Kappa -Symmetry of $SL(2, \mathbb{R})$ -Invariant Super IIB Branes

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Introduction

Unifying all fundamental interactions in one theory is one of the goals of physics research. In this course, symmetries have been playing and will still play important roles. We are on the way to the ultimate theory, if it exists, though we do not know when we can reach our destination. Many stories have been happening on this journey.

About 150 hundred years ago, Maxwell found his famous equations and thus gave an unified description of the electric and magnetic interactions, which are two sides of what we call the electromagnetic interaction today. This is the first unification in human history. Maxwell equations are linear equations and they are very elegant. It was found that the electromagnetic field can be described by a vector, namely, gauge potential. The electric and magnetic fields are field strength of this gauge potential. Maxwell theory has gauge symmetry, which means that the equations of motion are invariant under gauge transformations. Later, physicists developed some kinds of nonlinear electromagnetic theories as modifications of the original Maxwell theory. One of these is Born-Infeld theory, and the Born-Infeld action becomes very important in string theory later.

At the end of 1940's, quantum electrodynamics(QED) was fully established, which is a good theory to describe the process of scattering and also the creation and annihilation of some elementary particles. This theory has been verified by experiments very well at high accurate level. Quantum electrodynamics is a kind of quantum field theory which can be considered as a combination of special relativity and quantum mechanics. After merging the electromagnetic interaction with another interaction, namely the weak interaction which is in charge of the process of decay of particles, we arrive at another unified theory which is called electroweak theory. Electroweak theory has $U(1) \times SU(2)$ symmetry. Here, the three gauge bosons which transfer weak interactions can get their masses through Higgs mechanism(Spontaneous Symmetry Breaking). By the mid of 1970's, Quantum chromodynamics(QCD) was constructed, which is a theory describing strong interactions, and which has $SU(3)$ symmetry. $SU(3)$ Lie algebra has 8 generators, so, we have 8 different gluons which transfer color force, i.e. strong interactions, between quarks. This symmetry has no spontaneous symmetry breaking in this case, because the gluons are massless particles. In the following years, people combined the electroweak and strong interactions together and got the so called Standard Model which has $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry. The standard model has achieved great success and has been verified by many experiments. But it still has some problems. One is that the standard model has too many parameters which should be fixed by experiments. This is not satisfactory as a theory. And the standard model gave the prediction of the existence of the Higgs particle which has not been found yet. Some people may think that the Higgs particle may be found on LHC in the coming years but we can not guarantee this. Furthermore, the standard model does not include the gravitational interaction, which might be the most unsatisfied thing since the gravitational interaction is one of the four fundamental interactions we have known. On the other hand, the gravitation appears in string theory naturally. String theory may unify the four fundamental interactions.

At first, string theory was proposed as a candidate for describing the strong interaction but was discarded later because of the success of quantum chromodynamics. After interpreting the massless spin 2 particle which appears in string theory as the graviton, the interest of string theory was revived. Another good thing is that the ultraviolet divergences disappear in string theory which occur in the quantum field theory of elementary particles. By including fermionic fields, we get the superstring theory. At the beginning, there are many versions of superstring theories. Things changed after the so-called "first superstring revolution" which started at about 1984, lasted for about 2 years. The famous Green-Schwarz anomaly cancellation played a very important role during this period and left us with only five superstring theories, namely Type I, Type IIA, Type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic. All of these five theories require the dimension of spacetime to be 10. The "second superstring revolution" started at about 1995. During this revolution, some dualities were found, such as T-duality and S-duality, and the five superstring theories can be related by these dualities. And also M-theory was found, which lives in a spacetime of 11 dimensions. The five superstring theories can be derived from M-theory. At that time, Dp-branes and p-branes were discovered.

Since there are other extended objects in string theory, studying only strings is not reasonable. In order to understand string theory better, we should also study the bound states of strings, Dp-branes and p-branes. It was found that fundamental strings(F-strings) can end on D-strings(D1-branes). And these bound states have an $SL(2,\mathbb{R})$ symmetry at classical level, which reduces to $SL(2,\mathbb{Z})$ symmetry at quantum level. Similar things happen to other type IIB branes. The D3-brane forms a singlet and the five-branes(the D5-brane and the NS5-brane) form a doublet under $SL(2,\mathbb{R})$ symmetry transformations. While the one-branes and five-branes both form doublets under $SL(2,\mathbb{R})$ symmetry transformations, the seven-branes and the nine-branes form a triplet and a quadruplet separately. The type IIA string theory has no such an $SL(2,\mathbb{R})$ symmetry. This is because that the type IIA string theory has spinors of opposite chirality in right- and left-moving sectors, i.e., which is a non-chiral theory.

Besides $SL(2,\mathbb{R})$ symmetry, there is also another symmetry named kappa-symmetry for these bound states. Kappa-symmetry was first found in the super-particle case and then was generalized in the cases of superstrings, super p-branes and super Dp-branes. Kappa-symmetry is a natural tool to reduce the fermionic degrees of freedom by half and through which(among other things) the supersymmetry on the worldvolume can be gotten. The main topic of this report is the kappa-symmetry property of the $SL(2,\mathbb{R})$-invariant super IIB branes.

The outline of this report is as following. In Chapter 1, some basic concepts of string theory will be introduced. There, we will not cover all important things in string theory, but will just consider the things that we may use in the later chapters. In Chapter 2, we will give a discussion about kappa-symmetry for the cases of the super particle, the super D-string and the super F-string. In Chapter 3, we will discuss the $SL(2,\mathbb{R})$ symmetry of IIB branes. In Chapter 4, the kappa-symmetry property of the $SL(2,\mathbb{R})$-symmetric IIB branes will be discussed. And then, the summary and conclusions will be given. In the appendices, we will give some conventions, identities and proofs.

Chapter 1

String Theory

In this chapter, we will give a basic description of string theory. The purpose is to give some feelings of the concepts that may relate to the contents in the later chapters. We will start with a simple introduction of the relativistic particle. Then we will discuss the bosonic open and closed strings. Here, we will not go to the details of the quantisations, but will just discuss some symmetry properties of the action, the solutions under different boundary conditions and the first few levels of the state spectrum. Superstrings will be discussed in an almost similar way. At the last of this chapter, T-duality, Dp-branes and p-branes will be discussed briefly. For these aspects, we refer to [1, 2, 3, 4].

1.1 The relativistic particle

Before discussing strings, it is useful to discuss the free relativistic particle. One reason for this is that there are some analogies between the action of relativistic particles and the action of relativistic strings, the other reason is that since we will discuss the kappa-symmetry property of the superparticle in the next chapter, it is better to give some preliminaries of relativistic particles in advance.

First, we should know the action of a free relativistic particle. Certainly, the action is not arbitrary and it must have some properties. One important property is that the action should be Lorentz invariant, i.e., which should be a Lorentz scalar. Imaging a relativistic particle moving freely in a spacetime(the dimension is not specified here), we ask: what is the Lorentz scalar? From the theory of relativity, we know that the quantity below is a Lorentz scalar:

$$ds^2 = -g_{\mu\nu} dX^\mu dX^\nu. \tag{1.1}$$

Here, we define the spacetime metric as $g_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$. The length of the path traced out by the particle is $\int ds$, with $ds = \sqrt{ds^2}$. We conjecture that the action is proportional to this quantity. If we take $\hbar = c = 1$, the action should be dimensionless. Since $\int ds$ has a dimension of length, i.e. the inverse of mass, it is natural to take the action as

$$S = -m \int ds = -m \int d\tau \sqrt{-\dot{X}^2}, \quad \text{with} \quad \dot{X}^2 = g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu \quad \text{and} \quad \dot{X}^\mu \equiv \frac{dX^\mu}{d\tau}. \tag{1.2}$$

Where, the minus sign is a correct choice. The parameter m is identified with the rest mass of the particle. And τ is a parameter. Since the path traced out by the particle is one dimensional, we need only one parameter. Choosing another parameter $\tau' = \tau'(\tau)$, if $X'^\mu(\tau') = X^\mu(\tau)$, we get

$$\begin{aligned} S' = -m \int ds &= -m \int d\tau' \sqrt{-g_{\mu\nu}\frac{dX'^\mu(\tau')}{d\tau'}\frac{dX'^\nu(\tau')}{d\tau'}} \\ &= -m \int d\tau \left|\frac{d\tau'}{d\tau}\right|\left|\frac{d\tau}{d\tau'}\right|\sqrt{-\dot{X}^2} = S. \end{aligned} \tag{1.3}$$

Therefore, the action is invariant under reparameterization of the parameter τ. And also, the action has a global Poincaré symmetry.

Besides the action (1.2), there is another form of action which seems more convenient. It is

$$S = \frac{1}{2}\int d\tau (e^{-1}\dot{X}^2 - em^2), \tag{1.4}$$

here $e(\tau)$ is an auxiliary field which has a dimension of length/mass and which can also be looked as an einbein on the world-line. This action has no square root term. The action (1.2) can be derived from (1.4). While the action (1.2) is meaningless for a massless particle, the action (1.4) can be used to describe a massless particle. Furthermore, the action (1.4) is easier to be generalized to the superparticle case which will be discussed in Section 2.1.

1.2 The bosonic string

1.2.1 Actions

In this section, we only consider bosonic strings. Since the action of a free relativistic particle is proportional to the length of the path traced out by the particle, it is natural to guess that the action of a free string is proportional to the area of surface traced out by the string(usually, the surface is called the world-sheet of the string). Thus, we achieve the famous Nambu-Goto action for the bosonic string:

$$S = -T \int d^2\sigma \sqrt{-g}. \tag{1.5}$$

Where $d^2\sigma = d\sigma^1 d\sigma^2$ with σ^i's are the parameters, just as the parameter τ in the point particle case. And $g = \det g_{ij}$ with $g_{ij} = g_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X^\nu}{\partial \sigma^j}$. The metric g_{ij} is called the induced metric on the world-sheet or the pullback of the spacetime metric $g_{\mu\nu}$. The $X^\mu(\sigma^1, \sigma^2)$'s are the embedding coordinates or string coordinates, which map the point (σ^1, σ^2) in the parameter space to the point $X^\mu(\sigma^1, \sigma^2)$ in the target space. Here $\mu = 0, 1, \ldots, D-1$ with D is the dimension of the target spacetime. T is the tension of the string, which has a dimension of mass/length, and for general p-branes, T has a dimension of mass$\times$length^{-p}. And also, for strings, we have following relations:

$$T = \frac{1}{2\pi\alpha'}, \qquad \alpha' = \ell_s^2. \tag{1.6}$$

Where, the parameter α' is called the "universal Regge slope" and ℓ_s is the "intrinsic" length of the string.

Similar to the case of relativistic particle, we can also introduce an auxiliary field in the string case and then we get the Polyakov action:

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{ij}\partial_i X^\mu \partial_j X_\mu, \tag{1.7}$$

here the auxiliary field h_{ij} is called the world-sheet metric and $h = \det h_{ij}$. The Nambu-Goto action can be derived from the Polyakov action. There are some symmetries for this action:

- World-sheet reparameterization invariance

$$\delta X^\mu = \xi^i \partial_i X^\mu, \qquad \delta h^{ij} = \xi^k \partial_k h^{ij} - \partial_k \xi^i h^{kj} - \partial_k \xi^j h^{ik}, \tag{1.8}$$

  where ξ^i is the infinitesimal shift of σ^i.

- Target space global Poincaré invariance

$$\delta X^\mu = \Lambda^\mu_\nu X^\nu + a^\mu, \qquad \delta h_{ij} = 0, \tag{1.9}$$

  with Λ^μ_ν and a^μ are constant.

- Local Weyl scaling or conformal invariance

$$\delta h_{ij} = \Omega(\sigma^i, \sigma^j)h_{ij}, \qquad \delta X^\mu = 0. \tag{1.10}$$

Note that, the local Weyl scaling invariance is only available for strings and is not available for other extended objects with more dimensions. Certainly, the Nambu-Goto action also has the world-sheet reparameterization and global Poincaré invariance, but we can not see the local Weyl scaling invariance from the Nambu-Goto action.

1.2.2 Solutions

The equations of motion of X^μ from the action (1.7) are very complicated if we do not choose a good parameterization. It is lucky that we do can choose such a parameterization that by which we can simplify the action and then get simpler equations. First, we can use the world-sheet reparameterization to make the world-sheet metric has the form

$$h_{ij} = e^{w(\sigma^i,\sigma^j)}\eta_{ij} \quad \text{with} \;\; \eta_{ij} = \text{diag}(-1,1). \tag{1.11}$$

This kind of gauge is usually called the conformal gauge or orthonormal gauge. The spacetime with a metric having the above form is conformally flat. Further, we can use the local Weyl scaling invariance and get

$$h_{ij} = \eta_{ij}. \tag{1.12}$$

Therefore, the Polyakov action (1.7) can be simplified as

$$S = -\frac{T}{2}\int d^2\sigma \eta^{ij}\partial_i X^\mu \partial_j X_\mu. \tag{1.13}$$

It is easy to get the equations of motion by using the principle of least action. The equations of motion are just the one-dimensional wave equations:

$$\Box X^\mu = 0, \tag{1.14}$$

with some boundary conditions. Where $\Box \equiv \partial_i\partial^i$. Usually, for strings, we let $\sigma^1 \equiv \tau$, $\sigma^2 \equiv \sigma$ and $\sigma \in [0,\pi]$.

Now, let us discuss the boundary conditions and solutions for closed and open strings.

- Closed strings

  For closed strings, the imposed boundary condition is

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi). \tag{1.15}$$

  This is natural for closed strings, since closed strings have no end. And the solution of the equation of motion (1.14) with the above boundary condition can be expressed as

$$X^\mu(\tau,\sigma) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma), \tag{1.16}$$

  with $X^\mu_L(\tau+\sigma)$ and $X^\mu_R(\tau-\sigma)$ are the left- and right-moving parts separately, which take the form:

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}q^\mu + \frac{1}{2\pi T}P^\mu(\tau+\sigma) + \frac{i}{2}\ell_s\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2i(\tau+\sigma)n}, \tag{1.17}$$

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}q^\mu + \frac{1}{2\pi T}P^\mu(\tau-\sigma) + \frac{i}{2}\ell_s\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2i(\tau-\sigma)n}, \tag{1.18}$$

where, q^μ is the position of the center of mass of the string and P^μ is the total momentum.

- Open strings
  For open strings, there are two types of boundary conditions. One is the Neumann boundary condition

  $$\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) = 0. \tag{1.19}$$

  And the solution of the equation of motion (1.14) with the Neumann boundary condition is

  $$X^\mu(\tau, \sigma) = q^\mu + \frac{1}{\pi T} P^\mu \tau + i\ell_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma). \tag{1.20}$$

  Another boundary condition is the Dirichlet boundary condition

  $$\partial_\tau X^\mu(\tau, 0) = \partial_\tau X^\mu(\tau, \pi) = 0. \tag{1.21}$$

  Here, we do not give the solutions for the boundary condition. We will discuss this in Section 1.4. Certainly, we can impose different boundary conditions on different ends and for different μ. If an end is imposed with the Neumann boundary condition in the direction μ, this end can move freely along this direction. If an end is imposed with the Dirichlet boundary condition in the direction μ, this end can not move along this direction.

Note that, we shall not forget that there are still some constraints on the solutions and these constraints come from the equation of motion with respect to h_{ij}.

