Gauging the half-maximal trombone in 4D

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Abstract

In order to get non-abelian gauge symmetries in supergravity theories one can gauge subgroups of the global symmetry groups inherited from their higher dimensional origins (compactified on n-tori). Apart from these large symmetry groups it is also possible to gauge the local scaling symmetry (the trombone) present in these theories. In 4D this has been done for maximal supergravity, here the half-maximal case is investigated. In particular the main constraints, following from the requirements of supersymmetry and consistency, are derived. Though the equations of motions are not derived here, it can be expected that it, as in the maximal case, will not be possible to formulate an action for this theory.

Contents

1 Introduction

In our search for a quantum theory of gravity, string theory still seems to be the most promising candidate. Though there is still no result obtained which enables an experimental testing for the theory, still many theoretical physicists are that much impressed by its elegance and richness to feel justified studying it. One reason for this is the elegant and natural way gravity is popping out of the theory.

It turned out that there is not only one string theory, but five different types. But all these types are related by dualities and are now thought of as species of one unified eleven dimensional theory, M-Theory. 11 dimensional supergravity then can be seen as the effective field theory of M-theory. But supergravity theories also can be build up starting from general relativity and combining it with supersymmetry. In particular one can get general relativity by making global supersymmetry local, i.e. gauging it.

The aim is to construct a theory combining quantum mechanics (i.e. the Standard Model described by quantum field theories) with gravity. Because in our daily life we do not see more than four space-time dimensions, we have to get rid of the extra dimensions of the eleven dimensional string and supergravity theories. This can be done by what is called dimensional reduction which turns space-time symmetries into internal (gauge) symmetries. In the end we hope to be able to find a way of turning the extra space-time dimensions into exactly the gauge symmetries of the Standard Model (or into something containing these).

Dimensional reduction, also called compactification, can be done in several ways, all giving different supergravity theories. Depending on the manifold you choose to compactify on, you get supergravity theories with different gauge symmetries. The simplest manifold you can think of is the n-torus, which is a generalisation of the circle and the 2-torus (a donut shaped manifold) in n dimensions. Compactification on this manifold will give a supergravity theory where there is only a $U(1)^n$ gauge symmetry. For reconstructing the Standard Model we need more, bigger and in particular non-abelian gauge groups. These can be derived from the higher dimensional theory by compactifying on more complicated manifolds, but also by gauging the simpler theory. The deformation parameters of the more complicated manifolds then are incorporated as gauge parameters in the gauging procedure.

Apart from the big global symmetry groups there is also a scaling symmetry present, which we also can gauge. The scaling, being a scaling of the Lagrangian as a whole, is an on-shell symmetry. Gauging this symmetry will give an additional positive contribution to the effective cosmological constant. This has been done in the four dimensional case for maximal supergravity [4], [5]. In this thesis the case of gauging this symmetry in half-maximal supergravity will be investigated. The main aim is to determine the different constraints put on the theory by demanding consistency and supersymmetry. In particular the linear and quadratic constraints will be derived. Though it will not be done in this thesis, one can try to find explicit solutions to these constraints and formulate, given these constraints, the equations of motion.

The scientific motivation for doing this kind of research is filling in the empty spots of the big puzzle. The hope is that the more we know about specific theories, the more we will be able to find regularities. Finding regularities makes it possible to generalize and to get more understanding of and insight in our theories. Also it can give direction to new research. In particular it can be checked if the results of this research fits with the results already found for the maximal case. In the maximal case gauging the trombone leads to a theory with equations of motions for which there can not be formulated an action. This highly interesting aspect is not investigated here for the half-maximal case, but there is no reason to think that it will differ from the maximal case in this respect.

Part I of this thesis will be about string theory. This discussion is mainly based on the well-known book of Green, Schwarz and Witten, volume 1 [2]. This part is meant to give the reader some insight in the higher dimensional origin of supergravity theories. In Part II we will turn our attention to supergravity. We start with some minimal supergravity, then go on to extended supergravity. First the ungauged supergravities are discussed and then the embedding tensor formalism for gauging is explained. Here we will follow Samtleben in his lecture notes on supergravity [8] and on gauged supergravities and flux compactifications [9]. When we go on to the half-maximal case the findings of Schön and Weidner [10], [12] are being used. Adding the gauging of the scaling symmetry to this theory is new work presented here. Le Diffon, Samtleben and Trigiante gave an analysis of the maximal case in [4], [5].

Part I
String Theory

2 Bosonic String Theory

The Nambu-Goto action of the bosonic string is proportional to the area of the world-sheet spanned by the string in space time. The coordinates on the string are denoted by $\sigma^\alpha = (\tau, \sigma)$ and the coordinates of the embedding space by X^μ, with $\mu = 1, \ldots, D-1$. $X^\mu(\sigma^\alpha)$ is a mapping from the string coordinates to the coordinates of the target space. The Nambu-Goto action is given by:

$$S = -T \int d^2\sigma \sqrt{-G} \tag{1}$$

The T here is associated with the string tension and is given by $T = (2\pi\alpha')^{-1}$. Where α' is the square root of the string length $\sqrt{2\alpha'} = l_s$. G is defined as $G = \det G_{\alpha\beta}$ with $G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$, which can be seen as the infinitely small unit area on the world-sheet.

But this form of the action is not easy to quantize, due to the square root in it. To quantize the square root of G we would have to expand it in an infinite series of operators. So therefore we would rather like to use another form of the action without this square root. We can get at such an action by introducing first an auxiliary field $h^{\alpha\beta}$ and then by gaugefixing this field using the symmetries of the new action. The new action with the auxiliary field is called the Polyakov action:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta}(\sigma) g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \tag{2}$$

Here $h^{\alpha\beta}$ can be considered as the inverse metric of the world-sheet and $g_{\mu\nu}(X)$ is the metric of the embedding space-time. It can be shown that this action is equivalent to (1).

From this we can easily derive the equations of motions, which turn out to be:

$$\Box X^\mu \equiv (\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}) X^\mu = 0$$

which is a common two dimensional wave equation. But $h^{\alpha\beta}$ is also a dynamical field in our theory, so it has an equation of motion too (which still should be satisfied even after gaugefixing). From this equation of motion follows that

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\alpha'\beta'} \partial_{\alpha'} X^\mu \partial_{\beta'} X_\mu = 0$$

which is a constraint on the solutions of the wave equations. This equation is equivalent to

$$\dot{X}^2 + X'^2 = 0 \tag{3}$$

where $\dot{X}$ is the derivative of the X-coordinate with respect to τ and X' the derivative with respect to σ.

We can write the general solution as a mode expansion given by

$$X^\mu(\sigma,\tau) = x^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma$$

Now we can use local symmetries possessed by the Polyakov action to choose the three components of $h^{\alpha\beta}$ so that $h^{\alpha\beta} = \eta^{\alpha\beta} = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$. The symmetries we use for this are the reparametrization invariance and the Weyl-scaling of (2):

$$\begin{aligned}
\delta X^\mu &= \xi^\alpha\partial_\alpha X^\mu \\
\delta h^{\alpha\beta} &= \xi^\gamma\partial_\gamma h^{\alpha\beta} - \partial_\gamma\xi^\alpha h^{\gamma\beta} - \partial_\gamma\xi^\beta h^{\gamma\alpha} \\
\delta(\sqrt{h}) &= \partial_\alpha(\xi^\alpha\sqrt{h})
\end{aligned} \tag{4}$$

and

$$\delta h^{\alpha\beta} = \Lambda h^{\alpha\beta} \tag{5}$$

respectively. Note that the main reason why we work with strings and not with higher dimensional objects is that we can, with the help of the mentioned symmetries, in 2 space-time dimensions gauge away the dependence on $h^{\alpha\beta}$, which of course should be the case because it is an auxiliary field. For higher dimensional objects this is not possible. The reparametrizations act on the coordinates and the Weyl-scaling on the metric, but it is possible to rescale the action (2) by using the reparametrization invariance. So it is possible to 'undo' a Weyl scaling with the help of a reparametrization. We will get back to this point later on. We now have the simple form of our action, the gauge fixed Polyakov form:

$$S = -\frac{T}{2}\int d^2\sigma\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu$$

2.1 Quantization

There are two ways of quantizing the string, namely the covariant and the light-cone gauge quantization. The difference between these two ways of quantizing is the moment you incorporate the constraints given by the equations of motion for the metric on the world-sheet. In the covariant quantization you first quantize and then impose the constraints, whereas in the light-cone gauge quantization you impose the constraints before quantizing the string modes. The light-cone approach is easier and gives us the right number of states, but the physical meaning of these states is not that clear. While in the covariant approach it is not that easy to get the right number of states (due to so-called null-states) but their meaning is clear. We will focus here on the light-cone gauge approach.

We first introduce the light-cone coordinates:

$$X^{\pm} = \frac{1}{2}(X^0 \pm X^{D-1})$$

for the coordinates of space time, and

$$\sigma^{\pm} = (\sigma \pm \tau)$$

for the coordinates of the world-sheet. The reason why we introduce these light-cone coordinates is that it turns the quadratic constraint equations into lineair ones. As mentioned before you can undo a Weyl-scaling of (2) by a reparametrization. So we still have some gauge freedom after setting $h^{\alpha\beta} = \eta^{\alpha\beta}$. This can be seen by the fact that any combined reparametrization and Weyl scaling $\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta}$ (in the 'old' non light-cone gauge coordinates) preserves the gauge choice.

We can use this freedom to set $X^+(\sigma, \tau) = x^+ + p^+\tau$. This boils down to setting all the α_n^+ coefficients of the oscillator to zero for $n \neq 0$. After doing this, the constraint equations $(\dot{X} \pm X')^2 = 0$ become

$$(\dot{X}^- \pm X'^-) = (\dot{X}^i \pm X'^i)^2/2p^+ \tag{6}$$

So we can express X^- in terms of X^i leaving only the transversal oscillator modes X^i to have independent oscillations.

Equation (6) gives constraints on the oscillator coefficients α_n^- of the the mode expansion for X^-

$$X^- = x^- + p^-\tau + i\sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma$$

This gives us (for $n = 0$) the mass formula

$$M^2 = \frac{1}{\alpha'}(p^+p^- - \frac{1}{2}p^ip^i) = \frac{1}{\alpha'}(N - a) \tag{7}$$

where $M^2 = -p_\mu p^\mu$ will play the role of the mass of the string in a specific mode of oscillation. Here

$$N = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i \tag{8}$$

This a of (7) is the so-called 'normal ordering constant'. We need this constant because at the moment we quantize our string theory, and thus replace the Fourier coefficients by operators, we have to make a decision about the order in which we place them. This because for operators the order matters. In general we choose in this situation what is called the 'normal ordering' with all annihilation operators to the right and all the creation operators to the left. But nothing guarantees that this is the choice which will reproduces the 'right' quantum analog of the classical description. And because of the commutation relations we impose on the creation and annihilation operators

$$[\alpha_n^i, \alpha_m^j] = n\delta_{n+m,0}^{ij} \rightarrow [\alpha_n^i, \alpha_{-n}^i] = n$$

the interchange of the operators α_n^i and α_{-n}^i will cost us only a constant. This is the reason why we introduced in the mass formula a later-to-be-determined-constant a. From the given formula for the mass (7) we see that it really matters for our physics what value of a follows from our theory. A different value for a means different masses for our particles.

But how do we now find out what the value of a should be? In what sense should the quantum description reproduce the classical one? Well, at least we want to get back a Lorentz invariant theory. For our theory to be Lorentz invariant we have to take the value for a to be 1. In the end it turns out that we get another condition on our theory from the requirement of Lorentz invariance, namely on the number of spacetime coordinates of the embedding space: $D = 26$ for bosonic string theories. This constraint follows from demanding the cancellation of an anomaly term in the commutator of two of the Lorentz generators, namely J^{i-} and J^{j-}. This commutator should be zero to let the theory be Lorentz invariant. This commutator turns out to be the only one which is giving any anomaly problems. That the J^{i-} is involved is due to the fact that this generator is acting on X^+ and thus on the gauge condition. It can be shown that the commutator will have the form

$$[J^{i-}, J^{j-}] = -\frac{1}{(p^+)^2} \sum_{m=1}^{\infty} \Delta_m (\alpha_{-m}^i \alpha_m^j - \alpha_{-m}^j \alpha_m^i)$$

Some subtleties are involved, mainly due to the non Lorentz invariant choice of our gauge, but in the end the vanishing of the anomaly term requires

$$\Delta_m = m(\frac{26 - D}{12}) + \frac{1}{m}(\frac{D - 26}{12} + 2(1 - a)) = 0$$

which is satisfied for $D = 26$ and $a = 1$. No further constraints follow from the other commutators. Note that in $D = 3$ this commutator is zero, so that it is also possible to write down a Lorentz invariant consistent string theory in two space- and one time dimensions because then the commutator is trivially zero.

2.2 Spectrum

We can now take a look at the spectrum of the bosonic string theory. In principle there can be two types of strings, open and closed ones. You can build a theory with only closed strings, but not one with only open strings. Open strings can always close, so a theory with open strings should always also include closed strings. We take a look first at only the open string spectrum. The spectrum is generated by applying the transversal mode creation operators α_n^i ($n < 0$ creation and $n > 0$ annihilation) to the ground state (i.e. a string with momentum p but without any oscillations on it). The lowest level is a state with negative mass, called the tachyon. Given the mass formula (7) it has $\alpha' M^2 = -1$. The next level is a massless vector $\alpha_{-1}^i|0\rangle$ with 24 components (the transverse polarizations). Above that we can make a tower of massive states. For $N = 2$ ($M^2 = 1$) we have $[(\alpha_{-1})^2 + \alpha_{-2}]|0\rangle$ as the most general state. This is a 324 dimensional representation corresponding to a symmetric traceless 2-form of SO(25). And so on. But we are not that interested in the massive states because when we look at the supergravity limit of string theory, i.e. the low energy limit, these massive states become too heavy to be physical relevant. Low energies means great distances and compared to this great distances the strings become 'pointlike'. Because α' is proportional to the string length it also becomes very small, while very small α' means very big masses as can be seen by (7).

Knowing the spectrum of the open string we can easily compute the spectrum of the closed string. This time we have two types of oscillators $\{\alpha_n^i\}$ and $\{\tilde{\alpha}_n^i\}$ for the left and right moving modes on the string respectively. These modes are independent up to one restriction, namely that they have to have an equal amount of excitations which follows from the fact that the $n = 0$ constraint (7) for both the directions are equal. So $N = \tilde{N}$ with

$$\tilde{N} = \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i$$

So the states of the closed string are product states of the open string. In particular is the ground state still a tachyon (with an even bigger negative mass). The massless states have the form

$$|\Omega_{ij}\rangle = \alpha_{-n}^i \tilde{\alpha}_{-n}^j |0\rangle$$

The representation of this state can be decomposed in irreducible $SO(24)$ representations: a symmetric and traceless massless 2-form $G_{\mu\nu}$ which can be identified with the graviton, the trace ϕ, which is a massless scalar called the dilaton and the anti-symmetric 2-form $B_{\mu\nu}$. So $\mathbf{24} \otimes \mathbf{24} = \mathbf{299} \oplus \mathbf{1} \oplus \mathbf{276}$ or in Young-tableaux[1]:

$$\square \otimes \square = \square\square \oplus \cdot \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$$

2.3 Strings in Background Fields

Until now we worked with strings in flat Minkowski space. The action is

$$S_0 = -\frac{1}{2\pi} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

Now we want to consider strings in background fields. From the spectrum of the closed string we got three different types of massless states: the symmetric $G_{\mu\nu}$, the antisymmetric

[1]See appendix A for explanation of Young-Tableaux

$B_{\mu\nu}$ and the scalarfield ϕ. These are states of individual strings, but if you have many of them they form a continuous field. We are now interested in the interaction between a 'test string' and these background fields. We start with the interaction with the 26 dimensional gravitational field.

To describe this interaction we simply replace $\eta_{\mu\nu}$ with $g_{\mu\nu}(X^\rho)$ in the action:

$$S_0 = -\frac{1}{2\pi}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X^\rho) \tag{9}$$

We can expand $g_{\mu\nu}$ around the Minkowski metric: $g_{\mu\nu}(X^\rho) = \eta_{\mu\nu} + f_{\mu\nu}(X^\rho)$. The first term gives us the free theory, the second term describes the interaction with the gravitational field.

We want this interacting action to be scale, or Weyl, invariant as it was in the free case. In the beginning we started off with the Nambu-Goto action (1). Then we introduced the auxiliary field $h^{\alpha\beta}$ and saw that, with the help of the reparametrization invariance and the Weyl scaling, we could gaugefix all its components. So it indeed was auxiliary, meaning that it didn't have any physical degrees of freedom. But at the moment we had not Weyl invariance anymore in our quantum theory, it would mean that we do not work anymore with an action which is the quantum equivalent of the Nambu-Goto action.

