Derivative Corrections to the Born-Infeld Lagrangian and Electromagnetic Duality Invariance*

Joost de Jong

Institute for Theoretical Physics
Nijenborgh 4
9747 AG Groningen
The Netherlands

5th July 2006

* R*u*G Internal Report # 345

ABSTRACT

The open superstring effective action is given by the Born-Infeld action with specific derivative correction terms ($\alpha'^m \partial^n F^p$) added. In particular, it is shown that the world volume of a D-brane is governed by the Born-Infeld theory. A property of the Born-Infeld theory is that it is electromagnetic duality invariant. In order to perform the duality invariance test, the condition for electromagnetic duality invariance of a general Lagrangian $\mathscr{L}(F)$ is derived.

This thesis will show that the expanded Born-Infeld theory is electromagnetic duality invariant up to and including $\mathscr{O}(\alpha'^6)$. The lowest order derivative correction term, $\alpha'^4 \partial^4 F^4$, is also shown to be electromagnetic duality invariant. A redefinition of the G $(= \frac{\partial \mathscr{L}}{\partial F})$ tensor within the duality invariance condition was necessary in this case. However, starting out with general $\alpha'^4 \partial^4 F^4$-terms (significantly reduced in number by means of a computer) does not necessarily give the derivative correction terms belonging to the open superstring effective action by demanding electromagnetic duality invariance.

CONTENTS

INTRODUCTION

When I was young, I always liked doing math. Furthermore, I liked reading popular scientific books about the solar system, but it was not after the fourth grade of high school that I decided to take on a study in physics. Why? When I attended introductory days of mathematics and physics, I discovered that the mathematics practiced at high school corresponded best with the way physicists use mathematics; physicists use mathematics as a tool in understanding physics and do not worry about what I like to call ”fuzzy little things”. Mathematicians have already proved (for the most part) that physicists are allowed to do what they do.

But why did I not choose to study astrophysics instead? This is due to the fact that I am a theoretician and like to solve problems. Astrophysics appeared too practical for my taste at the time. So I started practicing physics. This thesis proofs I do not regret my choice.

Before my study I had no particular job occupation or goal in mind. I would worry about that after my study. However, during your study you get to hear things and the particular thing I heard was that there exists a theory called string theory. I was also told its alias is ”the theory of everything”. That sounded like magic to me. I decided that the goal of my study became to learn and understand more about string theory, because if it finally will be found to be a valid theory, I will at least know something about ”the theory of everything”. Therefore I chose a subject in string theory as my thesis project. I also thought this was a good opportunity to find out if I enjoy doing research.

I went to the theoretical physics department and asked what topics as a thesis project they got in string theory. I got an offer which had something to do with cosmic strings and an offer in doing something with Born-Infeld and derivative corrections added to it appearing in string theory. The latter sounded more like doing research to my ears. Less string theory to learn, but doing real research.

As is clear by now, this thesis is about Born-Infeld and derivative corrections added to it. The phenomenon of electromagnetic duality is discussed and applied to Born-Infeld and its derivative corrections. The chapters in summary are:

- Maxwell theory and its incompleteness. The incompleteness involves the self-energy of a charged particle. After that why the Born-Infeld theory is what it is will be discussed; motivations for its explicit form are explained.

- The Born-Infeld theory enters string theory in several ways. I decided to show one of the ways Born-Infeld enters by deriving the Lagrangian for the world volume of a D-brane.

- Chapter 3 is about electromagnetic duality invariance of Maxwell and Born-Infeld theory. In order to show duality invariance a general electromagnetic duality invariance condition is derived.

- The final chapter is the most important chapter. The first part is about trying to obtain independent derivative correction terms ($\partial^4 F^4$) from all possible $\partial^4 F^4$-terms. I spend a great deal of time on writing a computer program which aids me with this process. These obtained independent terms are subjected to the duality invariance condition in the next part of the chapter. The last part is about checking the derivative correction terms arising in string theory and subjecting them to the duality invariance condition and what needed to be done in order to let them be duality invariant.

- At last, the appendices are there to aid the reader and avoiding large and annoying little calculations within the main text. Where necessary the reader will be directed to the appendices.

It is important to note that electromagnetic duality invariance only works in four dimensions. Although string theory has $d > 4$, everything we do involving duality is therefore considered in four dimensions. If we are working in any other number of dimensions, the reader will be notified explicitly.

Thank you for at least considering to read my thesis report and I hope you will enjoy it all.

1. CLASSICAL ELECTROMAGNETISM

1.1 Maxwell Theory

Electricity and magnetism were discovered somewhere in the nineteenth century. Various scientists became acquainted with these new phenomena (some in a painful way). As time proceeded, more and more knowledge about electricity and magnetism was gathered. This led to the final theory of classical electromagnetism, the Maxwell equations. A funny thing is that vector calculus was not yet known at that time, so Maxwell necessarily published his equations written out in components. It must have been a tedious job which did not provide much insight. Nevertheless Maxwell did it. His equations in the absence of any sources and current densities are

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \\ \nabla \cdot \vec{B} &= 0, \\ \nabla \cdot \vec{E} &= 0, \\ \nabla \times \vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \end{aligned} \tag{1.1}$$

Or in covariant notation

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0, \\ \partial_\mu \star F^{\mu\nu} &= 0, \end{aligned} \tag{1.2}$$

where the star denotes the Hodge dual[1] of the field tensor. The second equation of (1.2) is a shorthand notation of the Bianchi identity. As said these equations constitute the classical theory of electromagnetism.

After Maxwell constructed his equations, people started realizing they contain unphysical results (at least what was considered unphysical at that time). First of all the Maxwell equations are not invariant under Galilean transformations. Secondly a point charge is predicted to have an infinite self-energy. These were/are considered major problems, but which are inherent properties of the theory.

We all know that with the discovery of Einstein's theory of relativity the first problem was no longer a real problem, because the Maxwell equations need to be invariant under Lorentz

[1] Appendix A

transformations rather then under Galilean transformations. Still there is the problem of the infinite self-energy of a charged particle. If one sticks to the Maxwell equation, this problem will not be solved. In order to fix it, one has to look further then the classical theory of electromagnetism. But before doing so let us show the calculation of the self-energy. The Lagrangian for the field spread by a charged point particle (cpp) in covariant form is just the vacuum Maxwell Lagrangian [1]

$$\mathscr{L}_{\text{Max}}(A_\nu, \partial_\mu A_\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1.3}$$

However we will need the Hamiltonian density, because after integrating it over all space it gives the total energy of the field produced by the cpp. The Hamiltonian density is given by (see for example [1])

$$\mathscr{H}_{\text{Max}}(A_\nu, \partial_\mu A_\nu) = \frac{1}{2}(\vec{E}^2 + \vec{B}^2). \tag{1.4}$$

The final part of the calculation is setting $\vec{B}$ equal to zero (particle is at rest) and integrating the Hamiltonian density over all space

$$\begin{aligned} H_{\text{Max}} &= \int_0^\infty d^3x (\frac{1}{2}\vec{E}^2), \\ &= \frac{1}{2}\int_0^\infty d^3x (k^2 \frac{q^2}{r^2}), \end{aligned}$$

which diverges. The self-energy of a charged particle is indeed infinite.

This section showed a brief summary of the main problem of the vacuum Maxwell equations. Due to this problem we would like to have a better theory than Maxwell. The next section will give an introduction to a new theory called the Born-Infeld theory which does not contain that problem.

1.2 Born-Infeld Theory

We have proven that classical electromagnetism leads to an infinite self-energy of a cpp in the previous section. One of the following methods can be used to solve this problem: a modification of the old theory or developing a new theory containing the old theory in a limiting case.

By a modification of a theory we mean keeping globally the same assumptions, but for example altered by adding more constraints or information. Consider the ideal gas law as an example. The assumption on which the law is based, is that the molecules of a gas behave like non-interacting hard spheres. This is of course true to some extent. If one desires more accuracy in some measurement, one should take the weak interactions between the molecules into consideration. This is incorporated in the law by altering the potential of a hard sphere by adding an attractive potential (the Van Der Waals potential). Thus we have modified the theory into a more accurate one by adding a piece of information (the altered potential).

An illustrative part of physics in which a new theory containing the old theory in a limiting case occurs, is gravity. Gravity was best described by Newton before the 1920s. We all know that Newton's theory is only valid when weak gravitational fields are involved. We know Einstein's theory of general relativity is the correct one nowadays, because it is correct for weak and strong gravitational fields and it predicts the bending of light by massive objects. Therefore Einstein's theory of gravity should reduce to Newton's theory of gravity in its weak gravitational field limit [2]. Both theories are clearly based on fundamental different "assumptions". Newton's theory is empirically discovered and Einstein's theory by the ansatz of the curvature of spacetime. We are going to find a new theory and make the same ansatz Einstein did; the presence of a charged particle influences the metric of space.

Maxwell's theory is valid for relativistic charged particles in a Minkowski space. In this framework the full Lagrangian density of a nonmoving cpp (and its accompanying electromagnetic field) heuristically is

$$\mathscr{L}_{\text{cpp}} = \mathscr{L}_{\text{rel}} + \mathscr{L}_{\text{Max}} \tag{1.5}$$

$$= -m\sqrt{-\det(\eta_{\mu\nu})} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.6}$$

The new Lagrangian can be deduced by the same arguments used in general relativity. Like with gravitation the particle spreads out an electromagnetic field. Therefore introduce a second, moving cpp. The moving cpp can be said to move freely through space, but then the space is not Minkowsky anymore due to the electromagnetic field of the nonmoving cpp. As for gravity, the particle being charged changes the metric of space. To push this argument further, a cpp changes the metric of space for itself! The Lagrangian of the nonmoving cpp should therefore contain an altered metric within the determinant and reduce to (1.6) in the weak field limit and to the Minkowski metric in the case when there is no field present. It is important to note that the new Lagrangian will only be of significance close to particles, because then the electric field is very strong ($\frac{1}{r}$).

What then do we claim the new Lagrangian to be? There are a lot of creative Lagrangians possible which have the proper weak field limit. We are interested in only one of them, the Born-Infeld (BI) Lagrangian which is given by

$$\mathscr{L}_{\text{BI}} = -b^2\sqrt{-\det(\eta_{\mu\nu} + \frac{1}{b}F_{\mu\nu})} + b^2, \tag{1.7a}$$

$$= -b^2\sqrt{1 - \frac{\vec{E}^2 - \vec{B}^2}{b^2} - \frac{(\vec{E}\cdot\vec{B})^2}{b^4}} + b^2. \tag{1.7b}$$

Equation (1.7b) is the BI Lagrangian in terms of $\vec{E}$ and $\vec{B}$-fields. It was constructed by Max Born and Leopold Infeld in the 1930s. A nice discussion about the way the BI Lagrangian was originally constructed, is given in [3]. The change of the metric is obvious. The constant b is included to fix a particular normalization. Note that adding an arbitrary constant to a Lagrangian does not change the equations of motion. Therefore we choose to omit the addition of the constant b^2 in equations (1.7a) and (1.7b)

$$\mathscr{L}_{\mathrm{BI}} = -b^2\sqrt{-\det(\eta_{\mu\nu} + \frac{1}{b}F_{\mu\nu})}, \tag{1.8a}$$

$$= -b^2\sqrt{1 - \frac{\vec{E}^2 - \vec{B}^2}{b^2} - \frac{(\vec{E}\cdot\vec{B})^2}{b^4}}. \tag{1.8b}$$

The constant was originally included to cancel the constant coming from the square-root upon expanding. The constant from the square-root is actually the rest-energy of the cpp and should not be excluded.

We are left with the task of showing that the BI Lagrangian indeed has the proper weak field limit and that it produces a finite self-energy of a cpp. We will use the BI Lagrangian density (1.8b) for this.

First the weak field limit. Neglecting terms of the order field to the power four, we can immediately write

$$\mathscr{L}_{\mathrm{BI}} = -b^2\sqrt{1 - \frac{\vec{E}^2 - \vec{B}^2}{b^2}}. \tag{1.9}$$

Expanding the square-root gives

$$\mathscr{L}_{\mathrm{BI}} = -b^2 - \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) = -b^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.10}$$

Hence the lowest order approximation of the BI Lagrangian is exactly the same as the Lagrangian (1.6), if b^2 is identified with the rest energy of a cpp. So the weak field limit is the one desired.

What about the self-energy of a charged particle? It can be argued to be finite by using a nice feature of BI, the existence of a maximum electric field strength. In the absence of a magnetic field (1.8b) can be simplified to

$$\mathscr{L}_{\mathrm{BI}} = -b^2\sqrt{1 - \frac{\vec{E}^2}{b^2}}. \tag{1.11}$$

Remember that $\vec{E}$ is the real electric field, with or without a medium present. Because of the fact that the metric is changed in the presence of an electromagnetic field, the vacuum can be considered a medium. A nonlinear medium in general. Therefore one can also define the field $\vec{D}$ in the usual way

$$\vec{D} = \frac{\partial \mathscr{L}}{\partial \vec{E}}. \tag{1.12}$$

This field is allowed to become arbitrary large and thus infinite as long as the physical field $\vec{E}$ stays finite. Now calculate $\vec{D}$ using (1.11) and (1.12)

$$\vec{D} = \frac{\vec{E}}{\sqrt{1 - \frac{\vec{E}^2}{b^2}}}. \tag{1.13}$$

Rewriting this to get an expression for $\vec{E}$ gives

$$\boxed{\vec{E}^2 = b^2 \left(\frac{\vec{D}^2}{\vec{D}^2 + b^2} \right)}. \tag{1.14}$$

It can be seen that however large the magnitude of $\vec{D}$ becomes, the magnitude of $\vec{E}$ never becomes larger than b. Therefore we have shown the existence of a maximum electric field.

Furthermore it can easily be argued that the self-energy of a cpp is finite at present. The problem in the classical theory of electromagnetism is that the electric field blows up close to the particle causing the self-energy to become infinite. With BI theory this is not the case, because when approaching a charged particle, the electric field at a certain point stops increasing rendering the self-energy finite!

We have shown that BI contains the desired properties which Maxwell does not have. Compare Maxwell and BI with Newton and Einstein respectively. Einstein describes gravity far more accurately then Newton. For electromagnetism the same is true for BI. Both BI's and Einstein's theory can be written in terms of the Minkowsky metric (flat space) with the influence of the fields incorporated by

$$\begin{aligned} \text{Einstein:} \quad & \eta_{\mu\nu} \to g_{\mu\nu}, \\ \text{BI:} \quad & \eta_{\mu\nu} \to \eta_{\mu\nu} + F_{\mu\nu}. \end{aligned} \tag{1.15}$$

We point out that we have introduced BI theory due to the fact that it arises in string theory. The next chapter will show the occurrence of BI in string theory.

2. THE BORN-INFELD LAGRANGIAN IN STRING THEORY

The bulk part of this chapter will be about D-branes and electromagnetic fields living on them in order to develop some background knowledge. After that the final goal of this chapter will be discussed, the Lagrangian governing the world volume of a D-brane. Therefore if the reader is already familiar with D-branes, the way electromagnetic fields live on D-branes and T-duality, it is suggested he or she continues with the last section of this chapter. As mentioned in the previous chapter, the specific Lagrangian governing the world volume of a D-brane will be shown to be the Born-Infeld Lagrangian.

2.1 D-branes

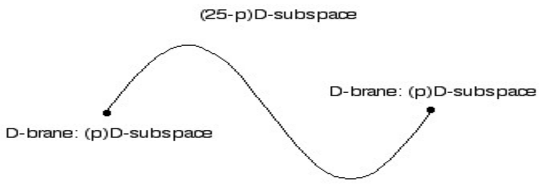
It is know from basic string theory that an open string endpoint (ose) has the possibility to obey Dirichlet boundary conditions[1]. These boundary conditions were a clue to the existence of D-branes. The physical interpretation of a Dirichlet boundary condition is that an ose must remain attached to some object in that direction. This object got the name D(irichlet)-brane.