1.2.3 Spectra

Now, let us go to the quantum theory of bosonic strings. There are three quantisation procedures: covariant quantisation, light-cone gauge quantisation and path integral quantisation. Each procedure has advantages and disadvantages, but we can get the same results by different procedures. Here, we do not go to the details of the Virasoro algebra. After quantisation, we get the state spectrum.

For open strings, the ground state is a tachyon scalar field, which has a imaginary mass and is not stable. The first excited states contain $D-2$ massless states. These $D-2$ massless states form a massless representation of D-dimensional Poincaré group. Furthermore, we can fix $D = 26$. This is because that we require an anomaly free theory. And also, we can get this critical dimension by comparing the results of covariant quantisation and light-cone gauge quantisation.

For closed strings, the ground state is still a tachyon. The first excited states contain $(D-2)^2$ massless states which form the massless representations of D-dimensional Poincaré group. These massless states can be split into three irreps: $\frac{D(D-3)}{2}$ states which form a symmetric and traceless tensor with spin 2, $\frac{(D-2)(D-3)}{2}$ states which form an antisymmetric tensor and one scalar.

1.3 The superstring

1.3.1 Actions

Bosonic strings do not include fermions but our real world do have fermions. So bosonic string theory is not sufficient and we should generalize it to the so-called superstring theory. We will see that the superstring theory has some advantages as compared to the bosonic string theory. For instance, superstrings has no tachyon and reduce the dimension of space-time to 10 which is much less than 26 and is closer to 4.

Clearly, for superstrings, the action should contains fermions. In the orthonormal gauge, the simplest action can be given by

$$S = -\frac{T}{2}\int d^2\sigma\big(\partial_i X^\mu \partial^i X_\mu - i\overline{\psi}^\mu \partial\!\!\!/\, \psi_\mu\big), \tag{1.22}$$

here ψ_μ are Majorana spinors on the world-sheet:

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}, \qquad \overline{\psi} = \psi^\dagger \gamma^0. \tag{1.23}$$

The action (1.22) is invariant under the following supersymmetry transformations:

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \qquad \delta\psi^\mu = -i\partial\!\!\!/ X^\mu \epsilon, \tag{1.24}$$

here ϵ is a constant spinor. This is just what we expected, if we require supersymmetry on world-sheet.

Now, let us try to get the equations of motion of ψ^μ and find the suitable boundary conditions. The variation of the action (1.22) with respect to the variation of ψ^μ is

$$\begin{aligned}\delta S &= \frac{iT}{2}\int d^2\sigma\big(\delta\overline{\psi}^\mu \partial\!\!\!/\,\psi_\mu + \overline{\psi}^\mu \partial\!\!\!/\,\delta\psi_\mu\big) \\ &= iT\int d^2\sigma\big[\delta\overline{\psi}^\mu \partial\!\!\!/\,\psi_\mu + \frac{1}{2}\partial_i(\overline{\psi}^\mu\gamma^i\delta\psi_\mu)\big]. \end{aligned}\tag{1.25}$$

It is obvious that the equation of motion is $\partial\!\!\!/\,\psi_\mu = 0$, as expected. If we choose $\gamma^1 = i\sigma_1$, we have following boundary conditions.

- For open superstrings
  Firstly, we can impose the Dirichlet condition. That is
$$\delta\psi_{\mu\pm}(\tau, \sigma = 0, \pi) = 0. \tag{1.26}$$
  Secondly, if we suppose that $\psi_+(\tau, 0) = \psi_-(\tau, 0)$ and $\delta\psi_{\mu+} = \pm\delta\psi_{\mu-}$ at the ends, we still have two choices:
$$\psi_{+\mu}(\tau, \pi) = \psi_{-\mu}(\tau, \pi) \qquad \text{(Ramond)} \tag{1.27a}$$
  or
$$\psi_{+\mu}(\tau, \pi) = -\psi_{-\mu}(\tau, \pi) \qquad \text{(Neveu-Schwarz).} \tag{1.27b}$$
  The equations of motion are given by
$$(\partial_\tau + \partial_\sigma)\psi_-^\mu = 0, \qquad (\partial_\tau - \partial_\sigma)\psi_+^\mu = 0. \tag{1.28}$$

- For closed superstrings
  Firstly, we have a periodic boundary condition(Ramond)
$$\psi_\mu(\sigma + \pi) = \psi_\mu(\sigma), \qquad \delta\psi_\mu(\sigma + \pi) = \delta\psi_\mu(\sigma). \tag{1.29}$$
  Secondly, we have a anti-periodic boundary condition(Neveu-Schwarz)
$$\psi_\mu(\sigma + \pi) = -\psi_\mu(\sigma), \qquad \delta\psi_\mu(\sigma + \pi) = -\delta\psi_\mu(\sigma). \tag{1.30}$$
  And we can impose Ramond and Neveu-Schwarz boundary conditions on the $\pm$ components separately.

As in the bosonic case, we should have some constraints to get the correct spectrum of states. One is that the energy momentum tensor $T_{ij} = 0$, which is the same as the bosonic string case, another is that the supersymmetry current $J^i \equiv (\not{\partial} X^\mu)\gamma^i\psi_\mu = 0$. And these two constraints are related to each other if requiring supersymmetry on the world-sheet.

1.3.2 Spectra

In the superstring theory, we can fix the dimension of space-time $D = 10$. As considering the spectrum of the states of open superstrings, the Ramond and Neveu-Schwarz sectors give different results. For the Neveu-Schwarz sector, the ground state is still a tachyon and the first excited states are $D-2$ massless states which is a massless vector, denoted by 8_v. The spectrum of the Ramond sector consists of no tachyon, and the ground states are massless and are degenerate. If we suppose that these spinors are Majorana-Weyl spinors, with other constraints, we can get that the degeneracy is 8. Because we can choose positive or negative chirality, we have two 8-dimensional spinors denoted by 8_s and 8_c respectively. The tachyon can be get rid of by the GSO(Gliozzi-Scherk-Olive) projection and at the same time we get supersymmetry.

For closed superstrings, we have four different sectors: (NS, NS), (NS, R), (R, NS) and (R, R) sectors. Here, (NS, R) means that Neveu-Schwarz condition is imposed on the +-component(left moving) and the Ramond condition is imposed on the −-component(right moving) and others are similar. This time, we can apply different GSO projections to the +-component and the −-component. As a result, we have type IIB superstring theory(the massless spinors have equal chirality in right- and left- moving sectors) and type IIA superstring theory(the massless spinors have opposite chirality in right- and left- moving sectors). Table 1.1 gives the spectra of the massless states of type IIA and type IIB superstring theories.

	IIB	IIA
(NS, NS)	$8_v \otimes 8_v = 1 \oplus 28 \oplus 35_v$	$8_v \otimes 8_v = 1 \oplus 28 \oplus 35_v$
(R, R)	$8_s \otimes 8_s = 1 \oplus 28 \oplus 35_s$	$8_s \otimes 8_c = 8_v \oplus 56_v$
(R, NS)	$8_s \otimes 8_v = 8_s \oplus 56_s$	$8_s \otimes 8_v = 8_s \oplus 56_s$
(NS, R)	$8_v \otimes 8_s = 8_s \oplus 56_s$	$8_v \otimes 8_c = 8_s \oplus 56_c$

Table 1.1: Spectra of the massless states of IIB and IIA superstring theories.

In this thesis, we will only consider IIB theory. The (NS, NS) sector of IIB theory contains a dilaton ϕ which controls the strength of string interactions, a Kalb-Ramond field $B_{\mu\nu}$ and a gravity field $G_{\mu\nu}$. The (R, R) sector of IIB theory contains a Ramond-Ramond 0-form $C_{(0)}$(or the axion ℓ), a Ramond-Ramond 2-form $C_{\mu\nu}$ and a Ramond-Ramond 4-form $C_{\mu\nu\rho\sigma}$. The (R, NS) and (NS, R) sectors have the same fields content: a gravitino field with spin $\frac{3}{2}$ and a dilatino field with spin $\frac{1}{2}$. And these fermionic states all have the same chirality.

1.4 The Dp-brane and p-brane

1.4.1 T-duality

Before discussing branes, let us give an introduction about T-duality for open strings. T-duality is a very important concept in string theory. It can give an equivalence between the string theory in a small scale and in a large scale. It interchanges the type IIA and type IIB superstring theories. And it also give the necessity for the existence of Dp-branes. Here we just give a short introduction. In the superstring theory, we have a spacetime of dimension $D = 10$. If one dimension is compactified, for example x^9, under T-duality, we have

$$X^9_L(\tau+\sigma) \longmapsto X^9_L(\tau+\sigma); \qquad X^9_R(\tau-\sigma) \longmapsto -X^9_R(\tau-\sigma), \tag{1.31}$$

and other string coordinates are not changed. Thus for open strings:

$$\begin{aligned} X^9(\tau,\sigma) \longmapsto X'^9(\tau,\sigma) &= X^9_L(\tau+\sigma) - X^9_R(\tau-\sigma) \\ &= \frac{1}{\pi T}\frac{n}{R}\sigma + \ell_s \sum_{n\neq 0} \frac{\alpha^9_n}{n} e^{-in\tau}\sin(n\sigma). \end{aligned} \tag{1.32}$$

Since $\sin(n\sigma) = 0$ at ends $\sigma = 0$ and $\sigma = \pi$, we get

$$\partial_\tau X'^9(\tau,0) = \partial_\tau X'^9(\tau,\pi) = 0, \tag{1.33}$$

which means that we have a Dirichlet boundary condition in this direction.

From the example of T-duality in the 9th direction above, we get a D8-brane, which is nine dimensional in a target spacetime. The ends of open strings can move freely in this D8-brane. Generally, if we T-dualize $(9-p)$ directions, we will get a Dp-brane.

1.4.2 Actions

The extended objects can also be gotten from supergravity. Supergravity theory is the low energy limit of superstring theory. In the (NS, NS) sector of type IIB supergravity, people found two solutions: the fundamental string and the non-singular solitonic five-brane(NS5-brane). These are extended objects but not Dp-branes, which are called p-branes. The fundamental string carries an electric string charge which is the source of $B_{\mu\nu}$. The non-singular solitonic 5-brane carries a magnetic charge with respect to the spacetime Hodge dual of $B_{\mu\nu}$. While the (NS, NS) sector have p-brane solutions, the (R, R) sector of IIB supergravity has Dp-brane solutions. It was found that these solutions are Dp-branes with $p = 1, 3, 5, 7$. The D1-brane has an electric charge and the D5- and D7-branes are magnetically charged. The D3-branes is a singlet which has both electric and magnetic charges.

Generally, a p-brane can couple to a NS-NS $(p+1)$-form and the action is given by

$$S = -T \int d^{p+1}\sigma[\sqrt{-g} + \sqrt{-g} \star B_{(p+1)}], \tag{1.34}$$

here, $B_{(p+1)}$ is a NS-NS $(p+1)$-form and the $\star$ is the Hodge dual operator. As an example, we give the action of the Fundamental string(F-string)

$$S = -T \int d^2\sigma[\sqrt{-g} + \frac{1}{2}\varepsilon^{ij} B_{ij}], \tag{1.35}$$

with $B_{ij} = \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}$ is called the pull back of $B_{\mu\nu}$.

And the Dp-brane action is given by

$$S = -T \int d^{p+1}\sigma [e^{-\phi}\sqrt{-\det(g+\mathcal{F})} - \mathcal{L}_{WZ}], \qquad (1.36)$$

here, ϕ is the dilaton and $\mathcal{F} = db - B_{(2)}$, with b is the Born-Infeld vector which lives on the world-volume and $B_{(2)}$ is a NS-NS 2-from or a Kalb-Ramond 2-form. The first part of the action is called Dirac-Born-Infeld(DBI) part. The second part is called Wess-Zumino(WZ) part which is

$$\mathcal{L}_{WZ} = C_{(p+1)} + C_{(p-1)} \wedge \mathcal{F} + \frac{1}{2!} C_{(p-3)} \wedge \mathcal{F} \wedge \mathcal{F} + \ldots, \qquad (1.37)$$

here, $C_{(p+1)}$ is the R-R $(p+1)$-form. For example, the action of the D-string is given by

$$S = T \int d^2\sigma [-e^{-\phi}\sqrt{-\det(g+\mathcal{F})} + \frac{1}{2}\varepsilon^{ij}(C_{ij} + \ell \mathcal{F}_{ij})], \qquad (1.38)$$

with $C_{ij} = \partial_i X^\mu \partial_j X^\nu C_{\mu\nu}$ and $\mathcal{F}_{ij} = d_{[i}b_{j]} - \partial_i X^\mu \partial_j X^\nu \mathcal{B}_{\mu\nu}$. Where $\ell = C_{(0)}$ is the axion.

Chapter 2

Kappa-Symmetry

Kappa-symmetry is a local fermionic symmetry and it is a natural approach to reduce half of the fermionic degrees of freedom. Thus it is important for keeping supersymmetry on the world-volume. We will give some simple examples of kappa-symmetry in this chapter. The kappa-symmetry properties of the superparticle, the super D-string and the super F-string will be discussed.

2.1 The superparticle

Kappa-symmetry was first found by Siegel [5] in the superparticle case. Following [6], we discuss kappa-symmetry of the superparticle in this section. In Section 1.1, we have discussed the relativistic bosonic point particle. In order to get the action of the superparticle, we need to generalize the Minkowski space which contains only bosonic coordinates to the so-called superspace which contains fermionic coordinates also. If we have N spinor coordinates $\theta^A (A = 1, 2, \ldots, N)$ in superspace, we say that we have N supersymmetries. Generally, the spinor θ can be Dirac spinors, but in this section, we restrict θ to be Majorana and the Γ matrices to be real.

For a massless particle, the supersymmetric generalization of the action (1.4) can be given as

$$\mathcal{L} = \frac{1}{2} e^{-1} (\dot{X} - \overline{\theta}^A \Gamma^\mu \dot{\theta}^A)^2, \tag{2.1}$$

which is obviously Lorentz invariant. And it is also invariant under the following supersymmetric transformations:

$$\delta_\epsilon \theta^A = \epsilon^A, \qquad \delta_\epsilon \overline{\theta}^A = \overline{\epsilon}^A, \qquad \delta_\epsilon X^\mu = \overline{\epsilon}^A \Gamma^\mu \theta^A, \qquad \delta_\epsilon e = 0, \tag{2.2}$$

with ϵ^A are infinitesimal constant spinors.

Now, let us define the kappa-symmetry transformations as following:

$$\delta_\kappa \theta^A = \Gamma\cdot\Pi\kappa^A, \quad \delta_\kappa \overline{\theta}^A = \overline{\kappa}^A \Gamma\cdot\Pi, \quad \delta_\kappa X^\mu = \overline{\theta}^A \Gamma^\mu \delta_\kappa \theta^A, \quad \delta_\kappa e = 4e\dot{\overline{\theta}}^A \kappa^A, \tag{2.3}$$

here $\kappa^A(\tau)$ are infinitesimal non-constant spinors and $\Pi^\mu \equiv \dot{X}^\mu - \overline{\theta}^A \Gamma^\mu \dot{\theta}^A$. So, the variation of the Lagrangian (2.1) under the kappa-symmetry transformations (2.3) is

$$\begin{aligned}
\delta_\kappa \mathcal{L} &= -\frac{1}{2e^2}(4e\dot{\overline{\theta}}^A \kappa^A)\Pi^2 + e^{-1}\Pi_\mu\left[\dot{\overline{\theta}}^A \Gamma^\mu \delta_\kappa \theta^A + \overline{\theta}^A \Gamma^\mu \delta_\kappa \dot{\theta}^A\right] \\
&\quad - e^{-1}\Pi_\mu\left[\delta_\kappa \overline{\theta}^A \Gamma^\mu \dot{\theta}^A + \overline{\theta}^A \Gamma^\mu \delta_\kappa \dot{\theta}^A\right] \\
&= -2e^{-1}\dot{\overline{\theta}}^A \kappa^A \Pi^2 + e^{-1}\Pi_\mu\left[\dot{\overline{\theta}}^A \Gamma^\mu \Gamma\cdot\Pi\kappa^A - \overline{\kappa}^A \Gamma\cdot\Pi\Gamma^\mu \dot{\theta}^A\right] \\
&= 0.
\end{aligned} \tag{2.4}$$

Therefore, the Lagrangian (2.1) has kappa-symmetry.