Hence, we have to impose this scale invariance by hand. Scale invariance in space-time is called conformal invariance and it means that the β-function must vanish. In QFT's the beta function describes the dependence of the coupling constant on the energy scale. It is defined as

$$\beta(g) = \frac{\partial g}{\partial log(\mu)}$$

with μ the energy scale and g the coupling constant. The fact that we can speak about a beta function, and about coupling constants, indicates a way of looking at the action (9). In this view we consider the X^μ's as scalarfields on the worldsheet. Because the metric also depends on these fields it can be considered as the coupling of the interaction. This is a quite different inperpretation of this action than when we just consider the X^μ as the coordinates of the target space where the strings are living in. Considered that way it is just a normal action of general relativity, for a string in a gravitational field.

Scale invariance of the action is broken by replacing $\eta_{\mu\nu}$ with $g_{\mu\nu}$, because there is no way to regularize the action while preserving the worldsheet scale invariance. But we have to regularize. For this reason we have to impose a condition to make the action Weyl invariant again. This turns out to be exactly the Einstein equations $R_{\mu\nu} = 0$. So from the fact that we want to have a conformal invariant theory, which means that the β-function has to set to zero, we get the classical vacuum Einstein equations.

Let's try to understand something of the regularization. First of all, we will choose the gauge $h_{\alpha\beta} = e^{(2+\epsilon)\phi}\eta_{\alpha\beta}$, and we will work in $2+\epsilon$ dimensions, with ϵ a small number of which we will take the limit to zero later on. We will consider $X^\mu(\sigma,\tau)$ as fields, and expand them around a vacuum expectation value: $X^\mu(\sigma,\tau) = X^\mu_o + x^\mu(\sigma,\tau)$. Now we want to expand the

metric accordingly. The most general form is quite complicated and ugly, but with the help of a redefinition of the field parameters $X^\mu \to \tilde{X}^\mu(X^\rho)$ we can get the coordinates X^μ to be locally inertial at the point X_0^μ.
With the help of this kind of field variables redefinition, and the use of Riemann normal coordinates, we can get an expression for the expansion of $g_{\mu\nu}$. If we insert this expression in the action and apply at the same time the expansion of $e^{\epsilon\phi} = 1 + \epsilon\phi + \ldots$ the action takes the form:

$$\begin{aligned}\tilde{S} = & - \frac{1}{2\pi} \int d^{2+\epsilon}\sigma \, [(\partial_\alpha x^\mu \partial^\alpha x^\nu)(1+\epsilon\phi)\eta_{\mu\nu} \\ & - \frac{1}{3} R_{\mu\lambda\nu\kappa}(X_0^\rho) x^\lambda x^\kappa \partial_\alpha x^\mu \partial^\alpha x^\nu (1+\epsilon\phi) + O(x^5)]\end{aligned}$$

With $R_{\mu\lambda\nu\kappa}(X_0^\rho)$ being the Riemann tensor on the space-time manifold at point X_0^ρ. We want this action to be not dependent on ϕ in the limit of $\epsilon \to 0$ and at the same time avoid any infinities due to poles. Some ϕ-dependences cancel each other out, but some others remain. And, as being said, we also have to renormalize the ϕ-independent terms with ϵ-poles. Introducing the appropiate renormalizations ($x^\mu \to x^\mu + \frac{1}{6\epsilon} R^\mu_\nu(X_0^\rho) x^\nu + O(x^2)$ and $g_{\mu\nu} \to g_{\mu\nu} - \frac{1}{2\epsilon} R_{\mu\nu}(X_0^\rho)$) gives again some ϕ-dependent terms. In the end, all the ϕ-dependent terms give an effective action of the form (for D=26)

$$\tilde{S} = -\frac{1}{4\pi} \int d^2\sigma \phi R_{\mu\nu}(X^\rho) \partial_\alpha X^\mu \partial^\alpha X^\nu$$

with $X^\mu(\sigma,\tau) = X_0^\mu + x^\mu(\sigma,\tau)$. So to this order, we only get a Weyl-invariant theory if $R_{\mu\nu}(X^\rho) = 0$, which is the vacuum Einstein equation!

Let us see if we can understand what the relation is between the Weyl invariance and the fact that we find the classical equations of motions of the string. On first instance it seems rather magical that the vanishing of the beta function happens to coincide with the Einstein equations (or actually with the constant part of the expected generalization as a series in α' of the Einstein equations). But actually this is exactly what you would expect.

In a quantum field theory with fields $\Phi^k, k = 1, \ldots, m$ we have scattering amplitudes of the form $A_n = \langle \phi_1^k \phi_2^k \ldots \phi_n^k \rangle$, where $\Phi^k = \Phi_0^k + \phi^k$. So, for $n = 1$ the conditon $A_1 = 0$ means that the Φ_0^k is the vacuum expectation value which satisfies the classical field equations, because the expectation value for the quantum fluctuations ϕ^k are zero. The analog in string theory is $A_n = \langle V_1^k V_2^k \ldots V_n^k \rangle$ with V^k the vertex operators corresponding to each field ϕ^k. This expectation value is computed on the worldsheet. Now $\langle V \rangle = 0$ gives again the condition to find the classical equations of motions. But this condition is also exactly the condition following from the conformal invariance of the worldsheet. We can see that as follows: conformal invariance of the action means invariance under the scaling transformations $\sigma^\alpha \to \lambda\sigma^\alpha$. Under this transformation the vertex operator of the closed string transforms as $V \to \lambda^{-2} V$. Invariance under this transformation thus implies that $\langle V \rangle = \langle \lambda^{-2} V \rangle$, so it must hold that $\langle V \rangle = 0$. So the condition of conformal invariance forces the theory to give the classical equations of motions, that is the Einstein equations.

From all this it follows that we can think of the string corrections to the Einstein equations in terms of a series in α'. The first term, linear in α' gives:

$$\beta_{\mu\nu}(X^\rho) = -\frac{1}{4\pi}(R_{\mu\nu} + \frac{\alpha'}{2}R_{\mu\kappa\lambda\tau}R_\nu{}^{\kappa\lambda\tau})$$

Until now we only discussed the gravitational background field. If we want to consider a more general action, containing besides the interaction with $g_{\mu\nu}(X^\rho)$ also the interactions with the antisymmetric $B_{\mu\nu}(X^\rho)$ and with the dilaton field $\Phi(X^\rho)$ we have to look after an action which is invariant under reparametrizations of the string world-sheet and also renormalizable by power counting[1].

The three terms we will find this way are:

$$\begin{aligned}
S_1 &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X^\rho) \\
S_2 &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X^\rho) \\
S_3 &= \frac{1}{4\pi}\int d^2\sigma\sqrt{h}\Phi(X^\rho)R^{(2)}
\end{aligned} \tag{10}$$

where $R^{(2)}$ is the world-sheet Ricci scalar. Note that the second term is a topological term because it contains no $h^{\alpha\beta}$ and that the last term does not contain an α'.

As we discussed before, the constraints we have to impose to get a Weyl invariant action are exactly the classical equations of motions. These take the form:

$$\begin{aligned}
0 &= R_{\mu\nu} + \frac{1}{4}H_\mu{}^{\lambda\rho}H_{\nu\lambda\rho} - 2D_\mu D_\nu\Phi \\
0 &= D_\lambda H^\lambda{}_{\mu\nu} - 2(D_\lambda\Phi)H^\lambda{}_{\mu\nu} \\
0 &= 2(D_\mu\Phi)^2 - 4D_\mu D^\mu\Phi + R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}
\end{aligned} \tag{11}$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$.

But, if these equations are really equations of motions, we can wonder is we are able to write down an action which would give these equations by demanding the variation to be zero. Suprisingly, equations (11) indeed turn out to be the Euler-Langrange equation of the following 26 dimensional supergravity action:

$$S_{26} = -\frac{1}{2\kappa^2}\int d^{26}x\sqrt{g}e^{-2\Phi}(R - 4D_\mu\Phi D^\mu\Phi + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho})$$

The first equation of (10) follows from $\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} = 0$, the second one from $\frac{\partial\mathcal{L}}{\partial B_{\mu\nu}} = 0$ and the last one from $\frac{\partial\mathcal{L}}{\partial\phi} = 0$. Interestingly we will see later on that it is not always possible to interpret supergravity equations of motions as Euler-Langrange equations of some action.

[1] The latter condition means that there must be precisely two world-sheet derivatives in each term

3 Superstring theory

Now we will add supersymmetry to our string theory. Supersymmetry couples bosons to fermions, making it possible to 'rotate' them into eachother. To be able to do that the transformations belonging to it have parameters with spin $\frac{1}{2}$. In having such a fermionic parameter, coming along with a so-called Lie superalgebra, it is circumventing the Coleman-Mandula theorem which states that for any symmetry with a Lie algebra associated to it, it is not possible to mix space-time and gauge symmetries in a non-trivial way.

There are some reasons why we need supersymmetry for letting string theory be a sensible theory. The two main reasons are:

- With super symmetry we can get rid of the so-called tachyon which is a state with negative mass which appears in bosonic string theory.
- We want to have a theory which describes bosons as well as fermions, beacuse we in the end are looking for a 'theory of everything' containing at least all the particles of our Standard Model

One thing being necessary (but not sufficient) to write down a supersymmetric theory is that the bosonic and fermionic degrees of freedom present in the theory are equal. This is just a matter of counting which bosons and fermions you have in the theory and how many degrees of freedom these particles have. And if the amount of fermionic degrees of freedom don't fit the bosonic ones, you have to impose more conditions (choose another representation) for your particles. Common ways to do that are for example by choosing Majorana, Weyl or Majarona-Weyl spinors instead of Dirac spinors.

It turns out that there are two different ways of getting supersymmetry in our string theory. What we are looking for is supersymmetry for the particles in our spectrum. These particles are living in the embedding 10D space-time. So we want to get supersymmetry in our target space and one thing we obviously can do is just imposing it there. But it turns out that if we instead impose it on the 2D world-sheet, we also end up with supersymmetry in the target space.

Nothing guarantees us that these two different ways of imposing supersymmetry will give us the same theory in the end. And in fact, for superstring theory it turns out that, without imposing certain extra conditions (the GSO projections, which are also for other reasons maybe a good thing to do) on the allowed states you won't end up with the same theory. But let us first take a look at supersymmetry on the world-sheet.

3.1 Supersymmetry on world-sheet and its quantization

So, what we now want to do is make our theory supersymmetric. We want to get fermions out of our theory the way we got the bosons, namely by investigating the states of the string when the Virasoro constraints are imposed. To get the right constraints on the states, we have to introduce fermionic fields in our two dimensional field theory on the world-sheet.

These fields can also be interpreted as fermionic, i.e. anti-commuting, coordinates in the target space.

The supersymmetric action we will work with is (with conformal gauge $h^{\alpha\beta} = e^{-\phi}\eta^{\alpha\beta}$):

$$S = -\frac{1}{2\pi}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \tag{12}$$

Here the ρ^α denotes the two dimensional Dirac matrices with their usual properties. This action is invariant under the following supersymmetric transformations:

$$\begin{aligned} \delta X^\mu &= \bar{\epsilon}\psi^\mu \\ \delta\psi &= -i\rho^\alpha \partial_\alpha X^\mu \epsilon \end{aligned}$$

As may be expected the commutator of two supersymmetry transformations gives a translation:

$$\begin{aligned} [\delta_1, \delta_2] X^\mu &= a^\alpha \partial_\alpha X^\mu \\ [\delta_1, \delta_2] \psi^\mu &= a^\alpha \partial_\alpha \psi^\mu \end{aligned}$$

where $a^\alpha = 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2$. But this is only true given that the equations of motions hold, i.e. the Dirac equation $\rho^\alpha\partial_\alpha\psi = 0$.

We can caculate the supercurrent by considering the variation under the given transformations when ϵ is not a constant. Its variation then will get the general form

$$\delta S = \frac{2}{\pi}\int d^2\sigma (\partial_\alpha \bar{\epsilon}) J^\alpha$$

where

$$J_\alpha = \frac{1}{2}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu \tag{13}$$

is de supercurrent.

We want to close the algebra also off-shell, i.e. without the need to satisfy the e.o.m.. To make it this way we can introduce an auxiliary field B^μ with the right kind of transformation.

You can just introduce this field B^μ but you also can use the so called superspace formalism where it is much more clear that the action you will be working with indeed is supersymmetric. In the superspace formalism two extra coordinates on the world-sheet, Grassmann coordinates θ^A, are introduced. A general function in superspace can be written as

$$Y^\mu(\sigma, \theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma)$$

If we now take as our action

$$S = \frac{i}{4\pi} \in d^2\sigma d^2\theta \bar{D}Y^\mu DY_\mu$$

with $D = \frac{\partial}{\partial\theta} - i\rho^\alpha\theta\partial_\alpha$ the superspace covariant derivative, and work out everything with the properties of the superfields, we get back our original action and transformations. But now with this B field present.

$$S_0' = -\frac{1}{2\pi}\int d^2\sigma(\partial_\alpha X^\mu\partial^\alpha X_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu - B^\mu B_\mu)$$

and

$$\begin{aligned} \delta X^\mu &= \bar{\epsilon}\psi^\mu \\ \delta\psi &= -i\rho^\alpha\epsilon\partial_\alpha X^\mu + B^\mu\epsilon \\ \delta B^\mu &= -i\bar{\epsilon}\rho^\alpha\partial_\alpha\psi^\mu \end{aligned}$$

The equation of motion for B^μ is simply $B^\mu = 0$, so we can set this field to zero and retrieve our original supersymmetric action.

Now we have a supersymmetric action we can go through the whole analysis again to get the spectrum. First we will take a look at the classical constraints. Then the constraints will be quantised in two different ways (covariant and light-cone) and after that we will impose them on the states and see that there are more constraints needed to get a sensible theory out of it.

The algebra of the light-cone components of the calculated supercurrent (13) is given by:

$$\begin{aligned} \{J_+(\sigma), J_+(\sigma')\} &= \pi\delta(\sigma-\sigma')T_{++}(\sigma) \\ \{J_-(\sigma), J_-(\sigma')\} &= \pi\delta(\sigma-\sigma')T_{--}(\sigma) \\ \{J_+(\sigma), J_-(\sigma')\} &= 0 \end{aligned}$$

As in the bosonic case the action (12) we are working with is a gauge fixed action. And because we are now working with a supersymmetric theory we expect to gauge fix not only bosonic but also fermionic degrees of freedom. The auxiliary fields of which we will gauge fix the degrees of freedom are the so called zweibein e^a_α (the vielbein on the world-sheet) and the Rarita-Schwinger field X_α. As should be expected we can gauge away this fields (i.e. $e^a_\alpha = \delta^a_\alpha, X_\alpha = 0$) with the help of the symmetries present in the Lagrangian (i.e. two world-sheet reparametrizations, one local Lorentz, one Weyl scaling, two supersymmetries and two superconformal symmetries). In the end we then get back our action (12), but from the equations of motion of the vielbein and the Rarita-Schwinger field we get the constraints:

$$0 = J_+ = J_- = T_{++} = T_{--}$$

These are the so called super-Virasoro constraints which lead to setting to zero the timelike components of ψ^μ and X^μ.

3.1.1 Boundary conditions

Apart from applying the constraints we also need to impose boundary conditions to get a physical solution for the fields on the strings. The analysis of the 'normal' bosonic X^μ coordinate is the same as in the non-supersymmetric theory, with boundary conditions for open and closed string etc.. Let's take a look at the fermionic coordinates. Vanishing of the surface terms requires that $\psi_+\delta\psi_+ - \psi_-\delta\psi_- = 0$ at each end of the string, so we should have $\psi_+ = \pm\psi_-$. Without loss of generality we can set $\psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau)$, because the relative sign between these two is a matter of convention.

