\partial^\mu M) = -e\mathrm{tr}(P_\mu P^\mu). \tag{A.11}$$

To summarize, we found that dimensional reduction of pure gravity on a torus T^q yields a D-dimensional theory which describes gravity coupled to q Kaluza-Klein vectors A^a_μ, one dilaton ρ and scalars $\mathcal{V}$ that parameterize an SL(q)/SO(q) coset. The global symmetry group is GL(q) = $\mathbb{R}^+\times$ SL(q).

Another important and common feature in dimensional reduction is the symmetry enhancement that takes place whenever p-form fields are dualized into scalars. Let's consider the d=3 case. The Kaluza-Klein vector fields A^a_μ can then be dualized into scalars A_a via the duality equation

$$\rho^{2+2/q} M_{ab} A^{b\mu\nu} = \epsilon^{\mu\nu\rho}\partial_\rho A_a, \tag{A.12}$$

where we use the covariant epsilon tensor, i.e. $\epsilon^{012} = e^{-1}$. Note that A.12 defines the scalars A_a only up to global shifts $A_a \to A_a + \kappa_a$. When formulating the theory without the vector fields, only in terms of the metric and the scalars, this shifts become global symmetries. One expects the global symmetry group the be $\mathrm{GL}(q)\times\mathbb{R}^q$ but a miraculous symmetry enhancement takes place and the complete global symmetry turns out to be $GL_0 = \mathrm{SL}(q+1)$. In other words, before dualization we have q^2 generators of $\mathrm{GL}(q)$ and after dualizing the vector fields we not only add q generators for the shifts but also extra q generators dual to the shift symmetries. We will find this to be a universal feature.

We now want to make the $\mathrm{SL}(q+1)$ symmetry explicit. The scalars ρ, A_a and $\mathcal{V}$ form an $\mathrm{SL}(q+1)/\mathrm{SO}(q+1)$ coset, the appropriate coset representative is defined as follows

$$\tilde{\mathcal{V}} = \begin{pmatrix} \rho^{-1} & 0 \\ \rho^{-1}A_a & \rho^{1/q}\mathcal{V} \end{pmatrix} \tag{A.13}$$

Using the scalar current $\tilde{P}_\mu$ of $\tilde{\mathcal{V}}$, defined analogously to (A.10), the effective action takes the compact form

$$\mathcal{L}_{\mathrm{eff,d=3}} = eR^{(3)} - e\mathrm{tr}\ (\tilde{P}_\mu \tilde{P}^\mu) + e\rho^{-3}\mathcal{L}_{\mathrm{M}} \tag{A.14}$$

The resulting equations of motion for A_a are the integrability equations needed to reintroduce the vector fields A_μ^a via the duality equation (A.12), because of these equations all other equations of motion become equivalent to those derived from the previous Lagrangian (A.5).

The $\mathrm{SL}(q+1)$ acts on $\tilde{\mathcal{V}}$ from the left, analogous to (A.9). The corresponding $\mathrm{SL}(q+1)$ matrices read

$$\tilde{\Lambda}(\Lambda,\lambda) = \begin{pmatrix} \lambda^{-1/q} & 0 \\ 0 & \lambda^{1/q}\Lambda \end{pmatrix}, \qquad \tilde{\Lambda}(\kappa) = \begin{pmatrix} 0 & 0 \\ \kappa_a & 0 \end{pmatrix}, \qquad \tilde{\Lambda}(\tau) = \begin{pmatrix} 0 & \tau^a \\ 0 & 0 \end{pmatrix}. \tag{A.15}$$

The transformations λ and Λ from (A.7) correspond to $\tilde{\Lambda}(\Lambda,\lambda)$, the shift symmetries κ act via $\tilde{\Lambda}(\kappa)$, and the symmetry enhancement is described by the additional $\mathrm{SL}(q+1)$ elements $\tilde{\Lambda}(\tau)$. Left action with $\tilde{\Lambda}(\tau)$ on the coset representative $\tilde{\mathcal{V}}$ destroys the block-form (A.14), and an appropriate $\mathrm{SO}(q+1)$ action is necessary to restore this form.

