Little Higgs Models: Effective Gauge Theories stabilizing the Electro-Weak Scale employing Collective Symmetry Breaking

Master Thesis

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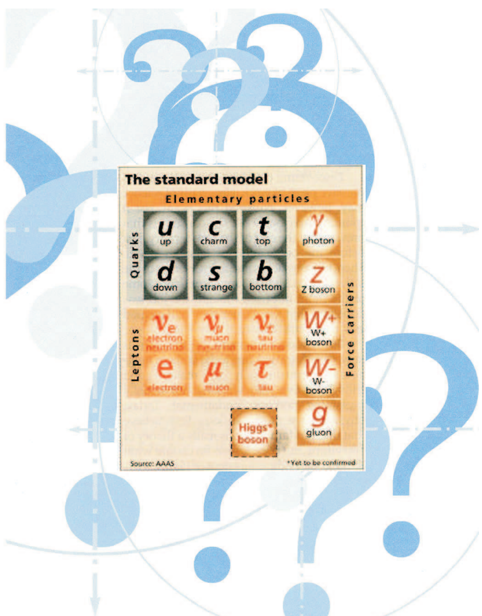

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General Introduction

In physics our goal is to clarify and understand the immense complexity of nature. Our present understanding of the theories of elementary particle physics is guided by the principles of symmetry and renormalizability, though the latter has been somewhat weakened over the last decades. The principle of symmetry, or invariance under certain transformations, became appreciated when Einstein identified the symmetry group of spacetime. Physics is invariant under Lorentz transformations.

In the 1930's Heisenberg proposed a symmetry called isospin. He was intrigued by the fact that the proton and the neutron have an almost identical mass. Isospin conservation is an (approximate) symmetry of the strong interactions which puts the proton and the neutron in a doublet (fund. rep.) of $SU(2)$. The three pions are grouped together in a triplet (adjoint rep.). In the 1950's, with new, more powerful accelerators, new particles were found and a new symmetry was discovered called *strangeness*. Strangeness appeared to be conserved by the strong interaction, but not by weak interactions. This discovery led to a generalization of the isospin idea with the classification of particles with representations of the group $SU(3)$. Later on this led to the successful proposal of the quark model by Gell-Mann. Symmetry is now regarded as an *a priori* principle.

The fundamental interactions however, seem to have different symmetries at their basis. In 1956 it was experimentally discovered that weak interactions do not conserve parity. First described in a field theoretical model by Fermi, the Lagrangian density of leptonic weak interactions has the form [1]:

$$\mathcal{L}_{\mathcal{F}} = -\frac{G_F}{\sqrt{2}} j^{\mu\dagger} j_\mu \tag{1.1}$$

Where nowadays j_σ is known to be:

$$j_\sigma = \bar{\psi}_e \gamma_\sigma (1 - \gamma^5) \psi_{\nu e} + \bar{\psi}_\mu \gamma_\sigma (1 - \gamma^5) \psi_{\nu \mu} \tag{1.2}$$

This Lagrangian is Lorentz invariant and expresses that only lefthanded particles (righthanded anti-particles) take part in weak interactions. This idea was motivated by experiment. This model is known as the V-A model, because (1.2) contains only *vector* and *axial vector* operators. Lorentz invariance allows for other operators such as tensor operators, but there is as yet no experimental motivation to include such operators. In the 1960's Glashow, Salam and Weinberg constructed a theory that unified weak- and electromagnetic interactions. This theory, known as the standard model or GSW model, is based on a local $SU(2) \times U(1)$ gauge symmetry. Lefthanded leptons and quarks are placed into doublets (fund. rep.) and interact through intermediate particles called vector bosons. The theory required, a by then unknown, neutral vector particle. This particle, the Z^0 boson, was experimentally confirmed in nucleon-neutrino reactions. In 1979 the Nobelprice was awarded to Glashow, Salam and Weinberg. Nowadays the standard model of electro-weak interactions and the $SU(3)$ theory of quantum chromo dynamics (QCD) stand firmly as theories of elementary particle physics.

Since its development in the late sixties, the standard model of electroweak interactions has been incredibly successful in describing the physics of elementary particles. Numerous theoretical predictions were experimentally confirmed to high accuracy. It seems, however, that the standard model has arrived at a pivotal moment in history. With the construction of the LHC, due to begin operation in 2007, the physics community seems convinced that new physics will soon be discovered. One important motivation for these expectations is known as the *hierarchy problem*.