Now we can impose two different boundary conditions:

- Ramond (R) boundary conditions: $\psi_+^\mu(\pi,\tau) = \psi_-^\mu(\pi,\tau)$
- Neveu-Schwarz (NS) boundary conditions: $\psi_+^\mu(\pi,\tau) = -\psi_-^\mu(\pi,\tau)$

These conditions give for the open string the following mode expansions of the Dirac equation:

- Ramond: $\psi_\pm^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\in Z} d_n^\mu e^{-in(\tau\pm\sigma)}$
- Neveu-Schwarz: $\psi_\pm^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in Z+1/2} b_r^\mu e^{-ir(\tau\pm\sigma)}$

and for closed strings these give the follwing expansions:

- Ramond:

$$\begin{aligned}
\psi_-^\mu(\sigma,\tau) &= \sum_{n\in Z} d_n^\mu e^{-2in(\tau-\sigma)} \\
\psi_+^\mu(\sigma,\tau) &= \sum_{n\in Z} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)}
\end{aligned}$$

- Neveu-Schwarz:

$$\psi_-^\mu(\sigma,\tau) = \sum_{n\in Z+1/2} b_r^\mu e^{-2ir(\tau-\sigma)} \tag{14}$$

$$\psi_+^\mu(\sigma,\tau) = \sum_{n\in Z+1/2} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)} \tag{15}$$

$$\tag{16}$$

So we now still have the well known bosonic α_m^μ modes which now are accompanied by the b_r^μ's which are bosonic too (and right-moving). The d_m^μ are the right moving fermionic modes. In the closed string case we also have the $\tilde{d}_n^\mu$ and the $\tilde{b}_r^\mu$ left-moving modes.

We can make four different combinations of these left- or right-moving modes: NS-NS, NS-R, R-NS, R-R. The first and the last of these combinations describe closed string states that are bosons, and the other two describe fermions.

Also the constraints have to be quantized. And if we want to write down the super-Virasoro operators, the Fourier transforms of $T_{\alpha\beta}$ and J_α, we get:

- for the open bosonic strings:

$$L_m = \frac{1}{\pi}\int_\pi^\pi d\sigma e^{im\sigma}T_{++}$$

- for the open super strings:

$$\begin{aligned}\text{R: } F_m &= \frac{\sqrt{2}}{\pi}\int_\pi^\pi d\sigma e^{im\sigma}J_+ \\ \text{NS: } G_r &= \frac{\sqrt{2}}{\pi}\int_\pi^\pi d\sigma e^{ir\sigma}J_+\end{aligned}$$

For the closed strings you have similar expressions, but a additional set of generators in terms of T_{--} and J_-.

3.1.2 Covariant quantization

Now we can turn on to quantizing the superstring. This time we first take a look at the covariant quantization, because in the end we want to make a connection between the covariant and the light-cone gauge approach. The commutators for the coordinates and their Fourier coefficients are respectively:

$$\begin{aligned}[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] &= -i\pi\delta(\sigma-\sigma')\eta^{\mu\nu} \\ [\alpha_m^\mu, \alpha_n^\nu] &= m\delta_{m+n}\eta^{\mu\nu} \\ \{\psi_A^\mu(\sigma,\tau), \psi_B^\nu(\sigma',\tau)\} &= \pi\delta(\sigma-\sigma')\eta^{\mu\nu}\delta_{AB} \\ \{b_r^\mu, b_s^\nu\} &= \eta^{\mu\nu}\delta_{r+s} \\ \{d_m^\mu, d_n^\nu\} &= \eta^{\mu\nu}\delta_{m+n}\end{aligned} \tag{17}$$

Here we denote with $A, B, \ldots$ world-sheet spinor indices. We again get the condition for the mass from the zero-frequency constraint (with again a normal ordering constant a).

$$\alpha' M^2 = N + a$$

Here $N = N^\alpha + N^d$ or $N = N^\alpha + N^b$ (depending on the choice of the boundary conditions) where

$$
\begin{aligned}
N^\alpha &= \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m \\
N^d &= \sum_{m=1}^{\infty} m d_{-m} \cdot d_m \\
N^b &= \sum_{r=1/2}^{\infty} r b_{-r} \cdot b_r
\end{aligned}
\tag{18}
$$

With similar tilded operators for the left moving part in the closed-string case. The quantized Virasoro operators are now given by:

$$
\begin{aligned}
L_m &= L_m^{(\alpha)} + L_m^{(b)} \text{ (NS)} \\
L_m &= L_m^{(\alpha)} + L_m^{(d)} \text{ (R)}
\end{aligned}
$$

where

$$
\begin{aligned}
L_m^{(\alpha)} &= \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{-n} \cdot \alpha_{m+n} : \\
L_m^{(b)} &= \frac{1}{2} \sum_{r=-\infty}^{\infty} (r + \frac{1}{2} m) : b_{-r} \cdot b_{m+r} : \\
L_m^{(d)} &= \frac{1}{2} \sum_{n=-\infty}^{\infty} (n + \frac{1}{2} m) : b_{-n} \cdot b_{m+n} : \\
G_r &= \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot b_{r+n} (NS) \\
F_m &= \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot d_{m+n} (R)
\end{aligned}
$$

The L_m's are the Fourier modes of T_{++} and the G_r and F_m of the J_+ under the different boundary conditions. And with this we can write down the super-Virasoro algebra in the bosonic (i.e. NS) sector:

$$
\begin{aligned}
[L_m, L_n] &= (m-n) L_{m+n} + A(m) \delta_{m+n} \\
[L_m, G_r] &= (\frac{1}{2} m - r) G_{m+r} \\
\{G_r, G_s\} &= 2 L_{r+s} + B(r) \delta_{r+s}
\end{aligned}
$$

and for the fermionic (R) sector:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + A(m)\delta_{m+n} \\
[L_m, F_n] &= (\frac{1}{2}m - n)G_{m+n} \\
\{F_m, F_n\} &= 2L_{m+n} + B(r)\delta_{m+n}
\end{aligned}
$$

Where in both cases the $A(m)$ and $B(r)$ are anomaly terms. We now impose the constraints by requiring physical states $|\phi\rangle$ to satisfy

$$
\begin{aligned}
G_r|\phi\rangle &= 0 \text{ for } r > 0 \\
L_n|\phi\rangle &= 0 \text{ for } n > 0 \\
(L_0 - a)|\phi\rangle &= 0
\end{aligned}
$$

Again we search for the value of a for which we get a consistent theory. It turns out that these values for a and D are respectively 1/2 and 10, which again can be seen better in the Light-Cone gauge quantization where it follows from demanding the theory to be Lorentz invariant (as in the bosonic case). So let's take a look at the Light-Cone quantization.

3.1.3 Light-Cone Gauge quantization

In the light cone gauge quantization of the bosonic theory we used the residual reparametrization invariance to gauge away the + components of all the nonzero modes, while preserving the covariant gauge. For the X^+ coordinate this is still possible in the supersymmetric case. In the supersymmetric theory we, besides that, use also the freedom of gauge choice preserving local supersymmetric transformations. With the help of these we turn out to be able to gauge away ψ^+ completely. So we set $\psi^+ = 0$.

If we rewrite the constraints in terms of the light cone coordinates and with taking into account our gauge choice, we can express the - ('minus') components in terms of the i components:

$$
\begin{aligned}
\alpha_n^- &= \frac{1}{2p^+}\sum_{i=1}^{D-2}(\sum_{m=-\infty}^{\infty} : \alpha_{n-m}\alpha_m : \\
&+ \sum_{r=-\infty}^{\infty}(r - n/2) : b_{n-r}^i b_r^i :) - \frac{a\delta_n}{2p^+} \\
b_r^- &= \frac{1}{p^+}\sum_{i=1}^{D-2}\sum_{s=-\infty}^{\infty}\alpha_{r-s}^i b_s^i
\end{aligned}
$$

After taking a look at both ways of quantizing we want to proof that these two ways give us the same theory. This can be done by the so called 'No-Ghost Theorem' which show that for $D = 10$ and $a = 1/2$ the manifestly covariant way of quantising (which is however not manifestly ghost free) is actually equivalent to the light-cone gauge quantization and therefore ghost free. With 'ghosts' are meant states with negative norm. One can show that these states are not present by introducing DDF operators, which describes the physical transverse excitations. With these operators they form DDF states which are 'spurious' (i.e. states $|\phi\rangle$ for which hold that $(L_0 - 1)|\phi\rangle = 0$ and $\langle\phi|\psi\rangle = 0$ for physical states $|\psi\rangle$) and physical ($L_m|\phi\rangle = 0$ and $(L_0 - a)|\phi\rangle = 0$ for $m > 0$). By showing that the complete basis of the bosonic and fermionic modes can be expressed by DDF states together with their orthogonal complements and that acting on these states with L_m and G_r again gives DDF states they show that there are no ghosts in this theory.

3.1.4 GSO conditions

But the model as described above gives an inconsistent quantum theory. In particular we still have the tachyons in our spectrum. To get rid of this and some other problems, we mod out the theory by a discrete symmetry, by the so called GSO projection. The GSO projection tells us to only keep the states with even quantum number $(-1)^F$ under which Fermi fields ψ^μ are odd and Bose fields X^μ even. So a general state has $(-1)^F = (-1)^n$ but we only keep the states of $(-1)^F = +1$. So we only keep the singlets of the symmetry. Modding out a theory in this way will give us a consistent theory.

One nice aspect of the GSO projection is that it gives us a supersymmetric theory in ten dimensions. You can show that if we impose both Weyl and Majorana constraints we get a theory with the same amount of fermionic degrees of freedom as bosonic degrees of freedom, which is a necessary condition for a supersymmetric theory. It turns out to be possible in a 10 dimensional theory to apply these constraints at the same time. The proof of supersymmetry is not yet given, but it is an "encouraging indication".

3.2 Green-Schwarz formalism

In the formalism described above it is not that clear that we indeed end up with a supersymmetry on the target space. And also the idea behind and the subtleties of the GSO projection are not that simple. Luckily there is another formalism which is manifest target space supersymmetric and which turns to be equivalent to the former. This is the Green-Schwarz (GS) formalism.

In the Green-Schwarz formalism we work with an action which is explicit invariant under supersymmetry. We will introduce an action with a lot of symmetries. These symmetries we will use for fixing the Light-cone gauge and for choosing the right representations for our fermionic particles in order to get an equal amount of fermionic and bosonic degrees of freedom. So let us take a look first at the symmetries which are present in the supersymmetric theory. Before going to a theory of superstrings we first will take a look at the classical superparticle.

3.2.1 The classical superparticle

For the classical superparticle we can write down an action which is the simplest generalization of the action for the bosonic point particle which is still Lorentz invariant:

$$S = \frac{1}{2}\int e^{-1}(\dot{x}^\mu - i\bar{\theta}^A)^2 d\tau$$

Here e, the vielbein on the worldline, is an auxilairy field. This action has a lot of symmetries:

- Local reparametrization $\tau \to f(\tau)$
- Poincaré

$$\begin{aligned} \delta x^\mu &= a^\mu + b^\mu_\nu x^\nu \\ \delta e &= 0 \end{aligned}$$

- Supersymmetry

$$\begin{aligned} \delta\theta^A &= \epsilon^A \\ \delta x^\mu &= i\bar{\epsilon}^A\Gamma^\mu\theta^A \\ \delta\bar{\theta}^A &= \bar{\epsilon}^A \\ \delta e &= 0 \end{aligned}$$

- κ-symmetry (local fermionic symmetry)

$$\begin{aligned} \delta\theta^A &= i\Gamma\cdot p\kappa^A \\ \delta x^\mu &= i\bar{\theta}^A\Gamma\delta\theta^A \\ \delta e &= 4e\dot{\bar{\theta}}^A\kappa^A \end{aligned}$$

- Local bosonic symmetry

$$\begin{aligned} \delta\theta^A &= \lambda\dot{\theta}^A \\ \delta x^\mu &= i\bar{\theta}^A\Gamma^\mu\delta\theta^A \\ \delta e &= 0 \end{aligned}$$

This local bosonic symmetry is not independent in that sense that it is not implying any new conditions beyond those that follow from the other symmetries. It doesn't change the on-shell number of degrees of freedom. But what is this κ-symmetry? This κ-symmetry can be seen in the equations of motion which we can compute for the given action:

$$p^2 = 0,\ \dot{p}^\mu = 0,\ \Gamma\cdot p\dot{\theta}$$

where $p^\mu = \dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A$. In the e.o.m's θ always appears in the combination $\Gamma \cdot p$, while this matrix (Γ can be seen as a vector with matrices in it) has only half the maximum rank because $(\Gamma \cdot p)^2 = -p^2 = 0$. So for half of the components of θ it does not really matter which value they take, the are 'decoupled' from theory. So the presence of the κ-symmetry costs half of the components of the fermionic coordinate.

3.2.2 The superstring

In analogy of the classical superparticle we can now formulate the action for a classical superstring. But the first guess we would make, i.e. the simplest Lorentz invariant supersymmetric action for a string, doesn't have the κ-symmetry which is present in the superparticle case. For this reason the θ then would describe twice as much degrees of freedom as in the particle case. This in principle is not a problem but it nevertheless turns out that we need this κ-symmetry to get the right amount of fermionic degrees of freedom.

It turns out to be possible to add a term to our first guess to make the action κ-symmetric. But this doesn't work for a general theory. It puts some constraints on our theory, in particular on the number of supersymmetries present in the theory, namely $N \leq 2$. Where N denotes the number of super symmetry parameters. After adding this term the action becomes:

$$\begin{aligned} S &= \frac{1}{2\pi}\int d2\sigma \Big[-\sqrt{h}h^{\alpha\beta}\Pi_\alpha \cdot \Pi_\beta \\ &+ 2\left\{-i\epsilon^{\alpha\beta}\partial_\alpha X^\mu \left(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2\right)\right. \\ &+ \left.\epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2\right\}\Big] \end{aligned} \tag{19}$$

The term between {} is the term which had to be added to get back the κ symmetry. This term is a so-called 'topological term' and does, due to the absence of $h_{\alpha\beta}$, not contribute anything to the energy-momentum tensor and has no influence on any of the other symmetries present in the action.

Now you can in principle check that there is $\mathcal{N} = 2$ supersymmetry in this action, but only under one of the following four conditions:

- D = 3 and θ is Majorana
- D = 4 and θ is Majorana or Weyl
- D = 6 and θ is Weyl
- D = 10 and θ is Majorana-Weyl

Only in these specific cases a term contributing to the variation of the action (19) becomes zero. In the end, by further conditions following from the quantization, we will see that from these four options the last one is singled out.

Now we can take a look at the symmetries of the classical superstring. By construction the action has the following symmetries, for the given conditions on the space-time dimension and the spinor representations:

- Local reparametrization
- Weyl invariance
- κ-symmetry (+ local bosonic symmetry)
- Poincaré

The first three symmetries are symmetries of the world-sheet which we will use to make a gauge choice for our metric: $h_{\alpha\beta} = \eta_{\alpha\beta}$. The last one is a symmetry in target space. But with this gauge choice we don't use all the freedom we had to make specific choices. It turns out that besides that we are able to enforce the following conditions:

- $\Gamma^+\theta^1 = \Gamma^+\theta^2 = 0$ (Light-cone gauge choice)
- $X^+(\sigma, \tau) = x^+ + p^+\tau$ (all α_n^+ for $n \neq 0$ set to zero)

In this way are able to reduce the components of θ from 32 complex components (a Dirac spinor in 10 dimensions) by first imposing the Majorana-Weyl condition ($\rightarrow$ 16 real components) and then the light-cone gauge condition to 8 real components. We then can identify these 8 components with the transversal directions and put them in an eight dimensional representation of the SO(8) group.

3.3 Type II superstring theory

In the given Lagrangian (19) we used two different θ's, θ^1, θ^2. These different θ's originate from the fact that this theory has two different supersymmetric transformations, with two different parameters ϵ^1, ϵ^2. These two different transformations lead to two sets of eight components which we can put in two different representations of the SO(8) group. The SO(8) group has actually three different eight-dimensional irreducible representations, the vector representation 8_v and two unequivalent spinor representations 8_s and 8_c.

It is a 'coincidence' that in the case of SO(8) the vector representation has as much components as the spinor representation. For general groups this is not the case, and this 'coincidence' goes under the name of 'triality'. Having three different representations means that these three different sets of eight numbers transform different under SO(8) transformations because they come along with different sets of generators. A group element of SO(8) can be represented by

$$g = e^{\overrightarrow{\alpha} \cdot \overrightarrow{T}_R}$$

Here $\overrightarrow{\alpha}$ are the parameters and $\overrightarrow{T}_R$ the generators belonging to a specific representation. The representations define the rules under which they transform, the parameters can be freely chosen. But it is not possible to find a specific set $\overrightarrow{\alpha}$ to make a linear combination of generators belonging to one of the representations which forms all the generators belonging to another representations.

So we can put our coordinates in three different representations:

- $X^{i=1,\ldots,8}$ (8_v)
- $\theta^1_{a=1,\ldots,8} \rightarrow S_{a=1,\ldots,8}$ (8_s) or $S_{\dot{a}=1,\ldots,8}$ (8_c)
- $\theta^2_{a=1,\ldots,8} \rightarrow \tilde{S}_{a=1,\ldots,8}$ (8_s) or $\tilde{S}_{\dot{a}=1,\ldots,8}$ (8_c)

We go from θ to S by multiplying with the number $\sqrt{p^+}$, just for convenience.