In a given spacetime (26 dimensional) a Dp-brane (if $p < 25$) is a lower dimensional object with p spatial dimensions on which an ose can end. Once an ose is attached to a Dp-brane, it can never become detached again. At least not without the influences of other D-branes and strings. The subspace of the ose coordinates coinciding with the coordinates of the whole space minus the space of the Dp-brane still have to obey the usual Neumann boundary conditions.

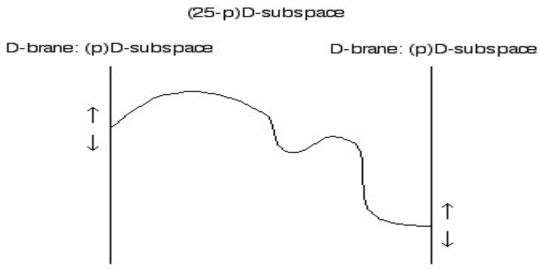
To visualize the concept of a Dp-brane consider two freely moving pins with a rubber band spanned in between in 2D-space. The rubber band and pins are analogous to a string with its endpoints fixed to two Dp-branes by 2 times 2 Dirichlet boundary conditions ($p = 23$). If one would look inside the pins one would see the endpoints of the string moving freely on the 23-dimensional manifold of the D23-brane (Neumann boundary conditions). Imagine that one dimension of the D23-brane is unwrapped, i.e. the pin becomes a nail. Figure 2.1 visualizes the unwrapping. Be careful not to consider the nail integrated into the 2D space. The endpoints of the rubber band can move freely over the nails just like open string endpoints can move freely within the world volume of a D23-brane.

Now that the concept of a D-brane is clarified, we can look at how one should quantize open strings ending on D-branes in order to see what particles the quantization process gives.

[1] $\partial_\tau X^\mu(\tau,0) = 0 = \partial_\tau X^\mu(\tau,l)$



(a) A string spanned between two pins with hidden extra dimensions.



(b) A string spanned between two pins with one of the invisible dimensions made visible.

Fig. 2.1: Visualization of a string spanned between two D-branes.

2.2 Quantizing Open Strings Ending on D-branes

We can ask the important question of the way an open string attached to a Dp-brane ($p < 25$) should be quantized. The question is not too difficult to answer, because one in fact already knows how a string attached to a D25-brane is quantized; it is just quantizing the open string with a Neumann boundary condition for every coordinate.

Consider a Dp-brane ($p < 25$) with a string attached to it. The endpoints of the string obey Neumann boundary conditions for the coordinates coinciding with the world volume of the Dp-brane. But as mentioned before, the rest of the coordinates obey Dirichlet boundary conditions. So the situation is slightly different then the known quantization of open strings.

In order to quantize divide the coordinates of the string into two parts. One part of the coordinates is strictly tangential and the other part strictly normal to the Dp-brane coordinates

$$x^0, x^1, \ldots, x^p \text{ (tangential)}, \qquad x^{p+1}, x^{p+2}, \ldots, x^{25} \text{ (normal)}.$$

Quantizing the coordinates transversal to the Dp-brane is already known and fortunately are the only ones of importance to us. Performing the light-cone quantization on the tangential coordinates gives $(p+1)-2$ massless states. These states live on the world volume of the Dp-brane. From quantum field theory it is known that if the number of independent states is two less then the dimension of spacetime, the states are massless vector states (Maxwell field) [4]. Therefore the final conclusion of the quantization procedure is that a Maxwell field lives on the world volume of a Dp-brane.

2.3 Electromagnetic Field Coupling to the Open String

We have acquired the knowledge of a Maxwell field living on the world volume of a D-brane. Therefore the validity of the boundary conditions of the open string have to be questioned. They should be altered if an ose carries charge and indeed it does carry charge (charge distribution of the string: $q(\sigma)$), but the how and why will not be shown in this thesis. See for example [5] for detailed information.

We continue to construct the new Lagrangian giving rise to new boundary conditions. The Lagrangian will be constructed in analogy to the Lagrangian of a charged particle for which the vector potential A_μ couples to the current. Therefore we can write the Lagrangian for the string with the endpoints coupling to a Maxwell field of a D-brane like

$$S = \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \mathscr{L}_{\mathrm{NG}}(\dot{X}, X') + \int \mathrm{d}\tau\, \mathrm{d}\sigma\, q(\sigma) A_m(X) \frac{\mathrm{d}X^m}{\mathrm{d}\tau}. \tag{2.1}$$

Where m denotes the coordinates of the world volume of the D-brane and runs from 0 to p with $p \leq 25$. The interaction terms only play a role at the endpoints of the string. The value of $q(\sigma)$ equals 1 for $\sigma = l$ and -1 for $\sigma = 0$. After having performed the σ-integration in the interaction part of the action we are left with

$$S = \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \mathscr{L}_{\mathrm{NG}}(\dot{X}, X') + \int \mathrm{d}\tau\, A_m(X) \frac{\mathrm{d}X^m}{\mathrm{d}\tau}\Big|_{\sigma=l} - \int \mathrm{d}\tau\, A_m(X) \frac{\mathrm{d}X^m}{\mathrm{d}\tau}\Big|_{\sigma=0}. \tag{2.2}$$

To simplify matters considerably choose a constant electromagnetic field (F_{mn} =cst, where $F_{mn} = \partial_m A_n - \partial_n A_m$ is the usual field tensor) on the world volume of the Dp-brane. The vector potential then becomes

$$A_n(X) = \frac{1}{2} F_{mn} X^m. \tag{2.3}$$

Proceed by varying the action (2.2) and using (2.3) to give

$$\begin{aligned} \delta S \;=\; & \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \left[\frac{\partial \mathscr{L}_{\mathrm{NG}}(\dot{X}, X')}{\partial \dot{X}^\mu} \partial_\tau \delta X^\mu + \frac{\partial \mathscr{L}_{\mathrm{NG}}(\dot{X}, X')}{\partial X'^\mu} \partial_\sigma \delta X^\mu \right] \\ & + \frac{1}{2} \int \mathrm{d}\tau\, F_{mn} \left[\delta X^m \partial_\tau X^n + X^m \partial_\tau \delta X^n \right] \Big|_{\sigma=l} \\ & - \frac{1}{2} \int \mathrm{d}\tau\, F_{mn} \left[\delta X^m \partial_\tau X^n + X^m \partial_\tau \delta X^n \right] \Big|_{\sigma=0}. \end{aligned} \tag{2.4}$$

The first integral contains the canonical momenta $\frac{\partial \mathscr{L}_{\mathrm{NG}}(\dot{X}, X')}{\partial \dot{X}^\mu} = \mathscr{P}^\tau_\mu$ and $\frac{\partial \mathscr{L}_{\mathrm{NG}}(\dot{X}, X')}{\partial X'^\mu} = \mathscr{P}^\sigma_\mu$. Without the endpoints of the string attached to a Dp-brain, the wave equation $\Box X^\mu$ is obtained from the Nambu-Goto action. The present situation is the same as the free string, except for the endpoints. Therefore the string itself obeys the wave equation as before, but the boundary conditions for the endpoints are changed. By using $\Box X^\mu = 0$ write (2.4) like

$$\begin{aligned}\delta S &= \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \Big[\partial_\sigma(\mathscr{P}^\sigma_\mu \delta X^\mu) + \partial_\tau(\mathscr{P}^\tau_\mu \delta X^\mu)\Big] \\ &\quad + \frac{1}{2}\int \mathrm{d}\tau\, F_{mn}\,[\delta X^m \partial_\tau X^n + X^m \partial_\tau \delta X^n]\Big|_{\sigma=l} \\ &\quad - \frac{1}{2}\int \mathrm{d}\tau\, F_{mn}\,[\delta X^m \partial_\tau X^n + X^m \partial_\tau \delta X^n]\Big|_{\sigma=0}. \end{aligned} \tag{2.5}$$

Obviously, coordinates normal to the brane obey the usual Neumann boundary conditions, but the transverse directions do not. To proceed, integrate over sigma and note that $\partial_\tau(\mathscr{P}^\tau_\mu \delta X^\mu)$ gives zero when integrating over τ. The result is

$$\delta S = \int \mathrm{d}\tau\; \delta X^m\,(\mathscr{P}^\sigma_m + F_{mn}\partial_\tau X^n)\Big|_{\sigma=l} - \int \mathrm{d}\tau\; \delta X^m\,(\mathscr{P}^\sigma_m + F_{mn}\partial_\tau X^n)\Big|_{\sigma=0}. \tag{2.6}$$

The boundary conditions for the endpoints thus are

$$\mathscr{P}^\sigma_m + F_{mn}\partial_\tau X^n = 0 \qquad\qquad \sigma = 0, l. \tag{2.7}$$

And using that $\mathscr{P}^\sigma_\mu = -\frac{1}{2\pi\alpha'}\partial_\sigma X_\mu$ for the boundary conditions (2.7) gives

$$\boxed{\partial_\sigma X_m - 2\pi\alpha' F_{mn}\partial_\tau X^n = 0 \qquad\qquad \sigma = 0, l}. \tag{2.8}$$

The boundary conditions for an open string attached to a Dp-brane are obtained. If there is no electromagnetic field, the boundary conditions reduce to the Neumann boundary conditions. On the other hand an infinitely strong field will completely freeze the endpoints of the open string which means we have Dirichlet boundary conditions. The upcoming section will be about the boundary conditions of the open string attached to a Dp-brane which has a constant electric field ($F_{ij} = 0;\ i$ or $j = 0$) living on its world volume.

2.4 D-branes Containing Electric Fields

About T-duality, T-duality is a transformation which leaves the physics invariant [5]. Due to this property of T-duality, two to the eye very different systems can be shown to be equivalent. Therefore new properties and/or insights into either of the systems can be derived from each other. This section will deal with the T-dual of a D-brane containing an electric field.

But before we go to the T-duality part, we will rewrite the boundary conditions (2.8) into a more convenient expression. To simplify the upcoming expressions, a constant electric field along the 10th compactified dimension of the Dp-brane is chosen. Choose

$$F_{10} = E. \tag{2.9}$$

The only relevant changes in the boundary conditions of the open string are for the X^0 and X^{10} coordinates. The boundary conditions (2.8) become (keep in mind we raised/lowered indices)

$$\partial_\sigma X^0 - \mathscr{E}\partial_\tau X^{25} = 0, \tag{2.10a}$$

$$\partial_\sigma X^{25} - \mathscr{E}\partial_\tau X^0 = 0. \tag{2.10b}$$

Where $\mathscr{E} = 2\pi\alpha' E$. The idea is to write these equations in a matrix, because it enables us to work with transformation matrices later on. In order to construct a matrix form we define

$$\partial_+ = \frac{1}{2}(\partial_\tau + \partial_\sigma), \qquad \partial_- = \frac{1}{2}(\partial_\tau - \partial_\sigma). \tag{2.11}$$

The boundary conditions can then be written as

$$\partial_+ X^0 - \mathscr{E}\partial_+ X^{10} = \partial_- X^0 + \mathscr{E}\partial_- X^{10}, \tag{2.12a}$$

$$-\mathscr{E}\partial_+ X^0 + \partial_+ X^{10} = \mathscr{E}\partial_- X^0 + \partial_- X^{10}. \tag{2.12b}$$

In matrix form this becomes

$$\partial_+ \begin{pmatrix} X^0 \\ X^{10} \end{pmatrix} = \begin{pmatrix} \frac{1+\mathscr{E}^2}{1-\mathscr{E}^2} & \frac{2\mathscr{E}}{1-\mathscr{E}^2} \\ \frac{2\mathscr{E}}{1-\mathscr{E}^2} & \frac{1+\mathscr{E}^2}{1-\mathscr{E}^2} \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X^{10} \end{pmatrix}. \tag{2.13}$$

It is known that a Dp-brane containing no electromagnetic field and a compact dimension of radius R is T-dual to a D$(p-1)$-brane attached to a point on the dual compact dimension of radius $\tilde{R} = \frac{\alpha'}{R}$. However, we have a D$p$-brane with a constant electric field along this compact dimension. It is not clear how a T-duality transformation should be performed in this situation. Therefore we will try some configuration of a D$(p-1)$-brane in the dual world and transform it by means of a T-duality transformation back to the present world. Explicitly, we will show that a D$(p-1)$-brane moving along the dual compact dimension with constant velocity leads to the same boundary conditions as the Dp-brane with the electric field. The moving D$(p-1)$-brane must therefore be the T-dual of the Dp-brane with the constant electric field, because as said a T-duality transformation is not supposed to change the physics. Coordinates without a prime are with respect to the frame S in the dual world which denotes the frame at rest at some point in the compact dimension. S' is the frame moving along with the D$(p-1)$-brane. Note that X'^0 is Neumann and $\tilde{X}'^{10}$ is Dirichlet. Consider the following expression

$$\partial_+ \begin{pmatrix} X'^0 \\ \tilde{X}'^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X'^0 \\ \tilde{X}'^{10} \end{pmatrix}. \tag{2.14}$$

At first sight this expression may look irrelevant; it just says $\partial_\tau X'^0 = \partial_\tau X'^0$. This is true, but we are looking for an expression that has the same form as (2.13) in order to be able to compare the two. Due to the matrix forms the two expressions will always be comparable. We want to know what expression (2.14) becomes in the present world. The only way we know how to get to the present world is by performing a T-duality transformation on (2.14) expressed in the S-frame.

Therefore the required transformations are first a boost from the S' to the S frame and second a T-duality transformation from the dual world to the present world.

As said, first perform the boost. Suppose the S'-frame has a velocity β. The 26 dimensional spacetime is still assumed to be flat, so the boost is performed in exactly the same manner as we are used to in 4D spacetime.

$$\begin{aligned} X'^{0} &= \gamma(X^{0} - \beta\tilde{X}^{10}), \\ \tilde{X}'^{10} &= \gamma(-\beta X^{0} + \tilde{X}^{10}). \end{aligned} \tag{2.15}$$

This is equivalent to

$$\begin{pmatrix} X'^{0} \\ \tilde{X}'^{10} \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} X^{0} \\ \tilde{X}^{10} \end{pmatrix} \equiv M \begin{pmatrix} X^{0} \\ \tilde{X}^{10} \end{pmatrix}. \tag{2.16}$$

Therefore (2.14) can be written like

$$\partial_{+} \begin{pmatrix} X^{0} \\ \tilde{X}^{10} \end{pmatrix} = M^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M \, \partial_{-} \begin{pmatrix} X^{0} \\ \tilde{X}^{10} \end{pmatrix}. \tag{2.17}$$

The first transformation is done. Performing the T-duality transformation is not difficult, because a T-duality transformation only changes a sign in the right movers of a string coordinate

$$\begin{aligned} \tilde{X}^{10} &= X_{L}^{10}(\tau+\sigma) + X_{R}^{10}(\tau+\sigma), \\ \tilde{X}^{10} &= X_{L}^{10}(\tau+\sigma) - X_{R}^{10}(\tau+\sigma). \end{aligned} \tag{2.18}$$

So the ∂_{+} and the ∂_{-} acting on the 10th coordinate transform like

$$\partial_{+}\tilde{X}^{10} = \partial_{+}X^{10}, \qquad\qquad \partial_{-}\tilde{X}^{10} = -\partial_{-}X^{10}. \tag{2.19}$$

From this T-duality transformation matrices $T_{\pm}$ belonging to the two-vectors $\partial_{\pm}(X^{0}, X^{10})$ can be constructed

$$T_{\pm} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{2.20}$$

Equation (2.17), after having applied the T-duality transformation, becomes

$$\partial_{+} \begin{pmatrix} X^{0} \\ X^{10} \end{pmatrix} = T_{+}^{-1} M^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M T_{-} \partial_{-} \begin{pmatrix} X^{0} \\ X^{10} \end{pmatrix}. \tag{2.21}$$

Working out the matrix multiplications gives

$$\partial_{+} \begin{pmatrix} X^{0} \\ X^{10} \end{pmatrix} = \begin{pmatrix} \frac{1+\beta^{2}}{1-\beta^{2}} & \frac{2\beta}{1-\beta^{2}} \\ \frac{2\beta}{1-\beta^{2}} & \frac{1+\beta^{2}}{1-\beta^{2}} \end{pmatrix} \partial_{-} \begin{pmatrix} X^{0} \\ X^{10} \end{pmatrix}. \tag{2.22}$$

The moment of comparison has arrived. We find ourselves in the present world and have two relations between the vectors $\partial_+(X^0, X^{25})$ and $\partial_-(X^0, X^{25})$, equation (2.13) and (2.22) respectively. We want the two expressions to be the same by T-duality arguments. In order to equalize the two equations, we make the following identification

$$\mathscr{E} = \beta. \tag{2.23}$$

With this relation it is easy to see that the two equations are equivalent in a consistent way. Therefore we have shown, by means of T-duality, that a D$(p-1)$-brane moving along the dual compact dimension gives the same physics as a Dp-brane with an electric field pointing along the direction of the compact dimension.