The equations of motion are

$$\Pi^2 = 0, \qquad \dot{\Pi}^\mu = 0, \qquad \Gamma\cdot\Pi\dot{\theta}^A = 0. \tag{2.5}$$

Because $(\Gamma\cdot\Pi)^2 = -\Pi^2 = 0$, the matrix $\Gamma\cdot\Pi$ has half the maximum possible rank. Thus, half of the components of θ^A are gauged away since which always appears multiplied by $\Gamma\cdot\Pi$. This is the consequence of that the Lagrangian (2.1) has kappa-symmetry.

2.2 The super D-string

2.2.1 The basic idea

Kappa-symmetry was discussed in the superstring case [7, 8]. And it was proved for the super Dp-branes in a flat background [9], in a bosonic background [10] and in a general background [11].

The basic idea to prove that there is kappa-symmetry for super Dp-branes is as follows [9]. The Lagrangian of a super Dp-brane contains two parts, namely the Dirac-Born-Infeld part and the Wess-Zumino part:

$$\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{WZ}. \tag{2.6}$$

Under kappa-symmetry transformations, if we can write the variations as

$$\delta_\kappa \mathcal{L}_{DBI} = 2\delta_\kappa \overline{\theta}\gamma^{(p)} T^j_{(p)} \partial_j \theta, \tag{2.7a}$$

$$\delta_\kappa \mathcal{L}_{WZ} = 2\delta_\kappa \overline{\theta} T^j_{(p)} \partial_j \theta, \tag{2.7b}$$

with some $\gamma^{(p)}$ and $T^j_{(p)}$ to be specified, and at the same time, $(\gamma^{(p)})^2 = 1$, and further, if we require that

$$\delta_\kappa \overline{\theta} = \overline{\kappa}(1 - \gamma^{(p)}), \tag{2.8}$$

we can get $\delta_\kappa \mathcal{L} = 0$. Since $(\gamma^{(p)})^2 = 1$, the eigenvalues of $\gamma^{(p)}$ are $+1$ or -1. And furthermore, if $\mathrm{tr}(\gamma^{(p)}) = 0$, half of the fermionic components will be gotten rid of.

2.2.2 Preliminaries

In this section, we will discuss kappa-symmetry for super D-strings in a flat background. We follow [9]. The Grassmann coordinates θ are spacetime spinors and world-volume scalars. In the Type IIB theory, we have two Majorana-Weyl spinors of the same chirality. The Dirac matrices defined here have a extra factor i as compared to the usual ones. The anticommutation relations are

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \tag{2.9}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. As in the superparticle case, we define a supersymmetric quantity

$$\Pi^\mu_i \equiv \partial_i X^\mu - \overline{\theta}\Gamma^\mu \partial_i \theta, \tag{2.10}$$

here i is the index of world-volume coordinates. Π^μ_i is a "pull back operator" in a superspace background just as $\partial_i X^\mu$ in a bosonic background. Then we can get the induced world-volume metric

$$g_{ij} = \Pi^\mu_i \Pi^\nu_j \eta_{\mu\nu}. \tag{2.11}$$

Sometimes, it is convenient to use the notations of differential forms, thus it is good to define

$$d\theta \equiv d\sigma^i \partial_i \theta = -\partial_i \theta d\sigma^i. \tag{2.12}$$

We see that $d\sigma^i$ is regarded as an odd element of the Grassmann algebra. The supersymmetric one-forms are $\Pi^\mu = dX^\mu + \overline{\theta}\Gamma^\mu d\theta$ and $d\theta$.

Just as in the superparticle case, the variation of the bosonic coordinate under kappa-symmetry transformations is defined as

$$\delta_\kappa X^\mu = \overline{\theta}\Gamma^\mu \delta_\kappa \theta = -\delta_\kappa \overline{\theta}\Gamma^\mu \theta. \tag{2.13}$$

Then the variation of Π^μ_i is

$$\begin{aligned} \delta_\kappa \Pi^\mu_i &= \delta_\kappa(\partial_i X^\mu) - \delta_\kappa(\overline{\theta}\Gamma^\mu \partial_i \theta) \\ &= -\partial_i(\delta_\kappa \overline{\theta}\Gamma^\mu \theta) - \delta_\kappa \overline{\theta}\Gamma^\mu \partial_i \theta - \overline{\theta}\Gamma^\mu \partial_i \delta_\kappa \theta \\ &= -2\delta_\kappa \overline{\theta}\Gamma^\mu \partial_i \theta. \end{aligned} \tag{2.14}$$

Defining the "induced γ matrix": $\gamma_i \equiv \Pi_i^\mu \Gamma_\mu$, we get

$$\begin{aligned}
\delta_\kappa g_{ij} &= \delta_\kappa(\Pi_i^\mu \Pi_j^\nu \eta_{\mu\nu}) \\
&= -2\delta_\kappa \overline{\theta}\Gamma^\mu \partial_i \theta \Pi_j^\nu \eta_{\mu\nu} - 2\Pi_i^\mu \delta_\kappa \overline{\theta}\Gamma^\nu \partial_j \theta \eta_{\mu\nu} \\
&= -2\delta_\kappa \overline{\theta}(\gamma_i \partial_j + \gamma_j \partial_i)\theta \\
&\equiv -2\delta_\kappa \overline{\theta}\gamma_{(i}\partial_{j)}\theta. \qquad (2.15)
\end{aligned}$$

We have known that there should be a Born-Infeld vector which lives on the world-volume. The field strength is defined by

$$\mathcal{F}_{ij} = F_{ij} - B_{ij}, \qquad (2.16)$$

where $F_{ij} = \partial_i b_j - \partial_j b_i$ with b is the BI field and $B_{ij} = \Pi_i^\mu \Pi_j^\nu B_{\mu\nu}$.

The supersymmetric version of the pull back of the NS-NS 2-form turns out to be [12]:

$$B_{(2)} = -\overline{\theta}\tau_3 \Gamma_\mu d\theta \big(dX^\mu + \frac{1}{2}\overline{\theta}\Gamma^\mu d\theta\big). \qquad (2.17)$$

Under kappa-symmetry transformations, $B_{(2)}$ varies as

$$\delta_\kappa B_{(2)} = -2\delta_\kappa \overline{\theta}\tau_3 \Gamma_\mu d\theta \Pi^\mu + d\Delta, \qquad (2.18)$$

with

$$\Delta = -\delta_\kappa \overline{\theta}\tau_3 \Gamma_\mu \theta \Pi^\mu + \frac{1}{2}\delta_\kappa \overline{\theta}\tau_3 \Gamma_\mu \theta \overline{\theta}\Gamma^\mu d\theta - \frac{1}{2}\delta_\kappa \overline{\theta}\Gamma^\mu \theta \overline{\theta}\tau_3 \Gamma_\mu d\theta. \qquad (2.19)$$

By requiring $\delta_\kappa b = \Delta$, we can get

$$\delta_\kappa \mathcal{F} = 2\delta_\kappa \overline{\theta}\tau_3 \Gamma_\mu d\theta \Pi^\mu, \qquad (2.20)$$

or in terms of components

$$\delta_\kappa \mathcal{F}_{ij} = 2\delta_\kappa \overline{\theta}\tau_3(\gamma_i \partial_j - \gamma_j \partial_i)\theta \equiv 2\delta_\kappa \overline{\theta}\tau_3 \gamma_{[i}\partial_{j]}\theta. \qquad (2.21)$$

2.2.3 The super D-string

Now, let us consider the super D-string case. The action of the super D-string can be written as

$$S = \int d^2\sigma \Big\{ -e^{-\phi}\sqrt{-\det(g+\mathcal{F})} + \frac{1}{2}\varepsilon^{ij}(C_{ij} + \ell \mathcal{F}_{ij})\Big\}, \qquad (2.22)$$

which is formally the same as the action given by (3.27). Since both the action (3.28) and (3.27) can be derived from the same action (3.25) by integrating out the auxiliary field t, we say that these two actions are equivalent. We will use the action (3.28). As we consider the D-string case, we will take $p = 0$ and $q = 1$ in the action (3.28)(notice that, the definitions of p and q are different from those in Chapter 4). Then, we have

$$\mathcal{L} = -\sqrt{e^{-2\phi} + \ell^2}\sqrt{-g} + \frac{1}{2}\varepsilon^{ij}C_{ij}, \tag{2.23}$$

which is formally looks the same as the action of the bosonic D-string, but the fields contents are different. Here $C_{ij} = \Pi_i^\mu \Pi_j^\nu C_{\mu\nu}$, but in the bosonic case, $C_{ij} = \partial_i X^\mu \partial_j X^\nu C_{\mu\nu}$. If we define

$$\begin{aligned} \rho_D &= \frac{1}{2}\sqrt{e^{-2\phi} + \ell^2}\varepsilon^{ij}\tau_3\gamma_i\gamma_j, & \text{(2.24a)} \\ T_D^j &= \sqrt{e^{-2\phi} + \ell^2}\varepsilon^{ij}\tau_3\gamma_i, & \text{(2.24b)} \end{aligned}$$

from (2.24a), we can get

$$\begin{aligned} (e^{-2\phi} + \ell^2)^{-1}\rho_D^2 &= \frac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\tau_3\gamma_i\gamma_j\tau_3\gamma_k\gamma_l \\ &= \frac{1}{4}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})\gamma_i\gamma_j\gamma_k\gamma_l \\ &= \frac{1}{4}\gamma_i\gamma_j\gamma^{[i}\gamma^{j]} \\ &= -\gamma_0\gamma_0\gamma_1\gamma_1 \\ &= -\det(g_{ij}). \end{aligned} \tag{2.25}$$

Now, let us prove that

$$\rho_D\gamma_i = T_D^j g_{ji}. \tag{2.26}$$

From (2.24a) and (2.24b), we have

$$\begin{aligned} (e^{-2\phi} + \ell^2)^{-1/2}\rho_D\gamma_k &= \frac{1}{2}\varepsilon^{ij}\tau_3\gamma_i\gamma_j\gamma_k \\ &= \tau_3\varepsilon^{ik}\gamma_i\gamma_k^2 \\ &= \tau_3\varepsilon^{ik}\gamma_i g_{kk}, \text{ (no summation on } k.) & \text{(2.27)} \\ (e^{-2\phi} + \ell^2)^{-1/2}T_D^j g_{jk} &= \varepsilon^{ij}\tau_3\gamma_i g_{jk} \\ &= \tau_3\varepsilon^{ik}\gamma_i g_{kk} \\ &= (e^{-2\phi} + \ell^2)^{-1/2}\rho_D\gamma_k. \text{ (no summation on } k.) & \text{(2.28)} \end{aligned}$$

Hence, (2.26) is proved. If let

$$\rho_D = \gamma_D\sqrt{e^{-2\phi} + \ell^2}\sqrt{-g}, \tag{2.29}$$

we can get

$$\gamma_D = \frac{1}{\sqrt{-g}}\frac{1}{2}\varepsilon^{ij}\tau_3\gamma_i\gamma_j, \tag{2.30}$$

and it is easy to check that

$$\gamma_D^2 = 1, \quad \mathrm{tr}\gamma_D = 0. \tag{2.31}$$

These conditions are important for kappa-symmetry.

Now, let us consider the variation of the DBI part under kappa-symmetry transformations:

$$\begin{aligned}
\delta_\kappa\Big[-\sqrt{e^{-2\phi}+\ell^2}\sqrt{-g}\Big] &= \frac{-\sqrt{e^{-2\phi}+\ell^2}}{2\sqrt{-g}}\delta_\kappa(-g) \\
&= -\frac{1}{2}\sqrt{e^{-2\phi}+\ell^2}\sqrt{-g}\, g^{ji}\delta_\kappa g_{ij} \\
&= 2\sqrt{e^{-2\phi}+\ell^2}\sqrt{-g}\delta_\kappa\overline{\theta}\gamma_i g^{ij}\partial_j\theta \\
&= 2\delta_\kappa\overline{\theta}\gamma_D T_D^j\partial_j\theta.
\end{aligned} \tag{2.32}$$

Where, we let $\delta_\kappa\phi = 0$ and $\delta_\kappa\ell = 0$. If we let

$$\delta_\kappa C_{ij} = 2\sqrt{e^{-2\phi}+\ell^2}\,\delta_\kappa\overline{\theta}\tau_3\gamma_{[i}\partial_{j]}\theta, \tag{2.33}$$

or in the form of forms

$$\delta_\kappa C_{(2)} = 2\sqrt{e^{-2\phi}+\ell^2}\,\delta_\kappa\overline{\theta}\tau_3\Gamma_\mu d\theta\Pi^\mu, \tag{2.34}$$

we can get $\delta_\kappa\mathcal{L}_{WZ} = 2\delta_\kappa\overline{\theta}T_D^j\partial_j\theta$. And if $\delta_\kappa\overline{\theta} = \overline{\kappa}(1-\gamma_D)$, the super D-string action will be invariant under kappa-symmetry transformations. Certainly, there could be some total derivative in the variation of $C_{(2)}$.