S denotes the right moving modes, $\tilde{S}$ the left moving ones. We can make different choices for which representation we want to use. This is connected with the choice we make for the chiralities of the fermions of our theory. It doesn't matter in which representation we put S, what matters is where we put $\tilde{S}$ after making a choice for S. So let's put S in 8_s. We have two different choices here, either we choose them to have the same chirality (we use S_a (8_v) and $\tilde{S}_a$ (8_v)) or we choose them to have opposite chirality (S_a (8_v) and $\tilde{S}_{\dot{a}}$ (8_c)). From this choice we get two different string theories, namely the two type II theories. If we choose them to have the same chirality we end up with type IIB and if we choose them to have opposite chirality we will get type IIA.

3.4 The supersymmetric string spectrum

As said before we are only interested in the massless states of the spectrum because for very small strings (i.e. in low energy limits) the masses of the massive states will become too big to be of physical relevance. In the supersymmetric case we don't have the tachyon, so the lowest state is the massless state. This massles state is degenerate and we can find all of them by applying the fermionic zero mode operators S_0^a (and $\tilde{S}_0^a$ in the closed string case). Only these modes give massless states because they are the only ones which are commuting with the mass operator:

$$M^2 = \frac{2(N+\tilde{N})}{\alpha'} \tag{20}$$

with

$$N \equiv \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + nS^a_{-n}S^a_n$$

and

$$\tilde{N} \equiv \sum_{n=1}^{\infty} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + n\tilde{S}^a_{-n}\tilde{S}^a_n$$

the level operators for the right- and left-moving excitations. Only the fermionic zero modes commutes with this operator because of the presence of the n in the 'fermionic part' of the level operators, which is zero for the lowest level.

In principle we should now be able to build the spectrum by making creation- and annihilation-operators out of the modes. But because all the modes are real we can't do that. For that reason we first decompose the SO(8) group in SU(4) $\times$ U(1) by making complex combinations of the modes ($S_0^1 \pm iS_0^2 \to b_1, b_1^\dagger$ etc.). With this operators we can make the spectrum which we can interpret if we undo the complexification to get back our SO(8) representations.

It in the end turns out that for the right-moving sector the NS sector of the RNS-formalism transforms like a (8_v) and the R sector like a (8_c). So the right-moving sector can be put in a $(8_v \oplus 8_c)$. It turns out that the left-moving sector generated by $\tilde{S}^{\dot{a}}$ furnishes a $(8_v \oplus 8_s)$. The total spectrum then is generated by the product of two of these. Depending on our choice of the chiralities this gives us $(8_v \oplus 8_c) \otimes (8_v \oplus 8_c)$ (Type IIB) or $(8_v \oplus 8_c) \otimes (8_v \oplus 8_s)$ (Type IIA).

The NS-NS sector which is giving bosonic states is common to both the two types: $(8_v \otimes 8_v) = 35_v \oplus 28_v \oplus 1 \to g_{\mu\nu} \oplus B_{\mu\nu} \oplus \phi$. Where $g_{\mu\nu}$ is the graviton (the metric) which is symmetric and traceless, the 2-form antisymmetric $B_{\mu\nu}$ and the dilaton ϕ which is a scalar. These three fields are called the universal bosonic sector because they appear in all the five different string theories.

The bosonic states of the R-R sector differs for the two types II string theory:

$$\begin{array}{rrrl} (8_c \otimes 8_s) = & 56_v \oplus 8_v \to & C_3 \oplus C_1 & (IIA) \\ (8_c \otimes 8_c) = & 35_c \oplus 28_c \oplus 1 \to & C_4 \oplus C_2 \oplus C_0 & (IIB) \end{array}$$

The fermionic content of the spectrum is given by the NS-R and the R-NS sectors. If we write down the full spectrum for the two types this gives us, for type IIA:

$$\begin{aligned} (8_v \oplus 8_c) \otimes (8_v \oplus 8_s) &= (35_c \oplus 28_c \oplus 1 \oplus 56_v \oplus 8_v)_B \\ &\quad +(56_s \oplus 8_s \oplus 56_c \oplus 8_c)_F \end{aligned}$$

and for type IIB:

$$\begin{aligned} (8_v \oplus 8_c) \otimes (8_v \oplus 8_c) &= (35_v \oplus 28_v \oplus 1 \oplus 35_c \oplus 28_c \oplus 1)_B \\ &\quad +(56_s \oplus 8_s \oplus 56_s \oplus 8_s)_F \end{aligned}$$

Part II
Supergravity

4 Ungauged supergravity

4.1 From strings to supergravity

As we have seen in section (2.3), we can derive the Einstein's equations by demanding the action of a particle in a gravitational background field to be scale invariant. We derived it in the bosonic string theory which leads to a 26-dimensional Einstein equation, but by adding supersymmetry we would end up in a 10-dimensional curved space-time. We, of course, would like to get a theory which describes our own 4-dimensional space-time. In order to get from 10 dimensions to 4, one needs to perform what is called a compactification. This compactification can be seen as rolling up some of the dimensions and in this way transforming space-time symmetries into internal, i.e. gauge symmetries. These rolled up dimensions can only be seen on a very small spatial scale, i.e. on a very high energy scale. The degrees of freedom of the fields in these dimensions get infinitely heavy when we take the limit of the radii of the compactified dimensions to zero. So for the low energy limit these degrees of freedom decouple from the theory. In that sense theories with these compactified dimensions are the low energy limit of certain string theories.

We will not go in any detail about these compactifications, but there are many ways to perfom one. The simplest possibility is the compactification of a $(D+n)$ space-time on a n-Torus. An n-torus is an n dimensional generalisation of the circle and the 2 dimensional torus (the well-known donut-shaped object embedded in three dimensional space).

But by using this n-torus we will not end up with the model we want to reproduce, namely the standard model combined with general relativity. We do not get the right gauge symmetries. Instead in general we get much simpler symmetry groups, in particular we end up with only a $U(1)^n$ gauge symmetry. To solve this problem, one can compactify on more complicated manifolds (on spheres S^n for example) or add fluxes to the higher dimensional theory. These fluxes are the analogues of the classical fluxes in electromagnetism, but are self maintaning (without the presence of charges) due to non trivial cycles of the manifold.

One difference between compactifications on T^n and on more complicated manifolds is that in the last case in general one gets non-abelian gauge symmetries under which the fields are charged, which are the so-called gauged supergravities. The theory one gets when compactifying on the n-torus is called 'ungauged'. These non-abelian gauge symmetries are one of the main reasons to be interested in gauged supergravities because if we want te reproduce the symmetries of the standard model we will need these bigger gauge symmetry groups.

Another interesting aspect is that these theories have a scalar potential, which is interesting because this scalar potential could support an effective cosmological constant, give mass terms for the fields, lead to spontaneous supersymmetry breaking, etc.. From a supergravity point of view these scalar potential terms are needed to restore supersymmetry.

Instead of doing compactifications of the more complicated manifolds or in the presence of fluxes, we can also first compactify on an n-Torus and then gauge this theory. In this way you also end up with gauged supergravity theories, where the deformation parameters of the more complicated manifolds are incorporated as gauge parameters in the gauging procedure. This is the path we will take, and in this chapter the ungauged supergravity theory is discussed.

To get a bit more feeling for what is happening when we compactify a theory, we can follow the sketch of Samtleben in [8] of the reduction of the pure gravity case in $(D+n)$ dimensions on an n-torus to D dimensions. Here we will use the following indices:

	curved indices (manifold)	flat indices (tangent space)
(D+n) dim. space time	M, N = 0, ..., D+n-1	A, B = 0, ..., D+n-1
D dim. space time	$\mu, \nu = 0, \ldots,$ D-1	$\alpha, \beta = 0, \ldots,$ D-1
n circles (T^n)	m = 1, ... n	a,b = 1, ..., n

In compactifying dimensions we for all fields take a normal mode expansion on an n-torus with radii $R_1, \ldots, R_n$

$$\phi(x,y) = \sum_{k_1,\ldots,\kappa_n \in \mathbb{Z}} \phi_{k_1 k_2 \ldots k_n}(x) e^{2\pi i k_1 y^1/R_1} e^{2\pi i k_2 y^2/R_2} \cdots e^{2\pi i k_n y^n/R_n} \tag{21}$$

where x, y are coordinates in the reduced space-time and the internal manifold respectively. Because M^2 of these fields are proportional to $1/R_i^2$, for $k_1, \ldots k_n \neq (00\ldots0)$ they become infinitely heavy when the limit of the R's going to zero is taken. So the fields with $k_1, \ldots, k_n \neq (00\ldots0)$ are decoupled and the left over fields are independent of y^m.

Using the Lorentz invariance in (D+n) dimensional space-time we can write its vielbein ${E_M}^A$ in terms of the D dimensional vielbein ${e_\mu}^\alpha$ and two matrices ${\mathcal{V}_m}^a$ and ${B_\mu}^m$ as

$${E_M}^A = \begin{pmatrix} \underset{(D\times D)}{e^{\kappa\phi}{e_\mu}^\alpha} & \underset{(D\times n)}{e^{\phi/n}{\mathcal{V}_m}^a {B_\mu}^m} \\ 0 & \underset{(n\times n)}{e^{\phi/n}{\mathcal{V}_m}^a} \end{pmatrix} \tag{22}$$

With $\kappa = -\frac{1}{D-2}$. In D dimensional space ϕ, ${\mathcal{V}_m}^a$ are scalar fields and ${B_\mu}^m$ are n vectors fields. Putting ${E_M}^A$ in this form breaks $SO(1, D+n-1)$ down to $SO(1, D-1) \times SO(n)$. The $SO(1, D-1)$ acts as a D dimensional Lorentz transformation on ${e_\mu}^\alpha$ and the $SO(n)$ is a local symmetry on $\mathcal{V}$. We can use this symmetry to remove $\frac{1}{2}n(n-1)$ components of $\mathcal{V}$. Also $\mathcal{V} \in SL(n)$, as we see later on, so the scalar fields transform under this global $SL(n)$ and the local $SO(n)$ as:

$$\delta\mathcal{V} = \Lambda\mathcal{V} + \mathcal{V}k(x) \tag{23}$$

We will come back to this later.

A diffeomorphism on $E_M{}^A$ is given by

$$\delta E_M{}^A = \xi^N \partial_N E_M{}^A + E_N{}^A \partial_M \xi^N . \tag{24}$$

From this we get under dimensional reduction again diffeomorphisms on the fields in D dimensions. But for the part with $\xi^m(x)$ (parameters living in the dimensions of the T^n but depending on the coordinates of the D dimensions) we get a transformation on $B_\mu{}^m$ of the following form:

$$\delta B_\mu{}^m = \partial_\mu \zeta^m(x) \tag{25}$$

while the graviton and the scalars are not transforming. This is the transformation of a $U(1)^n$ symmetry with gauge fields $B_\mu{}^m$. Then we have diffeomorphisms of the form $\xi^m(y) = g^m{}_n y^n$ which induces the global $SL(n)$ transformation on the scalar fields given in (23) and one on the vector fields given by:

$$\delta B_\mu{}^m = -g^m{}_n B_\mu{}^n \tag{26}$$

Here we have taken the matrix $g^m{}_n$ to be traceless: $g^m{}_m \equiv 0$. This trace is also acting on the vielbein $e_\mu{}^\alpha$ and has for that reason to be accompanied by a Weyl scaling to get a proper of-shell symmetry:

$$\delta\phi = n(D-2), \quad \delta B_\mu{}^m = -(D+n-2)B_\mu{}^m \tag{27}$$

These are all the symmetries inherited from the diffeomorphisms in $(D+n)$ dimensions. This case was for pure supergravity. If we add more supersymmetry, and thus more p-forms, we end up with bigger global symmetry groups as we will see later for $\mathcal{N} = 4$.

4.2 Minimal supergravity

Supergravity is a supersymmetric theory containing a super with a spin-2 state in it which can be identified with the gravitational field. To let this theory really be about gravity we need to get invariance under diffeomorphisms. This request leads in the non-supersymmetric case to a dynamical metric and gives us General Relativity. It turns out that by making a supersymmetry of a theory local, we force it to be invariant under diffeomorphisms. This can be seen by considering the local supersymmetric transformations on the bosonic (B) and fermionic (F) field. Schematically these are as follows:

$$\delta_\epsilon B = \bar{\epsilon}(x)F,\ \delta_\epsilon F = \epsilon(x)\partial B \rightarrow [\delta_{\epsilon_1}, \delta_{\epsilon_2}]B = (\bar{\epsilon}_1\gamma^\mu\epsilon_2)(x)\partial_\mu B \tag{28}$$

Here the right hand side is a local diffeomorphism in space-time. So a local supersymmetric theory automatically includes local diffeomorphism invariance and thus will have a dynamical metric and includes gravity in this way.

If we consider a $\mathcal{N} = 1$ supergravity theory (i.e. a theory where we have only one supersymmetric parameter) the graviton is forming a supermultiplet with a spin-$\frac{3}{2}$ field ψ_μ^a which has a spinor as well as a vector index. This field is called the 'gravitino' and such a supermultiplet is called simply a supergravity multiplet. Besides being the superpartner of the graviton it is also the gauge field coming from the gauging of the supersymmetry. We do expect a spin-$\frac{3}{2}$ state to be the gauge field because the supersymmetry transformation has a spin-$\frac{1}{2}$ parameter.

Due to the presence of the gravitino we have problems to describe the theory on a curved background because transformations of spinors are not well defined there. The solution to this problem goes by introducing the vielbein e_μ^a which takes lower space-time curved vector indices μ to lower flat 'Lorentz' indices a.

The vielbein is defined as

$$e_\mu{}^a(x_0) := \frac{\partial y^a(x_0;x)}{\partial x^\mu}\Big|_{x=x_0} \tag{29}$$

Where $y^a(x_0;x), a = 0,\ldots,3$ denotes a coordinate frame which is inertial at space-time point x_0. The vielbein is related to the metric by:

$$g_{\mu\nu}(x) = e_\mu{}^a(x)e_\nu{}^b(x)\eta_{ab} \tag{30}$$

with $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ the Minkowski metric[1].

4.2.1 The Lagrangian

The full supersymmetric lagrangian is be given by [8]:

$$\begin{aligned}\mathcal{L}_0[e,\psi,\omega] = \; & -\frac{1}{4}|e|e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5 D_\rho\psi_\sigma \\ & -\frac{1}{4}|e|(K_a{}^{ac}K_b{}^b{}_c + K^{abc}K_{cab})\end{aligned} \tag{31}$$

Where

[1]For more on the Vielbein formalism, see e.g. [11]

$$K^{a}_{\ \mu}{}^{b} = -i(\bar{\psi}^{[a}\gamma^{b]}\psi_\mu + \frac{1}{2}\bar{\psi}^a\gamma_\mu\psi^b) \tag{32}$$

This Lagrangian can be derived in different ways, for example by starting with a global supersymmetric theory and then applying the Noether procedure to it. A point of attention here is that because of the relation between supersymmetry transformations and the diffeomorphisms (28) we will have a dynamical metric in our theory. For that reason we have to introduce not only a kinetic term for the gravitino field but also for the gravitational field in order to get a supersymmetric theory because they live together in a superdoublet. This makes the procedure a bit lengthy, but it can be done. We will just state here the result and refer to [8] for a derivation (though he is also not deriving the lagrangian by the full Noether procedure).

The first term of 31)is the well known Einstein-Hilbert term rewritten in vielbein quantities. By setting ψ to zero we would get from this the Einstein equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$. The second term denotes a kinetic term for the gravitino ψ, with in the third term explicitly the terms quartic in the fermions which come along with introducing the second term.

The supersymmetry transformations are given by:

$$\begin{aligned} \delta_\epsilon e^a_\mu &= -\bar{\epsilon}\gamma^a\psi_\mu \\ \delta_\epsilon \psi_\mu &= \hat{D}_\mu \epsilon \end{aligned} \tag{33}$$

with $\hat{D}_\mu = D_\mu(\hat{\omega})$, where D_μ is the covariant derivative given by:

$$D_\mu = \partial_\mu + \frac{1}{2}\omega^{ab}_\mu M_{ab} \tag{34}$$

This derivative is covariant under local Lorentz transformations, i.e. under rotations and boosts of our local Lorentz frame. $\hat{\omega}$ then is defined as:

$$\hat{\omega}_\mu{}^{ab} = \hat{\omega}_\mu{}^{ab}[e,\psi] = \omega_\mu{}^{ab}[e] + K^{a}_{\ \mu}{}^{b}, \tag{35}$$

ω^μ_{ab} are the gaugefields of the gauging of the Lorentz group (which together are called the spin connection) and M_{ab} the generators of these transformations. The spin connection is defined by demanding the invariance of the vielbein under the local Lorentz transformations:

$$D_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda = 0 \rightarrow D_{[a} e^a_{\nu]} = \frac{1}{2}T^a_{\mu\nu} \tag{36}$$

With $T^a_{\mu\nu}$ the torsion tensor. This equation can be solved for ω which is, in case of zero torsion, only depending on the vielbein. For the full supersymmetric action (31) the torsion is non-zero due to the presence of the second term in (31), the kinetic term for ψ. Then the solution is given by (35).