Although the identification of the electric field with the velocity was easily done, it has major implications. We have actually shown the existence of a maximum electric field strength in string theory, because β depends linearly on the velocity and obeys $\beta \leq 1$ always. Consequently, this yields for E

$$\boxed{E_{\text{max}} = \frac{1}{2\pi\alpha'}}. \tag{2.24}$$

Keep in mind that this is in the absence of a magnetic field. Having a general electromagnetic field living on the Dp-brane only results in a different value of the maximum electric field strength, higher or lower. The point is, there exists a maximum electromagnetic field strength. The maximum electric field property of string theory points in the direction for the need of a different theory of electromagnetism, because electromagnetism in string theory is certainly not fully described by Maxwell's theory.

2.5 D-branes Containing Magnetic Fields

We have obtained surprising physics by finding the T-dual of a Dp-brane containing a constant electric field on its world volume. Therefore it may also be interesting to find out what physics the T-dual of a Dp-brane with a constant magnetic field gives. We expect to recover the maximum electromagnetic field strength property in any case. The space we will consider has the 3rd dimension compactified. Furthermore we choose the vector potential such that its 3rd component is the only nonzero one. We have

$$A_2 = 0, \quad A_3 = Bx_2 \qquad \Rightarrow \qquad F_{23} = B, \quad F_{32} = -B. \tag{2.25}$$

The T-duality transformation back to the present world will be performed along the compact dimension. But the burning question again is: what is in this case the T-dual of the Dp-brane containing the magnetic field? Again we state the solution and show that it *is* the solution. The T-dual brane of the given Dp-brane is a D$(p-1)$-brane located at some point x_3 and tilted by an angle ω. We therefore define an auxiliary coordinate system S' which has its basis transverse to the tilted D$(p-1)$-brane.

The time has come to start worrying about boundary conditions. In complete analogy to the Dp-brane with the electric field, we can write boundary conditions for the Dp-brane containing the magnetic field as

$$\partial_+ \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} \frac{1-\mathscr{B}^2}{1+\mathscr{B}^2} & \frac{2\mathscr{B}}{1+\mathscr{B}^2} \\ -\frac{2\mathscr{B}}{1+\mathscr{B}^2} & \frac{1-\mathscr{B}^2}{1+\mathscr{B}^2} \end{pmatrix} \partial_- \begin{pmatrix} X^2 \\ X^3 \end{pmatrix}. \tag{2.26}$$

Where $\mathscr{B} = 2\pi\alpha' B$. As with the electric field case we need an expression for the string coordinates in the dual world that looks similar to the boundary conditions for the string coordinates of the present world. Again in complete analogy to the previous section we can write the identity

$$\partial_+ \begin{pmatrix} X^{2'} \\ \tilde{X}^{3'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X^{2'} \\ \tilde{X}^{3'} \end{pmatrix}. \tag{2.27}$$

Two transformations are again needed. One to rotate the S'-frame back by an angle ω to the S-frame and after that the T-duality transformation, but first things first. The rotation matrix of a point around some origin in 2D-space by an angle ω is given by

$$\begin{pmatrix} X^{2'} \\ \tilde{X}^{3'} \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} \equiv R \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix}. \tag{2.28}$$

Like in the case of the electric field we write

$$\partial_+ \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} = R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R\, \partial_- \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix}. \tag{2.29}$$

To see what this equation gives for conditions on the X^2 and X^3 coordinates, T-dual transform back to the present world. It is known from equation (2.20) of the previous section how to perform the T-duality transformation. Insert the matrices into the proper places to get

$$\partial_+ \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} = T_+^{-1} R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R T_- \partial_- \begin{pmatrix} X^2 \\ X^3 \end{pmatrix}. \tag{2.30}$$

Working out the matrix manipulations gives us the matrix which must be identified with the matrix of equation (2.26)

$$\partial_+ \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} = \begin{pmatrix} \cos 2\omega & -\sin 2\omega \\ \sin 2\omega & \cos 2\omega \end{pmatrix} \partial_- \begin{pmatrix} X^2 \\ X^3 \end{pmatrix}. \tag{2.31}$$

In the electric field case it was quite clear which identification had to be made to link all the elements [x,y] of the matrices without contradictions. In the present case it is not clear that if one makes an identification of for example the entries of [1,1], the identification yields for all entries [x,y]. Lets just try the identification for the [1,1] entries and see if it is consistent with other entries as well. Equating the [1,1] entries give

$$\frac{1-\mathscr{B}^2}{1+\mathscr{B}^2} = \cos 2\omega \qquad \Rightarrow \qquad \mathscr{B}^2 = \tan^2\omega. \tag{2.32}$$

We would like this to be consistent with

$$\frac{2\mathscr{B}}{1+\mathscr{B}^2} = -\sin 2\omega. \tag{2.33}$$

One can easily check by the use of some geometric identities that the above is consistent with (2.32).

First of all, does equation (2.32) stroke with the notion of a maximum electromagnetic field? Naively spoken one can raise the value of ω all the way to $\frac{1}{2}\pi$, the value where the tangent becomes infinite. However if $\omega = \frac{1}{2}\pi$ we have effectively interchanged dimensions x^2 and x^3, because $x^{'2}$ can be identified with x^3 and $x^{'3}$ with $-x^2$. Then we are back to the situation in which the $\mathrm{D}(p-1)$-brane is not tilted. Because of the previous argument the only relevant interval for ω giving different physical situations is $[-\frac{1}{4}\pi, \frac{1}{4}\pi]$. The maximum value for B is therefore

$$\boxed{B_{\max} = \frac{1}{2\pi\alpha'}}. \tag{2.34}$$

Where are we so far? We have learned that a Dp-brane containing a constant magnetic field is T-dual to a rotated $\mathrm{D}(p-1)$-brane giving rise to a maximum magnetic field strength. Furthermore from the tilting of the $\mathrm{D}(p-1)$-brane some interesting conclusions regarding the vector potential and therefore the x^2 dimension can be drawn. These conclusions will be used in the next section which deals with the occurrence of Born-Infeld theory in string theory.

The vector potential was chosen as $(A_2, A_3) = (0, Bx^2)$ with x^3 compact with radius R_3. It is known from string theory that in this case the vector potential is quantized [5]. We have

$$A_3 \sim A_3 + \frac{n}{R^3}, \qquad\qquad n \in \mathbb{Z}. \tag{2.35}$$

The interesting thing is that if one starts for example at $x^2 = 0$ and proceeds in the positive x^2 direction, the value of A_3 increases until the value of $\frac{n}{BR_3}$ is reached. Then A_3 jumps back to its previous value it had at $x^2 = 0$. Therefore we have a periodic vector potential and so it can be claimed that there is a repetition of the physics when proceeding in the x^2 direction. This has to have some impact in the dual world. To find the implications define as the period of the vector potential in the present world as

$$\triangle x^2 = \frac{n}{BR_3}. \tag{2.36}$$

In the dual world, using $n \to -n$, (2.36) becomes

$$\triangle x^2 = -\frac{n\tilde{R}_3}{\alpha' B} = \frac{2\pi n\tilde{R}_3}{\tan\omega} \qquad \Rightarrow \qquad \tan\omega = \frac{2\pi n\tilde{R}_3}{\triangle x^2}. \tag{2.37}$$

Where we have used equation (2.32). Walking along x^2 gives the same physics after a period of $\triangle x^2$. We can therefore write an effective domain for x^2

$$x^2 \ \in \ \mathbb{R} \ \bmod \frac{\tilde{R}_3}{\alpha' B}. \tag{2.38}$$

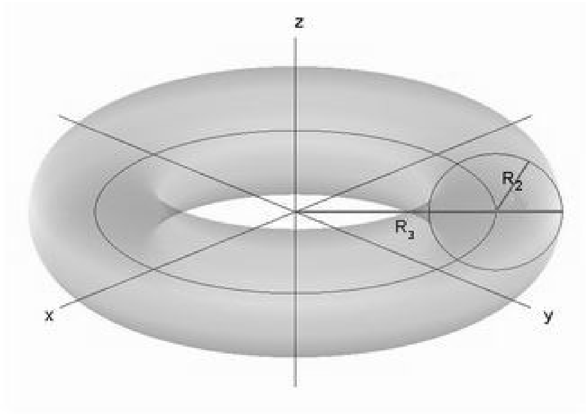


Fig. 2.2: Visualization of two compactified dimensions x^2 and x^3. Other dimensions are contained in the surface of the two-torus.

In other words we have compactified the x^2 dimension. Suppose the radius of this compact dimension is given by R_2. Then $\triangle x^2 = 2\pi R_2$ gives the period of the dimension. Using the result of equation (2.37) gives

$$\boxed{\tan\omega = n\frac{\tilde{R}_3}{R_2}}. \tag{2.39}$$

Equation (2.39) tells us that ω is quantized. The quantization means physically that somebody walking along the two-torus by an angle ω, arrives exactly at his starting point. The D$(p-1)$-brane lies on the diagonal of a two-torus determined by the radii R_2 and R_3 (figure 2.2). This result is very important, because we will need it when making the connection of the Lagrangian of the world volume of a D-brane with the Born-Infeld Lagrangian in the next section.

2.6 Lagrangian Governing the World Volume of a D-brane

This is the final section of this chapter. We will show by means of T-duality the way the BI Lagrangian enters string theory. To sketch the configuration we will use for this section, the dual world has $p-1$ compact spatial dimensions and one time dimension which spans the D$(p-1)$-brane's world volume. Furthermore, the dimensions x^2 and x^3 are both compact and form a two-torus as in the previous section. To begin with the analysis, we equate the mass of a D$(p-1)$-brane in the dual world with the mass of a Dp-brane. The mass of a general D-brane is given by its tension times its volume. Therefore we have for the D$(p-1)$-brane that the volume V_{p-1} enclosed by the D$(p-1)$-brane is equal to L_{diag}, the length of the diagonal on the two-torus times V_{p-2}. The tension of the D$(p-1)$-brane in the dual world we define as $T_{p-1}(\tilde{g})$. In formula

$$M_{p-1} = T_{p-1}(\tilde{g})V_{p-1} = T_{p-1}(\tilde{g})V_{p-2}L_{\text{diag}}. \tag{2.40}$$

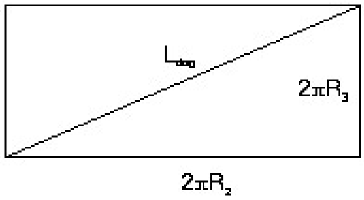


Fig. 2.3: A two-torus unfolded with radii R^2 and R^3.

How to construct a Lagrangian from this? Actions are based on the fact that a particle, string or even a membrane traces out the minimum volume in the target space during the evolution of the parameters of the world space. The volume needs to be multiplied by a constant in order to fix the dimensionality of the action. In string theory this factor is the tension of the string, which multiplied by the volume, gives the string mass. Extrapolating this argument to D-branes, the Lagrangian of a D-brane is given by

$$L_{\mathrm{p}-1} = -T_{p-1}(\tilde{g})L_{\mathrm{diag}}V_{p-2}. \tag{2.41}$$

We are only interested in what the Lagrangian looks like in the present world. Remember that x^3 is compact, so it is necessary to know what L_{diag} and $T_{p-1}(\tilde{g})$ transforms like under a T-duality transformation along this dimension (V_{p-2} does not include the dual dimension and is therefore equal in both worlds).

First we deal with $T_{p-1}(\tilde{g})$. As mentioned before, T-duality is a transformation which leaves the physics invariant. Due to this fact the mass of a Dp-brane must be the same as the mass of its dual D$(p-1)$-brane. Therefore equating the masses of the different D-branes gives

$$2\pi R_3\, V_{p-1}T_p(g) = V_{p-1}T_{p-1}(\tilde{g}) \tag{2.42}$$

Equation (2.42) gives the T-duality relation between the tensions in the two worlds. Next we calculate L_{diag}. Consider figure (2.3) which is the torus of figure (2.2) unwrapped. It follows from the Pythagorean theorem that

$$L_{\mathrm{diag}} = \sqrt{(2\pi R_2)^2 + (2\pi\tilde{R}_3)^2}. \tag{2.43}$$

Using equation (2.42) and (2.43) together gives for the Lagrangian of the D$(p-1)$-brane

$$L_{\mathrm{p}-1} = -V_{p-2}(2\pi R_2)(2\pi R_3)\sqrt{1+\left(\frac{R_3}{R_2}\right)^2}\,T_p(g). \tag{2.44}$$

Performing the last T-duality transformation on the quotient inside the square root using the important result (2.39) of the previous section gives

$$L_{\mathrm{p}} = -V_pT_p(g)\sqrt{1+(2\pi\alpha' B)^2}. \tag{2.45}$$

Or as a Lagrangian density

$$\mathscr{L}_{\text{Dp}-\text{brane}} = -T_p(g)\sqrt{1+(2\pi\alpha')^2(B)^2}. \tag{2.46}$$

We have arrived at the final part of this section. We obtained a Lagrangian density for the world volume of a Dp-brane. The only thing left to do is to show that the Lagrangian density is analogous to the Born-Infeld Lagrangian density (1.8b). Therefore set the $\vec{E}$-field of equation (2.46) equal to zero to give

$$\mathscr{L}_{\text{BI}} = -b^2\sqrt{1+\frac{B^2}{b^2}}+b^2. \tag{2.47}$$

As one can see, the two Lagrangian densities do not look entirely the same. The strategy we will use to show that they nonetheless are, is to start with the Born-Infeld Lagrangian density and argue it equivalent to the Dp-brane Lagrangian density.

First of all, one can add an arbitrary constant to a Lagrangian density without changing the equations of motions. Therefore the extra constant b^2 can be omitted in the Born-Infeld Lagrangian; it was originally included to cancel the constant coming from the square-root (upon expanding it). The constant is now needed to represent the rest energy of the Dp-brane. Then there is the matter of the constant b^2 in front of the square-root. The b^2 in the Born-Infeld was included in order to obtain the Maxwell theory with a particular chosen normalization in the low field strength limit. The constant in equation (2.46) is clearly different. How to handle this? We just obtained the constant T_p and there is nothing we can do about it. So we have to define

$$T_p(g) = \frac{1}{2\pi\alpha'} \equiv b^2. \tag{2.48}$$

By those arguments we have obtained our first piece of evidence that the lagrangian density of the Dp-brane is governed by the Born-Infeld lagrangian density.

Another piece of evidence is to consider the following Lagrangian for a moving D$(p-1)$-brane in the dual space

$$L_{p-1} = -T_{p-1}(\tilde{g})V_{p-1}\sqrt{1-\frac{v^2}{c^2}}. \tag{2.49}$$

The relativistic factor comes from the fact that we have a moving D$(p-1)$-brane observed by a nonmoving observer. Using the relation (2.23) obtained in section 2.4 for (2.49) gives

$$L_{p-1} = -T_{p-1}(\tilde{g})V_{p-1}\sqrt{1-(2\pi\alpha')^2E^2}. \tag{2.50}$$

Next, T-dual transform the tension back to the present world by using (2.42) to obtain the Lagrangian

$$L_p = -2\pi R_3 T_p(g)V_{p-1}\sqrt{1-(2\pi\alpha')^2E^2}. \tag{2.51}$$

Using (2.42) gives

$$L_p = -T_p(g)V_p\sqrt{1-(2\pi\alpha')^2E^2}. \tag{2.52}$$

Or as a lagrangian density

$$\mathscr{L}_p = -T_p(g)\sqrt{1-(2\pi\alpha')^2E^2}. \tag{2.53}$$

Behold, this is the same Lagrangian density as equation (1.8b) by letting $B \to 0$. We now have two pieces of evidence that world volumes of Dp-branes are governed by Born-Infeld lagrangian densities.