One should, however not think that the standard model is incorrect. It has been incredibly successful in precisely describing numerous phenomena in elementary particle physics. It has quite some free parameters, but once these parameters are measured they are fixed for all processes, essentially leaving no freedom. All but one of these parameters have been measured, some extremely accurate others somewhat less. The greatest unknown is the precise value of the Higgs boson mass. One might argue that this precise value is a matter of time, but the situation is actually somewhat more subtle than that. Moreover, there are questions regarding physical and theoretical phenomena to which the standard model provides no answer. Only one example of this is for example the *strong CP problem*. CP violating terms appear in the QCD Lagrangian, but they are unbelievably much smaller than one would assume i.e. order one.

What physicists expect is that the standard model is an incomplete theory. This means that at a certain energy scale it must give way to a more fundamental theory, which must of course reproduce the standard model at low energy. Attempts to construct such a more fundamental theory have been made and some of them have stood the test of theoretical inconsistencies and precision measurements. A popular theory, that can also be tested in the near future, is the *Minimal Supersymmetric Standard Model* (MSSM). It is an extension of the standard model involving additional particles which are related to the standard model particles by a very special symmetry. In this thesis, however, we will not discuss the MSSM, but a different class of extensions of the standard model known as *Little Higgs models*. Before elaborating on these models we will clarify an important motivation for expecting (spectacular) new physics beyond the standard model, not too far from the energy scales physicists can reach nowadays. This is a problem known as the hierarchy problem.

2

Hierarchy

2.1 The Hierarchy Problem

In this world virtually everyone is accustomed to hierarchy. However, hierarchy is not something that only occurs in the world of politics or other organizational structures. It is also an important concept in the mathematical structure of physical theories. The word 'hierarchy' comes from the fact that in physics one always speaks of energy scales. Almost every physical theory has a specific domain of validity or application. Examples of various theories or areas of physics and their energy scales are shown in table (2.1).

Atomic Physics	$\sim$ 1 eV-10 keV
Nuclear Physics	$\sim$ 1 MeV-1 GeV
Electro-Weak theory	$\sim 10^2$ GeV -(?)GeV
Grand Unified Theories (GUTS)	$\sim 10^{14}$ GeV
String Theory	$\sim 10^{19}$ GeV

Tab. 2.1: Table of physical theories and their relevant domain.

One might ask: what is the problem? The problem is one of naturalness. The standard model of electro-weak interactions appears to describe physics extremely well up to the smallest distance scales yet probed. Ignoring gravity, it is a highly accurate theory of physics, however it still needs experimental input. There are still many questions to which the standard model has no answers. For example one might wonder about neutrino masses, values of fundamental constants and couplings, the origin of families or the exact charge relations between various particles. An even greater question might be: what is the role of gravity in all this? For now it seems that only general relativity correctly describe the physics of gravity and perhaps string theory. What we do know is that the standard model is certainly not the last theory of physics. At some energy it must give way to a more fundamental theory. This is perhaps a grand unified theory or string theory or something else (not yet considered).

There are a number plausible reasons to expect that a grand unified theory could appear at a scale of 10^{14} GeV. These expectations come from considerations on the running of coupling constants in the SM and from the lifetime of the proton. This GUT scale is a lot larger than the known scale of weak interactions, which is approximately $v \sim 250$ GeV. One might ask: what lies beyond the scale of weak interactions? The simplest answer would be: simply nothing. Except for the known standard model particles there is an energy desert all the way up a scale of 10^{14} GeV. To physicists this scenario seems highly unlikely. The motivation for rejecting this 'energy desert' hypothesis is the hierarchy problem or the problem of 'naturalness'. The problem arises due to a number of assumptions:

- The standard model remains valid up to a scale M, which is a lot larger than the scale $v \sim 250$ GeV.
- New, beyond the standard model, physics does not occur until a large scale of grand unification or perhaps (quantum) gravity theories.
- Physics at ordinary (low) energies is not extremely sensitive to the particular values of fundamental constants.

Suppose the standard model has a cutoff Λ. Unknown physics resides in the gauge couplings, Yukawa couplings and the Higgs potential. These parameters depend on the structure of a more complete theory at an energy scale M. The energy scale of the standard model or the mass of every ordinary particle in the standard model is proportional to the vacuum expectation value of the Higgs field. This $\langle\phi\rangle$ is not a new parameter, but is obtained by minimizing the Higgs potential.

$$V(\phi) = -\mu^2\phi^2 + \lambda\phi^4 \quad (2.1)$$

$$\langle\phi\rangle = \frac{\mu}{\sqrt{2\lambda}} \quad (2.2)$$

These values are the renormalized mass and couplings. Let us have a look at the parameter μ. The relevant diagrams contributing to the renormalized value of μ are shown in figure (2.1). They include vacuum corrections to the

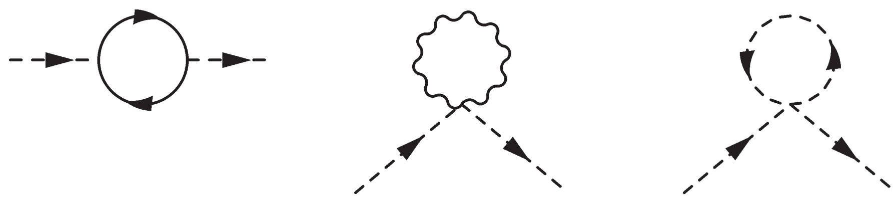


Fig. 2.1: Quadratically divergent contributions to the Higgs mass squared in the standard model.