In principle it is possible to show that the given lagrangian (31) has a local supersymmetry and thus gives really a supergravity theory. But because of the presence of $K^{a}{}_{\mu}{}^{b}$ and its dependence on ψ this is quite a tough task. For this reason other formalisms are developed to make the computations easier. The most common used one is called the "1.5th order formalism" because it uses aspect of the second order formalism (doing the full calculation starting from (31)) and the first order formalism where the spin connection is treated as independent field.

We can shortly state the idea behind the first order formalism. As we stated, ω is a dependent field. It is totally determined by the values of the vielbein. Nevertheless you can, as a formal trick, consider ω as a independent field and then you will get two first order field equations instead of the second order Einstein equation. We then still get the Einstein equations but these become linear because we express it in terms of ω and its derivative. The second equation we will find, gives us precisely (36) for zero torsion.

Together with the Lorentz transformations (M_{ab}) and the translations, the supersymmetry transformations (33) form a supersymmetry algebra. This supersymmetry algebra should close on the fields of our theory e, ψ, ω. This turns out to give constraints on the gravitino field which can be interpreted as equations of motion. This means that the supersymmetry algebra only closes on-shell. Schematically this super-Poincaré algebra is given by:

$$
\begin{aligned}
[M, M] &\propto M, & [P, M] &\propto P, \\
[M, Q] &\propto Q, & \{Q, Q\} &\propto P
\end{aligned}
\tag{37}
$$

4.3 Extended supergravity

Untill now we discussed supergravity theories with only one supersymmetric parameter, the so called 'minimal supersymmetric' theories, denoted by the number of supersymmetric generators: $\mathcal{N} = 1$. By extending the number of supersymmetries, we have to use bigger supermultiplets. In the end we are interested in the 4 dimensional, half maximal supergravity theory. Maximal supersymmetry theories in 4 dimensions have $\mathcal{N} = 8$, so half-maximal means $\mathcal{N} = 4$.

In the $\mathcal{N} = 1$ case we got a multiplet with in it a graviton and a gravitino. However we also could have made multiplets with other field content, for example a vector multiplet (χ^M, A_μ^M) with one spin-1 gauge field A_μ^M and one spin-1/2 fermions χ^M for each M, or the gravitino multiplet which consists of the gravitino and a spin-1 field.

However adding this last multiplet to a supergravity multiplet implies introducing another local supersymmetry. We then get an $\mathcal{N} = 2$ supergravity theory, with a supermultiplet which is the combination of the two supergravity $\mathcal{N} = 1$ multiplets, consisting of one spin-2, two spin-3/2 and one spin-1 fields. To get to $\mathcal{N} = 4$ we have to do this trick another two times to get a multiplet with one spin-2, four spin 3/2, six spin-1, four spin 1/2 and one spin-0 fields.

Apart from this supergravity multiplet we also could think of other kinds of multiplets in $\mathcal{N} = 4$, for example the vector multiplet consisting of one field with helicity 1 and -1, four with helicity 1/2 and -1/2, six spin-0 fields. The possible multiplets are limited by the fact that we are not able to write down a consistent theory in 4D containing particles with higher spin than 2. In particular, this is the reason why we can not have a theory with more supersymmetry than $\mathcal{N} = 8$, and that for $\mathcal{N} = 8$ there exists only one possible multiplet.

4.3.1 $\mathcal{N} = 4, D = 4$

In our $\mathcal{N} = 4$ supergravity theory in four dimensions, we have a supergravity multiplet and we can add as many vector multiplets as we want. From the compactification procedure follows that het global symmetry group for this theory is $G = SL(2) \times SO(6, 6+n)$. Here n is the number of vector multiplets we added. For (half) maximally theories the scalars end up in a non-linear representation of the global symmetry group which is isomorphic to a G/K coset space symmetry group. This follows from the compactification procedure as we saw in section (4.1) and origninates from the inheritance of the local symmetry group from the higher dimensional Lorentz symmetry. The coset space where the scalars of the four dimensional half maximal supergravity theory live in is given by

$$G/K = \frac{SL(2)}{SO(2)} \times \frac{SO(6, 6+n)}{SO(6) \times SO(6+n)} \tag{38}$$

It appears to be convenient to put the scalars in a matrix which is an element of $G : \mathcal{V} \in G$. From this $\mathcal{V}$ we can construct an element of the Lie Algebra of G: $J_\mu = \mathcal{V}^{-1}\partial_\mu \mathcal{V} \in \text{Lie } G$. This 'current' can be decomposed into a compact and a non-compact part:

$$J_\mu = Q_\mu + P_\mu, \; Q_\mu \in \text{Lie } K, \; P_\mu \in \text{complement of Lie } K \tag{39}$$

The compact part is modded out of the G symmetry, as can be seen at (38), so the non-compact part of J_μ is being used in the kinetic term of the action for the scalars:

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2} e \text{Tr}[P_\mu P^\mu] \tag{40}$$

which can be written equivalently in terms of

$$\mathcal{M} \equiv \mathcal{V} \Delta \mathcal{V}^T, \tag{41}$$

with Δ a constant K-invariant positive definite matrix, as

$$\mathcal{L}_{\text{scalar}} = \frac{1}{8} \text{Tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}) \tag{42}$$

which is invariant under the following global G and local K transformation:

$$\delta\mathcal{V} = \Lambda\mathcal{V} - \mathcal{V}k(x); \ \Lambda \in \text{Lie } G, \ k \in \text{Lie } K \tag{43}$$

This K-symmetry can be fixed by choosing a specific form of the matrix $\mathcal{V}$, i.e. to choose a set of representatives. But when we fix $\mathcal{V}$ we have to take care of the invariance of the action under the given transformation. To do this we have to make k dependent on Λ and on the fields in order to make sure that the transformed $\mathcal{V}$ has still the right form of our gauge choice. Making k depending on the fields makes the transformation non-linear. We also could stay with the K-symmetry without gauging it. In that case we would keep linear transformations but also would stay with a lot of unphysical degrees of freedom in our theory.

4.3.2 Coset space SL(2)/SO(2)

Let us see how this looks like for the coset space $SL(2)/SO(2)$ which is part of the coset space of the theory we are interested in. Lie $SL(2)$ has three generators, given by:

$$\mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \ \boldsymbol{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \ \boldsymbol{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{44}$$

while the only generator of Lie SO(2) is:

$$\boldsymbol{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \tag{45}$$

which of course equals $(\mathbf{e} - \boldsymbol{f})$.

Now, let us put $\mathcal{V}$ in what is called the triangular gauge, which can be done by constructing it in the following way:

$$\mathcal{V} = e^{c\boldsymbol{e}} e^{\phi\boldsymbol{h}} = \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix} = \begin{pmatrix} e^{\phi} & Ce^{\phi} \\ 0 & e^{-\phi} \end{pmatrix} \tag{46}$$

if we now would apply an operator of the form $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right) \in SL(2)$ we see that it does not stay in the gauge:

$$\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & Ce^{\phi} \\ 0 & e^{-\phi} \end{pmatrix} = \begin{pmatrix} e^{\phi} & Ce^{\phi} \\ ne^{\phi} & ne^{\phi} + e^{-\phi} \end{pmatrix} \tag{47}$$

In the matrix notation we let the $SL(2)$ generator (from the global G symmetry) working from the left, and the $SO(2)$ (the K-symmetry) from the right $\delta\mathcal{V}_{\alpha}{}^{\underline{\alpha}} = \Lambda_{\alpha}{}^{\beta}\mathcal{V}_{\beta}{}^{\underline{\alpha}} + \mathcal{V}_{\alpha}{}^{\underline{\beta}}k_{\underline{\beta}}{}^{\underline{\alpha}}$. The $SO(2)$ transformation has the form

$$e^{\omega\boldsymbol{g}} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \tag{48}$$

If we apply this one from the right to (46) we get:

$$\begin{pmatrix} e^{\phi} & Ce^{\phi} \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \tag{49}$$

$$= \begin{pmatrix} e^{\phi}\cos\omega - Ce^{\phi}\sin\omega & e^{\phi}\sin\omega + Ce^{\phi}\cos\omega \\ -e^{\phi}\sin\omega & e^{-\phi}\cos\omega \end{pmatrix} \tag{50}$$

So for $\mathcal{V}$ to get back in the right form (46) after the transformation by both $SO(2)$ and $SL(2)$, we have to relate ω and ϕ:

$$ne^{\phi} - \sin\omega e^{-\phi} = 0 \rightarrow \omega = \sin^{-1}(ne^{2\phi})$$

If we put this back in the K-transformation we see that the result indeed is a non-linear transformation due to the dependence of the transformation parameter ω on the fields ϕ.

We now can calculate J_μ and decompose it as in (39):

$$\begin{aligned} \mathcal{V}^{-1}\partial\mathcal{V} &= \begin{pmatrix} \partial_\mu\phi & e^{-2\phi}\partial_\mu C \\ 0 & -\partial_\mu\phi \end{pmatrix} \\ &= \partial_\mu\phi\boldsymbol{h} + (e^{-2\phi}\partial_\mu C)\boldsymbol{e} + 0\cdot\boldsymbol{f} \\ &= \underbrace{\partial_\mu\phi\boldsymbol{h} + \frac{1}{2}(e^{-2\phi}\partial_\mu C)(\boldsymbol{e}+\boldsymbol{f})}_{=P_\mu} + \underbrace{\frac{1}{2}(e^{-2\phi}\partial_\mu C)(\boldsymbol{e}-\boldsymbol{f})}_{=Q_\mu} \end{aligned} \tag{51}$$

where we have isolated the non-compact and the compact part respectively. Now we are able to write down the Lagrangian explicitly from (40):

$$e^{-1}\mathcal{L}_{\text{scalar}} = -\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{-4\phi}\partial_\mu C\partial^\mu C = -\frac{1}{4(\Im\tau)^2}\partial_\mu\tau\partial^\mu\tau^* \tag{52}$$

where we expressed it in terms of a complex scalar field $\tau = C + ie^{2\phi}$.

In terms of τ a SL(2) transformation can be given as:

$$\tau \rightarrow \frac{a\tau+b}{c\tau+d}, \quad \text{for } \exp(\Lambda) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2). \tag{53}$$

where we easily can recognize the non-linear character of the transformation.

4.3.3 Vectors

Vectors live just in the normal linear representation of G, in our case SL(2) $\times$ SO(6,6+n). They transform as follows:

$$\delta A_{\mu}^{M} = -\Lambda^{A}(t_{A})_{N}{}^{M} A_{\mu}^{N} \tag{54}$$

Here the $(t_A)_N{}^M$ are the generators of Lie G in the fundamental representation. In terms of the field strength $F_{\mu\nu}^{M} \equiv \partial_\mu A_\nu^M - \partial_\nu A_\mu^M$ we can give the action for the vector fields:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} e \mathcal{M}_{MN} F_{\mu\nu}^{M} F^{\mu\nu N} \tag{55}$$

In the matrix $\mathcal{M}$ the dependency on the scalar fields of the compactified field strength $F_{\mu\nu A}$ is parametrized. For general p-forms $B^{I}_{\nu_1 \cdots \nu_p}$ the action is given by:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2(p+1)!} \mathcal{M}_{IJ} F^{I}_{\nu_1 \ldots \nu_{p+1}} F^{\nu_1 \ldots \nu_{p+1} J} \tag{56}$$

4.3.4 Dualities

For the field strenght corresponding to a p-form you can define a field strength corresponding to a (D-2-p) form which is equivalent in terms of physics. This can be done only on-shell. What is called the dual field strength is defined as

$$G_{\mu_1 \ldots \mu_{(D-p-1)} I} \equiv \frac{e}{(p+1)!} \epsilon_{\mu_1 \ldots \mu_{D-p-1} \nu_1 \ldots \nu_{p+1}} \mathcal{M}_{IJ} F^{\nu_1 \ldots \nu_{p+1} J} \tag{57}$$

In terms of this field strenght the equations of motions and the Bianchi identity change role:

$$\partial^{\mu}(\mathcal{M}_{IJ} F^{J}_{\mu\nu_1 \ldots \nu_p}) = 0 \rightarrow \partial_{[\mu_1} G_{\mu_2 \ldots \mu_{D-p} I} = 0 \tag{58}$$

and

$$\partial_{[\nu_1} F^{I}_{\nu_2 \ldots \nu_{p+2}]} = 0 \rightarrow \partial^{\mu}(\mathcal{M}^{IJ} G_{\mu\nu_1 \ldots \nu_{D-p-2} J}) = 0 \tag{59}$$

Locally we then can define a $(D-p-2)$ form C_I of which G_I the fieldstrength is:

$$G_{\mu_1 \ldots \mu_{D-p-1} I} \equiv (D-p-1) \partial_{[\mu_1} C_{\mu_2 \ldots \mu_{D-p-1}] I} \tag{60}$$

In even dimension we have that $(\frac{1}{2}D-1)$-forms are self-dual. They form a pair $(B^{\Lambda}, B_{\Lambda})$, but only half of them appear in the Lagrangian.

The self-duality equation is given by

$$F^P_{\nu_1\dots\nu_K} = -\frac{e}{K!}\epsilon_{\nu_1\dots\nu_K\mu_1\dots\mu_K}\Omega^{PQ}\mathcal{M}_{MR}F^{\mu_1\dots\mu_K R} \tag{61}$$

The action of this dual field, in terms of its field strength, mixes the components of the vector (B^Λ, B_Λ). So, in even dimension, G is only realized on-shell because only the pair as a whole transforms in an irreducible representation of G. In analogy of the classical electric and magnetic fields, which are also dual in this sense, we can choose different frames to describe the same physical situation. Different splittings correspond to different electric frames which are related by sympletic/orthogonal rotations.

In the end we will be interested in the the half-maximal theory in four dimensions. In this case the vectors $A^{M\alpha}_\mu$ live in bifundamental representations of SO(6,6+n) $\times$ SL(2), with indices M and α respectively. Now the electric/magnetic split can be chosen such that it breaks the SL(2) doublet index: $A^{m\alpha} \to (A^{m+}, A^{m-})$.

In this case the lagrangian can be given by

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}\left(e\Im(\tau)\mathcal{M}_{mn}F^{m+}_{\mu\nu}F^{\mu\nu n+} + \frac{1}{2}\epsilon^{\mu\nu\sigma\tau}\Re(\tau)\eta_{mn}F^{m+}_{\mu\nu}F^{n+}_{\sigma\tau}\right) \tag{62}$$

Where the matrix (41) is factorized according to $\mathcal{M}_{M\alpha N\beta} = \mathcal{M}_{MN}\mathcal{M}_{\alpha\beta}$, η_{mn} the metric of SO(6,6+n), and τ given above (53).

5 Gauging Supergravity

By gauging a supergravity theory we mean promoting a subgroup of the global symmetry group $G_0 \subset G$ into a local symmetry. To do this we select generators of G of which we want to gauge the associated symmetries by defining what is called an embedding tensor:

$$X_M \equiv \Theta_M{}^A t_A \tag{63}$$

Here X_M are the generators of G_0 and t_A the generators of G. This embedding tensor Θ is a matrix with its indices living in the fundamental ($M = 1, \dots, n_v$), with n_v the number of vector fields in the theory, and adjoint representation ($A = 1, \dots,$dim G). A specific choice of Θ is a choice for a specific gauging. The main advantage of the embedding tensor formalism is that it gives a way for doing the whole analysis G covariant, untill a choice for a specific Θ is being made.[1]

5.1 Quadratic constraints

When we gauge our theory, we also should introduce a covariant derivative. In our case the covariant derivative is defined as:

[1]More on the embedding tensor formalism and uses of it can be found in [9],[1]

$$D_\mu \equiv \partial_\mu - gA_\mu^M X_M \tag{64}$$

where X_M is given by (63). Given this derivative, the theory should be invariant under

$$\begin{aligned} \partial\mathcal{V} &= g\Lambda^M X_M \mathcal{V} & (65)\\ \partial A_\mu^M &= D_\mu\Lambda^M \equiv \partial_\mu\Lambda^M + gA_\mu^N {X_{NP}}^M \Lambda^P & (66) \end{aligned}$$

Where $\Lambda^M = \Lambda^M(x)$ is a local parameter and ${X_{NP}}^M$ is defined by

$${X_{NK}}^M \equiv {\Theta_N}^A {(t_A)_K}^M \tag{67}$$

So, due to the appearance of Θ in the transformation rules it determines the couplings in the theory. Different Θ will give different gaugings, different interactions and different equations of motions for the fields.