2.3 The super F-string

Now, let us prove that the action of the super F-string has kappa-symmetry in a flat background. The procedure is almost the same as that in last section. The action of the super F-string is given by

$$S = -\int d^2\sigma[\sqrt{-g} + \frac{1}{2}\varepsilon^{ij}B_{ij}], \tag{2.35}$$

where $B_{ij} = \Pi_i^\mu\Pi_j^\nu B_{\mu\nu}$. If we let

$$\rho_F = \frac{1}{2}\varepsilon^{ij}\tau_3\gamma_i\gamma_j, \tag{2.36a}$$

$$T_F^j = \varepsilon^{ij}\tau_3\gamma_i, \tag{2.36b}$$

similar to the case of super D-string in last section, we can get

$$\rho_F^2 = -\det g. \tag{2.37}$$

$$\rho_F \gamma_k = T_F^j g_{jk}. \tag{2.38}$$

If γ_F is defined by

$$\rho_F = \gamma_F \sqrt{-\det g}\ , \tag{2.39}$$

we have

$$\gamma_F = \frac{1}{\sqrt{-\det g}} \frac{1}{2} \varepsilon^{ij} \tau_3 \gamma_i \gamma_j. \tag{2.40}$$

Then, we can get

$$\gamma_F^2 = 1, \quad \mathrm{tr}\gamma_F = 0. \tag{2.41}$$

Hence

$$\begin{aligned} \gamma_F T_F^j &= \gamma_F \rho_F \gamma_i g^{ij} \\ &= \sqrt{-\det g}\ \gamma_i\, g^{ij}. \end{aligned} \tag{2.42}$$

Therefore, the variation of the DBI part is

$$\begin{aligned} \delta_\kappa[-\sqrt{-\det g}\,] &= \frac{-1}{2\sqrt{-\det g}} \delta_\kappa(-\det g) \\ &= \frac{-\sqrt{-\det g}}{2} g^{ji} \delta_\kappa g_{ij} \\ &= \frac{-\sqrt{-\det g}}{2} g^{ji} [-2\delta_\kappa \overline{\theta}(\gamma_i \partial_j + \gamma_j \partial_i)\theta] \\ &= 2\delta_\kappa \overline{\theta} \sqrt{-\det g} \gamma_i g^{ij} \partial_j \theta \\ &= 2\delta_\kappa \overline{\theta} \gamma_F T_F^j \partial_j \theta. \end{aligned} \tag{2.43}$$

And we already have [9](ignoring the total derivative)

$$\begin{aligned} \delta_\kappa \frac{1}{2} \varepsilon^{ij} B_{ij} &= \frac{1}{2} \varepsilon^{ij} [-2\delta_\kappa \overline{\theta} \tau_3 (\gamma_i \partial_j - \gamma_j \partial_i)\theta] \\ &= -\delta_\kappa \overline{\theta} \tau_3 \varepsilon^{ij} (\gamma_i \partial_j - \gamma_j \partial_i)\theta \\ &= -2\delta_\kappa \overline{\theta} \tau_3 \varepsilon^{ij} \gamma_i \partial_j \theta \\ &= -2\delta_\kappa \overline{\theta} T_F^j \partial_j \theta. \end{aligned} \tag{2.44}$$

So, the Lagrangian varies as

$$\delta_\kappa \mathcal{L} = 2\delta_\kappa \overline{\theta}(\gamma_F + 1) T_F^j \partial_j \theta, \tag{2.45}$$

which is zero if $\delta_\kappa \overline{\theta} = \overline{\kappa}(1 - \gamma_F)$. Therefore, the super F-string really has kappa-symmetry as expected.

So far, we have discussed the kappa-symmetry properties for the simple cases. Before considering kappa-symmetry in the more complicated situations, we will first discuss $SL(2, \mathbb{R})$ symmetry of IIB brane actions in next chapter.

Chapter 3

$SL(2,\mathbb{R})$-Invariant IIB Branes

The Type IIB superstring theory has $SL(2,\mathbb{R})$ symmetry at the classical level and $SL(2,\mathbb{Z})$ symmetry at the quantum level. This symmetry was first proposed by [13]. But, type IIA theory has no such symmetry. The reason is that type IIA theory is a non-chiral theory. Type IIB supergravity theory is the low energy limit of type IIB superstring theory. The F-string and the D-string are the solutions of type IIB supergravity. They form a doublet under $SL(2,\mathbb{R})$ transformations [14]. The non-singular solitonic 5-brane(NS5-brane) and the D5-brane, which are also the solutions of type IIB supergravity, form a doublet under $SL(2,\mathbb{R})$ transformations [15]. Unlike the cases of one-branes and five-branes, the D3-brane solution is a singlet under these transformations. In this chapter, we will consider the $SL(2,\mathbb{R})$ symmetry of the brane actions. The brane actions which are invariant under the transformations of $SL(2,\mathbb{R})$ symmetry have been constructed for the strings [16, 17], the D3-brane [18] and the five-branes [19]. Recently, all $SL(2,\mathbb{R})$-invariant IIB brane actions have been constructed by an elegant way [20].

In this chapter, first, we will discuss the $SL(2,\mathbb{R})$ symmetry for the string action by following [16, 21]. Then, we will follow [20] to give an universal description for all type IIB branes.

3.1 The action of one-branes

First, let us consider the string case. The $SL(2,\mathbb{R})$-invariant action can be constructed as(in Einstein frame) [16]

$$S = -\int d^2\sigma \frac{1}{2\upsilon}\Big\{ -g_E + \big[e^{-\phi}(\star\mathcal{F}_b)^2 + e^{\phi}(\star\mathcal{F}_c - \ell\star\mathcal{F}_b)^2\big]\Big\}, \tag{3.1}$$

here $g_E = \det[(g_E)_{ij}]$ with $(g_E)_{ij} = e^{-\phi/2}g_{ij}$. $(g_E)_{ij}$ and g_{ij} are the pullbacks of the Einstein metric and the string metric respectively. The field strength are defined by

$$\mathcal{F}_b = F_b - B_{(2)} = db - B_{(2)}, \tag{3.2a}$$
$$\mathcal{F}_c = F_c - C_{(2)} = dc - C_{(2)}. \tag{3.2b}$$

Where b and c are the vector fields on the worldvolumes of the F-string and the D-string respectively. $B_{(2)}$ and $C_{(2)}$ are the NS-NS and R-R 2-forms which couple to the F-string and the D-string respectively. υ is an introduced auxiliary field. It is easy to check that the above action (3.1) has the rescaling invariance:

$$X^\mu \longrightarrow \lambda X^\mu, \quad \upsilon \longrightarrow \lambda^4 \upsilon, \quad \mathcal{F}_b \longrightarrow \lambda^2 \mathcal{F}_b, \quad \mathcal{F}_c \longrightarrow \lambda^2 \mathcal{F}_c. \tag{3.3}$$

Now, let us rewrite the action (3.1) in a manifestly $SL(2,\mathbb{R})$-invariant way. If we define

$$\overrightarrow{\mathcal{F}}^T \equiv \begin{pmatrix} \star\mathcal{F}_c & \star\mathcal{F}_b \end{pmatrix}, \qquad \mathcal{M} \equiv e^\phi \begin{pmatrix} |\lambda|^2 & \ell \\ \ell & 1 \end{pmatrix}, \tag{3.4}$$

with $\lambda \equiv \ell + ie^{-\phi}$, we can get

$$\begin{aligned}
\overrightarrow{\mathcal{F}}^T \mathcal{M}^{-1} \overrightarrow{\mathcal{F}} &= \begin{pmatrix} \star\mathcal{F}_c & \star\mathcal{F}_b \end{pmatrix} e^\phi \begin{pmatrix} 1 & -\ell \\ -\ell & |\lambda|^2 \end{pmatrix} \begin{pmatrix} \star\mathcal{F}_c \\ \star\mathcal{F}_b \end{pmatrix} \\
&= e^\phi \begin{pmatrix} \star\mathcal{F}_c - \ell\star\mathcal{F}_b & |\lambda|^2\star\mathcal{F}_b - \ell\star\mathcal{F}_c \end{pmatrix} \begin{pmatrix} \star\mathcal{F}_c \\ \star\mathcal{F}_b \end{pmatrix} \\
&= e^\phi \Big[(\star\mathcal{F}_c)^2 - 2\ell(\star\mathcal{F}_b)(\star\mathcal{F}_c) + |\lambda|^2(\star\mathcal{F}_b)^2\Big] \\
&= e^\phi \Big[(\star\mathcal{F}_c)^2 - 2\ell(\star\mathcal{F}_b)(\star\mathcal{F}_c) + (\ell^2 + e^{-2\phi})(\star\mathcal{F}_b)^2\Big] \\
&= e^{-\phi}(\star\mathcal{F}_b)^2 + e^\phi(\star\mathcal{F}_c - \ell\star\mathcal{F}_b)^2.
\end{aligned} \tag{3.5}$$

So, the action (3.1) can be written as

$$S = -\int d^2\sigma \frac{1}{2\upsilon}\Big[-g_E + \overrightarrow{\mathcal{F}}^T \mathcal{M}^{-1} \overrightarrow{\mathcal{F}}\Big]. \tag{3.6}$$

If $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an $SL(2,\mathbb{R})$ matrix, it is obvious that the action (3.6) is invariant under the following $SL(2,\mathbb{R})$ transformations:

$$\mathcal{M} \longrightarrow \Lambda\mathcal{M}\Lambda^T, \qquad \overrightarrow{\mathcal{F}} \longrightarrow \Lambda\overrightarrow{\mathcal{F}}, \qquad (g_E)_{ij} \longrightarrow (g_E)_{ij}. \tag{3.7}$$

We will see that the Nambu-Goto action can be derived from the action (3.1). First, we need the equation of motion of c. Because

$$\mathcal{F}_{cij} = d_{[i} \wedge c_{j]} - C_{ij} \;\longrightarrow\; \star\mathcal{F}_c = \frac{1}{2}\,\varepsilon^{ij}\big[d_{[i} \wedge c_{j]} - C_{ij}\big], \tag{3.8}$$

from the action (3.1) and the Euler-Lagrange equation we can get

$$\partial_i\left\{\frac{1}{\upsilon}e^{\phi}(\star\mathcal{F}_c-\ell\star\mathcal{F}_b)\frac{1}{2}\,\varepsilon^{ij}\right\}=0, \tag{3.9}$$

then

$$\begin{aligned}\frac{1}{\upsilon}\,e^{\phi}(\star\mathcal{F}_c-\ell\star\mathcal{F}_b)\frac{1}{2} &= \text{constant}\equiv T_D/2\\ \longrightarrow\ \star(\mathcal{F}_c-\ell\mathcal{F}_b) &= \upsilon\,e^{-\phi}\,T_D.\end{aligned} \tag{3.10}$$

The constant T_D will be identified as the tension of D-string later. Similarly, we can get the equation of motion of b:

$$\begin{aligned}\partial_i\left\{\frac{1}{\upsilon}\left[e^{-\phi}(\star\mathcal{F}_b)\frac{1}{2}\,\varepsilon^{ij}+e^{\phi}\star(\mathcal{F}_c-\ell\mathcal{F}_b)(-\ell)\frac{1}{2}\,\varepsilon^{ij}\right]\right\} &= 0\\ \longrightarrow\ \frac{1}{\upsilon}\left[e^{-\phi}(\star\mathcal{F}_b)\frac{1}{2}+e^{\phi}\star(\mathcal{F}_c-\ell\mathcal{F}_b)(-\ell)\frac{1}{2}\right] &= \text{constant}\\ &\equiv -T_F/2\\ \longrightarrow\ e^{\phi}\ell\star(\mathcal{F}_c-\ell\mathcal{F}_b)-e^{-\phi}\star\mathcal{F}_b=\upsilon\,T_F.\end{aligned} \tag{3.11}$$

The constant T_F will be identified as the tension of F-string later. We see that the tensions can be generated from the equations of motion of one-forms c and b. From (3.10) and (3.11), it is easy to get

$$\star\mathcal{F}_b = -\upsilon\,e^{\phi}\big(T_F-\ell\,T_D\big), \tag{3.12a}$$
$$\star\mathcal{F}_c = \upsilon\,e^{-\phi}T_D-\ell\upsilon\,e^{\phi}\big(T_F-\ell\,T_D\big). \tag{3.12b}$$

If we let $T_D=0$, the action (3.1) becomes

$$S=-\int d^2\sigma\frac{1}{2\upsilon}\Big[-g_E+e^{\phi}\,\upsilon^2\,T_F^2\Big]. \tag{3.13}$$

Solving for υ, we can get the Nambu-Goto action

$$\begin{aligned}S &= -T_F\int d^2\sigma\,e^{\phi/2}\sqrt{-g_E}\\ &= -T_F\int d^2\sigma\,\sqrt{-g},\end{aligned} \tag{3.14}$$

where we have used the relation $(g_E)_{ij}\equiv e^{-\phi/2}g_{ij}$. Further, if we add an interaction term $-\frac{T_F}{2}\,\varepsilon^{ij}B_{ij}$ to the action (3.13), we get

$$S=-\int d^2\sigma\left\{\frac{1}{2\upsilon}\big[-g_E+e^{\phi}\,\upsilon^2\,T_F^2\big]+\frac{1}{2}T_F\,\varepsilon^{ij}B_{ij}\right\}. \tag{3.15}$$

Solving for υ again, we recover the fundamental string action

$$S = -T_F \int d^2\sigma \, [\sqrt{-g} + \frac{1}{2}\varepsilon^{ij} B_{ij}]. \tag{3.16}$$

Now, we want to show that the equations of motion for the embedding coordinates from the action (3.15) and from the action (3.1) are the same in the situation of $T_D = 0$. First, let us try to find the equations of motion for the embedding coordinates from the action (3.1). The action (3.1) can be written as

$$\begin{aligned} S = \quad & -\int d^2\sigma \frac{1}{2\upsilon}\Big\{ -\det\big(\partial_i X^\mu \partial_j X_\mu\big) + \frac{1}{2} e^{-\phi}(F_{bij} - \partial_i X^\mu \partial_j X^\nu B_{\mu\nu})^2 \\ & + \frac{1}{2} e^{\phi}\Big[(F_{cij} - \partial_i X^\mu \partial_j X^\nu C_{\mu\nu}) - \ell(F_{bij} - \partial_i X^\mu \partial_j X^\nu B_{\mu\nu})\Big]^2\Big\}. \end{aligned} \tag{3.17}$$

The equations of motion for embedding coordinates from the action (3.17) is

$$\begin{aligned} & \partial_i \frac{1}{2\upsilon}\Big\{ DET - e^{-\phi}(F_b^{ij} - \partial^i X^\mu \partial^j X^\nu B_{\mu\nu})\partial_j X^\nu B_{\mu\nu} \\ & - e^{\phi}\Big[(F_c^{ij} - \partial^i X^\mu \partial^j X^\nu C_{\mu\nu}) - \ell(F_b^{ij} - \partial^i X^\mu \partial^j X^\nu B_{\mu\nu})\Big] \\ & \big(\partial_j X^\nu C_{\mu\nu} - \ell \partial_j X^\nu B_{\mu\nu}\big)\Big\} \\ & = 0, \end{aligned} \tag{3.18}$$

here, $\partial_i \equiv \frac{\partial}{\partial \sigma^i}$, and DET means the contribution from $-g_E$. If $T_D = 0$, the above equations of motion become

$$\partial_i \frac{1}{2\upsilon}\Big\{ DET - e^{-\phi}(F_b^{ij} - \partial^i X^\mu \partial^j X^\nu B_{\mu\nu})\partial_j X^\nu B_{\mu\nu}\Big\} = 0, \tag{3.19}$$

which coincides with the equations of motion for the embedding coordinates from the following action

$$S = -\int d^2\sigma \Big\{ \frac{1}{2\upsilon}\big[-g_E + e^{\phi}\,\upsilon^2\, T_F^2 \big] - \frac{1}{2} T_F\, \varepsilon^{ij} \mathcal{F}_{bij}\Big\}. \tag{3.20}$$

By integrating out b from the action (3.20), we recover the action (3.15).