Higgs mass from predominantly top quark loops, loops from $SU(2) \times U(1)$ gauge bosons and from the Higgs itself. Since all these diagrams are formally infinite we have to regularize them in some way. Incidently, renormalization (not regularization) is also required in finite results. However, our only purpose here is to show the type of divergence that occurs. We could choose to use a gauge invariant regularization scheme such as dimensional regularization or Pauli-Villars regularization, but then a $1/\epsilon$ pole would correspond to a quadratical dependence on the cut-off or a quadratical dependence on the Pauli-Villars mass. Here we simply choose to cut the integrals of at a scale Λ. This type of regularization is not gauge invariant, but that is of no concern to us. The diagrams in figure (2.1) are all quadratically dependent on the cutoff[5]. Thus μ^2 has the following form:

$$\mu^2 = \mu_{bare}^2 + \Lambda^2(\lambda_t^2 c_1 + g^2 c_2 + \lambda c_3 + \text{ higher order terms}) \qquad (2.3)$$

The value of μ is $\sqrt{2\lambda}\langle\phi\rangle$, which is approximately $\langle\phi\rangle$ for λ of order unity. From electro-weak measurements on gauge boson masses we know that $\langle\phi\rangle \approx$ 250 GeV. Suppose we had considered some grand unified theory to replace the standard model at a scale of $\Lambda = 10^{14}$ GeV.

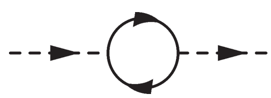
$$\frac{\mu^2}{\Lambda^2} = 10^{-24} = \frac{\mu_{bare}^2}{\Lambda^2} + (\lambda_t^2 c_1 + g^2 c_2 + \lambda c_3 + \text{ higher order terms}) \qquad (2.4)$$

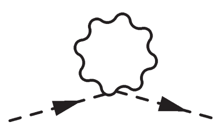
So the bare Higgs mass parameter would have to be finely tuned to 26 decimal places in order to cancel the complicated series and so realizing a low energy scale and a small physical Higgs mass and the light masses of the known standard model particles. In addition, one would expect the fraction μ_{bare}^2/Λ^2

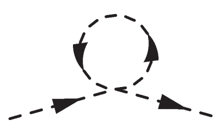
to depend on the structure of the high energy scale theory and not that it is finely tuned to produce low energy scale physics. So it seems unnatural that a theory such as the standard model with an energy scale v has a more fundamental energy scale which is many orders of magnitude higher. This is known as the hierarchy problem.

Given this hierarchy problem one would expect that new physics should appear on an energy scale not too far away from that of the standard model. It is possible to make a rough estimate of the scale at which new physics may appear and one could even ask the question of wether or not it is plausible that we have not seen any spectacular new physics at todays most powerful particle accelerators.

First we calculate the corrections to the Higgs mass coming from the quadratically divergent diagrams. Assuming that the standard model has a cut-off Λ we do not know how to compute loop diagrams with momenta higher than Λ, because beyond that scale new physics takes over. So we will cut the integrals of at momentum Λ. The calculation of the three diagrams can be found in Appendix A. The contributions are the following:

$-iM_1^2 =$  $= -i\frac{-3}{8\pi^2}\lambda_t^2\Lambda^2 \Rightarrow \sim$ -(2000 GeV)^2

$-iM_2^2 =$  $= -i\frac{9}{64\pi^2}g^2\Lambda^2 \Rightarrow \sim$ +(750 GeV)^2

$-iM_3^2 =$  $= -i\frac{3}{16\pi^2}\lambda\Lambda^2 \Rightarrow \sim$ +(450 GeV)^2

Where the cut-off Λ was taken to be 10 TeV. The $-i$ factor is convenient because it lets the M_i's appear in the same as way a mass does in the propagator [2]. Assuming a cut-off of 10 TeV to the standard model therefore leads to a fine tuning of approximately one part in a hundred of the Higgs mass and thus of the electroweak scale. This is illustrated in figure (2.2) on the next page. Todays most powerful particle accelerators have probed up to energies of approximately 1 TeV. Taking Λ to be 1 TeV we find corrections of approximately -(200 GeV)^2 from the top loop, (75 GeV)^2 from $SU(2)$ gauge boson loops and (45 GeV)^2 from the Higgs self interaction diagram. One can conclude that with a cut-off of 1 TeV there is no need for