However, if we want to get a consistent theory, we can't choose the embedding tensor arbitrarily. To let the theory be consistent and supersymmetric, it has to satisfy two non-trivial constraints, namely a linear and a quadratic constraint. The quadratic constraint follows from the fact that Θ should be invariant under the local gauge symmetry:

$$\delta_P {\Theta_M}^A \equiv {\Theta_P}^B \delta_B {\Theta_M}^A = {\Theta_P}^B {(t_B)_M}^N {\Theta_N}^A - {\Theta_P}^B {(t_B)_C}^A {\Theta_M}^C = 0 \tag{68}$$

Because the gauge transformation is both defined by and applied on the embedding it is quadratic in it. By contracting this expression with a generator t_A we get the quadratic constraint in the following form:

$$[X_M, X_N] = -{X_{MN}}^P X_P \equiv -{\Theta_M}^A {(t_A)_N}^P X_P \tag{69}$$

which looks like a normal commutator of a Lie algebra, but is not exactly this as we will see soon.

5.2 Linear constraints

Let us now go on to the linear constraints. The linear constraints puts conditions on the representations in which the embedding tensor can live. They project out some of the representations because these would not lead to consistent theories. For our theory, there are two ways in which we can see that the linear constraints are necessary. In one way it follows from a purely bosonic condition for being able to write down a consistent action. But the constraint also follows from the requirement that the theory, after gauging, should still be invariant under supersymmetry.

5.2.1 Bosonic argument for linear constraints

Having defined the covariant derivative (64), we want to construct a field strength for A_μ^M. It is not straigthforward to define a covariant field strength due to the presence of a symmetric part in the 'structure constant' of the commutator relation (69):

$$X_{MN}{}^P \equiv X_{[MN]}{}^P + Z^P{}_{MN} \tag{70}$$

where, of course, $Z^P{}_{MN} = X_{(MN)}{}^P$. This problem originates from the fact that we chose to work in a G-covariant formulation, because of what we have defined our embedding tensor in terms of all the n_v generators corresponding to G. The embedding tensor can then be seen as selecting a subset of the vector fields to be the gauge fields of our gauged theory. But because in general n_v is bigger then the dimension of G_0 not all the components of Θ are non-zero and thus not all the X_M are linearly independent. So the vector fields A_μ^M split into two sets:

- $A_\mu^m \rightarrow$ transforming in the adjoint of G_0: the gauge vectors
- $A_\mu^i \rightarrow$ transforming in some representation of G_0: the remaining vectors

If we now would formulate just a normal field strength in terms of both these sets vector fields, we would not end up with a covariant field strength. We can solve this problem in a very general way by defining a full covariant field strength:

$$\mathcal{H}_{\mu\nu}^M = \mathcal{F}_{\mu\nu}^M + gZ^M{}_{PQ}B_{\mu\nu}^{PQ} \tag{71}$$

where we have introduced a two-form $B_{\mu\nu}^{PQ} = B_{[\mu\nu]}^{(PQ)}$, and $\mathcal{F}$ is the 'normal' field strength:

$$\mathcal{F}_{\mu\nu}^M = \partial_\mu A_\nu^M - \partial_\nu A_\mu^M + gX_{[NP]}{}^M A_\mu^N A_\nu^P \tag{72}$$

By introducing extra fields you alway can make your theory covariant. This is called the "Stückelbergtrick". Z is called the intertwining tensor. The idea is that when we define the appropiate transformation rules for these new two-forms they can cancel the bad transformational behavior of the non-gauge vector fields. The appearance of $Z^M{}_{PQ}$ follows from the fact that the transformations of the remaining vector fiels also are projected with Z. In particular this above defined field strength is covariant under the following set of gauge transformations:

$$\begin{aligned} \delta A_\mu^M &= D_\mu \Lambda^M - gZ^M{}_{PQ}\Xi_\mu^{PQ} \\ \delta B_{\mu\nu}^{MN} &= 2D_{[\mu}\Xi_{\nu]}^{MN} - 2\Lambda^{(M}\mathcal{H}_{\mu\nu}^{N)} + 2A_{[\mu}^{(M}\partial A_{\nu]}^{N)} \end{aligned} \tag{73}$$

Where Ξ_μ^{PQ} are the parameters of the gauge transformation associated with the two-forms.

But these two-forms $B_{\mu\nu}^{MN}$ should be found between the two-forms which are already present in the theory in order to stay covariant under supersymmetry. This gives conditions on $Z^M{}_{PQ}$ because this tensor should only project onto the representations filled by the two-forms of the ungauged theory. And because $Z = Z(\Theta)$, from this the lineair constraint on Θ follows.

5.2.2 Supersymmetry argument for linear constraint

The linear constraint also follows from the fact that we want to end up with a theory which is supersymmetric, which is not necessarily the case anymore after the gauging we did. First we will try to find a lagrangian which is compatible with the new transformation rules (73). Note that these rules imply a non-trivial coupling between the fieldstrengths of a p and a $(p-1)$-form since the full set of gauge transformations are (symbolically) [9]:

$$\begin{aligned} \partial\mathcal{V} =& \quad \Theta\Lambda\mathcal{V} \\ \partial A_\mu =& \quad D_\mu\Lambda - g\Theta\Xi_\mu \\ \partial B_{\mu\nu} =& \quad 2D_{[\mu}\Xi_{v]} + \cdots - g\Theta\Phi_{\mu\nu} \\ \partial C_{\mu\nu\rho} =& \quad 3D_{[\mu}\Phi_{\nu\rho]} + \cdots - g\Theta\Sigma_{\mu\nu\rho} \\ & etc... \end{aligned} \tag{74}$$

So due to this transformation rules, we have more fields in our gauged theory than we had in our ungauged theory. And in even dimensions we also get extra fields in our Lagrangian from the self-dualities which we can not leave out because in the gauged theory we are working with a G covariant action. This could cause problems in terms of inconsistent field equations, but this turns out to be not the case. Suprisingly, the topological terms, which are necessary from a supersymmetric point of view, make the field equations of these additional fields consistent. This gives a purely bosonic reason for the appearence of these topological terms. Which is rather suprising. In D=4 for example, gauge invariance of the Lagrangian requires topological terms which make the field equations of the magnetic duals exactly the duality equation!

In this way we are able to construct a gauge invariant Lagrangian. But this Lagrangian is not invariant under supersymmetry anymore. This can be restored by introducing fermionic mass terms of the form (schematically [9]):

$$\mathcal{L}_{fm} = g(\bar{\psi}^i A_{ij}\psi^j + \bar{\chi}^A B_{Ai}\psi^i + \bar{\chi}^A C_{AB}\chi^B) + h.c. \tag{75}$$

where ψ^i and χ^A are gravitons and spin-1/2 fermions respectively and where the indices i and A refer to their transformation character under the local K-transformation. These A_{ij}, B_{Ai} and C_{AB} are constrained by the facts that:

- $\mathcal{L}_{fm}$ must be invariant under K-transformations

- They must be expressed in terms of ${\Theta_M}^A$ in order to be able to cancel the supersymmetry violating terms in the Lagrangian

Specifically they are defined as components of the following T-tensor:

$$ {T_{\underline{N}}}^{\underline{B}} \equiv {\Theta_M}^A \mathcal{V}^M{}_{\underline{N}} {\mathcal{V}_A}^{\underline{B}} \tag{76} $$

where the underlined indices refer to the transformation behaviour under the local subgroup K. M, N refer to the fundamental and A, B to the adjoint representation. For cancelling the supersymmetry violating terms, the mass tensors in $\mathcal{L}_{fm}$ should live in representations which are present in the decomposition of the G-representation of Θ under the compact subgroup K.

So for the theory to be supersymmetric, the components of the mass tensors which are not present in the decomposition of Θ should be projected out. This exactly is been done by the linear constraint. From the constraint on T orginating from this demand the linear constraint on Θ follows.

5.3 $\mathcal{N} = 4, D = 4$

5.3.1 Linear constraint

Now, let's turn back to our theory with $\mathcal{N} = 4$ and see what kind of conditions are imposed by the linear constraint for that case. For this theory, the fundamental and the adjoint indices take value both in $SO(6, 6+n)$ and $SL(2)$. We denote this by curly indices: $\mathcal{M} = M\alpha$, $\mathcal{N} = N\beta, \ldots$. Here $M, N, \ldots$ are the indices of the fundamental representation of $SO(6, 6+n)$ and α, $\beta, \ldots$ of $SL(2)$. Because the adjoint of $SO(6, 6+n) \times SL(2)$ is the sum of the adjoints of the two groups, and the generators are given by

$$ {(t_{QR})_N}^P = \delta^P_{[Q}\eta_{R]N}, \qquad {(t_{\delta\epsilon})_\beta}^\gamma = \delta^\gamma_{(\delta}\epsilon_{\epsilon)\beta} \tag{77} $$

the adjoint indices $A, B, \ldots$ are summed over pairs in $SO(6, 6+n)$ and in $SL(2)$. So:

$$ {X_{\mathcal{M}\mathcal{N}}}^{\mathcal{P}} = {\Theta_{\alpha M}}^{QR} {(t_{QR})_N}^P \delta^\gamma_\beta + {\Theta_{\alpha M}}^{\delta\epsilon} {(t_{\delta\epsilon})_\beta}^\gamma \delta^P_N \tag{78} $$

In this theory the embedding tensor is living in a tensor product of the fundamental and the adjoint representation of $SO(6, 6+n) \times SL(2)$, which can be decomposed into irreducible (traceless) representations according to[1]

$$ \begin{aligned} & (\ydiagram{1}, \ydiagram{1}) \otimes \Big((\ydiagram{2}, \cdot) + (\cdot, \ydiagram{1,1}) \Big) \\ = \; & (\ydiagram{3}, \ydiagram{1}) \oplus (\ydiagram{2,1}, \ydiagram{1}) \oplus (\ydiagram{1}, \ydiagram{2,1}) \oplus (\ydiagram{1}, \ydiagram{1}) \oplus (\ydiagram{1}, \ydiagram{1,1,1}) \\ = \; & 2 \cdot (\ydiagram{1}, \ydiagram{1}) \oplus (\ydiagram{3}, \ydiagram{1}) \oplus (\ydiagram{1}, \ydiagram{2,1}) \oplus (\ydiagram{1}, \ydiagram{1,1,1}) \end{aligned} \tag{79} $$

[1]See for explanation on the Young tableaux appendix A

This last equality is due to the fact that for $SL(2)$ the $\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ representation is equivalent to $\square$. So for example the dimensions of these representations are the same, because $\frac{2\cdot 3\cdot 1}{3\cdot 1\cdot 1} = 2$. The fourth representation of the second line is the trace of the third one.

The linear constraint for this theory is often given in the following form (e.g. in [9], [10]):

$$X_{(\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\Omega_{\mathcal{K})\mathcal{P}} = 0 \tag{80}$$

with $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ defined in (68) and Ω the symplectic matrix by which we lower and raise indices given by

$$\Omega_{\mathcal{M}\mathcal{N}} = \Omega_{M\alpha N\beta} \equiv \eta_{MN}\epsilon_{\alpha\beta} \tag{81}$$

That taking the linear constraint having this form is a sufficient condition to fulfill the condition imposed by the the bosonic argument, we can see from the following considerations. The only two-forms we have at our disposal in our four dimensional theory are the ones which are (on-shell) dual to scalar fields. The duality of scalar fields with (D-2)-forms is a bit different from the dualities discussed in section 4.3.4 due to the non-linear coupling of the scalar fields as discussed in section 4.3.1. The duality then becomes

$$G_{\mu_1\ldots\mu_{D-1}A} \equiv e\epsilon_{\mu_1\ldots\mu_{D-1}\nu}j^{\nu}_A \tag{82}$$

Where j^{ν}_A is the conserved Noether current of the symmetry generated by t_A. From this we see that our two-forms have to transform in the adjoint representation. So we should have an intertwining tensor $Z^{\mathcal{K}}{}_{\mathcal{M}\mathcal{N}} = X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{K}}$ which only projects onto two-forms $B_{\mu\nu A}$ like

$$Z^{\mathcal{K}}{}_{\mathcal{M}\mathcal{N}}B^{\mathcal{M}\mathcal{N}}_{\mu\nu} \propto \Theta^{\mathcal{K}A}B_{\mu\nu A}, \text{ with } B_{\mu\nu A} = (t_A)_{\mathcal{M}\mathcal{N}}B^{\mathcal{M}\mathcal{N}}_{\mu\nu} \tag{83}$$

That is $Z^{\mathcal{K}}{}_{\mathcal{M}\mathcal{N}}$ should be of the form

$$Z^{\mathcal{K}}{}_{\mathcal{M}\mathcal{N}} \propto \Theta^{\mathcal{K}A}(t_A)_{\mathcal{M}\mathcal{N}} \tag{84}$$

We now can easily see that the given linear constraint (80) is a sufficient condition for this to be the case because we can rewrite Z as follows:

$$\begin{aligned}
Z^{\mathcal{K}}{}_{\mathcal{M}\mathcal{N}} &= X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{K}} \\
&= \frac{1}{2}\Theta_{\mathcal{M}}{}^{A}(t_A)_{\mathcal{N}}{}^{\mathcal{K}} + \frac{1}{2}\Theta_{\mathcal{N}}{}^{A}(t_A)_{\mathcal{M}}{}^{\mathcal{K}} \\
&= \frac{3}{2}\Theta_{(\mathcal{M}}{}^{A}(t_A)_{\mathcal{N}\mathcal{L})}\Omega^{\mathcal{K}\mathcal{L}} - \frac{1}{2}\Theta^{\mathcal{K}A}(t_A)_{\mathcal{M}\mathcal{N}} \\
&= \frac{3}{2}X_{(\mathcal{M}\mathcal{N}\mathcal{L})}\Omega^{\mathcal{K}\mathcal{L}} - \frac{1}{2}\Theta^{\mathcal{K}A}(t_A)_{\mathcal{M}\mathcal{N}}
\end{aligned} \tag{85}$$

So, setting $X_{(\mathcal{MNL})}$ to zero gives us the required result.

That this is also a necessary condition we can see by checking the $X_{(\mathcal{MNL})}$ for 'undercover' terms of the form (84). If we write out the embedding tensor in terms of the possible representations (79) we can check whether $X_{(\mathcal{MNL})} = 0$ implies the same constraints as (84). We can take our full embedding tensor to be:

$$\begin{aligned} \Theta_{\alpha M}{}^{NP} &= f_{\alpha M}{}^{NP} + \delta_M^{[N} \xi_{1\alpha}^{P]} + \Lambda_{\alpha M}{}^{NP} && (86) \\ \Theta_{\alpha M}{}^{\beta\gamma} &= \xi_{2\delta M} \epsilon^{\delta(\beta} \delta_\alpha^{\gamma)} + \zeta_{\alpha M}{}^{\beta\gamma} && (87) \end{aligned}$$

Where the given tensors have the following symmetries:

$$\begin{array}{rlll} f_{\alpha MNQ}: & f_{\alpha MNQ} = f_{\alpha[MNQ]} & \left(\Box, \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}\right) & (88) \\ \zeta_{M\alpha\beta\gamma}: & \zeta_{M\alpha\beta\gamma} = \zeta_{M(\alpha\beta\gamma)} & (\Box\Box\Box, \Box) & \\ \Lambda_{\alpha MNQ}: & M, N, Q \text{ living in } \begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix} & \left(\Box, \begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}\right) & \\ \xi_{1\alpha M}, \xi_{2\alpha M}: & - & (\Box, \Box) & \end{array}$$

Let us now check term by term what kind of constraints are implied by the given conditions. The contribution of $f_{\alpha MNQ}$ to $X_{(\mathcal{MNL})}$ is

$$\propto f_{\alpha MNQ}\epsilon_{\beta\phi} + f_{\beta QMN}\epsilon_{\phi\alpha} + f_{\phi NQM}\epsilon_{\alpha\beta} \tag{89}$$

Because $f_{[\alpha MNQ}\epsilon_{\beta\phi]} = 0$ (SL(2) can not have three antisymmetric indices) we can rewrite the first two terms to a term of the same form of the third one. This last term has the right form according to (84) because we can rewrite it to the form

$$f_{\phi NQM}\epsilon_{\alpha\beta} = f_{\phi Q}{}^{RS}\epsilon_{\alpha\beta}\eta_{N[R}\eta_{S]M} = f_{\phi Q}{}^{RS}\epsilon_{\alpha\beta}[t_{RS}]_{MN} \tag{90}$$

So (84) does not imply any constraint on f. Nor does $X_{(\mathcal{MNL})} = 0$ because all three terms of (89) add up to a term of the form $f_{[\alpha MNQ}\epsilon_{\beta\phi]}$ and this is anyway zero.