The pure gravity case we just discussed already shows some universal features used in this thesis. In particular, it is characteristic of maximal (and also half-maximal) supergravities that the scalars of the theory arrange in a coset structure G_0/H, where G_0 is the global symmetry group and H is its maximal compact subgroup. Also the occurrence of an enhanced symmetry group in the lower dimensional theory after appropriate dualization of gauge fields is an important characteristic of Supergravities. In the pure gravity case only vector fields appear as gauge fields in the lower dimensional theory but in general higher rank fields will appear as gauge fields in Supergravities. The symmetry enhancement always takes place when dimensional reduction gives rise to scalars fields in the lower dimensional theory either by dualization of any p-form or directly from the reduction of a p-form in the higher dimensional theory.

D	G_0	H	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
8	SL(2)×SL(3)	SO(2)×SO(3)	(2,3)	(1,$\overline{3}$)	(2,1)	(1,3)	(2,$\overline{3}$)	(3,1)⊕(1,8)
7	SL(5)	SO(5)	$\overline{10}$	5	$\overline{5}$	10	24	–
6	SO(5,5)	SO(5)×SO(5)	16_s	10	16_c	45	–	–
5	$E_{6(6)}$	Usp(8)	$\overline{27}$	27	78	–	–	–
4	$E_{7(7)}$	SU(8)	56	133	–	–	–	–
3	$E_{8(8)}$	SO(16)	248	–	–	–	–	–
2	$E_{9(9)}$	$K(E_9)$	–	–	–	–	–	–

Table A.1: Table of the global symmetry groups and representations of the p-form gauge fields for the maximal supergravities in D dimensions.

A.2 Reduction of maximal supergravities

Although it is not the purpose of this thesis to review explicitly the origin of the dimensionally reduced $D = 8$ Supergravity, for completeness we will give a little taste of it. The unique supergravity theory in $D = 11$ space-time dimensions contains as bosonic degrees of freedom the metric and a three-form gauge field $C_{\hat{\mu}\hat{\nu}\hat{\rho}}$ with field strength $G_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = 4\partial_{[\hat{\mu}}C_{\hat{\nu}\hat{\rho}\hat{\lambda}]}$ and gauge symmetry $\delta C_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3\partial_{[\hat{\mu}}\Lambda_{\hat{\nu}\hat{\rho}]}$. The bosonic part of the Lagrangian reads

$$\mathcal{L}_{D=11} = \sqrt{-g}\left(R - \frac{1}{12}G_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}G^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} + \frac{2}{72^2}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}\hat{\sigma}\hat{\tau}\hat{\kappa}\hat{\gamma}\hat{\chi}\hat{\theta}\hat{\xi}}G_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}G_{\hat{\sigma}\hat{\tau}\hat{\kappa}\hat{\gamma}}C_{\hat{\chi}\hat{\theta}\hat{\xi}}\right). \tag{A.16}$$

We dimensional reduce this theory on a torus T^q down to $D = 11 - q$ dimensions, i.e. we make the Ansatz (A.3) for the metric and demand $C_{\hat{\mu}\hat{\nu}\hat{\rho}}$ to be constant along the torus coordinates y^a, i.e.

$$\frac{\partial}{\partial y^a}C_{\hat{\mu}\hat{\nu}\hat{\rho}} = 0 \tag{A.17}$$

In D-dimensions the three-form then yields $q(q-1)(q-2)/6$ scalars, $q(q-1)/2$ vector gauge fields, q two-form gauge fields and one three-form gauge field. Plus the q gauge vector fields and the $q(q+1)/2$ scalars from gravity. So for the $D = 8$ case this gives: (1+6) scalars, (3+3) gauge vector fields, 3 two-form gauge fields and one three-form gauge fields. Recall the scalars organize in a coset structure SL(2)×SL(3)/SO(2)×SO(2) which has dimension 7. The vector fields transform under the representation (2,3), dimension 6. The two-forms transform in the (1,$\bar{3}$), dimension 3. And finally the three-form field is self dual so it transforms under (2,1). Now becomes clear the relation between the field content of the theory and the representations under which the different p-form fields transform with their dimensional reduction origin. Table A.1 sums up the different p-form representations and global symmetry groups for various dimensions.

Appendix B

Dualities

This appendix is not meant to give a deep knowledge on dualities. It is just a quick review on the issue as throughout this work we referred to dualities in several ocasions. Only some simple concepts and basic rules will be depicted.

To introduce the concept of duality we will use the example of Electromagnetism. Here duality means that the equations that govern Electromagnetism are invariant under the interchange of magnetic and electric fields, thus these fields are dual to each other. This duality is not just any duality as it will extend to Poincaré dualities which are the ones referred to in this thesis.