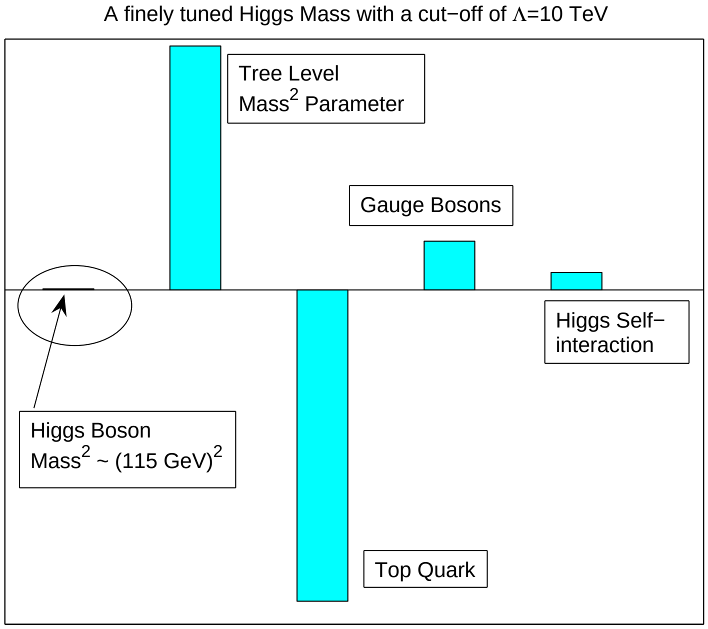


Fig. 2.2: A graphical illustration of the fine tuning of the Higgs boson mass in a standard model with a cut-off of $\Lambda = 10$ TeV.

fine tuning. So it does not seem strange that we have not yet found any new physics beyond the standard model. An estimate of the scale at which new physics should appear in order to avoid fine tuning of the electro-weak scale can be made by proposing a fine tuning of no more than 10 percent. This leads to a cut-off Λ of approximately 2 TeV. So it is to be expected that new particles exist and that these particles have masses of approximately 2 TeV. In order for these particles to cancel the quadratic divergences caused by standard model corrections one might suggest that these new particles are related to the standard model particles by a symmetry. One popular and (partly) successful solution to the hierarchy problem is *supersymmetry*. In supersymmetric theories the quadratically divergent diagrams are cancelled by diagrams with superpartners running in the loops. Speaking colloquially, these cancellations occur because superpartners have opposite spin-statistics.

2.2 The Little Hierarchy Problem

Combining the theory of the standard model and supersymmetry results in a theory known as the minimal supersymmetric standard model (MSSM). The MSSM however, does not appear to be a perfect solution to the hierarchy problem. At low, ordinary energies nature is not supersymmetric, superpartners have thus far not been detected. This means that supersymmetry must be broken at some scale M_{SUSY}. If supersymmetry were unbroken the quadratically divergent diagrams would cancel completely. If weak scale SUSY exists, superpartner masses weakly break supersymmetry. In that case there are only cancellations above the superpartner mass scale M_{SUSY}. This is not a problem since super partner masses could occur in the range of a few hundred GeV. Incidently, Standard Model particles are not suitable as superpartners. What is a problem in the MSSM is that the tree level Higgs mass μ is bounded from above by the Z boson mass (91 GeV)[4]. Radiative corrections from the top quark superpartner, *stop*, can raise the Higgs mass to 115 GeV, however this requires a rather large mass for the *stop*. This large mass however reintroduces a certain amount of fine tuning.

Experimental bounds suggest that the value of the Higgs mass is about 10 percent of the loop corrections. So in the MSSM model a small amount of fine tuning seems necessary. This is known as the *Little Hierarchy Problem*.

There are a number of other solutions to the hierarchy problem such as *Technicolor* models and *Little Higgs* models. This thesis is concentrated on the last of these.

3

Nambu-Goldstone Bosons

3.1 Symmetry

In contemporary physics we hold symmetry to be an important principle. The goal in physics is to see the huge amount of phenomena occurring in nature in a unified manner, guided by a few basic principles. In classical physics for example one holds, virtually unaware, parity to be a symmetry of nature. Under a parity transformation, physics does not change. As we mentioned, symmetry became more appreciated when Einstein introduced special relativity in the beginning of the twentieth century. Physics is invariant under the group of Lorentz transformations. Gradually physicists understood that symmetry was an important guiding principle. Symmetry was understood not be an accidental fact of nature, but nature seems *a priori* based on symmetry. Also in particle physics symmetries were identified, such as conservation of lepton number and baryon number.