If we want to use a trick like this for checking the constraint for ζ we see that in its contribution to

$$X_{(\mathcal{MNL})} \propto \zeta_{M\alpha\beta\phi}\eta_{NQ} + \zeta_{N\alpha\beta\phi}\eta_{QM} + \zeta_{Q\alpha\beta\phi}\eta_{MN} \tag{91}$$

the third term is again of the desired form. This time we can express it as a term proportional to the generators of SL(2):

$$\zeta_{Q\alpha\beta\phi}\eta_{MN} = -\zeta_{Q\phi}{}^{\delta\epsilon}\epsilon_{\alpha(\delta}\epsilon_{\epsilon)\beta}\eta_{MN} = -\zeta_{Q\phi}{}^{\delta\epsilon}[t_{\delta\epsilon}]_{\alpha\beta}\eta_{MN} \tag{92}$$

But we can not get rid of the first two terms of (91) by any other means than setting it to zero. This also is implied by $X_{(\mathcal{MNL})} = 0$ because the terms of (91) add up to $\zeta_{(M\alpha\beta\phi}\eta_{NQ)}$ and to fulfill this constraint again ζ has to be set to zero.

Let us now take a look at the third representation, $\Lambda_{\alpha MNQ}$. We will start with a Λ having the following symmetry property: $\Lambda_{\alpha MNQ} = \Lambda_{\alpha M[NQ]}$. If we work out $X_{(\mathcal{MNL})}$ we get three terms:

$$\Lambda_{\beta NMQ}\epsilon_{\phi\alpha} + \Lambda_{\alpha MQN}\epsilon_{\beta\phi} + \Lambda_{\phi QNM}\epsilon_{\alpha\beta} \tag{93}$$

The last term is of the desired form (SO(6,6+n) generator), the second term we again can change into two terms due to the vanishing of three antisymmetric SL(2) indices:

$$\Lambda_{\beta NMQ}\epsilon_{\phi\alpha} - \Lambda_{\beta MQN}\epsilon_{\phi\alpha} - \Lambda_{\phi MQN}\epsilon_{\alpha\beta} \tag{94}$$

We now can sum the first two as $-2\Lambda_{\beta(MN)Q}\epsilon_{\alpha\phi}$. The last term can be decomposed into irreducible representations in the following way:

$$-\Lambda_{\phi MQN}\epsilon_{\alpha\beta} = +\Lambda_{\phi MNQ}\epsilon_{\alpha\beta} = +\Lambda_{\phi[MN]Q}\epsilon_{\alpha\beta} + \Lambda_{\phi(MN)Q}\epsilon_{\alpha\beta} \tag{95}$$

of which the last term is again of the right form (SL(2)). So, now we have:

$$-2\Lambda_{\beta(MN)Q}\epsilon_{\alpha\phi} + \Lambda_{\phi(MN)Q}\epsilon_{\alpha\beta} \tag{96}$$

Multiplying these terms with e.g. $\epsilon^{\alpha\phi}$ shows that $\Lambda_{\beta(NM)Q}$ should be zero. Because we took Lambda to be antisymmetric in its last indices, we showed in this way that tensors having the symmetries of $\yng(2,1)$ have to be set to zero by the linear constraint. This is also implied by $X_{(\mathcal{MNL})} = 0$.

From the linear constraint also follows a condition on ξ_1 and ξ_2, namely that there can be only one ξ. So the two ξ's should be equal. That this indeed follows from (84), we will show in the case with the trombone.

Now we know what the linear constraint is implying, we know which representations of (79) can be used in the theory. In our case the constraint tells us to use only the following representations:

$$(\Box,\Box) \oplus (\Box,\yng(1,1,1)) \tag{97}$$

So the non-zero components of the embedding tensor are of the form:

$$\begin{aligned} \theta_{M\alpha}{}^{NP} &= f_{\alpha M}{}^{NP} + \frac{1}{2}\delta_M^{[N}\xi_\alpha^{P]} &(98)\\ \theta_{M\alpha}{}^{\beta\gamma} &= \frac{1}{2}\xi_{\delta M}\epsilon^{\delta(\beta}\delta_\alpha^{\gamma)} &(99)\end{aligned}$$

where $f_{\alpha MNP} = f_{\alpha[MNP]}$.

5.3.2 Quadratic constraints

Let us now turn to the quadratic constraints. We start with checking in what representations these constraints will live. The quadratic constraints are, of course, quadratic in Θ. So they should live in the symmetric product of the representation of (97) with themselves. This is given by:

$$
\begin{aligned}
& \left((\yng(1),\yng(1)) \oplus (\yng(1),\yng(1,1))\right) \otimes_s \left((\yng(1),\yng(1)) \oplus (\yng(1),\yng(1,1))\right) \\
= \; & (\yng(1),\yng(1)) \otimes_s (\yng(1),\yng(1)) \oplus (\yng(1),\yng(1,1)) \otimes_s (\yng(1),\yng(1,1)) \oplus (\yng(1),\yng(1)) \otimes (\yng(1),\yng(1,1)) \\
= \; & (\yng(2),\cdot) \oplus (\yng(2),\yng(2)) \oplus (\cdot,\yng(1,1)) \\
& \oplus (\yng(2),\yng(2,2,2)) \oplus (\yng(2),\yng(2,2)) \oplus (\yng(2),\yng(2)) \oplus (\yng(2),\cdot) \oplus (\yng(2),\yng(2,1,1,1,1)) \oplus (\yng(2),\yng(1,1,1,1)) \\
& \oplus (\cdot,\yng(1,1,1,1,1,1)) \oplus (\cdot,\yng(2,2,1,1)) \oplus (\cdot,\yng(2,1,1)) \oplus (\cdot,\yng(1,1)) \\
& \oplus (\yng(2),\yng(1,1,1,1)) \oplus (\yng(2),\yng(2,1,1)) \oplus (\yng(2),\yng(1,1)) \oplus (\cdot,\yng(1,1,1,1)) \oplus (\cdot,\yng(2,1,1)) \oplus (\cdot,\yng(1,1))
\end{aligned}
\tag{100}
$$

But besides this they should live also in the tensor product of a vector, living in the fundamental, and the embedding tensor, because it is a gauge transformation on Θ. This product decomposes as follows:

$$
\begin{aligned}
& (\yng(1),\yng(1)) \otimes \left((\yng(1),\yng(1)) \oplus (\yng(1),\yng(1,1))\right) \\
= \; & (\yng(2),\yng(2)) \oplus (\yng(2),\cdot) \oplus (\cdot,\cdot) \oplus (\cdot,\yng(1,1)) \oplus (\yng(2),\yng(1,1)) \oplus (\cdot,\yng(2)) \\
& \oplus (\yng(2),\yng(2,1,1)) \oplus (\cdot,\yng(2,1,1)) \oplus (\yng(2),\yng(1,1)) \oplus (\cdot,\yng(1,1)) \oplus (\yng(2),\yng(1,1,1,1)) \oplus (\cdot,\yng(1,1,1,1))
\end{aligned}
\tag{101}
$$

So the constraints live in the intersection of these sets, which is given by:

$$
\begin{aligned}
& (\yng(2),\yng(2)) \oplus (\yng(2),\cdot) \oplus 2(\cdot,\yng(1,1)) \oplus (\yng(2),\yng(1,1)) \\
& \oplus (\yng(2),\yng(2,1,1)) \oplus (\cdot,\yng(2,1,1)) \oplus (\yng(2),\yng(1,1,1,1)) \oplus (\cdot,\yng(1,1,1,1))
\end{aligned}
\tag{102}
$$

Which we can identify in the following way with the constraints found by Schön and Weidner [10]:

$$
\begin{aligned}
\xi_\alpha^M \xi_{\beta M} &= 0 && (\yng(2), \cdot) \\
\xi_{(\alpha}^P f_{\beta)PMN} &= 0 && (\yng(2), \yng(1,1)) \\
3 f_{\alpha R[MN} f_{\beta PQ]}{}^R + 2\xi_{(\alpha[M} f_{\beta)NPQ]} &= 0 && (\yng(2), \yng(1,1,1,1)) \\
\epsilon^{\alpha\beta}(\xi_\alpha^P f_{\beta PMN} + \xi_{\alpha M}\xi_{\beta N}) &= 0 && (\cdot, \yng(1,1)) \\
\epsilon^{\alpha\beta}(f_{\alpha MNR} f_{\beta PQ}{}^R - \xi_\alpha^R f_{\beta R[M[P}\eta_{Q]N]} & && (\cdot, \yng(2,1,1)) \oplus (\cdot, \yng(1,1)) \\
-\xi_{\alpha[M} f_{N][PQ]\beta} + \xi_{\alpha[P} f_{Q][MN]\beta}) &= 0
\end{aligned}
\tag{103}
$$

We see that not all the possible representations are used; $(\cdot, \yng(1,1,1,1))$, $(\yng(2), \yng(2,1,1))$ and $(\yng(2), \yng(2))$ are not. From the analysis we did it is not clear why we could expect these representations not being used.

Now we have reconstructed the constraints for the four dimensional half-maximal gauged supergravity, we can turn to adding the gauging of the scaling symmetry.

6 Gauging the trombone

Now we have seen what the linear and quadratic constraints are for the half-maximal supersymmetric theory in $D = 4$ we will turn our attention to the modifications thereof which are imposed when we also gauge the global scaling symmetry. The conformal scaling symmetry, which also is present in general relativity, scale different fields with different weight. To be precise, p-forms scale with weight p. So for example scalars are invariant and the metric transforms as

$$
g_{\mu\nu} \to \lambda^2 g_{\mu\nu} \tag{104}
$$

with constant λ. In general for the bosonic fields:

$$
\delta A_{\mu_1 \ldots \mu_p} = p\lambda A_{\mu_1 \ldots \mu_p} \tag{105}
$$

Because of this behaviour, bigger weights for higher p-forms, it is often called the trombone gauging due to it having a kind of similarity to the way tones on a trombone are distributed. For the fermionic fields we have

$$
\delta\psi_\mu = \frac{1}{2}\lambda\psi_\mu, \;\; \delta\chi = -\frac{1}{2}\lambda\chi \tag{106}
$$

for the gravitino and the spin-1/2 fermionfield respectively. For supergravity theories, it turns out that under this symmetry the full Langrangian scale with the same weigth:

$$\delta\mathcal{L} = (D-2)\lambda\mathcal{L} \tag{107}$$

As we did with the big symmetry groups inherited from dimensional reduction of the 10 dimensional string theories, we also could gauge this symmetry. Gauging of this symmetry can be done by the normal procedure, starting with the definition of the covariant derivative. For this we again use the formalism using the embedding tensor and we extend the definition of the covariant derivative (64) as

$$\hat{D}_\mu \equiv \partial_\mu - gA_\mu^M \hat{\Theta}_M{}^A t_A - gA_\mu^M \hat{\Theta}_M{}^0 t_0 = \partial_\mu - gA_\mu^M \hat{\Theta}_M{}^{\hat{A}} t_{\hat{A}} \tag{108}$$

where t_0 is the generator of the scaling symmetry, $\hat{A} = \{A, 0\}$ and $\hat{\Theta}$ the to be determined extended embedding tensor. We will see that the linear and quadratic constraints will be changed by adding the scaling symmetry $\mathbb{R}^+$

6.1 Linear constraint

We will now investigate what the conditions are, imposed by the linear constraints. Because we still are limited in the fields we have to our desposal to counteract the bad transformational behavoir of the vector fields which not become gauge fields, we still are bound by the condition (84). The field content for the trombone gauged theory should be the same as for the case without the trombone, because we still want to have a supersymmetric theory. That is, the intertwining tensor $\hat{Z}^{\mathcal{K}}{}_{\mathcal{MN}}$ should be of the form

$$\hat{Z}^{\mathcal{K}}{}_{\mathcal{MN}} \propto \hat{\Theta}^{\mathcal{K}A}(t_A)_{\mathcal{MN}} \tag{109}$$

Though this is the same condition for the case without the trombone, we will see that the implications are somewhat different due to the difference in Θ. In particular the conditions implied are not the same as what would be implied by $\hat{X}_{(\mathcal{MNL})} = 0$.

In principle the embedding tensor could now live in the representations of the following product:

$$(\square,\square)\otimes\left((\square\square,\cdot)\oplus\left(\cdot,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\oplus(\cdot,\cdot)\right) = 3(\square,\square)\oplus(\square\square\square,\square)\oplus\left(\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right)\oplus\left(\square,\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) \tag{110}$$

Comparing to (79) shows that the only new possibility is another $(\square,\square)$. So the constraint still tells us to dismiss the middle two representation and does not imply any constraint on the last one. We now want to check what kind of constraints are put on the representations of the form $(\square,\square)$.

For this reason we take the components of the embedding tensor to be

$$\hat{\Theta}_{\alpha M}{}^{NP} = f_{\alpha M}{}^{NP} + \frac{1}{2}\delta_M^{[N}\xi_{1\alpha}^{P]} + \delta_M^{[N}\kappa_{1\alpha}^{P]} \tag{111}$$

$$\hat{\Theta}_{\alpha M}{}^{\beta\gamma} = \frac{1}{2}\xi_{2\delta M}\epsilon^{\delta(\beta}\delta_\alpha^{\gamma)} + \kappa_{2\delta M}\epsilon^{\delta(\beta}\delta_\alpha^{\gamma)} \tag{112}$$

$$\hat{\Theta}_{\alpha M} = \kappa_{3\alpha M} \tag{113}$$

In the calculation we can leave out f. Furthermore we can do the calculation with only κ_1, κ_2 and κ_3 (so leaving out the ξ's) because we will see that we in that way will find two solutions, one corresponding to the case where κ_3 is set to zero. The relation we then find between κ_1 and κ_2 is exactly the one we already imposed on ξ_1 and ξ_2.

So let's start again with the intertwining tensor and do the change of indices trick in order to take out the part which anyway had the rigth form:

$$\begin{aligned}
\hat{Z}_{\mathcal{MN}}{}^{\mathcal{P}} &= \hat{\Theta}_{(M}{}^{\hat{A}}[t_{\hat{A}}]_{\mathcal{N})}{}^{\mathcal{P}} \\
&= \hat{\Theta}_{(M}{}^{A}[t_A]_{\mathcal{N})}{}^{\mathcal{P}} + \hat{\Theta}_{(M}{}^{0}[t_0]_{\mathcal{N})}{}^{\mathcal{P}} \\
&= \big(\frac{3}{2}\hat{\Theta}_{(M}{}^{A}[t_A]_{\mathcal{NQ})} - \hat{\Theta}_{Q}{}^{A}[t_A]_{\mathcal{MN}} + \hat{\Theta}_{(M}{}^{0}[t_0]_{\mathcal{N})\mathcal{Q}}\big)\Omega^{\mathcal{PQ}}
\end{aligned} \tag{114}$$

From this, the second term again has the rigth form. The first and the third term will give us conditions on the κ's. The first term turns out to contain the following terms proportional to κ_1:

$$\begin{aligned}
\frac{1}{2}\big(\eta_{MQ}\epsilon_{\beta\phi}\kappa_{1\alpha N} &- \eta_{QN}\epsilon_{\phi\alpha}\kappa_{1\beta M} \\
+\eta_{QN}\epsilon_{\alpha\beta}\kappa_{1\phi M} &- \eta_{MQ}\epsilon_{\alpha\beta}\kappa_{1\phi N} \\
+\eta_{MN}\epsilon_{\phi\alpha}\kappa_{1\beta Q} &- \eta_{MN}\epsilon_{\beta\phi}\kappa_{1\alpha Q}\big)
\end{aligned} \tag{115}$$

For κ_2 we get exactly the same terms out of this part but with an overall minus sign, what shows that the condition for the case without the trombone indeed implies that $\xi_1 = \xi_2$ (because only looking at these two terms is the same as setting κ_3 to zero). If we also take into account the last term of (114) we have to be a bit more carefull. First we can recognise that the terms of the second line can be rewritten in terms of the $SO(6, 6+n)$ generators as can the last line in terms of $SL(2)$ generators:

$$\begin{aligned}
\eta_{QN}\epsilon_{\alpha\beta}\kappa_{1\phi M} - \eta_{MQ}\epsilon_{\alpha\beta}\kappa_{1\phi N} &= 2\kappa_1^R\delta_Q^S\epsilon_{\alpha\beta}\eta_{S[N}\eta_{M]R} \\
&= 2\kappa_1^R\delta_Q^S\epsilon_{\alpha\beta}[t_{SR}]_{MN}
\end{aligned} \tag{116}$$

$$\begin{aligned}
\eta_{MN}\epsilon_{\phi\alpha}\kappa_{1\beta Q} - \eta_{MN}\epsilon_{\beta\phi}\kappa_{1\alpha Q} &= -2\kappa_{1Q}^\delta\eta_{MN}\delta_\phi^\epsilon\epsilon_{\epsilon(\alpha}\epsilon_{\beta)\delta} \\
&= 2\kappa_{1Q}^\delta\eta_{MN}\delta_\phi^\epsilon[t_{\delta\epsilon}]_{\alpha\beta}
\end{aligned} \tag{117}$$

The first line of (115) does contain terms which are not of the desired form. This can be seen best if we decompose the terms in irreducible representations.

$$\eta_{MQ}\epsilon_{\beta\phi}\kappa_{1\alpha N} - \eta_{QN}\epsilon_{\phi\alpha}\kappa_{1\beta M} = -2\eta_{Q[M}\epsilon_{\phi[\beta}\kappa_{1\alpha]N]} - 2\eta_{Q(M}\epsilon_{\phi(\beta}\kappa_{1\alpha)N)} \tag{118}$$

Where we decomposed both terms in their symmetric and antisymmetric parts in M, N and α, β. The terms with mixed symmetries cancel out due to the symmetry of $\mathcal{MN}$. The first term is proportional to $\epsilon_{\alpha\beta}\eta_{R[M}\eta_{N]S}$ because $\epsilon_{\alpha\beta}$ is the only antisymmetric two-form of SL(2). So this term has the right form and we are left with only the last term. This term can not be written in the form of (109).