We will construct the $SL(2,\mathbb{R})$-invariant action for the (p,q)-string. First, it can be verified that the equations of motion for the embedding coordinates from the following action are also (3.18).

$$
\begin{aligned}
S &= -\int d^2\sigma \frac{1}{2\upsilon}\Big\{-g_E + \upsilon^2\, e^{\phi}\,(\ell\, T_D - T_F)^2 + \upsilon^2\, e^{-\phi}\, T_D^2 \\
&\quad -2\upsilon\big(T_F \star \mathcal{F}_b - T_D \star \mathcal{F}_c\big)\Big\} \\
&= -\int d^2\sigma \Big\{\frac{-g_E}{2\upsilon} + \frac{\upsilon}{2}\big[e^{\phi}(\ell\, T_D - T_F)^2 + e^{-\phi}\, T_D^2\big] \\
&\quad -\frac{1}{2}\varepsilon^{ij}\big(T_F \mathcal{F}_{bij} - T_D \mathcal{F}_{cij}\big)\Big\}.
\end{aligned} \tag{3.21}
$$

By integrating out b and c, we have

$$
\begin{aligned}
S &= -\int d^2\sigma \frac{1}{2\upsilon}\Big\{-g_E + \upsilon^2\, e^{\phi}\,(\ell\, T_D - T_F)^2 + \upsilon^2\, e^{-\phi}\, T_D^2 \\
&\quad +2\upsilon\big(T_F \star B_{(2)} - T_D \star C_{(2)}\big)\Big\} \\
&= -\int d^2\sigma \Big\{\frac{-g_E}{2\upsilon} + \frac{\upsilon}{2}\big[e^{\phi}(\ell\, T_D - T_F)^2 + e^{-\phi}\, T_D^2\big] \\
&\quad +\frac{1}{2}\varepsilon^{ij}\big(T_F B_{ij} - T_D C_{ij}\big)\Big\}.
\end{aligned} \tag{3.22}
$$

Solving for υ, we get

$$
\begin{aligned}
S &= -\int d^2\sigma \Big\{\sqrt{-g_E}\sqrt{(\ell\, T_D - T_F)^2 e^{\phi} + T_D^2 e^{-\phi}} \\
&\quad +\frac{1}{2}\varepsilon^{ij}\big[T_F B_{ij} - T_D C_{ij}\big]\Big\} \\
&= -T\int d^2\sigma \Big\{\sqrt{-g_E}\sqrt{(p - q\ell)^2 e^{\phi} + q^2 e^{-\phi}} \\
&\quad +\frac{1}{2}\varepsilon^{ij}\big[pB_{ij} - qC_{ij}\big]\Big\} \\
&\equiv -\int d^2\sigma \Big\{T_{(p,\,q)}\sqrt{-g_E} + \frac{T}{2}\varepsilon^{ij}\big[pB_{ij} - qC_{ij}\big]\Big\},
\end{aligned} \tag{3.23}
$$

here we have defined that $T_F = p\,T$ and $T_D = q\,T$ with p and q are the charges of F-string and D-string respectively. T_F and T_D are the tensions of F-string and D-string respectively. $T_{(p,\,q)} = T\sqrt{(p - q\ell)^2 e^{\phi} + q^2 e^{-\phi}}$ is the tension of the (p, q)-string, which coincides with the result given by [14]. The action (3.23) describes a (p, q)-string and it is $SL(2, \mathbb{R})$-invariant. If $\ell = 0$, $\phi = \phi_0$ and $g_s = e^{\phi_0}$, in the string frame we have

$$
T^{(s)}_{(p,\,q)} = T\sqrt{p^2 + \frac{q^2}{g_s^2}}. \tag{3.24}
$$

Where ϕ_0 is the asymptotic value of the dilaton. It is easy to see that the tension of F-string is $T^{(s)}_{(1,0)} = T$, and the tension of D-string is $T^{(s)}_{(0,1)} = T/g_s$. There is a strong-weak duality between the F-string and D-string.

There is another approach to get an $SL(2,\mathbb{R})$-invariant action for the (p,q)-string. Following [22], we define the action as

$$S = -T\int d^2\sigma\sqrt{-g}\Big\{e^{-\phi}\sqrt{q^2+(t\,e^{\phi})^2} - \star\big[(q\ell - t)\mathcal{F}_b + qC_{(2)}\big]\Big\}, \tag{3.25}$$

here t is an auxiliary scalar. The action (3.23) can be derived from the action (3.25). First, from (3.25), the equation of motion of t is

$$\frac{te^{\phi}}{\sqrt{q^2+(te^{\phi})^2}} = -\star\mathcal{F}_b. \tag{3.26}$$

Substituting this in the action (3.25), we get

$$\begin{aligned}
S &= -T\int d^2\sigma\sqrt{-g}\Big\{e^{-\phi}\sqrt{q^2+(te^{\phi})^2} + t\star\mathcal{F}_b - q\star\big[C_{(2)}+\ell\mathcal{F}_b\big]\Big\} \\
&= -T\int d^2\sigma\sqrt{-g}\Big\{e^{-\phi}\sqrt{q^2+\frac{q^2(\star\mathcal{F}_b)^2}{1-(\star\mathcal{F}_b)^2}} \\
&\quad -\frac{q\star\mathcal{F}_b}{\sqrt{1-(\star\mathcal{F}_b)^2}}e^{-\phi}\star\mathcal{F}_b - q\star\big[C_{(2)}+\ell\mathcal{F}_b\big]\Big\} \\
&= -Tq\int d^2\sigma\sqrt{-g}\Big\{e^{-\phi}\Big[\sqrt{\frac{1}{1-(\star\mathcal{F}_b)^2}} - \frac{(\star\mathcal{F}_b)^2}{\sqrt{1-(\star\mathcal{F}_b)^2}}\Big] - \star\big(C_{(2)}+\ell\mathcal{F}_b\big)\Big\} \\
&= -Tq\int d^2\sigma\sqrt{-g}\Big\{e^{-\phi}\sqrt{1-(\star\mathcal{F}_b)^2} - \star\big(C_{(2)}+\ell\mathcal{F}_b\big)\Big\} \\
&= -qT\int d^2\sigma\Big[e^{-\phi}\sqrt{-g+g(\star\mathcal{F}_b)^2} - \sqrt{-g}\star\big(C_{(2)}+\ell\mathcal{F}_b\big)\Big] \\
&= -qT\int d^2\sigma\Big[e^{-\phi}\sqrt{-g-\frac{1}{2}\mathcal{F}_b^2} - \sqrt{-g}\star\big(C_{(2)}+\ell\mathcal{F}_b\big)\Big] \\
&= -qT\int d^2\sigma\Big[e^{-\phi}\sqrt{-\det(g+\mathcal{F}_b)} - \sqrt{-g}\star\big(C_{(2)}+\ell\mathcal{F}_b\big)\Big],
\end{aligned} \tag{3.27}$$

here we used $\star\mathcal{F}_b = \frac{1}{2\sqrt{-g}}\epsilon^{ij}\mathcal{F}_{bij}$. Obviously, the action (3.27) is the q times the D-string action. The equation of motion of b from the action (3.25) imposes that $q\ell - t = \text{constant} \equiv p$. Substituting this in the action (3.25), and integrating out b, we can achieve

$$S = -T\int d^2\sigma\sqrt{-g}\Big[\sqrt{(qe^{-\phi})^2+(p-\ell q)^2} + p\star B_{(2)} - q\star C_{(2)}\Big], \tag{3.28}$$

which is the same as the action (3.23) if go into the Einstein frame.

As we have seen, the construction of the $SL(2,\mathbb{R})$-invariant action for one-branes is not so difficult. For the case of D3-brane, it is also not so difficult because it is a singlet. But for the case of five-branes, things become complicated. The reason is that we can only have one Born-Infeld vector on the word-volume otherwise we will break the balance of the degrees of freedom between the bosonic and the fermionic parts. In the string case, we can introduce another vector to form a doublet, since a vector has no physical degree of freedom in this case. But in the case of five-branes we can not do this because a vector will have some physical degrees of freedom in this case. Though an action was constructed for this case [19], it was not complete. Things become more complicated for the cases of seven- and nine-branes, because there are many types of seven-branes and nine-branes. Difficulties are overcome in [20]. We will discuss this briefly in next section.

3.2 The universal formula

In this section, we follow [20] to give an universal description of the property of the $SL(2,R)$ symmetry of type IIB branes.

As we know that a charged p-brane can couple to a $(p+1)$-form gauge field. It turns out that in the IIB case, these gauge fields are as follows:

$$A^{\alpha}_{(2)}, \quad A_{(4)}, \quad A^{\alpha}_{(6)}, \quad A^{\alpha\beta}_{(8)}, \quad A^{\alpha\beta\gamma}_{(10)}, \quad A^{\alpha}_{(10)}, \tag{3.29}$$

which are a doublet of 2-forms, a singlet of 4-form, a doublet of 6-forms, a triplet of 8-forms, a quadruplet of 10-forms and a doublet of 10-forms respectively. The $SU(1,1)$ indices $\alpha,\beta,\gamma = 1,2$. And also, the vector fields $V^{\alpha}_{(1)}$ is introduced. But in order to have supersymmetry on the world-volume, we can only have one vector on it. This is satisfied by requiring that only the combination $q_\alpha V^{\alpha}_{(1)}$ lives on the world-volume. Here q_α are the charges of F-string and D-string.

By requiring that the WZ term has a single world-volume vector field for the cases of $p>1$ and has target space gauge invariance, the WZ term can be formally written as

- $p=1, \quad \mathcal{L}_{WZ} = \widetilde{q}_\alpha \mathcal{C}^{\alpha}_{(2)};$
- $p \neq 1, \quad \mathcal{L}_{WZ} = q \cdot \mathcal{C} e^{q\mathcal{F}_{(2)}}.$

Where $\mathcal{C}$ is the formal sum of some forms(for details, see [20]). For the definitions of others, we refer to the reference and the appendices. These WZ terms are obviously $SL(2,\mathbb{R})$-invariant. The DBI part of the Lagrangian is suggested to be

$$\mathcal{L}_{DBI} = -\tau_{p,E}\sqrt{-\det\left(g_E + \frac{q\mathcal{F}_{(2)}}{(qq\mathcal{M})^{1/2}}\right)}, \tag{3.30}$$

with the tension $\tau_{p,E}$:

- $p = 1, \quad \tau_{1,E} = (\widetilde{q}\,\widetilde{q}\mathcal{M})^{1/2}$;
- $p \neq 1, \quad \tau_{p,E} = (qq\mathcal{M})^{\frac{p-3}{4}}$.

It is obvious that the DBI part is also $SL(2,\mathbb{R})$-invariant. Therefore, the total Lagrangian $\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{WZ}$ is $SL(2,\mathbb{R})$-invariant.

As we said, for the string case, we have the F-string and the D-string, which form a doublet. For the three-brane case, we have the D3-brane, which form a singlet. For the five-brane case, we have NS5-brane and D5-brane, which form a doublet. The cases of seven-branes and nine-branes are more complicated. For the seven-branes, it turns out that there are three conjugacy classes[23] labeled by det $q_{\alpha\beta}$ which can take the values: $\det q_{\alpha\beta} = 0$, det $q_{\alpha\beta} > 0$ or det $q_{\alpha\beta} < 0$. The later two classes can not be gotten from the first one by any $SL(2,\mathbb{R})$ transformations. Another important fact is that there is no action with $\det q_{\alpha\beta} \neq 0$ which contains only a single Born-Infeld vector. Thus in this report, for seven-branes, we only consider the conjugacy class with $\det q_{\alpha\beta} = 0$, which contains the D7-brane. For the nine-branes, there are also some conjugacy classes. We only consider the conjugacy class containing the D9-brane since the other conjugacy classes are not supersymmetric.

We leave the detail discussions about the actions for special cases to the reference.

In the next chapter, we will use the similar formula of the Lagrangian here but with some changes by redefining the fields. The property of $SL(2,\mathbb{R})$ symmetry is not changed yet.

Chapter 4

Kappa-Symmetry of $SL(2, \mathbb{R})$-Invariant Super IIB Branes

Since super Dp-branes and spuer p-branes are all have kappa-symmetry, it is believed that the bound states of these objects also have kappa-symmetry. In this chapter, we will discuss the kappa-symmetry property of $SL(2, \mathbb{R})$-invariant super IIB branes in a flat background and will work in the Einstein frame. We will use the method proposed by [9]. First, we will consider the super one-branes case, then we will discuss the cases of super IIB p-branes with $p > 1$ uniformly. At last we will give examples for some special cases.

4.1 The super one-branes

The $SL(2, \mathbb{R})$-invariant Lagrangians for general super IIB p-branes are formally similar to those given in [20].

In this section, let us consider the case of one-branes. The Lagrangian of the $SL(2, \mathbb{R})$-invariant super one-branes is

$$\begin{aligned} \mathcal{L}_{1-brane} &= -(\widetilde{q}\,\widetilde{q}\mathcal{M})^{1/2}\sqrt{-\det g_{ij}} + \widetilde{q}_\alpha \mathcal{C}^\alpha_{(2)} \\ &= -\sqrt{(q+p\ell)^2 e^{\phi} + p^2 e^{-\phi}}\sqrt{-g} + p\, C_{(2)} + qB_{(2)}, \end{aligned} \tag{4.1}$$

where g_{ij} is the pull back of the Einstein metric. It is understood that there should be a Hodge operator which operates on the WZ term if we do not write the WZ term in form of forms. And also note that the above action is not exactly as that in Chapter 3. This is just because that the definitions of charges or fields are different.

Under kappa-symmetry transformations, the variations of g_{ij} is the same as that in [9]:

$$\delta_\kappa g_{ij} = -2\delta_\kappa \overline{\theta}\gamma_{(i}\partial_{j)}\theta, \tag{4.2}$$

since here γ_i has the same definition as in [9]. We assume that

$$\begin{aligned}\delta_\kappa[\frac{1}{2}\varepsilon^{ij}(pC_{ij}+qB_{ij})] &= \frac{1}{2}\varepsilon^{ij}(p\delta_\kappa C_{ij}+q\delta_\kappa B_{ij}) \\ &= \sqrt{(q+p\ell)^2e^{\phi}+p^2e^{-\phi}}\varepsilon^{ij}\delta_\kappa\overline{\theta}\tau_3\gamma_{[i}\partial_{j]}\theta \\ &= (\widetilde{q}\widetilde{q}\mathcal{M})^{1/2}\varepsilon^{ij}\delta_\kappa\overline{\theta}\tau_3\gamma_{[i}\partial_{j]}\theta,\end{aligned} \tag{4.3}$$

which can also be written in the form of forms as

$$\delta_\kappa(\widetilde{q}_\alpha\mathcal{C}^\alpha_{(2)}) = 2(\widetilde{q}\widetilde{q}\mathcal{M})^{1/2}\delta_\kappa\overline{\theta}\tau_3\Gamma_\mu d\theta\Pi^\mu. \tag{4.4}$$

Where, $\delta_\kappa\overline{\theta}$, θ and Π^μ are invariant under $SL(2,\mathbb{R})$ transformations. We can recover (2.18) and (2.34) if we take $\widetilde{q}_{\alpha'}=(0,-1)$ and $\widetilde{q}_{\alpha'}=(1,0)$ in (4.4) respectively and at the same time go into the string frame. The contributions from the total derivatives of $\delta_\kappa C_{(2)}$ and $\delta_\kappa B_{(2)}$ should be cancelled in (4.4). If we let

$$T^j_{(1)} = (\widetilde{q}\widetilde{q}\mathcal{M})^{1/2}\tau_3\varepsilon^{ij}\gamma_i, \tag{4.5}$$

we can get

$$\delta_\kappa[\frac{1}{2}\varepsilon^{ij}(pC_{ij}+qB_{ij})] = 2\delta_\kappa\overline{\theta}T^j_{(1)}\partial_j\theta. \tag{4.6}$$

It is obvious that $T^j_{(1)}$ is invariant under $SL(2,\mathbb{R})$ transformations. We can recover (2.24b) and (2.36b) if we take $\widetilde{q}_{\alpha'}=(1,0)$ and $\widetilde{q}_{\alpha'}=(0,-1)$ in (4.5) respectively and at the same time go into the string frame. This is not strange, since we can get the actions of D-string and F-string from the action of (p,q)-string in the same situation. Now, let us define that

$$\rho^{(1)} = \frac{1}{2}(\widetilde{q}\widetilde{q}\mathcal{M})^{1/2}\tau_3\varepsilon^{ij}\gamma_i\gamma_j. \tag{4.7}$$

Then, it is easy to check that

$$(\rho^{(1)})^2 = (\widetilde{q}\widetilde{q}\mathcal{M})(-g). \tag{4.8}$$

Further, if we define

$$\rho^{(1)} = \gamma^{(1)}(\widetilde{q}\widetilde{q}\mathcal{M})^{1/2}\sqrt{-g}, \tag{4.9}$$

we have

$$\gamma^{(1)} = \frac{1}{\sqrt{-g}}\frac{1}{2}\varepsilon^{ij}\tau_3\gamma_i\gamma_j. \tag{4.10}$$

It is obvious that $\gamma^{(1)}$ is invariant under $SL(2,\mathbb{R})$ transformations. Also, we can get $(\gamma^{(1)})^2 = 1$ and $\mathrm{tr}\gamma^{(1)} = 0$. It can be proved that

$$\rho^{(1)}\gamma_k = T^j_{(1)} g_{jk}. \tag{4.11}$$

The variation of the DBI part under kappa-symmetry transformations is given by

$$\begin{aligned} & \delta_\kappa\big[-(\widetilde{q}\widetilde{q}\mathcal{M})^{1/2}\sqrt{-g}\big] \\ = \;& 2(\widetilde{q}\widetilde{q}\mathcal{M})^{1/2}\sqrt{-g}\delta_\kappa\overline{\theta}\gamma_i g^{ij}\partial_j\theta \\ = \;& 2\delta_\kappa\overline{\theta}\gamma^{(1)}T^j_{(1)}\partial_j\theta. \end{aligned} \tag{4.12}$$

Therefore

$$\delta_\kappa\mathcal{L}_{1-brane} = 2\delta_\kappa\overline{\theta}(\gamma^{(1)}+1)T^j_{(1)}\partial_j\theta. \tag{4.13}$$

So, the $SL(2,\mathbb{R})$-invariant super one-branes have kappa-symmetry in a flat background if we require that $\delta_\kappa\overline{\theta} = \overline{\kappa}(1-\gamma^{(1)})$.