All the symmetries just mentioned are known as global symmetries, they do not explicitly depend on space and time. With the development of quantum electrodynamics (QED) it was observed that QED possessed a much more fundamental and powerful symmetry. This is a symmetry which depends on space and time, or a 'local' symmetry. This symmetry consists out of introducing an arbitrary phase change, which is a function of space and time, in the electron field. The EM vector potential undergoes a compensating gauge transformation. Today this is known as a $U(1)$ gauge symmetry.

$$\psi \Rightarrow \psi^{'} = e^{i\theta(x)}\psi \tag{3.1}$$

$$A_\mu \Rightarrow A_\mu^{'} = A_\mu - \frac{1}{e}\partial_\mu\theta(x) \tag{3.2}$$

If this symmetry is imposed, the combined Dirac Lagrangian with a covariant derivative:

$$D_\mu = \partial_\mu + ieA_\mu \qquad (e = -|e|) \tag{3.3}$$

and the Maxwell Lagrangian, result in interactions between charged spin 1/2 fermions and fotons follow. These interactions are precisely the ones obtained by using the principle of minimal substitution from classical electrodynamics. These ideas can be extended to more elaborate (non-abelian) groups and this was done in the 1950's by C.N. Yang and R. Mills. They attempted to generalize the gauge invariance idea from QED to construct a theory for the strong interactions of nucleons[7]. This theory was based on the group $SU(2)$ working on the isospin nucleon doublet. B vector states, comparable to pions, were treated as the gauge bosons of $SU(2)$. It was later found that this was not a correct approach for describing nucleonic interactions, however the idea of non-abelian gauge invariance proved very useful in the theories of elementary particle physics. The principle is the same as in QED, only now the fermion field transforms under the action of the (representation of a) non-abelian group element and the gauge fields (one for every generator) undergo a generalized gauge transformation. This is expressed in the following equations:

$$\psi \Rightarrow \quad \psi' = U\psi \tag{3.4}$$

$$A^a_\mu T^a \Rightarrow A^{a'}_\mu T^a = UA^a_\mu T^a U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \tag{3.5}$$

$$D_\mu = \partial_\mu - igA^a_\mu T^a \tag{3.6}$$

For the group $U(1)$ one finds the QED result.

3.2 Broken Symmetry

It appears that not only symmetry itself is very important in physics but also the phenomenon of symmetry breaking. While the fundamental laws of nature appear to be symmetric, nature itself is not always perfectly symmetric. The symmetry breaking mechanism is in fact a crucial idea in the breaking of the electro-weak symmetry. There are many ways to break a symmetry, but in nature symmetries appear to be broken only very subtle. Consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}[(\partial\phi)^2 - \mu^2\phi^2] - \frac{\lambda}{4}\phi^4 \tag{3.7}$$

Here ϕ=$(\phi_1, \ldots, \phi_N)$. This Lagrangian possesses an $O(N)$ symmetry. ϕ transforms as an N-vector of $O(N)$. If one adds for example a term ϕ_1^2 to the Lagrangian, then the $O(N)$ symmetry is broken down to an $O(N-1)$ symmetry under which $(\phi_2, \ldots, \phi_N)$ transforms as an $(N-1)$-vector. This type of symmetry breaking is known as explicit symmetry breaking because we just added a term to the Lagrangian which does not respect the $O(N)$ symmetry.

We can break the symmetry in a more subtle way. Consider the Lagrangian (3.7) but now with the difference that the sign of the $\mu^2\phi^2$ term is positive:

$$\mathcal{L} = \frac{1}{2}[(\partial\phi)^2 + \mu^2\phi^2] - \frac{\lambda}{4}\phi^4 = \frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{3.8}$$

$$V(\phi) = -\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 \tag{3.9}$$

Where $V(\phi)$ is a potential for the ϕ field. Let's take as an example $N = 2$. The form of the potential $V(\phi)$ is shown in figure (3.1). The potential has

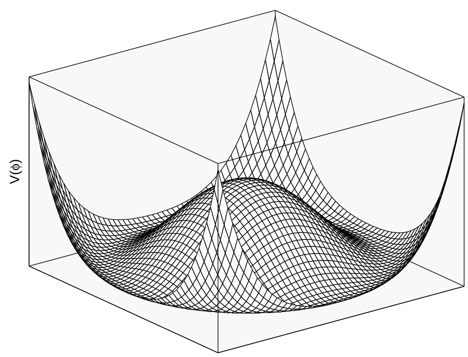


Fig. 3.1: Potential experienced by the ϕ field (N=2).