As a final argument why the Born-Infeld lagrangian density is the correct one, consider the lagrangian density

$$\boxed{\mathscr{L}_\mathrm{p} = -T_p(g)\sqrt{-\det\left(\eta_{mn}+2\pi\alpha' F_{mn}\right)}} \tag{2.54}$$

Remember section 2 of the previous chapter. There was a discussion about how the electric field changes the metric of spacetime. The same discussion applies to the world volume of a D-brane. Furthermore the mass of a D-brane arises from its tension. By these heuristics the lagrangian density (2.54) should be the correct one.

After a great deal of analysis we have arrived at the final goal of this chapter, the lagrangian density governing the world volume of a Dp-brane. Keep in mind that that this lagrangian density only includes massless interactions, but more on this later on. In the next chapter the Born-Infeld lagrangian density will be discussed in more detail.

3. ELECTROMAGNETIC DUALITY

In this chapter we will deal with the possibility of containing electromagnetic duality invariance by some theories of electromagnetism. In particular the electromagnetic duality invariance of the Born-Infeld theory will be shown. First we will show what electromagnetic duality is all about, both for the equations of motion and for the Lagrangian themselves by using Maxwell as an example. After that a general condition for a Lagrangian to be electromagnetic duality invariant is derived. Born-Infeld theory will be shown to be electromagnetic duality invariant using the obtained invariance condition in the final part of this chapter.

3.1 Maxwell and Electromagnetic Duality Invariance

As mentioned above, the first thing we will do is show that Maxwell's theory is electromagnetic duality invariant. Therefore we write down the Maxwell equations (1.1) in terms of $\vec{E}$ and $\vec{B}$ fields again

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \\ \nabla \cdot \vec{B} &= 0, \\ \nabla \cdot \vec{E} &= 0, \\ \nabla \times \vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \end{aligned} \tag{3.1}$$

These equations harbor another symmetry besides the familiar Lorentz symmetry. Consider the transformation that transforms electric into magnetic fields and vice versa

$$\begin{aligned} \vec{E} &\rightarrow a\,\vec{E} - b\,\vec{B}, \\ \vec{B} &\rightarrow a\,\vec{B} + b\,\vec{E}. \end{aligned} \tag{3.2}$$

This is the most general transformation for the $\vec{E}$ and $\vec{B}$ fields which leaves the equations of motions invariant. The transformation is equivalent to the transformation of the group $SO(2)$, if we demand the determinant of the transformation matrix to be one. Therefore the final electromagnetic duality transformation becomes

$$\begin{aligned} \vec{E} &\rightarrow \cos\alpha\,\vec{E} - \sin\alpha\,\vec{B}, \\ \vec{B} &\rightarrow \cos\alpha\,\vec{B} + \sin\alpha\,\vec{E}. \end{aligned} \tag{3.3}$$

Equation (3.3) is the duality transformation (duality stands for electromagnetic duality from now on) of the Maxwell theory in vector notation.

Next we are going to look at the duality transformation of the covariant Maxwell equations. We write the Maxwell equations in covariant form, equation (1.2)

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0, \\ \partial_\mu \star F^{\mu\nu} &= 0. \end{aligned} \tag{3.4}$$

It can easily be checked that the equations of motions are invariant under a $GL(2)$ transformation

$$\begin{aligned} F'_{\mu\nu} &= a\, F_{\mu\nu} + b\, \star F_{\mu\nu}, \\ \star F'_{\mu\nu} &= c\, \star F_{\mu\nu} + d\, F_{\mu\nu}. \end{aligned} \tag{3.5}$$

But the $GL(2)$ transformation in (3.5) reduces to the $SO(2)$ transformation, because it is equivalent to equation (3.3).

To conclude, the invariant duality transformation of the Maxwell theory in covariant notation is

$$\begin{aligned} F'_{\mu\nu} &= \cos\alpha\, F_{\mu\nu} + \sin\alpha\, \star F_{\mu\nu}, \\ \star F'_{\mu\nu} &= \cos\alpha\, \star F_{\mu\nu} - \sin\alpha\, F_{\mu\nu}. \end{aligned} \tag{3.6}$$

The duality transformation applied to the Maxwell theory is sometimes called a Hodge rotation. We may ask ourselves the question if the Lagrangians giving rise to duality invariant equations of motions are duality invariant themselves. The next two sections will answer this question.

3.2 $\mathscr{L}$(F)'s Equations of Motion Electromagnetic Duality Invariant

This section will show that the equations of motion of a general Lagrangian are duality invariant. To this end we have to know how equations of motion are derived. Consider a general Lagrangian of electromagnetism which is a function of the vector potential and its derivative

$$\mathscr{L} = \mathscr{L}(A, \partial A). \tag{3.7}$$

However, theories of electromagnetism are always expressed in terms of the field tensor F. Consequently, we rephrase the notion of general into

$$\mathscr{L} = \mathscr{L}(\partial A) = \mathscr{L}(F). \tag{3.8}$$

We will consider really general Lagrangians (for example $\mathscr{L}(F, \partial F)$) later on. The equations of motion belonging to a Lagrangian (3.8) are derived by varying the action

$$S[F] = \int \mathrm{d}^4 x \mathscr{L}(F). \tag{3.9}$$

The variation of the action gives

$$
\begin{aligned}
\delta S &= \frac{1}{2}\int \mathrm{d}^4x \left(\frac{\partial \mathscr{L}}{\partial F_{\mu\nu}}\delta F_{\mu\nu}\right), \\
&= \frac{1}{2}\int \mathrm{d}^4x \left(\frac{\partial \mathscr{L}}{\partial F_{\mu\nu}}\right)\delta F_{\mu\nu}, \\
&= \int \mathrm{d}^4x \, \left(\partial_\mu \frac{\partial \mathscr{L}}{\partial F_{\mu\nu}}\right)\delta A_\nu, \\
&= 0.
\end{aligned}
\tag{3.10}
$$

The expression between brackets must be set equal to zero, giving part of the equations of motion. For convenience define the G tensor as

$$
G_{\mu\nu} \equiv -\frac{\partial \mathscr{L}}{\partial F^{\mu\nu}}. \tag{3.11}
$$

This definition is a correct one, because if the Lagrangian is taken to be the Maxwell Lagrangian, we have

$$
G^{\mu\nu} = F^{\mu\nu}, \tag{3.12}
$$

which gives us Maxwell's equations of motion. By equation (3.10) and (3.11), we state that the complete set of equations of motion are

$$
\begin{aligned}
\partial_\mu G^{\mu\nu} &= 0, \\
\partial_\mu \star F^{\mu\nu} &= 0.
\end{aligned}
\tag{3.13}
$$

The Bianchi identity always has to be included, because it is an inherent property of the field tensor. These equations look like the Maxwell equations of motion, equation (1.2), but they are valid for a general Lagrangian. The next question to ask is: are these equations of motion duality invariant? In order to answer this question, consider the following $GL(2)$ transformation

$$
\begin{aligned}
G'_{\mu\nu} &= a\, G_{\mu\nu} + b\, \star F_{\mu\nu}, \\
\star F'_{\mu\nu} &= c\, \star F_{\mu\nu} + d\, G_{\mu\nu}.
\end{aligned}
\tag{3.14}
$$

These transformations are the duality transformations of the equations of motion of any general Lagrangian dependent only on F due to the definition of $G_{\mu\nu}$. An additional constraint on the constants can arise. For example the $GL(2)$ symmetry reduces to a $SO(2)$ symmetry in the case of the Maxwell Lagrangian. The $GL(2)$ symmetry can also be restricted to the $SO(2)$ symmetry in the case of the BI theory, because in lowest order it reduces to the Maxwell theory. In fact, any potential physical Lagrangian trying to describe high field strengths has to reduce to the Maxwell Lagrangian in the low field limit. Consequently, this gives the duality transformation of the equations of motion of any physically possible Lagrangian dependent only on F

$$
\begin{aligned}
G'_{\mu\nu} &= \cos\alpha\ G_{\mu\nu} - \sin\alpha\ \star F_{\mu\nu}, \\
\star F'_{\mu\nu} &= \cos\alpha\ \star F_{\mu\nu} + \sin\alpha\ G_{\mu\nu}.
\end{aligned} \tag{3.15}
$$

We have shown that a general physical Lagrangian has accompanying equations of motion which are duality invariant. Specifically, the equations of motion contain the $SO(2)$ duality symmetry as the Born-Infeld and Maxwell theory do. But what can be said about the possibility of having the Lagrangian duality invariant itself? A Lagrangian can be an arbitrary polynomial of F, which most likely does not inhabit duality invariance. However, in certain cases it does. The next section will derive a condition on the Lagrangian for it to be duality invariant.

3.3 Electromagnetic Duality Invariant Lagrangians

This section will deal with the duality invariance of Lagrangians. We must stress that by the duality invariance of a Lagrangian, we mean the way the Lagrangian should transform in contrast with the transformation of the fields in order to produce the invariant equations of motion. The simplest example of a duality invariant Lagrangian is the Maxwell Lagrangian

$$
\mathscr{L}_{\text{Max}} = \frac{1}{2}(-\vec{E}^2 + \vec{B}^2). \tag{3.16}
$$

If we apply the duality transformation, equation (3.3), we obtain the Maxwell Lagrangian once again, but with a factor in front. This does not matter, because the dual Maxwell Lagrangian still gives rise to the same equations of motion as the Maxwell Lagrangian. Therefore we say that the Maxwell Lagrangian is duality invariant, while we keep in mind it is not; its transformation gives rise to the same equations of motion.

One can check a Lagrangian on duality invariance by just applying the transformations. On the other hand, one can also derive a duality condition which a Lagrangian has to satisfy in order to contain duality invariance. The way to find this condition is to consider infinitesimal changes of the fields

$$
\begin{aligned}
F'_{\mu\nu} &= F_{\mu\nu} + \delta F_{\mu\nu}, \\
G'_{\mu\nu} &= G_{\mu\nu} + \delta G_{\mu\nu}.
\end{aligned} \tag{3.17}
$$

Next compare (3.17) with an infinitesimal electromagnetic duality transformation of equation (3.15)

$$
\begin{aligned}
\star F'_{\mu\nu} &= \star F_{\mu\nu} - \alpha\ G_{\mu\nu}, \\
G'_{\mu\nu} &= G_{\mu\nu} + \alpha\ \star F_{\mu\nu}.
\end{aligned} \tag{3.18}
$$

So if a Lagrangian is to be duality invariant, we have the following condition on the fields F and G and therefore on $\mathscr{L}$

$$
\begin{aligned}
\delta F_{\mu\nu} &= \alpha \star G_{\mu\nu}, & (3.19)\\
\delta G_{\mu\nu} &= \alpha \star F_{\mu\nu}. & (3.20)
\end{aligned}
$$

Equations (3.19) and (3.20) will be the basis of our derivation of the electromagnetic duality invariance condition on any Lagrangian $\mathscr{L}(F)$. We would like to get an expression in terms of $\mathscr{L}$ and F only, because then one can substitute the explicit expression for $\mathscr{L}$ and easily check if the condition is satisfied.

The condition is derived by writing (3.20) in the following way

$$
\frac{1}{2}\alpha\ \varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} = -\frac{\partial}{\partial F_{\mu\nu}}(\delta\mathscr{L}). \tag{3.21}
$$

The Lagrangian is considered a function of $F_{\mu\nu}$. Therefore we have

$$
\frac{1}{2}\alpha\ \varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} = -\frac{\partial}{\partial F_{\mu\nu}}\left(\frac{1}{2}\frac{\partial\mathscr{L}}{\partial F_{\sigma\tau}}\delta F_{\sigma\tau}\right). \tag{3.22}
$$

Use equation (3.19) to get rid of the varied field tensor and interchange the derivatives. The result is

$$
\begin{aligned}
\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} &= -\frac{1}{4}\varepsilon_{\sigma\tau\alpha\beta}G^{\alpha\beta}\frac{\partial}{\partial F_{\mu\nu}}\left(\frac{\partial\mathscr{L}}{\partial F_{\sigma\tau}}\right) - \frac{1}{4}\varepsilon_{\sigma\tau\alpha\beta}\left(\frac{\partial\mathscr{L}}{\partial F_{\sigma\tau}}\frac{\partial G^{\alpha\beta}}{\partial F_{\mu\nu}}\right),\\
&= \frac{1}{2}\varepsilon_{\sigma\tau\alpha\beta}\frac{\partial\mathscr{L}}{\partial F_{\alpha\beta}}\frac{\partial}{\partial F_{\sigma\tau}}\left(\frac{\partial\mathscr{L}}{\partial F_{\mu\nu}}\right).
\end{aligned} \tag{3.23}
$$

Where we have used the fact that the Levi-Chivita tensor is symmetric under the interchange of the first pair of indices and the last pair. The condition is already in the desired form, but it can be rewritten in a more manageable way

$$
\begin{aligned}
\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} &= \varepsilon_{\sigma\tau\alpha\beta}\frac{\partial}{\partial F_{\mu\nu}}\left(\frac{\partial\mathscr{L}}{\partial F_{\sigma\tau}}\frac{\partial\mathscr{L}}{\partial F_{\alpha\beta}}\right), & (3.24)\\
\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}F^{\mu\nu} &= \varepsilon_{\sigma\tau\alpha\beta}\frac{\partial\mathscr{L}}{\partial F_{\sigma\tau}}\frac{\partial\mathscr{L}}{\partial F_{\alpha\beta}} + C. & (3.25)
\end{aligned}
$$

We already know that the Maxwell Lagrangian is duality invariant up to a minus sign. Fortunately, equation (3.25) is insensitive to this change in minus sign. It can easily be seen that the Maxwell Lagrangian satisfies equation (3.25) up to the integration constant C. Consequently, it must be set equal to zero. The condition for a Lagrangian to be duality invariant is thus

$$\boxed{\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}F^{\mu\nu}=\varepsilon_{\sigma\tau\alpha\beta}\frac{\partial\mathscr{L}}{\partial F_{\alpha\beta}}\frac{\partial\mathscr{L}}{\partial F_{\sigma\tau}}}. \tag{3.26}$$

We have derived a condition which a Lagrangian has to obey in order to be duality invariant. It is also nice to write the duality condition in a compact way as

$$\boxed{\star FF=\star GG}. \tag{3.27}$$

Having obtained the duality condition for a Lagrangian, we will show that BI is duality invariant up to and including order $\mathscr{O}(\alpha'^6)$ in the next section.