4.2 The universal formula for the super IIB p-branes with $p > 1$

In this section, let us consider the cases of $p > 1$. The Lagrangian of the $SL(2,\mathbb{R})$-invariant super IIB p-branes is similar to that of [20]:

$$\mathcal{L} = -(qq\mathcal{M})^{\frac{p-3}{4}}\sqrt{-\det\Big(g+\frac{q\mathcal{F}_{(2)}}{(qq\mathcal{M})^{1/2}}\Big)} + q\cdot\mathcal{C}e^{q\mathcal{F}_{(2)}}. \tag{4.14}$$

If we assume that the variation of the combination $q\mathcal{F}_{(2)}$ under kappa-symmetry transformations is

$$\begin{aligned} \delta_\kappa[q\mathcal{F}_{(2)}]_{ij} &= \delta_\kappa(-q\mathcal{F}_{cij}+p\mathcal{F}_{bij}) \\ &= 2\sqrt{(p-q\ell)^2e^{\phi}+q^2e^{-\phi}}\,\delta_\kappa\overline{\theta}\tau_3\gamma_{[i}\partial_{j]}\theta \\ &= 2(qq\mathcal{M})^{1/2}\delta_\kappa\overline{\theta}\tau_3\gamma_{[i}\partial_{j]}\theta, \end{aligned} \tag{4.15}$$

we can get the variation of the DBI part:

$$\begin{aligned} & \delta_\kappa\Big[-(qq\mathcal{M})^{\frac{p-3}{4}}\sqrt{-\det\Big(g+\frac{q\mathcal{F}_{(2)}}{(qq\mathcal{M})^{1/2}}\Big)}\;\Big] \\ = \;& 2(qq\mathcal{M})^{\frac{p-3}{4}}\sqrt{-\det\Big(g+\Phi\,q\mathcal{F}_{(2)}\Big)}\delta_\kappa\overline{\theta}\gamma_i\{(g+\tau_3\Phi\,q\mathcal{F}_{(2)})^{-1}\}^{ij}\partial_j\theta, \end{aligned} \tag{4.16}$$

where $\Phi = (qq\mathcal{M})^{-1/2}$. Similar as in [9], we let $\psi \equiv \gamma_i d\sigma^i$ and define

$$\begin{aligned}
\rho_B &= \sum_{p\ odd} \rho_{p+1} = \sum_{p\ odd} \frac{1}{(p+1)!} \rho_{i_1 \ldots i_{p+1}} d\sigma^{i_1} \ldots d\sigma^{i_{p+1}} \\
&= (qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} C_B(\psi)\tau_1, \qquad (4.17a)\\
T_B &= \sum_{p\ odd} T_p = \sum_{p\ odd} \frac{1}{p\,!} T_{i_1 \ldots i_p} d\sigma^{i_1} \ldots d\sigma^{i_p} \\
&= (qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} S_B(\psi)\tau_1, \qquad (4.17b)
\end{aligned}$$

with

$$\begin{aligned}
C_B(\psi) &= \tau_3 + \frac{1}{2!}\psi^2 + \frac{1}{4!}\tau_3\psi^4 + \frac{1}{6!}\psi^6 + \ldots, \qquad (4.18a)\\
S_B(\psi) &= \psi + \frac{1}{3!}\tau_3\psi^3 + \frac{1}{5!}\psi^5 + \frac{1}{7!}\tau_3\psi^7 + \ldots. \qquad (4.18b)
\end{aligned}$$

And let

$$\begin{aligned}
\rho^{(p)} &= \frac{1}{(p+1)!} \varepsilon^{i_1 \ldots i_{p+1}} \rho_{i_1 \ldots i_{p+1}}, \qquad (4.19a)\\
T^j_{(p)} &= \frac{1}{p!} \varepsilon^{i_1 \ldots i_p j} T_{i_1 \ldots i_p}. \qquad (4.19b)
\end{aligned}$$

From the above definitions, we give some ρ_p's and T_p's here explicitly:

$$\begin{aligned}
\rho_2 &= (qq\mathcal{M})^{-1/2}\big[\frac{1}{2}\tau_1\psi^2 + i\tau_2\Phi\, q\mathcal{F}_{(2)}\big],\\
\rho_4 &= \frac{1}{4!} i\tau_2\psi^4 + \frac{1}{2}\tau_1\Phi\, q\mathcal{F}_{(2)}\psi^2 + \frac{1}{2} i\tau_2(\Phi\, q\mathcal{F}_{(2)})^2,\\
\rho_6 &= (qq\mathcal{M})^{1/2}\Big[\frac{1}{6!}\tau_1\psi^6 + \frac{1}{4!} i\tau_2\Phi\, q\mathcal{F}_{(2)}\psi^4\\
&\quad + \frac{1}{4}\tau_1(\Phi\, q\mathcal{F}_{(2)})^2\psi^2 + \frac{1}{3!} i\tau_2(\Phi\, q\mathcal{F}_{(2)})^3\Big],\\
&\ldots, \qquad (4.20a)\\
T_1 &= (qq\mathcal{M})^{-1/2}\tau_1\psi,\\
T_3 &= \frac{1}{3!} i\tau_2\psi^3 + \tau_1\Phi\, q\mathcal{F}_{(2)}\psi,\\
T_5 &= (qq\mathcal{M})^{1/2}\Big[\frac{1}{5!}\tau_1\psi^5 + \frac{1}{3!} i\tau_2\Phi\, q\mathcal{F}_{(2)}\psi^3 + \frac{1}{2}\tau_1(\Phi\, q\mathcal{F}_{(2)})^2\psi\Big],\\
&\ldots. \qquad (4.20b)
\end{aligned}$$

ρ_B and T_B can also be written as

$$\rho_B = (qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} \begin{pmatrix} 0 & \cosh\psi \\ -\cos\psi & 0 \end{pmatrix}, \tag{4.21}$$

$$T_B = (qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} \begin{pmatrix} 0 & \sinh\psi \\ \sin\psi & 0 \end{pmatrix}. \tag{4.22}$$

It is obvious that ρ_p and $T^j_{(p)}$ are invariant under $S(2,\mathbb{R})$ transformations. Also, we can get

$$\rho^{(p)} = (qq\mathcal{M})^{\frac{p-3}{4}} \sum_{n=0}^{(p+1)/2} \frac{\tau_3^{\frac{p-2n+3}{2}}\tau_1}{2^n n!(p-2n+1)!} \varepsilon^{i_1 j_1 \dots i_n j_n i_{n+1} \dots i_{p-n+1}} \Phi^n (q\mathcal{F}_{(2)})_{i_1 j_1} \dots (q\mathcal{F}_{(2)})_{i_n j_n} \gamma_{i_{n+1}} \dots \gamma_{i_{p-n+1}}, \tag{4.23}$$

$$T^{i_{p-n+1}}_{(p)} = (qq\mathcal{M})^{\frac{p-3}{4}} \sum_{n=0}^{(p-1)/2} \frac{\tau_3^{\frac{p-2n+3}{2}}\tau_1}{2^n n!(p-2n)!} \varepsilon^{i_1 j_1 \dots i_n j_n i_{n+1} \dots i_{p-n+1}} \Phi^n (q\mathcal{F}_{(2)})_{i_1 j_1} \dots (q\mathcal{F}_{(2)})_{i_n j_n} \gamma_{i_{n+1}} \dots \gamma_{i_{p-n}}, \tag{4.24}$$

where $(q\mathcal{F}_{(2)})_{i_n j_n} = (-q\mathcal{F}_{c i_n j_n} + p\mathcal{F}_{b i_n j_n})$. It can be proved that(see Appendix B)

$$(\rho^{(p)})^2 = (qq\mathcal{M})^{\frac{p-3}{2}} \Big[-\det\Big(g + \Phi\, q\mathcal{F}_{(2)}\Big)\Big], \tag{4.25}$$

$$\rho^{(p)}\gamma_i = T^j_{(p)} \Big(g + \tau_3 \Phi\, q\mathcal{F}_{(2)}\Big)_{ji}. \tag{4.26}$$

And, if we let

$$\rho^{(p)} = \gamma^{(p)} (qq\mathcal{M})^{\frac{p-3}{4}} \sqrt{-\det\Big(g + \Phi\, q\mathcal{F}_{(2)}\Big)}, \tag{4.27}$$

we will have $(\gamma^{(p)})^2 = 1$, $\mathrm{tr}\gamma^{(p)} = 0$ and

$$\gamma^{(p)} = \frac{1}{\sqrt{-\det\Big(g + \Phi\, q\mathcal{F}_{(2)}\Big)}} \sum_{n=0}^{(p+1)/2} \frac{\tau_3^{\frac{p-2n+3}{2}}\tau_1}{2^n n!(p-2n+1)!} \varepsilon^{i_1 j_1 \dots i_n j_n i_{n+1} \dots i_{p-n+1}} \Phi^n (q\mathcal{F}_{(2)})_{i_1 j_1} \dots (q\mathcal{F}_{(2)})_{i_n j_n} \gamma_{i_{n+1}} \dots \gamma_{i_{p-n+1}}. \tag{4.28}$$

It is obvious that $\rho^{(p)}$ and $\gamma^{(p)}$ are invariant under $SL(2,\mathbb{R})$ transformations. The variation of the DBI part under kappa-symmetry transformations is

$$\begin{aligned}
\delta_\kappa \mathcal{L}_{DBI} &= -(qq\mathcal{M})^{\frac{p-3}{4}} \delta_k \sqrt{-\det\left(g + \Phi\, q\mathcal{F}_{(2)}\right)} \\
&= 2(qq\mathcal{M})^{\frac{p-3}{4}} \sqrt{-\det\left(g + \Phi\, q\mathcal{F}_{(2)}\right)} \delta_\kappa \overline{\theta} \gamma_i \{(g + \tau_3 \Phi\, q\mathcal{F}_{(2)})^{-1}\}^{ij} \partial_j \theta \\
&= 2\delta_\kappa \overline{\theta} \gamma^{(p)} \rho^{(p)} \gamma_i \{(g + \tau_3 \Phi\, q\mathcal{F}_{(2)})^{-1}\}^{ij} \partial_j \theta \\
&= 2\delta_\kappa \overline{\theta} \gamma^{(p)} T^k_{(p)} \left(g + \tau_3 \Phi\, q\mathcal{F}_{(2)}\right)_{ki} \{(g + \tau_3 \Phi\, q\mathcal{F}_{(2)})^{-1}\}^{ij} \partial_j \theta \\
&= 2\delta_\kappa \overline{\theta} \gamma^{(p)} T^j_{(p)} \partial_j \theta. \qquad (4.29)
\end{aligned}$$

The variation of the WZ part is

$$\delta_\kappa \mathcal{L}_{WZ} = \delta_\kappa [q \cdot \mathcal{C} e^{q\mathcal{F}_{(2)}}] = e^{q\mathcal{F}_{(2)}} [\delta_\kappa (q \cdot \mathcal{C}) + (q \cdot \mathcal{C}) \delta_\kappa (q\mathcal{F}_{(2)})]. \qquad (4.30)$$

On the other hand, if requiring kappa-symmetry, we should have

$$\delta_\kappa \mathcal{L}_{WZ} = 2\delta_\kappa \overline{\theta} T^j_{(p)} \partial_j \theta, \qquad (4.31)$$

or in terms of forms

$$\delta_\kappa \mathcal{L}_{WZ} = (-1)^{p+1} 2\delta_\kappa \overline{\theta} T_p d\theta. \qquad (4.32)$$

Note that in the IIB case, p is always odd. Formally, we can get

$$\begin{aligned}
& e^{q\mathcal{F}_{(2)}} [\delta_\kappa (q \cdot \mathcal{C}) + (q \cdot \mathcal{C}) \delta_\kappa (q\mathcal{F}_{(2)})] \\
& = 2\delta_\kappa \overline{\theta} (qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} S_B(\psi) \tau_1 d\theta. \qquad (4.33)
\end{aligned}$$

Because

$$\begin{aligned}
\delta_\kappa (q\mathcal{F}_{(2)}) &= 2(qq\mathcal{M})^{1/2} \delta_\kappa \overline{\theta} \tau_3 \Gamma_\mu d\theta \Pi^\mu \\
&= 2(qq\mathcal{M})^{1/2} \delta_\kappa \overline{\theta} \tau_3 \Gamma_\mu \Pi^\mu d\theta, \qquad (4.34)
\end{aligned}$$

where $\Pi^\mu = dX^\mu + \overline{\theta} \Gamma^\mu d\theta = \Pi^\mu_i d\sigma^i$, we get

$$\begin{aligned}
e^{q\mathcal{F}_{(2)}} \delta_\kappa (q \cdot \mathcal{C}) &= 2\delta_\kappa \overline{\theta} \Big[(qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} S_B(\psi) \tau_1 \\
&\quad - e^{q\mathcal{F}_{(2)}} (q \cdot \mathcal{C}) (qq\mathcal{M})^{1/2} \tau_3 \Gamma_\mu \Pi^\mu \Big] d\theta. \qquad (4.35)
\end{aligned}$$

So, for the cases of $p > 1$, the variation of $(q \cdot \mathcal{C})$ under kappa-symmetry transformations is given by (4.35). On both sides of this equation, there are sums of different forms. For the p-branes case, we should just consider the sum of $(p+1)$-forms on both sides.