its minimum at $\phi^2 = \frac{\mu^2}{\lambda}$. As one can see, 'rolling down the hill', the field ϕ can go into an infinite number of vacuum states. Each state with a different 'direction' of the vector ϕ. Physically these states are all equivalent because the Lagrangian possesses an $O(2)$ symmetry. Since all these vacuum states are equivalent physical results should not depend on the choice of a particular vacuum. Let us make the following particular choice:

$$\phi = \begin{pmatrix} \sqrt{\frac{\mu^2}{\lambda}} \\ 0 \end{pmatrix} \equiv \begin{pmatrix} v \\ 0 \end{pmatrix} \tag{3.10}$$

The ϕ field has acquired a vacuum expectation value (VEV). Now consider fluctuations of the field around this vacuum configuration:

$$\phi_1 = v + \phi_1^{'} \qquad \phi_2 = \phi_2^{'} \tag{3.11}$$

If these equations for ϕ_1 and ϕ_2 are substituted back into the Lagrangian (3.8) one gets a surprising result:

$$\mathcal{L} = \frac{1}{2}(\partial\phi^{'})^2 - \lambda v^2 \phi_1^{'2} + \frac{1}{4}\lambda v^4 + O(\phi^{'3}) \tag{3.12}$$

An arbitrary constant can always be added to the Lagrangian so we lose the third term. Dropping the primes and substituting for v one finds:

$$\mathcal{L} = \frac{1}{2}[(\partial\phi_1)^2 + (\partial\phi_2)^2] - \mu^2\phi_1^2 + O(\phi^3) \tag{3.13}$$

The field ϕ_2 is massless! This fact can be understood by looking at graph of the potential shown in figure (3.1). The massive field ϕ_1 is the mode fluctuating in the radial direction of the potential, the fluctuations require energy, as they roll up and down 'the hill'. The ϕ_2 field, which is massless, is the angular mode. Obviously it costs no energy to move along the minimum of the potential.

The massless particle that has appeared is known as a Nambu-Goldstone boson (NGB). The potential $V(\phi)$ induced a vacuum expectation value for the ϕ field. The $O(2)$ symmetry is 'spontaneously' broken in the vacuum as a result of the theory itself. This type of symmetry breaking is known as *spontaneous symmetry breaking*. It is in fact a general result that whenever a continuous symmetry is spontaneously broken a massless particle, a Nambu-Goldstone boson, emerges. This is known as Goldstone's theorem. We will make this more precise later on. It is important wether the symmetry is discrete or continuous. NGB's appear only with the spontaneous breaking of continuous symmetries. An example of this can be seen by inspecting the $N = 1$ case as in the foregoing discussion. The graph of the potential is shown in figure (3.2). This potential spontaneously breaks the discrete reflection symmetry. The vacuum state is either $\phi = +v$ or $\phi = -v$. Suppose one chooses the $\phi = +v$ vacuum. The field is again redefined in the following way:

$$\phi = v + \phi^{'} \tag{3.14}$$

When this is substituted in the Lagrangian (3.8), however now in the case of $N = 1$, one finds:

$$\mathcal{L} = \frac{1}{2}(\partial\phi^{'})^2 - \lambda v^2 \phi^{'2} + \frac{1}{4}\lambda v^4 + O(\phi^{'3}) \tag{3.15}$$

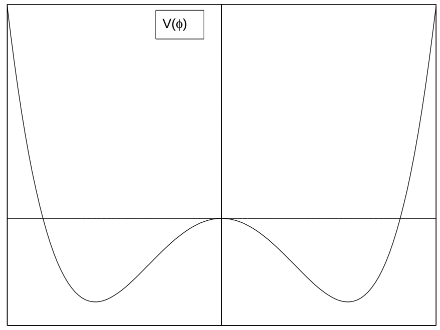


Fig. 3.2: Potential experienced by the ϕ field ($N = 1$).

Dropping the primes, the constant term and substituting for v we arrive at:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \mu^2\phi^2 + O(\phi^3) \tag{3.16}$$

The field ϕ has become massive with mass $\sqrt{2}\mu$, however there is no massless Nambu-Goldstone boson as anticipated.

3.3 Complex Fields and Non-Abelian Symmetries

In addition to real fields, many field theories contain complex fields. We can obtain the Lagrangian (3.8) in terms of complex fields by constructing the following complex field:

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{3.17}$$

The Lagrangian (3.8) can then be written in a complex formulation:

$$\mathcal{L} = (\partial\phi^\dagger)(\partial\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{3.18}$$

This Lagrangian is invariant under $U(1)$ transformations. In group theory the groups $U(1)$ and $O(2)$ are known as isomorphic. Informally this means that there is a kind of mapping between the respective elements which has certain properties. The potential in figure (3.1) again spontaneously breaks the $U(1)$ symmetry in the vacuum. Just as one can write a complex field in cartesian coordinates, it can also be expressed in a polar form. Taking into account the acquired VEV, one can express the field in terms of small oscillations around its VEV:

$$\phi(x) = \frac{1}{\sqrt{2}}(v + r(x))e^{i\theta(x)/v} \tag{3.19}$$

$r(x)$ is the 'massive radial mode' and θ(x) is the Nambu-Goldstone boson. If one considers the effect of a $U(1)$ transformation on this field, it can be seen that $r(x)$ is invariant under such a transformation, but the Nambu-Goldstone boson shifts. This is sometimes called a non-linear symmetry. This shift symmetry ensures only derivative interactions and also as a result ensures the masslessness of the NGB.