Let's now turn our attention to the last term of (114). Using the same kind of decomposition as before (again the mixed terms cancel) we get:

$$\hat{\Theta}_{(M}{}^{0}[t_0]_{\mathcal{N})\mathcal{Q}} = 2\kappa_{3[\alpha[M}\epsilon_{\beta]\phi}\eta_{N]Q} + 2\kappa_{3(\alpha(M}\epsilon_{\beta)\phi}\eta_{N)Q} \tag{119}$$

Here again the first term has the right form. So in the end we only have three terms which are not of the desired form, namely:

$$-2\eta_{Q(M}\epsilon_{\phi(\beta}\kappa_{1\alpha)N)} + 2\eta_{Q(M}\epsilon_{\phi(\beta}\kappa_{2\alpha)N)} + 2\kappa_{3(\alpha(M}\epsilon_{\beta)\phi}\eta_{N)Q} \tag{120}$$

As being said, the contribution of κ_2 is exactly the one of κ_1, but with different sign. This equation is simply stating that the following condition should be satisfied:

$$\kappa_3 = \kappa_2 - \kappa_1 \tag{121}$$

From this we can find easily two solutions. The first one being setting κ_3 to zero and κ_1 and κ_2 equal. This is the solution we imposed on the ξ's in the case without the trombone. For the second one we set $\kappa_1 = \kappa_3 = \frac{1}{2}\kappa_2$.

Given these two solutions we can now write down the full embedding tensor for the trombone case:

$$\begin{aligned}
\hat{\Theta}_{\alpha M}{}^{NP} &= f_{\alpha M}{}^{NP} + \frac{1}{2}\delta_M^{[N}\xi_\alpha^{P]} + \frac{1}{2}\delta_M^{[N}\kappa_\alpha^{P]} && (122)\\
\hat{\Theta}_{\alpha M}{}^{\beta\gamma} &= \frac{1}{2}\xi_{\delta M}\epsilon^{\delta(\beta}\delta_\alpha^{\gamma)} + \kappa_{\delta M}\epsilon^{\delta(\beta}\delta_\alpha^{\gamma)} && (123)\\
\hat{\Theta}_{\alpha M} &= \frac{1}{2}\kappa_{\alpha M} && (124)
\end{aligned}$$

This gives us the following expression for the generators of our gauged theory:

$$
\begin{aligned}
\hat{X}_{\mathcal{MN}}{}^{\mathcal{P}} &= X_{\mathcal{MN}}{}^{\mathcal{P}} + \frac{1}{2}(\delta^P_M\delta^\gamma_\beta\kappa_{\alpha N} - \delta_\beta\gamma\eta_{MN}\kappa^P_\alpha) \\
&\quad -\delta^P_N\delta^\gamma_\alpha\kappa_{\beta M} + \epsilon_{\alpha\beta}\delta^P_N\kappa_{\delta M}\epsilon^{\delta\gamma} + \frac{1}{2}\delta^\gamma_\beta\delta^P_N\kappa_{\alpha M} \\
&= -\delta^\gamma_\beta f_{\alpha MN}{}^P \\
&\quad +\frac{1}{2}(\delta^P_M\delta^\gamma_\beta\xi_{\alpha N} - \delta_{\beta\gamma}\eta_{MN}\xi^P_\alpha - \delta^p_N\delta^\gamma_\alpha\xi_{\beta M} + \epsilon_{\alpha\beta}\delta^P_N\xi_{\delta M}\epsilon^{\delta\gamma}) \\
&\quad +\frac{1}{2}(\delta^P_M\delta^\gamma_\beta\kappa_{\alpha N} - \delta^\gamma_\beta\eta_{MN}\kappa^P_\alpha) - \delta^P_N\delta^\gamma_\alpha\kappa_{\beta M} + \epsilon_{\alpha\beta}\delta^P_N\kappa_{\delta M}\epsilon^{\delta\gamma} \\
&\quad +\frac{1}{2}\delta^\gamma_\beta\delta^P_N\kappa_{\alpha M}
\end{aligned}
$$

6.2 Quadratic constraint

With this expression for the generators of our gauged theory we can now take a look at the quadratic constraints. The embedding tensor now lives in the following representation:

$$(\square,\square) \oplus (\square,\square) \oplus (\square,\ydiagram{1,1}) \tag{125}$$

So, if we want to do the same group theoretical calculation of the quadratic constraint as in the case without the trombone, this time we have to work out the following products:

$$((\square,\square) \oplus (\square,\square) \oplus (\square,\ydiagram{1,1})) \otimes_s ((\square,\square) \oplus (\square,\square) \oplus (\square,\ydiagram{1,1})) \tag{126}$$

and

$$(\square,\square) \otimes ((\square,\square) \oplus (\square,\square) \oplus (\square,\ydiagram{1,1})) \tag{127}$$

The second product gives us exactly the same kind of representations as in the case without the trombone, but more of them. In the first product we also get a lot of the same terms as before, but besides that the following two representations:

$$(\cdot,\cdot) \oplus (\cdot,\ydiagram{1,1}) \tag{128}$$

When we look again at the intersection, we get the following representations:

$$2(\ydiagram{2},\ydiagram{2}) \oplus 2(\ydiagram{2},\cdot) \oplus (\cdot,\ydiagram{2}) \oplus (\cdot,\cdot) \oplus 3(\ydiagram{2},\ydiagram{1,1}) \oplus 3(\cdot,\ydiagram{1,1}) \tag{129}$$

$$\oplus(\ydiagram{2},\ydiagram{2,1,1}) \oplus (\cdot,\ydiagram{2,1,1}) \oplus (\ydiagram{2},\ydiagram{1,1,1,1}) \oplus (\cdot,\ydiagram{1,1,1,1}) \tag{130}$$

These are the same as the ones used in the quadratic constraints in the case without the trombone, with the following extra representations:

$$2(\yng(2),\yng(2)) \oplus (\yng(2),\cdot) \oplus (\cdot,\yng(2)) \oplus (\cdot,\cdot) \oplus 2(\yng(2),\yng(1,1)) \oplus (\cdot,\yng(1,1)) \oplus (\cdot,\yng(1,1,1,1)) \tag{131}$$

Calculations by the not well understood method of the reduction of the infinite dimensional Kac-Moody algebra, which in the case without the trombone gives the right result, suggests that we should look for constraints in the represenations as in the case without the trombone supplemented by ones in the following representations:

$$(\yng(2),\cdot) \oplus (\cdot,\yng(2)) \oplus 2(\cdot,\cdot) \oplus (\yng(2),\yng(1,1)) \oplus (\cdot,\yng(1,1)) \oplus (\cdot,\yng(1,1,1,1)) \tag{132}$$

i.e. all the new representations in the intersection apart from the $2(\yng(2),\yng(2))$ and one of the $(\yng(2),\yng(1,1))$, supplemented with another $(\cdot,\cdot)$ and a $(\cdot,\yng(1,1,1,1))$. This last representation was already present in the intersection of the two sets without the trombone but did not appear as new possibility when adding the trombone.

To see what the real constraints are we can work out the explicit expression of the quadratic constraint which can be written as:

$$[\hat{X}_{\mathcal{M}}, \hat{X}_{\mathcal{N}}] = -\hat{X}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \hat{X}_{\mathcal{P}} \tag{133}$$

Where $\hat{X}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \equiv \hat{\Theta}_{\mathcal{M}}{}^{A} (t_A)_{\mathcal{N}}{}^{\mathcal{P}}$. Working out this expression in terms of the components of the embedding tensor (125) and projecting out all the irriducible representations gives us the following constraints[1]:

[1]For these calculation use has been made of the computerprogram Cadabra: [6], [7]

$$
\begin{aligned}
4\,\kappa_{(\alpha[M}f_{\beta)NPQ]} + 2\,\xi_{(\alpha[M}f_{\beta)NPQ]} + 3\,f^{R}_{(\alpha[MN}f_{\beta)PQ]R} &= 0 &\qquad& \left(\square\square, \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}\right)\\
\epsilon^{\phi\psi}\kappa_{\phi[M}f_{\psi NPQ]} &= 0 && \left(\cdot, \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}\right)\\
2\eta_{[M[P}\kappa^{R}_{(\alpha}f_{\beta)Q]N]R} + \eta_{[M[P}\xi^{R}_{(\alpha}f_{\beta)Q]N]R} &= 0 && \left(\square\square, \begin{smallmatrix}\square&\square\\\square&\\\square&\end{smallmatrix}\right)\\
2\,\kappa_{(\alpha}{}^{P}f_{\beta)MNP} - \xi_{(\alpha}{}^{P}f_{\beta)MNP} &= 0 && 2\left(\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\\
\kappa_{(\alpha[M}\xi_{\beta)N]} - \kappa_{(\alpha}{}^{P}f_{\beta)MNP} &= 0 &&\\
\epsilon^{\phi\psi}(4\,\kappa_{\phi[M}f_{\psi N][PQ]} - 4\,\kappa_{\phi[P}f_{\psi Q][MN]} + \xi_{\phi[M}f_{\psi N][PQ]} - \xi_{\phi[P}f_{\psi Q][MN]} & && \left(\cdot, \begin{smallmatrix}\square&\square\\\square&\\\square&\end{smallmatrix}\right)\\
-f_{\phi MN}{}^{R}f_{\psi PQR} - 4\,\eta_{[M[P}\xi_{\psi Q]}\kappa_{\phi N]} + 4\,\eta_{[M[P}\kappa_{\phi Q]}\xi_{\psi N]} & &&\\
-\eta_{[M[P}\xi_{\psi Q]}\xi_{\phi N]} - 12\,\eta_{[M[P}\kappa_{\psi Q]}\kappa_{\phi N]}) &= 0 &&\\
\epsilon^{\phi\psi}(16\,\kappa_{\phi}{}^{R}f_{\psi MNR} + 4\,\xi_{\phi}{}^{R}f_{\psi MNR} + 4\,f_{\phi[M}{}^{RS}f_{\psi N]RS} & && 3\left(\cdot, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\\
+8\,\xi_{\psi[M}\kappa_{\phi N]} - \xi_{\phi M}\xi_{\psi N} - 12\,\kappa_{\phi M}\kappa_{\psi N}) &= 0 &&\\
\epsilon^{\phi\psi}(6\,\kappa_{\phi[M}\xi_{\psi N]} + \xi_{\phi M}\xi_{\psi N} + \xi_{\phi}{}^{Q}f_{\psi MNQ}) &= 0 &&\\
\epsilon^{\phi\psi}(\kappa_{\phi[M}\xi_{\psi N]} + 6\,\kappa_{\phi M}\kappa_{\psi N} + \kappa_{\phi}{}^{Q}f_{\psi MNQ}) &= 0 &&\\
2\,\kappa^{M}_{(\alpha}\xi_{\beta)M} + \xi^{M}_{\alpha}\xi_{\beta M} &= 0 && 2\left(\square\square, \cdot\right)\\
2\,\kappa^{M}_{\alpha}\kappa_{\beta M} + \kappa^{M}_{(\beta}\xi_{\alpha)M} &= 0 &&\\
\epsilon^{\alpha\beta}\kappa_{\alpha(M}\xi_{\beta N)} &= 0 && \left(\cdot, \square\square\right)\\
\epsilon^{\alpha\beta}\kappa^{M}_{\alpha}\xi_{\beta M} &= 0 && (\cdot, \cdot)
\end{aligned}
$$

If we look at these constraints we conclude that they live in exactly the representations which were predicted by Kac-Moody, apart from the singlet $(\cdot, \cdot)$. It is not clear why Kac-Moody is giving this extra singlet. This also means that not all the representations of the intersection are being used. In particular, the representations $2(\square\square, \square\square)$,$\left(\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$ are not. Again, from our analysis we can not see the deeper reason for this.

7 Conclusion and outlook

Having found the linear and quadratic constraints for the half-maximal 4D gauged supergravity with trombone, we have determined the conditions which this theory has to fulfill to be consistent and supersymmetric. The next step now would be to find different solutions of the constraints, by finding appropiate embedding tensors. This has been done for maximal supergravity in [4] for different dimensions and particular choices of the gauge group. For D=4 there the pure trombone gauge condition $\theta^{A}_{M} = 0$ is been taken. In [5] a more general case is studied. There an elegant way of parametrizing the quadratic constraints in presence of the trombone has been found. In particular this makes it much easier to find solutions to these constraints and a specific example is giving. Also the equations of motions of general solutions are formulated. One peculiarity of these theories is that it is not possible to formulate an action principle from which these equations of motions follow. This in constrast with the situation discussed in section (2.3) where we were able to find such an action.

The general strategy in these references is to decompose the representations, and thus the embedding tensor, with respect to a specific subgroup of the global symmetry group. For example, for maximal supergravity in 4D (of which the global symmetry group is $E_{7(7)}$) they decompose all the objects with respect to $E_{6(6)} \times SO(1,1)$. One could investigate the possibility of using the same procedure to find solutions for the half-maximal case. It then can be expected that also in these cases one would find equations of motions which can not be derived from any action principle.

This rises interesting questions about the meaning of the action principle. What are we exactly doing by formulating an action and minimizing it, and why, in general, does nature satisfy the equations of motions derived from this? And why can we get equations of motions which can not derived from any action principle? What does that tell us exactly (about this theory, about the action principle)? Studying this can give us some fundamental insight in the ideas behind this widely used and fundamental formalism of the action principle.

Appendices

A Young tableaux and their dimensions

To calculate the dimensions of representations it is convenient to make use of Young-tableaux. Young tableaux are a way of depicting representations of specific symmetry groups. Each young tableau represents one specific representation. Boxes which are horizontally connected mean symmetric indices, vertically means anti-symmetric. So, for example: $T_{[\alpha\beta]} \rightarrow \begin{smallmatrix}\square\\\square\end{smallmatrix}$, $T_{(\alpha\beta\gamma)} \rightarrow \square\square\square$. $\square$ is just the fundamental representation.

The dimension of the representations are easily calculated using these tableaux. This can be done in the following way:

- put in each $\square$ a number due to the following rules:
  - put in the upper-left $\square$ the dimension of the fundamental representation of the group you are working with
  - for each $\square$ you go to the right you raise the number with one
  - for each $\square$ you go to beneath you lower the number with one
- multiply all these numbers and divide it by another number which you get by:
  - put in each $\square$ the sum of the $\square$'s which are to the right of it and the $\square$'s which are beneath it plus one (for itself)
  - multiply these numbers to get the number by which you have to divide
- to get the dimension of the irreducible representation subtract all possible traces by contraction of two appropiate indices and calculation of the dimension of the left over representation

So, for example, in $SO(6,6)$:

$$\begin{smallmatrix}\square\square\\\square\end{smallmatrix} \rightarrow \frac{12 \cdot 13 \cdot 11 \cdot 10}{4 \cdot 2} - \frac{12 \cdot 11}{2} = 2079 \tag{134}$$

We denote a tensor having indices in two different symmetry groups by a pair of tableaux. e.g. $(\square, \begin{smallmatrix}\square\square\\\square\end{smallmatrix})$. Where in our case the first refers to the transformation character under $SL(2)$ and the second under $SO(6,n)$. See [3] for more on young tableaux.

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