4.3 Special cases

Redefinitions of fields

Before considering special cases, let us redefine the fields. The reason that we do this is that we want to have a WZ term which is really the expansion of $q \cdot \mathcal{C}e^{q\mathcal{F}_{(2)}}$. Table 4.1 gives the definitions of fields in [20, 24] and in this thesis.

Definitions of fields in [20, 24]	Redefinitions of fields in this paper
$B_{(2)} = \frac{1}{2}(A^1_{(2)} + A^2_{(2)})$	$B'_{(2)} = -\frac{\sqrt{2}}{2}i(A^1_{(2)} - A^2_{(2)})$
$C_{(2)} = -\frac{i}{4}(A^1_{(2)} - A^2_{(2)})$	$C'_{(2)} = \frac{\sqrt{2}}{2}(A^1_{(2)} + A^2_{(2)})$
$A_{(4)}$	$A'_{(4)} = \frac{4}{3}A_{(4)}$
$\mathcal{C}_{(4)} = A_{(4)} - \frac{3}{8}\widetilde{q}_\alpha q_\beta A^\alpha_{(2)}A^\beta_{(2)}$	$\mathcal{C}'_{(4)} = \frac{4}{3}A_{(4)} - \frac{1}{2}\widetilde{q}_\alpha q_\beta A^\alpha_{(2)}A^\beta_{(2)}$
$A^\alpha_{(6)}$	$A'^\alpha_{(6)} = -\frac{1}{45}A^\alpha_{(6)}$
$\mathcal{C}^\alpha_{(6)} = A^\alpha_{(6)} + 20A_{(4)}A^\alpha_{(2)} - \frac{15}{2}q_\beta\widetilde{q}_\gamma A^\alpha_{(2)}A^\beta_{(2)}A^\gamma_{(2)}$	$\mathcal{C}'^\alpha_{(6)} = -\frac{1}{45}A^\alpha_{(6)} - \frac{4}{9}A_{(4)}A^\alpha_{(2)} + \frac{1}{6}q_\beta\widetilde{q}_\gamma A^\alpha_{(2)}A^\beta_{(2)}A^\gamma_{(2)}$
$A^{\alpha\beta}_{(8)}$	$A'^{\alpha\beta}_{(8)} = \frac{1}{315}A^{\alpha\beta}_{(8)}$
$\mathcal{C}^{\alpha\beta}_{(8)} = A^{\alpha\beta}_{(8)} + \frac{7}{4}A^{(\alpha}_{(6)}A^{\beta)}_{(2)} + 35A_{(4)}A^\alpha_{(2)}A^\beta_{(2)} - \frac{105}{8}q_\gamma\widetilde{q}_\delta A^\alpha_{(2)}A^\beta_{(2)}A^\gamma_{(2)}A^\delta_{(2)}$	$\mathcal{C}'^{\alpha\beta}_{(8)} = \frac{1}{315}A^{\alpha\beta}_{(8)} + \frac{1}{180}A^{(\alpha}_{(6)}A^{\beta)}_{(2)} + \frac{1}{9}A_{(4)}A^\alpha_{(2)}A^\beta_{(2)} - \frac{1}{24}q_\gamma\widetilde{q}_\delta A^\alpha_{(2)}A^\beta_{(2)}A^\gamma_{(2)}A^\delta_{(2)}$
$A^{\alpha\beta\gamma}_{(10)}$	$A'^{\alpha\beta\gamma}_{(10)} = \frac{1}{4725}A^{\alpha\beta\gamma}_{(10)}$
$\mathcal{C}^{\alpha\beta\gamma}_{(10)} = A^{\alpha\beta\gamma}_{(10)} - 3A^{(\alpha\beta}_{(8)}A^{\gamma)}_{(2)} - \frac{21}{4}A^{(\alpha}_{(6)}A^\beta_{(2)}A^{\gamma)}_{(2)} - 105A_{(4)}A^\alpha_{(2)}A^\beta_{(2)}A^\gamma_{(2)} + \frac{315}{8}q_\delta\widetilde{q}_\epsilon A^\alpha_{(2)}A^\beta_{(2)}A^\gamma_{(2)}A^\delta_{(2)}A^\epsilon_{(2)}$	$\mathcal{C}'^{\alpha\beta\gamma}_{(10)} = \frac{1}{4725}A^{\alpha\beta\gamma}_{(10)} - \frac{1}{1575}A^{(\alpha\beta}_{(8)}A^{\gamma)}_{(2)} - \frac{1}{900}A^{(\alpha}_{(6)}A^\beta_{(2)}A^{\gamma)}_{(2)} - \frac{1}{45}A_{(4)}A^\alpha_{(2)}A^\beta_{(2)}A^\gamma_{(2)} + \frac{1}{120}q_\delta\widetilde{q}_\epsilon A^\alpha_{(2)}A^\beta_{(2)}A^\gamma_{(2)}A^\delta_{(2)}A^\epsilon_{(2)}$

Table 4.1: Redefinitions of fields.

The fields $\mathcal{C}^\alpha_{(2)}$, $A^\alpha_{(2)}$, $\mathcal{C}_{(0)}$, $F^\alpha_{(2)}$, $V^\alpha_{(1)}$, Σ^α and $\Lambda^\alpha_{(1)}$ are the same as those in [20] and so are the charges(the definition of $\mathcal{F}^\alpha_{(2)}$ is different). The fields $B_{(2)}$ and $C_{(2)}$ defined in [24] do not form a doublet under S-duality transformation. In order to preserve gauge invariance, the corresponding gauge parameters and gauge transformations should be changed as compared with [20, 24]. For example, the variations of $A'_{(4)}$ and $\mathcal{C}'_{(4)}$ under gauge transformations should be

$$\delta_g A'_{(4)} = 4\partial\Lambda'_{(3)} - \frac{i}{3}\varepsilon_{\alpha\beta}\Lambda^\alpha_{(1)}F^\beta_{(3)}, \tag{4.36}$$

$$\delta_g \mathcal{C}'_{(4)} = 4\partial\Lambda'_{(3)} + \frac{2}{3}q_\alpha F^\alpha_{(3)}\widetilde{q}_\beta\Lambda^\beta_{(1)}. \tag{4.37}$$

Where, $\Lambda'_{(3)} = \frac{4}{3}\Lambda_{(3)}$ with $\Lambda_{(3)}$ is the same as that in [20, 24]. Certainly, the supersymmetric transformations of these fields should also be changed. The field strength $F^\alpha_{(3)}$ is the same as that in [20, 24], but the definitions of other high rank field strengths should be changed in order to keep them gauge invariant. Note that, the primes here just mean the redefinitions of fields, which do not mean that we use $SU(1,1)$ basis here. And we use notations without primes after redefinitions. And also notice that, we discuss super IIB branes here, so the fields are not only the bosonic parts though formally they are the same.

Super three-branes

Now, let us take the example of $p = 3$. After redefinitions, the Lagrangian becomes

$$\mathcal{L}_{3-brane} = -\sqrt{-\det\Big(g + \frac{q\mathcal{F}_{(2)}}{(qq\mathcal{M})^{1/2}}\Big)} + \mathcal{C}_{(4)} + \widetilde{q}_\alpha q_\beta \mathcal{C}^\alpha_{(2)}\mathcal{F}^\beta_{(2)} + \frac{1}{2}\mathcal{C}_{(0)}(q\mathcal{F}_{(2)})^2, \quad (4.38)$$

with $\mathcal{C}_{(4)} = A_{(4)} - \frac{1}{2}\widetilde{q}_\alpha q_\beta A^\alpha_{(2)} A^\beta_{(2)}$. We have

$$\begin{aligned} e^{q\mathcal{F}_{(2)}}\delta_\kappa(q\cdot\mathcal{C}) &= 2\delta_\kappa\overline{\theta}\Big[e^{\Phi\, q\mathcal{F}_{(2)}}S_B(\psi)\tau_1 \\ &\quad - e^{q\mathcal{F}_{(2)}}(q\cdot\mathcal{C})(qq\mathcal{M})^{1/2}\tau_3\Gamma_\mu\Pi^\mu\Big]d\theta \longrightarrow \\ \delta_\kappa\mathcal{C}_{(4)} + q\mathcal{F}_{(2)}\delta_\kappa(\widetilde{q}_\alpha\mathcal{C}^\alpha_{(2)}) &= 2\delta_\kappa\overline{\theta}\Big[\big(\frac{1}{3!}i\tau_2\psi^3 + \tau_1\Phi\, q\mathcal{F}_{(2)}\psi\big) \\ &\quad - \big(q\mathcal{F}_{(2)}\mathcal{C}_{(0)} + \widetilde{q}_\alpha\mathcal{C}^\alpha_{(2)}\big)(qq\mathcal{M})^{1/2}\tau_3\Gamma_\mu\Pi^\mu\Big]d\theta. \end{aligned} \quad (4.39)$$

Therefore

$$\begin{aligned} \delta_\kappa\mathcal{C}_{(4)} &= 2\delta_\kappa\overline{\theta}\Big[\big(\frac{1}{3!}i\tau_2\psi^3 + \tau_1\Phi\, q\mathcal{F}_{(2)}\psi\big) - q\mathcal{F}_{(2)}(\widetilde{q}\widetilde{q}\mathcal{M})^{1/2}\tau_3\Gamma_\mu\Pi^\mu \\ &\quad - \big(q\mathcal{F}_{(2)}\mathcal{C}_{(0)} + \widetilde{q}_\alpha\mathcal{C}^\alpha_{(2)}\big)(qq\mathcal{M})^{1/2}\tau_3\Gamma_\mu\Pi^\mu\Big]d\theta. \end{aligned} \quad (4.40)$$

It is obvious that eq. (4.40) is invariant under $SL(2,\mathbb{R})$ transformations.

So, the kappa-symmetry property of the Lagrangian (4.38) requires $\mathcal{C}_{(4)}$ to vary as (4.40) under kappa-symmetry transformations.

Super five-branes

We can use the same procedure as above for other cases. For the case of $p = 5$, the Lagrangian can be written as

$$\begin{aligned} \mathcal{L}_{5-brane} &= -(qq\mathcal{M})^{1/2}\sqrt{-\det\Big(g + \frac{q\mathcal{F}_{(2)}}{(qq\mathcal{M})^{1/2}}\Big)} + q_\alpha\mathcal{C}^\alpha_{(6)} \\ &\quad + \mathcal{C}_{(4)}q_\alpha\mathcal{F}^\alpha_{(2)} + \frac{1}{2}\widetilde{q}_\beta\mathcal{C}^\beta_{(2)}(q\mathcal{F}_{(2)})^2 + \frac{1}{6}\mathcal{C}_{(0)}(q\mathcal{F}_{(2)})^3. \end{aligned} \quad (4.41)$$

Since we have known how $\widetilde{q}_\alpha \mathcal{C}^\alpha_{(2)}$ and $\mathcal{C}_{(4)}$vary under kappa-symmetry transformations, we can get the variation of $q_\alpha \mathcal{C}^\alpha_{(6)}$ from eq. (4.35).

Super seven-branes and super nine-branes

After redefinitions, the Lagrangians of the super seven-branes and super nine-branes are given by

$$\begin{aligned}\mathcal{L}_{7-brane} &= -(qq\mathcal{M})\sqrt{-\det\left(g + \frac{q\mathcal{F}_{(2)}}{(qq\mathcal{M})^{1/2}}\right)} + q_\alpha q_\beta \mathcal{C}^{\alpha\beta}_{(8)} \\ &\quad + q_\alpha q_\beta \mathcal{C}^\alpha_{(6)} \mathcal{F}^\beta_{(2)} + \frac{1}{2}\mathcal{C}_{(4)}(q\mathcal{F}_{(2)})^2 + \frac{1}{6}\widetilde{q}_\gamma \mathcal{C}^\gamma_{(2)}(q\mathcal{F}_{(2)})^3 \\ &\quad + \frac{1}{24}\mathcal{C}_{(0)}(q\mathcal{F}_{(2)})^4,\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}\mathcal{L}_{9-brane} &= -(qq\mathcal{M})^{3/2}\sqrt{-\det\left(g + \frac{q\mathcal{F}_{(2)}}{(qq\mathcal{M})^{1/2}}\right)} + q_\alpha q_\beta q_\gamma \mathcal{C}^{\alpha\beta\gamma}_{(10)} \\ &\quad + q_\alpha q_\beta q_\gamma \mathcal{C}^{\alpha\beta}_{(8)} \mathcal{F}^\gamma_{(2)} + \frac{1}{2} q_\alpha \mathcal{C}^\alpha_{(6)}(q\mathcal{F}_{(2)})^2 + \frac{1}{6}\mathcal{C}_{(4)}(q\mathcal{F}_{(2)})^3 \\ &\quad + \frac{1}{24}\widetilde{q}_\epsilon \mathcal{C}^\epsilon_{(2)}(q\mathcal{F}_{(2)})^4 + \frac{1}{120}\mathcal{C}_{(0)}(q\mathcal{F}_{(2)})^5.\end{aligned} \tag{4.43}$$

Similarly, we can get the variations of $q_\alpha q_\beta \mathcal{C}^{\alpha\beta}_{(8)}$ and $q_\alpha q_\beta q_\gamma \mathcal{C}^{\alpha\beta\gamma}_{(10)}$ under kappa-symmetry transformations from previous results and eq. (4.35).

Notice that, the above two actions belong to the two conjugacy classes which contain the super D7-brane and the super D9-brane respectively. The actions belonging to the other conjugacy classes might also have kappa-symmetry but do not have $SL(2,\mathbb{R})$ symmetry.

Summary and Conclusions

The subject of this report is the kappa-symmetry of $SL(2,\mathbb{R})$-invariant super IIB branes.

A simple description of string theory was given in Chapter 1. Before discussing strings, we gave a short introduction to the relativistic particle at the beginning of this chapter. Then, both bosonic strings and superstrings were considered. We got the spectra of states for the first levels and the corresponding fields. At the end of this chapter, Dp-branes and p-branes were introduced. And we knew the necessity of the existence of Dp-branes in string theory by the T-duality of open strings.

Chapter 2 gave the proofs of kappa-symmetry for the cases of the superparticle, the super D-string and the super F-string. The later two cases were discussed in a flat background. We knew that kappa-symmetry can reduce half of the degrees of freedom of the femionic part. Thus it is a natural tool to keep supersymmetry on the world-volume.

The topic of Chapter 3 is the $SL(2,\mathbb{R})$ symmetry of IIB branes. But there, we only considered the bosonic part. The example of one-branes case was discussed in details. A very elegant way of constructing the $SL(2,\mathbb{R})$ symmetric actions for IIB branes was introduced simply at the end of this chapter.