3.4 Electromagnetic Duality Invariance of Born-Infeld Lagrangian

In this section we will show the duality invariance of the expanded BI Lagrangian up to and including $\mathscr{O}(\alpha'^6)$. The Lagrangian with rescaled field tensors[1] is given by

$$\begin{aligned}\mathscr{L}_{\mathrm{BI}} &= -I+\frac{1}{4}\left(\mathrm{Tr}\,F^2\right)-\frac{1}{32}\left((\mathrm{Tr}\,F^2)^2-4\mathrm{Tr}\,F^4\right)\\ &\quad -\frac{1}{384}\left(12\mathrm{Tr}\,F^2\mathrm{Tr}\,F^4-(\mathrm{Tr}\,F^2)^3-32\mathrm{Tr}\,F^6\right)+\mathscr{O}(F^8)\end{aligned} \tag{3.28}$$

In order to use the duality invariance condition (3.26) to check if the BI Lagrangian is duality invariant, it is convenient to have calculated the following

$$\frac{\partial\mathrm{Tr}F^2}{\partial F_{\sigma\tau}} = -4F^{\sigma\tau}, \tag{3.29}$$

$$\frac{\partial(\mathrm{Tr}F^2)^2}{\partial F_{\sigma\tau}} = -8\mathrm{Tr}F^2\,F^{\sigma\tau}, \tag{3.30}$$

$$\frac{\partial\mathrm{Tr}F^4}{\partial F_{\sigma\tau}} = -8F^{\sigma\mu}F_{\mu\nu}F^{\nu\tau}, \tag{3.31}$$

$$\frac{\partial\mathrm{Tr}F^6}{\partial F_{\sigma\tau}} = -12F^{\sigma\mu}F_{\mu\lambda}F^{\lambda\rho}F_{\rho\nu}F^{\nu\tau}. \tag{3.32}$$

Knowing these derivatives we can begin to test BI for duality invariance. Up to order F^8 we have for $\frac{\partial\mathscr{L}_{\mathrm{BI}}}{\partial F_{\sigma\tau}}$ and $\varepsilon_{\sigma\tau\alpha\beta}\frac{\partial\mathscr{L}_{\mathrm{BI}}}{\partial F_{\alpha\beta}}$

[1] Appendix B

$$\frac{\partial \mathscr{L}_{\text{BI}}}{\partial F_{\sigma\tau}} = -F^{\sigma\tau} + \frac{1}{4}\text{Tr}F^2\, F^{\sigma\tau} - (F^3)^{\sigma\tau}$$
$$\frac{1}{32}\left[4\text{Tr}F^4 F^{\sigma\tau} + 8\text{Tr}F^2 (F^3)^{\sigma\tau} - 32(F^5)^{\sigma\tau} - (\text{Tr}F^2)^2 F^{\sigma\tau}\right], \tag{3.33}$$

$$\varepsilon_{\sigma\tau\alpha\beta}\frac{\partial \mathscr{L}_{\text{BI}}}{\partial F_{\alpha\beta}} = -\star F_{\sigma\tau} + \frac{1}{4}\text{Tr}F^2\ \star F_{\sigma\tau} - (\star F^3)_{\sigma\tau}$$
$$\frac{1}{32}\left[4\text{Tr}F^4 \star F_{\sigma\tau} + 8\text{Tr}F^2 (\star F^3)_{\sigma\tau} - 32(\star F^5)_{\sigma\tau} - (\text{Tr}F^2)^2 \star F_{\sigma\tau}\right]. \tag{3.34}$$

Let us see whether we satisfy the duality invariance condition by using equations (3.33) and (3.34).

$$\begin{aligned}\varepsilon_{\sigma\tau\alpha\beta}\frac{\partial \mathscr{L}_{\text{BI}}}{\partial F_{\sigma\tau}}\frac{\partial \mathscr{L}}{\partial F_{\alpha\beta}} = &- \text{Tr}\star FF \\ &\frac{1}{2}\text{Tr}F^2\text{Tr}\star FF - 2\text{Tr}\star FF^3 \\ &\underbrace{-\frac{1}{16}(\text{Tr}F^2)^2\text{Tr}\star FF + \frac{1}{2}\text{Tr}F^2\text{Tr}\star FF^3 - \text{Tr}(\star F^3)(F^3)}_{F^4 \text{ terms } \times\, F^4 \text{ terms}} \\ &\underbrace{-\frac{1}{16}(\text{Tr}F^2)^2\text{Tr}\star FF + \frac{1}{2}\text{Tr}F^2\text{Tr}\star FF^3 - 2\text{Tr}\star FF^5 + \frac{1}{4}\text{Tr}F^4\text{Tr}\star FF}_{F^2 \text{ terms } \times\, F^6 \text{ terms}}.\end{aligned} \tag{3.35}$$

The lowest order term has the desired form in order to have duality invariance. We are therefore left with showing that the remaining higher order terms cancel each other. But first we make a simplification using the fact that $\star FF$ is diagonal[2]

$$\star FF = \lambda(A, \partial A)\, I. \tag{3.36}$$

We use this property to rewrite equation (3.35) into

$$\varepsilon_{\sigma\tau\alpha\beta}\frac{\partial \mathscr{L}_{\text{BI}}}{\partial F_{\sigma\tau}}\frac{\partial \mathscr{L}}{\partial F_{\alpha\beta}} = -\text{Tr}\star FF + \frac{1}{2}\lambda(\text{Tr}F^2)^2 - \text{Tr}(\star F^3)(F^3) - \lambda\text{Tr}F^4. \tag{3.37}$$

Everything works out alright except for the highest order terms. It is not possible to see immediately why they should cancel due to the $\star F^3F^3$-term. Higher order terms contain even more terms of this kind (for example $\star F^5F^3$). This problem can be solved by defining the following Lorentz invariants

[2] Appendix C

$$P = F_{\mu\nu}F^{\mu\nu} = -2(F_{01}^2 + F_{02}^2 + F_{03}^2 - F_{12}^2 - F_{13}^2 - F_{23}^2) \tag{3.38a}$$
$$Q = F_{\mu\nu}\star F^{\mu\nu} = 4(F_{03}F_{12} - F_{02}F_{13} + F_{01}F_{23}) \tag{3.38b}$$

If one writes for example $\mathrm{Tr}F^4$ out, one will see that (this is best done by using a computer program)

$$\mathrm{Tr}F^4 = \frac{1}{2}P^2 + \frac{1}{4}Q^2. \tag{3.39}$$

To make a complete list for traces of the field tensor in terms of P and Q up to and including $\mathscr{O}(F^6)$:

$$\mathrm{Tr}F^2 = -P, \tag{3.40a}$$
$$\mathrm{Tr}\star FF = -Q, \tag{3.40b}$$
$$\mathrm{Tr}F^4 = \frac{1}{2}P^2 + \frac{1}{4}Q^2, \tag{3.40c}$$
$$\mathrm{Tr}\star FF^3 = -\frac{1}{4}PQ, \tag{3.40d}$$
$$\mathrm{Tr}F^6 = -\frac{1}{4}P(P^2 + \frac{3}{4}Q^2), \tag{3.40e}$$
$$\mathrm{Tr}(\star F^3)(F^3) = \frac{1}{16}Q^3, \tag{3.40f}$$
$$\mathrm{Tr}\star FF^5 = \frac{1}{8}Q(P^2 + \frac{1}{2}Q^2). \tag{3.40g}$$

Using these equations we can rewrite the Born-Infeld Lagrangian as

$$\mathscr{L}_{\mathrm{BI}} = -\frac{1}{4}P + \frac{1}{32}(P^2 + Q^2) - \frac{1}{128}(PQ^2 + P^3). \tag{3.41}$$

In order to check if the Born-Infeld theory written in this form is duality invariant, we need the derivatives of P and Q with respect to the field tensor F

$$\frac{\partial P}{\partial F_{\sigma\tau}} = 4F^{\sigma\tau}, \tag{3.42}$$
$$\frac{\partial Q}{\partial F_{\sigma\tau}} = 4\star F^{\sigma\tau}. \tag{3.43}$$

This gives for the derivatives of the BI Lagrangian with respect to the field tensor

$$\frac{\partial \mathscr{L}_{\mathrm{BI}}}{\partial F_{\sigma\tau}} = -F^{\sigma\tau} + \frac{1}{4}(PF^{\sigma\tau} + Q\star F^{\sigma\tau}) \\ - \frac{1}{32}(Q^2F^{\sigma\tau} + 2PQ\star F^{\sigma\tau} + 3P^2F^{\sigma\tau}) \tag{3.44}$$

$$\varepsilon_{\sigma\tau\alpha\beta}\frac{\partial \mathscr{L}_{\mathrm{BI}}}{\partial F_{\alpha\beta}} = -\star F_{\sigma\tau} + \frac{1}{4}(P\star F_{\sigma\tau} - QF_{\sigma\tau}) \\ - \frac{1}{32}(Q^2\star F_{\sigma\tau} - 2PQF_{\sigma\tau} + 3P^2\star F_{\sigma\tau}) \tag{3.45}$$

Collecting same orders gives

$$F^2: \qquad Q \tag{3.46a}$$

$$F^4: \qquad \frac{1}{4}(-PQ + QP - PQ + QP) = 0 \tag{3.46b}$$

$$F^6: \qquad -\frac{1}{16}(P^2Q + Q^3) + \frac{1}{32}(Q^3 - 2P^2Q + 3P^2Q + Q^3 - 2P^2Q + 3P^2Q) = 0 \tag{3.46c}$$

We have proved that the Born-Infeld Lagrangian is duality invariant up to $\mathscr{O}(F^8)$. If one goes to higher and higher orders, we recommend using the method of the Lorentz invariants P and Q. The point is arrived where we have a condition for duality invariance of a Lagrangian dependent only on the field tensor which we used to show the duality invariance of BI up to $\mathscr{O}(F^8)$. The next two chapters will deal with interaction terms added to the Born-Infeld Lagrangian and whether that Lagrangian is still duality invariant or not.

4. CORRECTIONS TO THE BORN-INFELD LAGRANGIAN

The Born-Infeld theory does not only describe the world volume of a D-brane, it is also part of the effective action of strings and superstrings. The first section deals with the notion of an effective action. Massive interaction terms (∂F) added to the Born-Infeld Lagrangian will be discussed in the rest of the chapter. However superstrings and supersymmetry will not be discussed here. We only claim that the effective Lagrangian of a superstring is the Born-Infeld Lagrangian with massive interaction terms which corresponds to the Lagrangian of a D9-brane. The original purpose of this thesis was to construct all independent $\partial^4 F^4$ terms and giving them arbitrary constants and check wether we can create a, unique or not, duality invariant combination of these terms. This procedure failed due to an incomplete definition of G. Lastly the known $\partial^4 F^4$ terms arising in string theory were taken and checked for duality invariance. If one demands duality invariance to be a property of the full effective action, one should redefine G. As a reminder, throughout this chapter we are working in four dimensional spacetime.

4.1 Effective Action

In string theory the full action of a bosonic string is given by the Nambu-Goto action (2.1) or equivalently the Polyakov action [6]. The string action gives the spectrum of the string from the massless states until the Planck mass states times infinity. So far there is no problem, but when one includes string interactions the equations of motion become unknown. On the other hand, we know that any theory reduces to a field theory at low energies [7]. Therefore the equations of motion at low energies can be obtained by using perturbation theory in the specific field theory.

Fortunately, because the Planck mass is too large to have any significance in daily life, we never observe the states/particles of a string higher then the massless ground state; these states only played a role at the time of the big bang. Therefore we claim there is an effective action of string theory which only includes the massless states (in-going and out-going states). When calculating interactions, there are also the virtual massive particles which play a role. Why they play a role will not be treated in this thesis. A detailed treatment of effective actions can be found in [7].

We are interested in the effective action for the open superstring, because, as mentioned before, it is equivalent to the BI action. This action should depend on the massless modes of the superstring (dimension is 10). The field of a massless state is a vector field which conventioanlly is called A. It can be proven that the Born-Infeld Lagrangian is part of the effective action of an open superstring by using the σ-model or by showing that D-branes interact with the massless modes of the open superstring (see for example [8]).

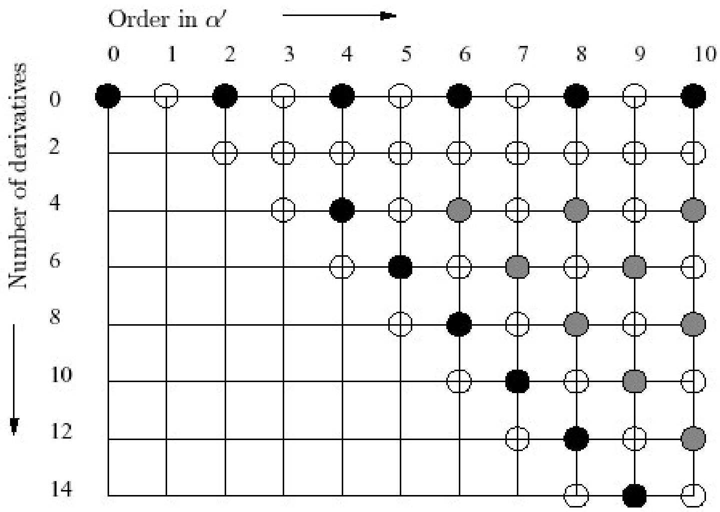


Fig. 4.1: This figure depicts all derivative terms included in the full superstring effective action. Terms represented by black dots have known explicit expressions. Grey dots denote terms we know exist, but for which we have no explicit expression whatsoever.

But as said the full effective action also consists of interactions with virtual massive modes of the superstring. To derive the full form of the effective action, we consider the propagator of a massive field and two vertices ($\sim g^2$)

$$\text{propagator and two vertices} \sim \frac{1}{\alpha'^2 M_{\text{planck}}^2} \frac{1}{\left(1 - \frac{p^2}{M_{\text{planck}}^2}\right)}. \tag{4.1}$$

Since the Planck mass is very large and proportional to $1/\alpha'$ (α' very small), we can write

$$\text{propagator and two vertices} \sim \left(1 + \alpha'^2 p^2 + \alpha'^4 p^4 + ...\right). \tag{4.2}$$

Note that this propagator is expressed in momentum space coordinates. In normal spacetime the momenta become derivatives. It is therefore, at least intuitively, clear that if one has an arbitrary number of external lines and internal lines/loops, the full effective Lagrangian for open superstrings is

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{BI}} + \mathscr{L}_{\text{der}}. \tag{4.3}$$

Where $\mathscr{L}_{\text{der}}$ consists of the terms given in figure (4.1). Therefore $\mathscr{L}_{\text{der}}$ can also be written as

$$\mathscr{L}_{\text{der}} = \alpha'^m \partial^n F^p. \tag{4.4}$$

Where m, n and p, by dimensional reasons, obey (see [9] for example)

$$2p - 2m + n - 4 \quad = \quad 0. \tag{4.5}$$

The next section tries to construct the lowest order derivative correction terms from scratch. We start with the most general derivative correction terms $\partial^4 F^4$ and try to obtain the terms by means of duality invariance obtained in [9].

4.2 Independent $\partial^4 F^4$ Corrections

As said earlier, this thesis is only concerned with the lowest order interaction term $\alpha'^4 \partial^4 F^4$. We want to find what the most general Lagrangian looks like in order to constrain it to duality invariance in 4 dimensions later on. This section deals with the way the independent terms of the first derivative correction is constructed. We start with the most general Lagrangian which includes the first derivative correction. It is written like

$$\mathscr{L} = \mathscr{L}_{\mathrm{BI}} + C_j \mathscr{L}^j_{\mathrm{der}}. \tag{4.6}$$

Where j runs from 1 to all possible ways to write a term of $\partial^4 F^4$ (51975 in all). The C_j are arbitrary constants. The number 51975 is calculated by multiplying the different ways of constructing partial derivative/field tensor combinations (5) by the number of ways to construct a set containing 12 indices which are contracted with each other (10395). A linear superposition of these terms constitute the most general expression for $\partial^4 F^4$. If all these terms would be independent of each other, we would have great computational problems. So we have to try to reduce the number of terms.

To start the elimination procedure, we write the so called bare terms (terms where indices have yet to be put in) in computer language

$$\begin{aligned}
\partial\partial\partial\partial F \quad FFF &= 00001111, && (4.7)\\
\partial\partial\partial F \quad \partial F \quad FF &= 00010111, && (4.8)\\
\partial\partial F \quad \partial\partial F \quad FF &= 00100111, && (4.9)\\
\partial\partial F \quad \partial F \quad \partial F \quad F &= 00101011, && (4.10)\\
\partial F \quad \partial F \quad \partial F \quad \partial F &= 01010101. && (4.11)
\end{aligned}$$

Where the ”bits” denote respectively the derivatives (0) and field tensors (1). Writing the bare terms like this has the nice property that one can conveniently store them in a computer.

Next, contractions need to be added to the bare terms. The total number of indices for $\partial^4 F^4$-terms is 12. Consequently there are 6 contractions present in each $\partial^4 F^4$-term. It was said that 10395 different contracted sets of indices can be constructed out of 12 indices. This number was obtained while systematically constructing and storing the contracted sets by means of the $\partial^4 F^4$ generator program in a computer.