Nambu-Goldstone bosons also emerge when continuous non-Abelian symmetries are spontaneously broken. In that case there will be a NGB for every broken generator of the symmetry group. The Goldstone theorem states precisely this fact: for every conserved charge that does not annihilate the vacuum state ($Q|0 >\neq 0$) a NGB will emerge. A proof of the Goldstone theorem is shown in the appendix on mathematical results. The number of conserved charges is equal to the number of symmetry group generators. The number of conserved charges that do not annihilate the vacuum is equal to the number of broken symmetry generators. If a Lagrangian is left invariant by a symmetry group S with $G(S)$ generators, and the vacuum state is left invariant only by a subgroup H, with G(H) generators, then the number of NGB is $G(S) - G(H)$. The shift symmetry persists in the non-Abelian case.

An important symmetry breaking pattern will be the breaking of an $SU(N)$ group to an $SU(N-1)$ group, by a VEV for a fundamental scalar field. For this purpose the NGB are conveniently parameterized in the following way (one of many ways):

$$\phi = \exp[\frac{i}{f}\begin{pmatrix} 0 & \vec{\pi} \\ \vec{\pi}^{\dagger} & \pi_0 \end{pmatrix}]\phi_0 = e^{\frac{i}{f}\Pi}\phi_0 \tag{3.20}$$

Where $\vec{\pi} = (\pi_1, \ldots, \pi_{N-1})^T$ and $\phi_0 = (0, \ldots, 0, f)^T$ a VEV configuration. π_0 is a real field and a singlet under these group transformations. We can find the transformation character under unbroken $SU(N-1)$ transformations of the NGB's $\vec{\pi}$ as follows:

$$\phi \Rightarrow \phi' = U_{N-1}\phi = U_{N-1}e^{(i/f)\Pi}U^{\dagger}_{N-1}U_{N-1}\phi_0 \tag{3.21}$$

ϕ_0 is invariant under U_{N-1} transformations, so we find:

$$\phi' = U_{N-1}e^{(i/f)\Pi}U^{\dagger}_{N-1}\phi_0 = e^{(i/f)U_{N-1}\Pi\ U^{\dagger}_{N-1}} = e^{(i/f)\Pi'} \tag{3.22}$$

The second equality can be proven by expanding the right side. The unitary matrices then factor out. These matrices are parameterized as:

$$U_{N-1} = \begin{pmatrix} U_{N-1\times N-1} & 0 \\ 0 & 1_{1\times 1} \end{pmatrix} \tag{3.23}$$

So under unbroken $SU(N-1)$ group transformations the NGB's transform as:

$$\pi \Rightarrow \pi' = U_{N-1}\vec{\pi}\, U^{\dagger}_{N-1} \tag{3.24}$$

Before we consider the transformation character of the NGB's under broken $SU(N)$ transformations, let us have a look at matrix exponents. From group theory we know that we can represent every group element as the exponent of a linear combination of the generators T^a of that group.

$$U_g = e^{iC\cdot T} \tag{3.25}$$

Where the coefficients C are usually infinitesimals. In general these generators are matrices and by the exponent of a matrix we mean the series expansion of the exponent.

$$U_g = e^{iC\cdot T} = 1 + iC\cdot T - \frac{1}{2!}(C\cdot T)(C\cdot T) - \frac{i}{3!}(C\cdot T)(C\cdot T)(C\cdot T) + \dots \tag{3.26}$$

It is not immediately obvious wether the rules for complex numbers in exponents can be carried over to matrices. For example, one might ask what the form of the matrix M is in the expression for combining two matrix exponents.

$$e^A \cdot e^B = e^M \tag{3.27}$$

M, in general, is not equal to the sum of A and B. In fact M is given by the expression:

$$M = A + B + \frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] + [[A,B],B]) + \dots \tag{3.28}$$

This result is known as the *Baker-Campbell-Hausdorff* formula. A proof is shown in the appendix on mathematical results. So the matrix M is only equal to first order to the sum of A and B, or exactly equal if the matrices A and B commute.