If we take Maxwell's equations in the vacua ($\rho_e = \vec{J}_e = 0$)

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} = 0, \qquad & \vec{\nabla} \cdot \vec{B} = 0; \\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0, \qquad & \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0; \end{aligned} \tag{B.1}$$

we can see they are invariant under the duality transformation

$$\vec{E} \to \vec{B}; \qquad \vec{B} \to -\vec{E}, \tag{B.2}$$

that is, the interchange of electric and magnetic fields. If we set $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk} B_k$, Maxwell's equations can be written in a covariant form as

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \qquad \partial_\mu \star F^{\mu\nu} = 0, \tag{B.3}$$

where $\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}$. The second equation allow us to write

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{B.4}$$

If A^μ is taken as the fundamental field of electromagnetism, then the second Maxwell equation follow as an identity rather than as a dynamical law. A^μ is called the vector potential and $F^{\mu\nu}$ its field strength. The duality transformation in the vacua now is given by

$$\begin{aligned} F^{\mu\nu} &\to \star F^{\mu\nu} \\ \star F^{\mu\nu} &\to -F^{\mu\nu}, \end{aligned} \tag{B.5}$$

that is, the rotation of the field strength and its dual into each other.

The vacua case is easier to see, however, this duality holds for the general Maxwell equations. The extensions to higher rank field strengths are called Poincaré dualities, those we used in this work.

Without proof we now want to give some simple rules to relate the different p-form fields to their duals. Let be D the dimension we are working in and p and p' the rank of the fields in question, they will be dual when

$$D - p - 2 = p' \tag{B.6}$$

if $p = p'$ we say the fields are self-dual. For a more detailed explanation see [4].

Appendix C

SL(2) structure and Borel Algebras

In this appendix we review the most relevant features of the group theory used in this thesis. In particular we give some characteristics of the algebras found in most solutions of the quadratic constraint.

C.1 SL(2)

In mathematics, the special linear group SL(2) (or $SL_2(R)$) is the group of all real matrices with determinant one

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ \mid \ a, b, c, d \ \in \ \mathbb{R} \text{ and } ad - bc = 1 \right\}. \tag{C.1}$$

Using the notation used previously, it is generated by the following set of generators

$$t_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{C.2}$$

Which close on themselves to form and algebra

$$[t_1, t_2] = t_2, \qquad [t_1, t_3] = -t_3. \tag{C.3}$$

The linear transformations they generate are of the kind

$$z \to \frac{az + b}{cz + d} \tag{C.4}$$

If we identify the non-zero values of the generators with the constants a, b, c and d, then t_1 generates rescalings, t_2 generates translations and t_3 is related to inversions. This can be easily shown. Let's see how a SL(2) matrix tranforms infinitesimally.

$$M' = (\mathbb{1}_2 + \sum_{i=1,3} \epsilon^i t_i) M = \begin{pmatrix} 1 + \frac{\alpha}{2} & \beta \\ \gamma & 1 - \frac{\alpha}{2} \end{pmatrix} M. \tag{C.5}$$

Now let's see what the variation of z looks like

$$\delta z = \frac{(1+\frac{\alpha}{2})z+\beta}{\gamma z + 1 - \frac{\alpha}{2}} - z \simeq \alpha z - \gamma z^2 + \beta. \tag{C.6}$$

It is straight forward how α and β generate rescalings and translations respectively. To see how γ involves inversions we can use the following example

$$\begin{aligned} z &\to z' = -\frac{1}{z} \\ z' &\to z'' = z' + \gamma \\ z'' &\to z''' = -\frac{1}{z''} = \frac{z}{1-\gamma z} \simeq z + \gamma z^2 \\ \delta z &= \gamma z^2. \end{aligned} \tag{C.7}$$

By the process of inversion-translation-inversion we obtain a term of the type z^2 in (C.6).

C.2 Borel Group and Algebra

The concepts of Borel subgroup and Borel subalgebra are not widely known nor easily described in the literature. Still we would like to give some basic definitions and a glance at their structure.

C.2.1 Borel Subgroup

In the case of a GL(n) group (n x n invertible matrices), the subgroup of upper triangular matrices is a Borel subgroup.

$$G \supset B = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \right\} \tag{C.8}$$

It is generated by a diagonal generator and the off-diagonal upper triangular generators.

C.2.2 Borel Algebra

For the special case of a Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$, given an ordering of $\mathfrak{h}$, the Borel subalgebra is the direct sum of $\mathfrak{h}$ and the weight subspaces of $\mathfrak{g}$ with positive weight, i.e. the upper triangular generators.

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \tag{C.9}$$

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