Take a contracted set containing only 2 indices, i.e. one contraction. There is only one contracted set of 2 indices possible (dots are contracted indices): $\cdot\ \cdot$. Next, construct contracted sets containing two contractions (4 indices) which contain the new indices contracted like (_ are empty spaces): $\cdot\ \cdot$ _ _, $\cdot$ _ $\cdot$ _ and $\cdot$ _ _ $\cdot$. Fill in all contracted sets with 2 indices less (in this case the contracted set $\cdot\ \cdot$) at the bars and all new possible contracted sets containing one more contraction are a fact. To extrapolate this method in order to obtain the contracted sets containing three contractions, put the contracted sets with 2 indices less at the bars of: $\cdot\ \cdot$ _ _ _ _, $\cdot$ _ $\cdot$ _ _ _, $\cdot$ _ _ $\cdot$ _ _, $\cdot$ _ _ _ $\cdot$ _ and $\cdot$ _ _ _ _ $\cdot$. It is now possible to construct all possible contracted sets with any even number of indices in this way using a computer. Furthermore, a formula to calculate the number of contracted sets of indices (P) given the number of contractions (N) can be constructed

$$P = \prod_{i=0}^{N-1} (1 + 2N) \,. \tag{4.12}$$

A contraction set can be stored in a computer for example like 121233 (contracted indices are represented by the same number) or like 341265 (position 1 is contracted with position 3 and vice versa). One method of storing has benefits over the other depending on the task at hand, but we will not discuss the details of the computer program.

We have a lot of terms (and this is just the lowest order derivative correction) stored in a computer. To reduce the number of terms, we start with extracting the *independent* terms by performing manipulation on the stored term using the $\partial^4 F^4$ generator computer program.

- First of all, the $\partial^4 F^4$ terms arise in an action. Therefore it is possible to carry out integration by parts. A term belonging to one type of bare term can be integrated by parts into another types of bare terms. We choose to keep bare terms of the form

$$\partial\partial F\ \partial\partial F\ F\ F, \tag{4.13}$$
$$\partial\partial F\ \partial F\ \partial F\ F. \tag{4.14}$$

- Second, a field tensor only has off-diagonal elements. Therefore terms containing a pair of contracted indices on a field tensor have to be eliminated.

- A field tensor is antisymmetric under the interchange of its two indices. Therefore interchanging two indices (call it a ”symmetry” transformation like it is called in the computer program) on a field tensor gives a new contracted set, but hence, not a new term. Therefore the terms obtained after symmetry transformations must be left out. Note that this also applies to terms containing a row of derivatives.

- Interchanging indices on a field tensor and the derivative operators in front of it with the indices on another field tensor with the same number of derivative operators in front (call it a ”supersymmetric” transformation), also produce equal terms which must be deleted.

Note that after performing a supersymmetric transformation it is still necessary to perform symmetric transformations to obtain more contracted sets which must be left out.

- As equation (1.2) shows, the factor $\partial_\mu F_{\mu\nu}$ corresponds to the vacuum Maxwell equations. Surely there has to be something which can be done with terms containing these factors? The Born-Infeld Lagrangian is equal to the Maxwell Lagrangian in lowest order. It is this term of the Born-Infeld expansion which will be used to get rid of terms containing the Maxwell equation-like factors. Therefore consider $F_{\mu\nu}F^{\mu\nu}$ and for example a term like $\partial_\mu F^{\mu\nu}\partial_\alpha F_{\beta\nu}F^{\alpha\rho}F_\rho^\beta$. We use integration by parts within the action to obtain

$$F_{\mu\nu}F^{\mu\nu} \to 4F^{\mu\nu}(\partial_\mu A_\nu + \partial_\mu \delta A_\nu) - 4\delta A_\nu \partial_\mu F^{\mu\nu}. \qquad (4.15)$$

  The only thing left to do is identifying δA_ν with $\partial_\alpha F_{\beta\nu}F^{\alpha\rho}F_\rho^\beta$ (field redefinition) and the term containing the Maxwell equation factor is absorbed into the lowest order term of the Born-Infeld expansion. This can be done with all terms containing factors $\partial_\mu F_{\mu\nu}$.

- For the next elimination step we use the Bianchi identities. Bianchi identities transform two contracted sets into a third one which consequently can be omitted. Because there are four partial derivatives, it is not trivial which Bianchi transformations should be performed. Furthermore, the resulting terms can also be manipulated by applying Bianchi identities and so on. The big question is thus which Bianchi transformations have to be performed and which ones not. For this reason it is very difficult to let a computer program do it efficiently. In the present case there are, relatively spoken, a few number of terms left and therefore it is possible to perform the Bianchi identities by hand in a smart way. After having checked all terms for Bianchi identities, there is a small number of terms left which is given in appendix D.1.

- The last thing that can be done is to check if a total derivative that has the form $\partial_a\left(\partial_a \partial_. F_{..}\partial_. F_{..}F_{..}F_{..}\right)$ or $\partial_a\left(\partial_. F_{..}\partial_. F_. \partial_. F_. F_{a.}\right)$ can be constructed out of a combination of terms which survived the elimination procedure. The a contraction is chosen such that only terms of the two selected bare terms are nonzero. The total derivatives are important, because they give zero in an action and therefore eliminate one term per total derivative. All independent total derivatives are found by looking at the terms of appendix D.1. Appendix D.2 contains all independent total derivatives. It also gives which terms of appendix D.1 are contained in each total derivative. The final set of independent terms are given in appendix D.3.

So there are 11 independent possible ∂^4F^4-terms which could constitute a duality invariant set in 4D. The next section will explain the procedure tried to obtain duality invariant combinations of derivative correction terms.

4.3 Duality Invariance Condition for $\partial^4 F^4$

In this section the duality invariance condition for the lowest order derivative correction will be derived. Let us write the duality invariance condition, equation (3.26), again

$$\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}F^{\mu\nu} \;\; = \;\; \varepsilon_{\sigma\tau\alpha\beta}\frac{\partial\mathscr{L}}{\partial F_{\alpha\beta}}\frac{\partial\mathscr{L}}{\partial F_{\sigma\tau}}. \tag{4.16}$$

The Lagrangian $\mathscr{L}$ is the one given by equation (4.6). To find the duality condition for the derivative corrections, we use that the Born-Infeld Lagrangian is duality invariant and that $\partial\mathscr{L}_{\text{der}}\partial\mathscr{L}_{\text{der}}$ terms are of too high an order. Therefore the duality invariance condition for the derivative correction part is

$$0 \;\; = \;\; \star F_{\sigma\tau}\frac{\partial\mathscr{L}_{\text{der}}}{\partial F_{\sigma\tau}} \tag{4.17}$$

Using this duality condition we silently assumed that $\mathscr{L}$ is only a function of F and consequently G is the same as in equation (3.11).

The only task left is to calculate $\frac{\partial\mathscr{L}_{\text{der}}}{\partial F_{\sigma\tau}}$ and giving values to the constants of appendix D.3 such that $\mathscr{L}$ is duality invariant. Therefore we have to know what $\frac{\partial\mathscr{L}_{\text{der}}}{\partial F}$ is, because it is not trivial what $\frac{\partial(\partial_a F_{bc})}{\partial F_{ef}}$ is. Consider

$$\delta\mathscr{L}_{\text{der}} = \frac{\partial\mathscr{L}_{\text{der}}}{\partial F_{\sigma\tau}}\delta F_{\sigma\tau} \tag{4.18}$$

This can be written as

$$\mathscr{L}_{\text{der}}(F_{\sigma\tau}+\delta F_{\sigma\tau}) - \mathscr{L}_{\text{der}}(F_{\sigma\tau}) = \frac{\partial\mathscr{L}_{\text{der}}}{\partial F_{\sigma\tau}}\delta F_{\sigma\tau} \tag{4.19}$$

In order to calculate the derivative it is sufficient to calculate the left side of equation (4.19) and keeping terms which contain only one δ. Calculating the derivatives like this gives terms containing factors like $\partial\delta F$ and $\partial\partial\delta F$. Looking at equation (4.19) the terms need to be transformed to terms containing only factors of δF.

The discrepancy of the whole calculation lies in the fact that is one allowed to transform $\partial\delta F$ into δF? If it is allowed, the terms we have obtained so far by (4.19) could also be obtained by taking $\mathscr{L}$ dependent on F, ∂F *and* $\partial\partial F$ and therefore defining G as

$$G^{\mu\nu} = -\partial_\tau\partial_\rho\frac{\partial\mathscr{L}}{\partial(\partial_\tau\partial_\rho F_{\mu\nu})} + \partial_\tau\frac{\partial\mathscr{L}}{\partial(\partial_\tau F_{\mu\nu})} - \frac{\partial\mathscr{L}}{\partial F_{\mu\nu}} \tag{4.20}$$

The corresponding duality condition still is $\star FF = \star GG$ (derive in the same way as equation (3.26)).But the discrepancy of getting from $\partial\delta F$ to δF still remains. The only manipulation possible to go from $\partial\delta F$ to δF is integration by parts. We justify the use of it by the fact that $\delta\mathscr{L} = 0$. Put

$$\int \mathrm{d}^4x\delta\mathscr{L} = Q \tag{4.21}$$

After having done the integrations by parts, perform a differentiation to obtain $\delta\mathscr{L}$ once again. Obtain $\frac{\partial\mathscr{L}_{\text{der}}}{\partial F_{\sigma\tau}}$ by taking the expression for $\delta\mathscr{L}$ without the δF.

The last step in the calculation is to check for duality invariance. It is obvious that a lot of terms are involved. Therefore we used a computer to aid us in the calculation. Eventually it was found that any combination of derivative correction terms does not lead to duality invariance. This is due to the fact that in the calculation of $\frac{\partial\mathscr{L}_{\text{der}}}{\partial F_{\sigma\tau}}$ all possible different bare terms (bare term is for example $\partial\partial\partial F\partial FFF$) arise which are not able to interact with each other; every linear combination of a single type of bare term is not duality invariant. However, there still is a possibility for duality invariance, but we need integration by parts to let different types of bare terms interact with each other.

Nevertheless, what we have is that there is no duality invariance among derivative correction terms using this section's method. If we are somehow allowed to use integration by parts, duality invariance is not yet ruled out by having other constraints on the constants. However, obtaining duality invariant combinations from the independent derivative correction terms stopped here, because we did not know how to allow ourselves to use integration by parts. From now on we will focus on the attempt to prove that the ∂^4F^4 derivative correction terms arising in string theory are duality invariant. The next two sections will deal with this.

4.4 Duality Invariance Condition Revisited

This section's derivation of the duality invariance condition will use an additional assumption upon the ones used in the derivation of the duality invariance condition (3.26). Furthermore, we need to introduce integration by parts. Otherwise no combination of derivative correction terms is duality invariant as will become clear later on. We let the integration by parts enter by defining G from the start as

$$G^{ab}(F) = -\frac{\delta S[F]}{\delta F_{ab}}. \tag{4.22}$$

From equation (3.18) in chapter 3 we know that the following yields for the transformation of G

$$G'_{ab}(F') = G_{ab}(F) - \alpha\star F_{ab} = G_{ab}(F) + \frac{\partial}{\partial F^{ab}}\left(-\frac{1}{4}\alpha F\cdot\star F\right). \tag{4.23}$$

By the new definition of G, equation (4.22), we obtain the expression for an infinitesimal transformation of G

$$G'(F') = -\frac{\delta S'[F']}{\delta F'} = -\frac{\delta S[F]}{\delta F'} - \frac{\delta}{\delta F'}\delta S = -\frac{\delta S[F]}{\delta F}\frac{\delta F}{\delta F'} - \frac{\delta}{\delta F}\delta S\ \frac{\delta F}{\delta F'}. \tag{4.24}$$

Note that we explicitly assume $\mathscr{L}' = \mathscr{L}$. This is the extra assumption with respect to the previous derivation. There we only used the way the fields should transform in order to obey duality invariance. Now we also use the constraint that the Lagrangian after the transformation equals the old Langrangian (self duality). To continue use also the fact that $F = F' - \alpha\star G$ from equation (3.18) to get to

$$
\begin{aligned}
G'(F') &= -\frac{\delta S[F]}{\delta F}\left(1-\alpha\frac{\delta\star G(F)}{\delta F'}\right)-\frac{\delta}{\delta F}\delta S\left(1-\alpha\frac{\delta\star G(F)}{\delta F'}\right)\\
&= G(F)-\alpha G(F)\frac{\delta\star G(F)}{\delta F'}-\frac{\delta}{\delta F}\delta S\left(1-\alpha\frac{\delta\star G(F)}{\delta F'}\right).
\end{aligned}
\tag{4.25}
$$

In obtaining the final expression for the transformation of G we ignore terms of order α^2 $(\alpha\delta S)$. This gives us

$$
G'(F') = G(F)-\alpha G(F)\frac{\delta\star G(F)}{\delta F}-\frac{\delta}{\delta F}\delta S = G(F)+\frac{\delta}{\delta F}\left(\delta S+\frac{1}{4}\alpha G(F)\cdot\star G(F)\right). \tag{4.26}
$$

Combining equations (4.23) and (4.26) gives

$$
\delta S = -\frac{1}{4}\alpha\left(\star F\ F+\star G\ G\right). \tag{4.27}
$$

Furthermore if we vary the action directly we obtain

$$
\delta S = \int d^4x\delta\mathscr{L} = \int d^4x\left(\frac{\partial\mathscr{L}}{\partial F}-\partial\frac{\partial\mathscr{L}}{\partial(\partial F)}\right)\delta F = -\frac{1}{2}\alpha\ \star G\ G. \tag{4.28}
$$

Low and behold if we require the expressions for δS to be consistent with each other we must have

$$
\boxed{\star FF = \star GG}. \tag{4.29}
$$

The duality condition (4.29) is derived in two different ways now. The last one being more general in the sense that it demands an additional constraint on the Lagrangian ($\mathscr{L}'=\mathscr{L}$). Furthermore we have learned how to obtain integration by parts by redefining the G-tensor. The next section will test the lowest order derivative correction to the Born-Infeld Lagrangian on duality invariance using equation (4.29) as the duality invariant criteria and equation (4.22) as the definition for G.