The main topic of this report, i.e., kappa-symmetry of $SL(2,\mathbb{R})$-invariant super IIB branes, came in Chapter 4. Since the case of one-branes differs from the cases of $p = 3, 5, 7, 9$, we considered it separately. By assuming how $\widetilde{q}_\alpha \mathcal{C}^\alpha_{(2)}$ varies under kappa-symmetry transformations, we proved that the action of super IIB one-branes, which has $SL(2,\mathbb{R})$ symmetry, has kappa-symmetry at the same time. After this, an universal discussion about the cases of $p = 3, 5, 7, 9$ was given. By assuming how $q_\alpha \mathcal{F}^\alpha_{(2)}$ varies under kappa-symmetry transformations, the general formulas of the variations of the DBI and WZ parts under kappa-symmetry transformations were given. We gave a general proof of the kappa-symmetry of the super IIB brane actions. The key steps of the proof were given in Appendix B. After the general description, the special case of $p = 3$ was discussed in detail. The variation of the $SL(2,\mathbb{R})$ singlet, i.e., the Ramond-Ramond four-form, under kappa-symmetry transformations was given. Following this, a simple discussion for other special cases was given. In this chapter, we also redefined the fields in order to get simpler formulae and at the same time make the proof natural and easier. However, the meaning of the redefinition is not yet understood. The discussion in this chapter is based on a flat background. A natural generalization is to consider the situation in a curved background. Then the situation will become more complicated and the IIB superspace constraints on field strength should be considered.

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Appendix A

Some Conventions

In this appendix, we give some conventions. Let the charges q_α form a doublet of $SU(1,1)$ and $q_{\alpha'}$ form a doublet of $SL(2,\mathbb{R})$ with $\alpha = 1,2$. If

$$\widetilde{q}_{\alpha'} = \begin{pmatrix} p \\ q \end{pmatrix}, \tag{A.1}$$

we can get

$$q_{\alpha'} = \begin{pmatrix} -q \\ p \end{pmatrix}, \quad q_\alpha = -\frac{\sqrt{2}}{2}\begin{pmatrix} q+ip \\ q-ip \end{pmatrix}, \quad \widetilde{q}_\alpha = \frac{\sqrt{2}}{2}\begin{pmatrix} p-iq \\ p+iq \end{pmatrix}, \tag{A.2}$$

by the transformations:

$$q_{\alpha'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\widetilde{q}_{\alpha'}, \quad q_\alpha = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} q_{\alpha'}, \quad q_\alpha = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\widetilde{q}_\alpha. \tag{A.3}$$

If we define

$$\frac{1}{2}(\widetilde{q}_\alpha q_\beta - \widetilde{q}_\beta q_\alpha) \equiv \widetilde{q}_{[\alpha} q_{\beta]} = \frac{i}{2}\varepsilon_{\alpha\beta}, \quad \text{with} \quad \varepsilon_{12} = 1 \quad \text{and} \quad \varepsilon_{1'2'} = -i, \tag{A.4}$$

we will get

$$p^2 + q^2 = 1. \tag{A.5}$$

Note that, (A.5) is just a normalization condition which does not mean that the charges are fractional. We can choose another normalization condition if we like. The $SU(1,1)$ doublet 2-forms potential fields are $A^\alpha_{(2)}$ with the constraint $(A^1_{\mu\nu})^* = A^2_{\mu\nu}$. The relations between the standard Kalb-Ramond 2-form potential $B_{(2)}$, the

Ramond-Ramond 2-form potential $C_{(2)}$ and the $SU(1,1)$ doublet 2-forms $A^{\alpha}_{(2)}$ are given as following:

$$B_{(2)} = -\frac{\sqrt{2}}{2} i(A^1_{(2)} - A^2_{(2)}), \tag{A.6a}$$

$$C_{(2)} = \frac{\sqrt{2}}{2}(A^1_{(2)} + A^2_{(2)}), \tag{A.6b}$$

which are different from those of [20, 24]. Given

$$\mathcal{M}^{\alpha'\beta'} = e^{\phi} \begin{pmatrix} \ell^2 + e^{-2\phi} & \ell \\ \ell & 1 \end{pmatrix}, \tag{A.7}$$

we have

$$\widetilde{q}_{\alpha}\widetilde{q}_{\beta}\mathcal{M}^{\alpha\beta} = (q + p\ell)^2 e^{\phi} + p^2 e^{-\phi}, \tag{A.8}$$

$$q_{\alpha}q_{\beta}\mathcal{M}^{\alpha\beta} = (p - q\ell)^2 e^{\phi} + q^2 e^{-\phi}. \tag{A.9}$$

For a special case, for instance, the D-brane case, we have

$$\widetilde{q}_{\alpha'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.10}$$

and then

$$\widetilde{q}_{\alpha}A^{\alpha}_{(2)} = C_{(2)}, \qquad q_{\alpha}A^{\alpha}_{(2)} = B_{(2)}, \qquad n = 2; \tag{A.11}$$

$$\widetilde{q}_{\alpha}A^{\alpha}_{(n)} = B_{(n)}, \qquad q_{\alpha}A^{\alpha}_{(n)} = C_{(n)}, \qquad n \neq 2. \tag{A.12}$$

It is convenient to define

$$A^{1'}_{(2)} = C_{(2)},\ A^{2'}_{(2)} = B_{(2)}; \quad A^{1'}_{(6)} = B_{(6)},\ A^{2'}_{(6)} = C_{(6)}, \tag{A.13}$$

which are useful in calculations. If we define

$$\mathcal{F}^{\alpha}_{(2)} = F^{\alpha}_{(2)} - A^{\alpha}_{(2)}, \quad F^{\alpha}_{(2)} = 2\partial V^{\alpha}_{(1)}, \tag{A.14}$$

and

$$\mathcal{F}^{\alpha'}_{(2)} = dV^{\alpha'}_{(1)} - A^{\alpha'}_{(2)}, \quad F^{\alpha'}_{(2)} = dV^{\alpha'}_{(1)}, \tag{A.15}$$

with

$$V^{1'}_{(1)} = c, \qquad V^{2'}_{(1)} = b, \tag{A.16}$$

we can get

$$\mathcal{F}^{1'}_{(2)} = dc - C_{(2)} \equiv \mathcal{F}_c, \qquad \mathcal{F}^{2'}_{(2)} = db - B_{(2)} \equiv \mathcal{F}_b. \tag{A.17}$$

If $\widetilde{q}_{\alpha'}$ is given by (A.1), we have

$$\begin{aligned} \widetilde{q}_{\alpha}\mathcal{C}^{\alpha}_{(2)} &= \widetilde{q}_{\alpha'} A^{\alpha'}_{(2)} \\ &= pA^{1'}_{(2)} + qA^{2'}_{(2)} \\ &= pC_{(2)} + qB_{(2)}, \end{aligned} \tag{A.18}$$

$$\begin{aligned} q_{\alpha} A^{\alpha}_{(2)} &= q_{\alpha'} A^{\alpha'}_{(2)} \\ &= -qA^{1'}_{(2)} + pA^{2'}_{(2)} \\ &= -qC_{(2)} + pB_{(2)}, \end{aligned} \tag{A.19}$$

$$\begin{aligned} q\mathcal{F}_{(2)} &= q_{\alpha'}\mathcal{F}^{\alpha'}_{(2)} \\ &= -q\mathcal{F}^{1'}_{(2)} + p\mathcal{F}^{2'}_{(2)} \\ &= -q\mathcal{F}_c + p\mathcal{F}_b. \end{aligned} \tag{A.20}$$

Appendix B

Proofs of Equations (4.25) and (4.26)

The proof of equation (4.25)

First, let us prove eq. (4.25). We follow the same method used by [9]. From eqs. (4.17a), (4.18a) and (4.19a), when $p = 2k+1$, we have

$$\rho_{2k+2} = (qq\mathcal{M})^{\frac{p-3}{4}} \sum_{n=0}^{k+1} \tau_3^{k-n+2} \frac{(\Phi\, q\mathcal{F}_{(2)})^n}{n!} \frac{\psi^{2(k-n+1)}}{[2(k-n+1)]!} \tau_1. \tag{B.1}$$

Defining $\gamma_i^{[2]} \equiv \gamma_{2i}\gamma_{2i+1}$, we can get

$$\begin{aligned}
\rho^{(2k+1)} &= (qq\mathcal{M})^{\frac{p-3}{4}} \\
&\quad \sum_{n=0}^{k+1} \tau_3^{k-n+2} \sum_{\substack{i_1<\ldots<i_n.\\ i_{n+1}<\ldots<i_{k+1}.}} \wedge_{i_1} \ldots \wedge_{i_n} \gamma_{i_{n+1}}^{[2]} \ldots \gamma_{i_{k+1}}^{[2]} \tau_1 \\
&= (qq\mathcal{M})^{\frac{p-3}{4}} \tau_3 \\
&\quad \sum_{n=0}^{k+1} \tau_3^{k-n+1} \sum_{\substack{i_1<\ldots<i_n.\\ i_{n+1}<\ldots<i_{k+1}.}} \wedge_{i_1} \ldots \wedge_{i_n} \gamma_{i_{n+1}}^{[2]} \ldots \gamma_{i_{k+1}}^{[2]} \tau_1 \\
&= (qq\mathcal{M})^{\frac{p-3}{4}} \tau_3 \prod_{i=0}^{k} (\wedge_i + \tau_3 \gamma_i^{[2]}) \tau_1 \\
&= (qq\mathcal{M})^{\frac{p-3}{4}} \tau_3\, \tau_1 \prod_{i=0}^{k} (\wedge_i - \tau_3 \gamma_i^{[2]}),
\end{aligned} \tag{B.2}$$

since in some Lorentz frame, we have

$$\Phi\, q\mathcal{F}_{(2)} = \sum_{i=0}^{k} \Lambda_i d\sigma^{2i} \wedge\, d\sigma^{2i+1}. \tag{B.3}$$

Where, $(i_1, ..., i_{k+1})$ is a permutation of the numbers $(0, ..., k)$. So

$$\begin{aligned}
(\rho^{(2k+1)})^2 &= (qq\mathcal{M})^{\frac{p-3}{2}} \prod_{i=0}^{k}(\wedge_i + \tau_3\gamma_i^{[2]})\tau_3\tau_2\tau_3\tau_2 \prod_{i=0}^{k}(\wedge_i - \tau_3\gamma_i^{[2]}) \\
&= -(qq\mathcal{M})^{\frac{p-3}{2}} \prod_{i=0}^{k}(\wedge_i + \tau_3\gamma_i^{[2]}) \prod_{i=0}^{k}(\wedge_i - \tau_3\gamma_i^{[2]}) \\
&= -(qq\mathcal{M})^{\frac{p-3}{2}} \prod_{i=0}^{k}(\wedge_i^2 - \gamma_i^{[2]^2}) \\
&= -(qq\mathcal{M})^{\frac{p-3}{2}} \det(g + \Phi\, q\mathcal{F}_{(2)}).
\end{aligned} \tag{B.4}$$

The proof of equation (4.26)

Now, let us prove eq. (4.26). It is equivalent to prove

$$\rho_{p+1}\gamma_i \;=\; T_p d\sigma^j \Big(g + \tau_3 \Phi\, q\mathcal{F}_{(2)}\Big)_{ji}. \tag{B.5}$$

Summing over odd p, we should prove that

$$\rho_B\gamma_i \;=\; T_B d\sigma^j \Big(g + \tau_3 \Phi\, q\mathcal{F}_{(2)}\Big)_{ji}. \tag{B.6}$$

We will use the following relations:

$$\frac{\psi^n}{n!}\gamma_i \;=\; \frac{\psi^{n-1}}{(n-1)!} d\sigma^j g_{ji} + \frac{(-1)^n}{(n+1)!} i_{e_i}(\psi^{n+1}), \quad \text{when}\;\; n \geq 1; \tag{B.7}$$

$$\gamma_i \;=\; i_{e_i}(\psi^1), \quad \text{when}\;\; n = 0. \tag{B.8}$$

Where i_{e_i} denotes the interior product operator induced by $e_i = \frac{\partial}{\partial \sigma_i}$. The general definition of i_X for an n-form ω and a vector field X is given by [9]

$$i_X\omega = \frac{1}{n!}\sum_{s=1}^{n}(-1)^{s-1} X^{\mu_s}\omega_{\mu_1...\mu_s...\mu_n} d\sigma^{\mu_1}...d\sigma^{\mu_{s-1}} d\sigma^{\mu_{s+1}}...d\sigma^{\mu_n}. \tag{B.9}$$

Therefore, for $n = 0$, we can get

$$e^{\Phi\, q\mathcal{F}_{(2)}}\tau_3\gamma_i \;=\; e^{\Phi\, q\mathcal{F}_{(2)}} i_{e_i}(\psi^1)\tau_3. \tag{B.10}$$

For $n = 2$, get

$$\begin{aligned}
\frac{\psi^2}{2!}\gamma_i &= \psi\, d\sigma^j g_{ji} + \frac{1}{3!} i_{e_i}(\psi^3) \longrightarrow \\
e^{\Phi\, q\mathcal{F}_{(2)}} \frac{\psi^2}{2!}\gamma_i &= e^{\Phi\, q\mathcal{F}_{(2)}} \psi\, d\sigma^j g_{ji} + e^{\Phi\, q\mathcal{F}_{(2)}} \frac{1}{3!} i_{e_i}(\psi^3).
\end{aligned} \tag{B.11}$$

For $n = 4$, get

$$\begin{aligned}
\frac{\psi^4}{4!}\gamma_i &= \frac{\psi^3}{3!} d\sigma^j g_{ji} + \frac{1}{5!} i_{e_i}(\psi^5) \longrightarrow \\
e^{\Phi\, q\mathcal{F}_{(2)}} \frac{\psi^4}{4!}\tau_3\gamma_i &= e^{\Phi\, q\mathcal{F}_{(2)}} \frac{\psi^3}{3!} d\sigma^j \tau_3\, g_{ji} + e^{\Phi\, q\mathcal{F}_{(2)}} \frac{1}{5!} i_{e_i}(\psi^5)\tau_3,
\end{aligned} \tag{B.12}$$

and so on. Summing over eqs. (B.10), (B.11), (B.12) and so on, and multiplying both sides by $(qq\mathcal{M})^{\frac{p-3}{4}}$ from left and τ_1 from right, we can get

$$\rho_B\gamma_i = T_B d\sigma^j g_{ji} + (qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} i_{e_i}(S_B(\psi))\tau_3\tau_1. \tag{B.13}$$

The both sides of the above equation are summations of $(p+1)$-forms. Since $e^{\Phi\, q\mathcal{F}_{(2)}} S_B(\psi)$ is a summation of $(p+2)$-forms, which should vanish. So we have

$$\begin{aligned}
& (qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} i_{e_i}(S_B(\psi))\tau_3\tau_1 \\
= & -(qq\mathcal{M})^{\frac{p-3}{4}} i_{e_i}(e^{\Phi\, q\mathcal{F}_{(2)}}) S_B(\psi)\tau_3\tau_1 \\
= & -(qq\mathcal{M})^{\frac{p-3}{4}} e^{\Phi\, q\mathcal{F}_{(2)}} i_{e_i}(\Phi\, q\mathcal{F}_{(2)}) S_B(\psi)\tau_3\tau_1 \\
= & -T_B i_{e_i}(\Phi\, q\mathcal{F}_{(2)})\tau_3 \\
= & -T_B d\sigma^j (\Phi\, q\mathcal{F}_{(2)})_{ij}\tau_3 \\
= & \; T_B d\sigma^j (\Phi\, q\mathcal{F}_{(2)})_{ji}\tau_3.
\end{aligned} \tag{B.14}$$

Therefore

$$\rho_B\gamma_i = T_B d\sigma^j \Big(g + \tau_3 \Phi\, q\mathcal{F}_{(2)}\Big)_{ji}, \tag{B.15}$$

hence

$$\rho^{(p)}\gamma_i = T^j_{(p)} \Big(g + \tau_3 \Phi\, q\mathcal{F}_{(2)}\Big)_{ji}. \tag{B.16}$$

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