4.5 ∂^4F^4-Terms Electromagnetic Duality Invariant

This section will show the duality invariance of the lowest order derivative correction [9]. It is given by

$$
\partial^4F^4 = t_{abcdefgh}\partial_kF^{ab}\partial^kF^{cd}\partial_lF^{ef}\partial^lF^{gh}. \tag{4.30}
$$

Where the t-tensor is to be expanded as

$$
\begin{aligned}
t_{abcdefgh} M_1^{ab} M_2^{cd} M_3^{ef} M_4^{gh} \quad = \quad & -2(\mathrm{Tr}\ M_1 M_2 \mathrm{Tr}\ M_3 M_4 + \mathrm{Tr}\ M_1 M_3 \mathrm{Tr}\ M_2 M_4 + \\
& \quad \mathrm{Tr}\ M_1 M_4 \mathrm{Tr}\ M_2 M_3) \\
& +8(\mathrm{Tr}\ M_1 M_2 M_3 M_4 + \mathrm{Tr}\ M_1 M_3 M_2 M_4 + \\
& \quad \mathrm{Tr}\ M_1 M_3 M_4 M_2).
\end{aligned} \tag{4.31}
$$

In section 4.3 we derived the duality condition for the lowest order derivative correction, equation (4.17). Using the new definition for the G-tensor we get

$$
0 = \star F_{\sigma\tau} \frac{\delta S_{\text{der}}}{\delta F_{\sigma\tau}}. \tag{4.32}
$$

The explicit form of $\frac{\delta S_{\text{der}}}{\delta F_{\sigma\tau}}$ is in the case of equation (4.30)

$$
\frac{\delta S_{\text{der}}}{\delta F_{\sigma\tau}} = \int d^4x \partial_\rho \frac{\partial \mathscr{L}}{\partial(\partial_\rho F_{\sigma\tau})}. \tag{4.33}
$$

The ingredients are ready, so we start the check for duality invariance by calculating

$$
\begin{aligned}
\star F_{\sigma\tau} \frac{\delta S_{\text{der}}}{\delta F_{\sigma\tau}} = \int d^4x (& t_{abcdefgh}\ \partial_k \partial^k F^{ab} \partial_l F^{cd} \partial^l F^{ef} \star F^{gh} \\
& + 2\ t_{abcdefgh}\ \partial_k \partial_l F^{ab} \partial^k F^{cd} \partial^l F^{ef} \star F_{gh}).
\end{aligned} \tag{4.34}
$$

Note that a total derivative term gives zero under the integral due to boundary conditions. Therefore we can write the previous equation as

$$
\star F_{\sigma\tau} \frac{\delta S_{\text{der}}}{\delta F_{\sigma\tau}} = \int d^4x \left(t_{abcdefgh}\ \partial_k F^{ab} \partial_l F^{cd} \partial^l F^{ef} \partial^k \star F^{gh} \right). \tag{4.35}
$$

The derivatives on the field tensors do not have any contraction with the t-tensor. So we write equation (4.35) in the most general way as (F_i is antisymmetric)

$$
\int d^4x \left(t_{abcdefgh}\ F_1^{ab} F_2^{cd} F_1^{ef} \star F_2^{gh} \right). \tag{4.36}
$$

The labels 1 and 2 are arbitrary and therefore can be interchanged simultaneously. Expand the t-tensor and after a few trivial simplifications we get

$$
\begin{aligned}
t_{abcdefgh}\ F_1^{ab} F_2^{cd} F_1^{ef} \star F_2^{gh} = & -4\mathrm{Tr}(F_1 F_2)\mathrm{Tr}(F_1 \star F_2) - 2\mathrm{Tr}(F_1 F_2)\mathrm{Tr}(\star F_1 F_2)) + \\
& 8\mathrm{Tr}(F_1 F_2 F_1 \star F_2) + 16\mathrm{Tr}(F_1 F_1 F_2 \star F_2)).
\end{aligned} \tag{4.37}
$$

In order to show that the above equation gives zero, we rewrite the product of traces to one big trace using the following

$$\star F_2 F_1 + \star F_1 F_2 = \frac{1}{2}\mathrm{Tr}(\star F_1 F_2) I, \tag{4.38}$$

$$\star F F = \frac{1}{4}\mathrm{Tr}(\star F F) I. \tag{4.39}$$

These relations are derived in appendix E. Applying them to equation (4.37) gives

$$t_{abcdefgh}\, F_1^{ab} F_2^{cd} F_1^{ef} \star F_2^{gh} = -12\,[\mathrm{Tr}(F_1 F_1 F_2 \star F_2) + 12\mathrm{Tr}(F_1 F_2 \star F_1 F_2)] + 8\mathrm{Tr}(F_1 F_2 F_1 \star F_2) + 16\mathrm{Tr}(F_1 F_1 F_2 \star F_2). \tag{4.40}$$

The only thing left to show is that $\mathrm{Tr}(F_1 F_2 \star F_1 F_2) = \mathrm{Tr}(F_1 F_1 F_2 \star F_2)$. This is a trivial task which can be acclompished by applying equation (4.38) two times on $\mathrm{Tr}(F_1 F_2 \star F_1 F_2)$. By this we have proven that

$$\boxed{t_{abcdefgh}\, \partial_k F^{ab} \partial_l F^{cd} \partial^l F^{ef} \partial^k \star F^{gh} = 0}. \tag{4.41}$$

The first derivative correction to the Born-Infeld Lagrangian is electromagnetic duality invariant! Actually, using equations (4.38) and (4.39) one can also show that an even more general expression of equation (4.36) gives zero, i.e.

$$t_{abcdefgh}\, F_1^{ab} F_2^{cd} F_3^{ef} F_4^{gh} = 0. \tag{4.42}$$

It is fortunate that this thesis can show the result of the first derivative correction being electromagnetic duality invariant. It is important to note that we needed a redefinition of the G-tensor in order to obtain integration by parts and hence duality invariance. These are the main result of this thesis. The next and last section of this thesis will be a summary of all the previous obtained results.

4.6 General Derivative Corrections Electromagnetic Duality Invariant?

To recapitulate, the duality invariance condition was derived in two ways. One using the way the fields F and G should transform in order to have duality invariance and one which uses besides the transformation property the assumption that $\mathscr{L}'(F') = \mathscr{L}(F)$. It is only valid in 4 dimensions and given by

$$\star F F = \star G G \tag{4.43}$$

Where we found it is necessary to define G as

$$G \equiv \frac{\delta S_{\mathrm{eff}}[F]}{\delta F} \tag{4.44}$$

The full effective open superstring Lagrangian is given by

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{BI}} + \mathscr{L}_{\text{der}}. \tag{4.45}$$

We have shown up to order α'^6 that the $\mathscr{L}_{\text{BI}}$-part of the effective Lagrangian is duality invariant in four dimensions. Furthermore, it can be shown that the Born-Infeld Lagrangian is invariant up to every order in α' [10]. The lowest order derivative correction of $\mathscr{L}_{\text{der}}$ has also been shown by us to be duality invariant. People of the group high energy physics are testing the second order derivative correction, $\partial^4 F^6$, on duality invariance at the institute for theoretical physics in Groningen right now. It is suspected by them that if the calculation works out, the term will be found to be duality invariant. It is even suspected that all derivative correction terms are duality invariant due to the fact that the derivatives are contracted only with each other.

But remember the gray dots in figure 4.1. The derivative correction terms the gray dots represent are terms which we know exist, but with explicit expressions which are a mystery to us. The original goal of this thesis was to see if we could uniquely obtain the lowest order derivative correction term by demanding duality invariance. This comes down to finding the t-tensor for the lowest order derivative correction terms. In the case of success we would know how to obtain the gray dot terms; only demand duality invariance and obtain t-tensors for higher order derivative corrections. This thesis proved that with the old definition of G this cannot be done. Unfortunately researchers of the institute also showed it is not possible to obtain unique higher order derivative correction terms despite using the redefined G-tensor; one always keeps free parameters. This will also be the case for the 11 independent derivative correction terms obtained in chapter 4.2 applied to the duality invariance condition with G defined as (4.44).

The conclusion of this thesis is that the BI part of the full effective open superstring Lagrangian is duality invariant. The lowest order derivative correction is also shown to be duality invariant. However one cannot demand duality invariance in order to construct the effective open superstring Lagrangian.

ACKNOWLEDGMENTS

I would like to thank my supervisor Dr. Mees de Roo for giving me this graduation project. I would like to thank him for his advise, guidance and patience. Furthermore, I would like to thank Wissam Chemissany for interesting conversations, discussions and for giving me new insights in the subject. I enjoyed doing research with them, but I also found out that research is not my calling. Therefore I will look for a job and not for a PhD position.

I thank my parents for giving me life and the brains necessary to do physics. My youth with them was a good one and they still support me in many ways.

Also, I want to give credit to all my roommates at the university (Seibo Holtmans, Robert Bloem, Duurt-Johan van der Hoek and Wijnand Broer) for nice conversations and joking around. The same credit is deserved by my fellow students (Jelle van der Zwaag, Jos Postma, Rachid el Yahyaoui, Oscar Versolato, Wilbert Kruithof and whomever I have forgotten to mention, but should be included).

Last but not least, the people who also play a (big) part in making my life interesting and fun: My girlfriend Cora Zwaagstra and all my friends back home in Friesland (Gerald Weidgraaf+Susan Scholtens, Peter Jansen, Eddy Zandstra, Sjoerd Wiersma and Jelmer Storm).

THANK YOU ALL!

APPENDIX

A. THE HODGE OPERATOR

This appendix chapter discusses some basic properties of what will be called forms. We use no fancy mathematical stuff, because the only purpose is clarifying what the Hodge operator applied to the field tensor is.

We begin with considering an abstract vector T in a n-dimensional vector space. T can be decomposed into its components contracted with a basis of the vector space

$$T = T_{\mu\ldots\nu}\; dx^{\mu} \wedge \ldots \wedge dx^{\nu}. \tag{A.1}$$

The $T_{\mu\ldots\nu}$ are the components of the vector T and the dx^{μ} are a basis in the vector space. They are called forms. The property of the strange $\wedge$ (wedge) is

$$dx^{\mu} \wedge dx^{\nu} = -dx^{\nu} \wedge dx^{\mu}. \tag{A.2}$$

If the situation occurs where T has k indices which has values running from 1 to k and $k < n$, then T is called a k-form. The basis of this subspace consequently consists of the forms

$$dx^1, \ldots,\; dx^k. \tag{A.3}$$

It is now time to introduce the Hodge operator: $\star$. The Hodge operator transforms a k-form into a $(n-k)$-form in the sense that the basis transforms like

$$\star dx^1 \wedge \ldots \wedge dx^k = dx^{k+1} \wedge \ldots \wedge dx^n. \tag{A.4}$$

This means that if a Hodge operator is applied, the basis of one subspace is transformed into the basis of the so called Hodge dual space. The Hodge dual space is also a subspace of the vector space and if those two subspaces are wedged together, the complete vector space is recovered.

Next we apply the Hodge operator to the electromagnetic field tensor. Note that the complete vector space is identified with the four dimensional spacetime now. The field tensor and the Hodge dual of it are

$$F = F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}, \tag{A.5}$$

$$\star F = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} dx^{\mu} \wedge dx^{\nu}. \tag{A.6}$$

The Levi-Chivita tensor appears, because it is in essence the Hodge operator. Why? The Hodge operator takes the basis of the subspace to the basis of the Hodge dual space. So we need that ρ and σ are not equal to μ and ν the Hodge operator is represented. The Levi-Chivita tensor accomplishes just that. Therefore $\star F$ in components is

$$\star F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{A.7}$$

Finally something about Bianchi identities. Consider the 1-form A. Applying two d's must give zero. The calculation goes as follows

$$\begin{aligned} dA &\equiv \partial_\nu A_\mu dx^\nu \wedge dx^\mu = \frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu)dx^\nu \wedge dx^\mu = F \\ ddA &= \partial_\sigma(\partial_\nu A_\mu - \partial_\mu A_\nu)dx^\sigma \wedge dx^\nu \wedge dx^\mu = 0 \end{aligned}$$

So we have $dF = 0$ and consequently $d \star F = 0$. More about forms and Hodge duality can be found for example in [12].

B. EXPANDING BORN-INFELD

This appendix chapter will show a way Born-Infeld can be expanded. There are multiple approaches to expand Born-Infeld, but this thesis considers the ”straightforward” approach. The downside is that it is much algebra; it needs three series expansions which need to be substituted into each other. At first sight we need the series expansion of only the square-root and determinant. But the series expansion of the determinant consists of two series expansions, the series expansion of the exponent and the natural logarithm.

Suppose we have a diagonalizable matrix A. In this case yields

$$\begin{aligned} A &= Q^{-1}DQ, \\ \det A &= \det D, \\ \det A &= e^{\ln(\det A)}. \end{aligned} \tag{B.1}$$

Using the above identities we write

$$\begin{aligned} \det A = e^{\ln(\det A)} = e^{\ln(\det D)} = e^{\sum_i (\ln D)_{ii}} = e^{\mathrm{Tr}(\ln D)} = e^{\mathrm{Tr}(\ln QAQ^{-1})}, \\ = e^{\mathrm{Tr}(\ln QQ^{-1} + \ln A)} = e^{\mathrm{Tr}(\ln IA)} = e^{\mathrm{Tr}(\ln A)}. \end{aligned} \tag{B.2}$$

By expanding the exponent we obtain

$$\det A = \sum_{n=0}^{\infty} \frac{\mathrm{Tr}(\ln A)^n}{n!} = I + \mathrm{Tr}(\ln A) + \frac{1}{2}(\mathrm{Tr}(\ln A))^2 + \frac{1}{6}(\mathrm{Tr}(\ln A))^3 + \ldots \tag{B.3}$$

In order to get a polynomial it is necessary to also know the expansion of the natural logarithm. Consider for small x

$$\ln(1-x) = \int_0^x \frac{-1}{1-y} \mathrm{d}y = -\int_0^x \sum_{n=0}^{\infty} y^n \mathrm{d}y = -\sum_{n=1}^{\infty} \frac{1}{n} x^n. \tag{B.4}$$

From this follows

$$\ln A = \sum_{n=1}^{\infty} \frac{-1}{n}(I-A)^n = -(I-A) - \frac{1}{2}(I-A)^2 - \frac{1}{3}(I-A)^3 - \ldots \tag{B.5}$$

The next step is to apply the expansions to the form of the Born-Infeld Lagrangian occurring in string theory. For convenience it is repeated with indices on the tensors

$$\begin{aligned}\mathscr{L}_{\mathrm{BI}} &= -T_p(g)\sqrt{-\det\left(\eta_{mn}+2\pi\alpha' F'_{mn}\right)},\\ &= -T_p(g)\sqrt{-\det\eta_{mj}}\sqrt{\det\left(\delta^j_n+2\pi\alpha'\, F'^{j}_{n}\right)},\\ &= -T_p(g)\sqrt{\det\left(I+2\pi\alpha' F\right)}.\end{aligned} \tag{B.6}$$

Where we have redefined F as (important!)

$$\boxed{F\equiv\eta F'} \tag{B.7}$$

Notice that F is still anti-symmetric. To continue the expansion of BI, we make the identification

$$A\equiv I+2\pi\alpha' F. \tag{B.8}$$

Using equations (B.5) and (B.8) together gives

$$\ln I+2\pi\alpha' F = 2\pi\alpha' F-\frac{1}{2}(2\pi\alpha' F)^2+\frac{1}{3}(2\pi\alpha' F)^3-\ldots \tag{B.9}$$

It can be expected from the series expansion that traces of all powers of F will occur in the expansion of BI. However, we can simplify the calculation by noting from the start that traces of any odd power of F can be eliminated. Consider the following for $\mathrm{Tr}\,F^3$ (generalizable argument)

$$\mathrm{Tr}\,F^3 = F^{ab}F_{bc}F^{ca} = -F^{ba}F_{cb}F^{ac} = -\mathrm{Tr}\,F^3 \qquad\Rightarrow\qquad \mathrm{Tr}\,F^3=0 \tag{B.10}$$

Knowing this we apply the series expansion of the natural logarithm to (B.3) and omit traces of odd powers of F to give

$$\begin{aligned}\det\left(I+2\pi\alpha' F\right) =&I+\\ &(2\pi\alpha')^2\left[-\frac{1}{2}\mathrm{Tr}\,F^2\right]+\\ &(2\pi\alpha')^4\left[\frac{1}{8}(\mathrm{Tr}\,F^2)^2-\frac{1}{4}\mathrm{Tr}\,F^4\right]+\\ &(2\pi\alpha')^6\left[\frac{3}{24}\mathrm{Tr}\,F^2\mathrm{Tr}\,F^4-\frac{1}{48}(\mathrm{Tr}\,F^2)^3-\frac{1}{6}\mathrm{Tr}\,F^6\right]+\\ &\mathscr{O}(2\pi\alpha'^8)\end{aligned} \tag{B.11}$$

We have arrived at the point of having obtained the series expansion of the determinant. The only thing left to do is to put the series expansion of the determinant into the series expansion of the square-root which is obtained by using Newton's binomial

$$(1+x)^{\frac{1}{2}} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\mathscr{O}(x^5) \tag{B.12}$$

Identify all terms of equation (B.11), excluding I, with x and substitute these terms for x into the series expansion of the square-root to get

$$\begin{aligned}\sqrt{\det\left(I+2\pi\alpha' F\right)} = I + \frac{1}{2}\Bigg[&(2\pi\alpha')^2\left(-\frac{1}{2}\mathrm{Tr}\,F^2\right)+(2\pi\alpha')^4\left(\frac{1}{8}(\mathrm{Tr}\,F^2)^2-\frac{1}{4}\mathrm{Tr}\,F^4\right)+\\&(2\pi\alpha')^6\left(\frac{3}{24}\mathrm{Tr}\,F^2\mathrm{Tr}\,F^4-\frac{1}{48}(\mathrm{Tr}\,F^2)^3-\frac{1}{6}\mathrm{Tr}\,F^6\right)\Bigg]-\\&\frac{1}{8}\Bigg[(2\pi\alpha')^2\left(-\frac{1}{2}\mathrm{Tr}\,F^2\right)+(2\pi\alpha')^4\left(\frac{1}{8}(\mathrm{Tr}\,F^2)^2-\frac{1}{4}\mathrm{Tr}\,F^4\right)+\\&(2\pi\alpha')^6\left(\frac{3}{24}\mathrm{Tr}\,F^2\mathrm{Tr}\,F^4-\frac{1}{48}(\mathrm{Tr}\,F^2)^3-\frac{1}{6}\mathrm{Tr}\,F^6\right)\Bigg]^2+\\&\frac{1}{16}\Bigg[(2\pi\alpha')^2\left(-\frac{1}{2}\mathrm{Tr}\,F^2\right)+(2\pi\alpha')^4\left(\frac{1}{8}(\mathrm{Tr}\,F^2)^2-\frac{1}{4}\mathrm{Tr}\,F^4\right)+\\&(2\pi\alpha')^6\left(\frac{3}{24}\mathrm{Tr}\,F^2\mathrm{Tr}\,F^4-\frac{1}{48}(\mathrm{Tr}\,F^2)^3-\frac{1}{6}\mathrm{Tr}\,F^6\right)\Bigg]^3\end{aligned} \tag{B.13}$$

Collect all orders of F up to and including order F^6 to get

$$\begin{aligned}\sqrt{\det\left(I+2\pi\alpha' F\right)} = I &- \frac{1}{4}(2\pi\alpha')^2\left(\mathrm{Tr}\,F^2\right)+\frac{1}{32}(2\pi\alpha')^4\left((\mathrm{Tr}\,F^2)^2-4\mathrm{Tr}\,F^4\right)+\\&\frac{1}{384}(2\pi\alpha')^6\left(12\mathrm{Tr}\,F^2\mathrm{Tr}\,F^4-(\mathrm{Tr}\,F^2)^3-32\mathrm{Tr}\,F^6\right)+\mathscr{O}(F^8)\end{aligned} \tag{B.14}$$

So the expanded Born-Infeld Lagrangian (B.6) is given by

$$\boxed{\begin{aligned}\mathscr{L}_{\mathrm{BI}} = -T\Bigg[I &- \frac{1}{4}(2\pi\alpha')^2\left(\mathrm{Tr}\,F^2\right)+\frac{1}{32}(2\pi\alpha')^4\left((\mathrm{Tr}\,F^2)^2-4\mathrm{Tr}\,F^4\right)+\\&\frac{1}{384}(2\pi\alpha')^6\left(12\mathrm{Tr}\,F^2\mathrm{Tr}\,F^4-(\mathrm{Tr}\,F^2)^3-32\mathrm{Tr}\,F^6\right)+\mathscr{O}(F^8)\Bigg]\end{aligned}} \tag{B.15}$$

If one looks at equation (3.26), it can be seen there is no room for the α''s contained in the Born-Infeld Lagrangian; the duality conditions is using a rescaled field and Lagrangian. Therefore we re-scale our Born-Infeld Lagrangian by

$$F' = 2\pi\alpha' F, \tag{B.16}$$

$$\mathscr{L}' = \frac{1}{2\pi\alpha'}\mathscr{L}. \tag{B.17}$$

After the re-scaling the varies forms of the Born-Infeld Lagrangians are (discard the primes)

$$\mathscr{L}_{\text{BI}} = -\sqrt{-\det\left(\eta_{\mu\nu} + F_{\mu\nu}\right)} \tag{B.18}$$

$$\begin{aligned} &= -I + \frac{1}{4}\left(\text{Tr } F^2\right) - \frac{1}{32}\left((\text{Tr } F^2)^2 - 4\text{Tr } F^4\right) \\ &\quad -\frac{1}{384}\left(12\text{Tr } F^2 \text{Tr } F^4 - (\text{Tr } F^2)^3 - 32\text{Tr } F^6\right) + \mathscr{O}(F^8) \end{aligned} \tag{B.19}$$

We were able to expand Born-Infeld and we will use the expansion to check BI for duality invariance up to order F^8. If one desires higher order terms of BI, one can easily find them by using the series expansions of this appendix. We could also have expanded BI by using the method of a generating function, but as said the method used is more straightforward.

C. PROOF THAT $\star FF$ IS DIAGONAL

We will show that $\star FF$ is diagonal. In components $\star FF$ looks like

$$\star FF \sim \varepsilon_{\mu\alpha\beta\sigma} F^{\alpha\beta} F^{\sigma\nu}. \tag{C.1}$$

The approach we use is that we take a value for μ and see what values ν can obtain. Take $\mu = 3$ and obtain the terms

$$\begin{aligned} &\varepsilon_{3012} F^{01} F^{2\sigma} + \varepsilon_{3120} F^{12} F^{0\sigma} + \varepsilon_{3201} F^{20} F^{1\sigma} + \\ &\varepsilon_{3021} F^{02} F^{1\sigma} + \varepsilon_{3102} F^{10} F^{2\sigma} + \varepsilon_{3210} F^{21} F^{0\sigma}. \end{aligned} \tag{C.2}$$

One can see that if for ν yields $\nu \neq \mu$, all terms cancel. This is due to the fact that the Levi-Chivita tensor is antisymmetric under the interchange of two indices next to each other. Only when $\mu = \nu$ do the terms add up. So $\star FF$ is diagonal and therefore can be written like

$$\star FF = \lambda(A, \partial A)\, I. \tag{C.3}$$

D. OBTAINING INDEPENDENT $\partial^4 F^4$ TERMS

D.1 Up To Bianchi Identities

$\partial\partial$ F $\partial\partial$ F F F

$$\partial_e\partial_f F_{ab} \quad \partial_e\partial_f F_{cd} \quad F_{ab} \quad F_{cd} \tag{D.1}$$
$$\partial_e\partial_f F_{bd} \quad \partial_e\partial_f F_{ac} \quad F_{ab} \quad F_{cd} \tag{D.2}$$
$$\partial_a\partial_e F_{fc} \quad \partial_b\partial_e F_{fd} \quad F_{ab} \quad F_{cd} \tag{D.3}$$
$$\partial_a\partial_e F_{fc} \quad \partial_b\partial_f F_{ed} \quad F_{ab} \quad F_{cd} \tag{D.4}$$
$$\partial_c\partial_d F_{ea} \quad \partial_c\partial_d F_{eb} \quad F_{ag} \quad F_{bg} \tag{D.5}$$
$$\partial_c\partial_d F_{ea} \quad \partial_c\partial_e F_{db} \quad F_{ag} \quad F_{bg} \tag{D.6}$$
$$\partial_a\partial_b F_{cd} \quad \partial_a\partial_b F_{cd} \quad F_{gh} \quad F_{gh} \tag{D.7}$$

$\partial\partial$ F ∂ F ∂ F F

$$\partial_d\partial_a F_{bc} \quad \partial_d F_{ef} \quad \partial_a F_{bc} \quad F_{ef} \tag{D.8}$$
$$\partial_d\partial_a F_{be} \quad \partial_d F_{ef} \quad \partial_a F_{bc} \quad F_{cf} \tag{D.9}$$
$$\partial_a\partial_b F_{cf} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{df} \tag{D.10}$$
$$\partial_a\partial_c F_{bf} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{df} \tag{D.11}$$
$$\partial_a\partial_b F_{ef} \quad \partial_c F_{db} \quad \partial_c F_{da} \quad F_{ef} \tag{D.12}$$
$$\partial_a\partial_b F_{ef} \quad \partial_d F_{cb} \quad \partial_c F_{da} \quad F_{ef} \tag{D.13}$$
$$\partial_d\partial_e F_{bc} \quad \partial_d F_{ef} \quad \partial_a F_{bc} \quad F_{af} \tag{D.14}$$
$$\partial_d\partial_b F_{ea} \quad \partial_d F_{ef} \quad \partial_a F_{bc} \quad F_{cf} \tag{D.15}$$
$$\partial_a\partial_b F_{df} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{cf} \tag{D.16}$$
$$\partial_a\partial_d F_{bf} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{cf} \tag{D.17}$$
$$\partial_b\partial_c F_{af} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{df} \tag{D.18}$$
$$\partial_b\partial_d F_{af} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{cf} \tag{D.19}$$

D.2 Independent Total Derivatives Terms

$$\partial_a \quad \left(F_{bc} \quad F_{bc} \quad \partial_a\partial_d F_{ef} \quad \partial_d F_{ef}\right) \tag{D.20}$$

$$\partial_a \quad \left(F_{bc} \quad F_{bd} \quad \partial_a\partial_c F_{ef} \quad \partial_d F_{ef}\right) \tag{D.21}$$

$$\partial_a \quad \left(F_{bc} \quad F_{bd} \quad \partial_a\partial_e F_{cf} \quad \partial_d F_{ef}\right) \tag{D.22}$$

$$\partial_a \quad \left(F_{bc} \quad F_{de} \quad \partial_a\partial_f F_{bc} \quad \partial_f F_{de}\right) \tag{D.23}$$

$$\partial_a \quad \left(F_{bc} \quad F_{de} \quad \partial_a\partial_f F_{bd} \quad \partial_f F_{ce}\right) \tag{D.24}$$

$$\partial_a \quad \left(F_{bc} \quad F_{de} \quad \partial_b\partial_f F_{cd} \quad \partial_f F_{ce}\right) \tag{D.25}$$

$$\partial_a \quad \left(F_{bc} \quad F_{de} \quad \partial_b\partial_f F_{ad} \quad \partial_c F_{fe}\right) \tag{D.26}$$

$$\partial_a \quad \left(F_{ae} \quad \partial_b F_{ef} \quad \partial_f F_{cd} \quad \partial_b F_{cd}\right) \tag{D.27}$$

$$\partial_a \quad \left(F_{ae} \quad \partial_e F_{bf} \quad \partial_f F_{cd} \quad \partial_b F_{cd}\right) \tag{D.28}$$

Expanding the Total Derivatives

(D.20) $\rightarrow$ (D.7),(D.8)
(D.21) $\rightarrow$ (D.5),(D.10), (D.11),(D.14)
(D.22) $\rightarrow$ (D.5),(D.6), (D.9),(D.15),(D.17), (D.18),(D.19)
(D.23) $\rightarrow$ (D.1),(D.8)
(D.24) $\rightarrow$ (D.2),(D.17), (D.18),(D.19)
(D.25) $\rightarrow$ (D.3),(D.4), (D.11),(D.18)
(D.26) $\rightarrow$ (D.4),(D.11), (D.18),(D.19)
(D.27) $\rightarrow$ (D.12),(D.13), (D.16),(D.17),(D.18), (D.19)
(D.28) $\rightarrow$ No new derivative terms

D.3 Independent $\partial^4 F^4$-Terms

$$A \quad \partial_e\partial_f F_{bd} \quad \partial_e\partial_f F_{ac} \quad F_{ab} \quad F_{cd} \tag{D.29}$$

$$B \quad \partial_c\partial_d F_{ea} \quad \partial_c\partial_d F_{eb} \quad F_{ag} \quad F_{bg} \tag{D.30}$$

$$C \quad \partial_c\partial_d F_{ea} \quad \partial_c\partial_e F_{db} \quad F_{ag} \quad F_{bg} \tag{D.31}$$

$$D \quad \partial_a\partial_b F_{cd} \quad \partial_a\partial_b F_{cd} \quad F_{gh} \quad F_{gh} \tag{D.32}$$

$$E \quad \partial_d\partial_a F_{be} \quad \partial_d F_{ef} \quad \partial_a F_{bc} \quad F_{cf} \tag{D.33}$$

$$F \quad \partial_a\partial_b F_{cf} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{df} \tag{D.34}$$

$$G \quad \partial_a\partial_b F_{ef} \quad \partial_c F_{db} \quad \partial_c F_{da} \quad F_{ef} \tag{D.35}$$

$$H \quad \partial_d\partial_e F_{bc} \quad \partial_d F_{ef} \quad \partial_a F_{bc} \quad F_{af} \tag{D.36}$$

$$I \quad \partial_d\partial_b F_{ea} \quad \partial_d F_{ef} \quad \partial_a F_{bc} \quad F_{cf} \tag{D.37}$$

$$J \quad \partial_a\partial_b F_{df} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{cf} \tag{D.38}$$

$$K \quad \partial_a\partial_d F_{bf} \quad \partial_c F_{de} \quad \partial_a F_{be} \quad F_{cf} \tag{D.39}$$

E. TRACE RELATIONS (ANTI)SYMMETRIC TENSORS

Start by writing the (anti)symmetric tensors F^1 and F^2 as

$$\begin{aligned} \star F^2 \star F^1 &= \star F^2_{ab} \star F^{1\ bc}, && \text{(E.1)} \\ &= \frac{1}{4}\varepsilon_{bcfg}\varepsilon^{abde}F^2_{de}F^{1\ fg}, && \text{(E.2)} \\ &= -\frac{1}{4}3!\ \delta^{ade}_{cfg}F^2_{de}F^{1\ fg}, && \text{(E.3)} \\ &= -\frac{1}{4}(2\delta^a_c F^2_{de}F^{1\ de} + 2\delta^a_n F^2_{cf}F^{1\ fg} + 2\delta^a_m F^2_{gc}F^{1\ fg}), && \text{(E.4)} \\ &= \frac{1}{4}(2\delta^a_c F^2_{de}F^{1\ de} + 4(F^1F^2)^a_c). && \text{(E.5)} \end{aligned}$$

From which follows

$$(\star F^2 \star F^1 - F^1F^2)^a_c = -\frac{1}{2}\delta^a_c F^2_{ij}F^{1\ ji}. \tag{E.6}$$

One can also write the previous expression like

$$\star F^2 \star F'^1 - F'^1F^2 = -\frac{1}{2}\mathrm{Tr}(\star F'^1F^2)I. \tag{E.7}$$

Next make the identification

$$F'^1 \equiv \star F^1. \tag{E.8}$$

Which gives us

$$\boxed{\star F^2F^1 + \star F^1F^2 = \frac{1}{2}\mathrm{Tr}(\star F^1F^2)I}. \tag{E.9}$$

Or if $F_1 = F_2$

$$\star FF = \frac{1}{4}\mathrm{Tr}(\star FF)I. \tag{E.10}$$

BIBLIOGRAPHY

[1] J.D. Jackson, ”Classical Electrodynamics” 3rd edtion, John Wiley & Sons (1999)

[2] R. D’Inverno, ”Introducing Einstein’s Relativity” 1st edtion, Oxford Univ. Press UK (1992)

[3] I. Bialynicki-Birula, ”Quantum Theory of Particles and Fields”, p.31 World Scientific, Singapore (1983)

[4] D. Atkinson and P.W. Johnson, ”Quantum Field Theory: A Self-Contained Course”, Rinton Press (2002)

[5] B. Zwiebach, ”A First Course in String Theory” 1st edition, Cambridge Univ. Press (2004)

[6] M. de Roo, ”Basic String Theory”, syllabus R*u*g (2004)

[7] M. Eenink, ”Dirichlet Branes, Effective Actions and Supersymmetry”, Ph. D thesis R*u*g (2004)

[8] R. Endo; R. Kuriki; S. Nojiri and S. Sugamoto, ”A Rule of Thumb Derivation of Born-Infeld Action for D-branes”, hep-ph/9801037 (1998)

[9] M. de Roo; M. Eenink, ”The Effective Action for the 4-point Function in Abelian Open Superstring Theory”, hep-th/0307211 (2003)

[10] G. Gibbons; D. Rasheed, ”Electromagnetic Duality Rotations in Nonlinear Electrodynamics”, hep-th/9506035 (2004)

[11] M. Kuzenko; S. Theisen, ”Nonlinear Self-Duality and Supersymmetry”, hep-th/0007231 (2000)

[12] F. Diamond; J. Shurman, ”A First Course in Modular Forms” 1st edition, Springer (2006)