Let us now have a look at the broken $SU(N)$ transformations and see how the NGB's transform. In the abelian case we found that the NGB's shift under the broken transformations. This is also true (to first order) for non-abelian groups. Every element g near the identity element of the group can be decomposed as follows:

$$g = e^{\theta\cdot U} \cdot e^{\xi\cdot V} \tag{3.29}$$

Where the U's are broken generators and the V's unbroken. So for every element g_0 one can write [6]:

$$g_0 e^{\pi \cdot U} = e^{\pi' U} \cdot e^{\xi' V} \tag{3.30}$$

So for the broken transformations we can write:

$$U_{N/N-1}\ e^{(i/f)\Pi}\phi_0 = \exp[\frac{i}{f}\begin{pmatrix} 0 & \vec{\theta} \\ \vec{\theta}^\dagger & 0 \end{pmatrix}]\exp[\frac{i}{f}\begin{pmatrix} 0 & \vec{\pi} \\ \vec{\pi}^\dagger & 0 \end{pmatrix}]\phi_0 \tag{3.31}$$

$$= \exp[\frac{i}{f}\begin{pmatrix} 0 & '\vec{\pi} \\ \vec{\pi}'^\dagger & 0 \end{pmatrix}]U_{N-1}\phi_0 \tag{3.32}$$

$$= \exp[\frac{i}{f}\begin{pmatrix} 0 & \vec{\pi}' \\ \vec{\pi}'^\dagger & 0 \end{pmatrix}]\phi_0 \tag{3.33}$$

Where in the first equality:

$$U_{N/N-1} = \exp[(i/f)\theta^a U^a_{N/N-1}] = \exp[\frac{i}{f}\begin{pmatrix} 0 & \vec{\theta} \\ \vec{\theta}^\dagger & 0 \end{pmatrix}] \tag{3.34}$$

In the second equality we used (3.30). π' is a function of θ and π, which to first order equals:

$$\pi' = \pi + \theta \tag{3.35}$$

So, as in the abelian case, we obtain the shift of the NGB's under broken transformations. This shift symmetry ensures that the NGB's have derivative interactions only, moreover it is massless and not subjected to a potential.

A last note on the mechanism of symmetry breaking: The symmetry manifests itself at high energy. The Lagrangian of the system is invariant under the symmetry operation, as such the symmetry cannot cease to exist, even if it appears to be violated. In fact it does not cease to exist, but the way in which it can identified is different. The system has many equivalent ground states that transform into each other under the symmetry operation. The system can only exist in one of these states, so one chooses a particular ground state, however the symmetry manifests itself in that one could have chosen any ground state, since they are all equivalent. Spontaneous symmetry breaking occurs not only in particle physics but also in other branches of physics. A particular insightful example is the ferromagnetic system, see for example [10]. Table (3.1) gives a few examples of systems in which spontaneous symmetry breaking occurs.

System	Broken Symmetry	Goldstone Boson
Crystal	Translation	Phonon
Ferromagnet	Rotation	Spin Wave
Superfluid	Global Symmetry	Phonon
Quant.Crom.Dyn.	Chiral Symmetry	Pion

Tab. 3.1: Systems with spontaneous symmetry breaking.

3.4 The Higgs mechanism

In the foregoing sections we mainly had global symmetries in mind. However, in nature, gauge symmetries are known to be very important. In fact, gauge theories provide a crucial understanding of particle physics. Gauge symmetries are local symmetries, depending on space and time as shown in equations (3.1) to (3.6). A problem, however, with gauge theories was that they seemingly did not allow for mass terms of any kind for any field. For example a term like $m^2 A_\mu A^\mu$ is not gauge invariant. Moreover, massless scalar particles like NGB's were not observed in nature. A mechanism known as the Higgs mechanism solves both these problems rather elegantly. If a gauge symmetry is spontaneously broken, Nambu-Goldstone bosons also emerge. In some sense they exist as particles, however they are not physical. Upon spontaneous symmetry breaking they become the longitudinal degrees of freedom of previously massless gauge fields in the theory. This is known as the Higgs mechanism. Let us illustrate this with a $U(1)$ example. Consider a locally $U(1)$ invariant Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{3.36}$$

Where we have:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.37}$$

$$D_\mu = \partial_\mu + ieA_\mu \tag{3.38}$$

The gauge invariance is expressed as:

$$\begin{aligned} \phi \Rightarrow \phi' &= e^{i\theta(x)}\phi \\ A_\mu \Rightarrow A'_\mu &= A_\mu - \frac{1}{e}\partial_\mu\theta(x) \end{aligned}$$

Expressing the scalar field in polar form, $\phi = \rho(x)e^{i\alpha(x)}$, the Lagrangian takes the form (3.39). Note that this parametrization has the same form as a $U(1)$ gauge transformation. This can be generalized to greater symmetries.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \rho^2(\partial_\mu\alpha - eA_\mu)^2 + \partial_\mu\rho\partial^\mu\rho + \mu^2\rho^2 - \lambda\rho^4 \tag{3.39}$$

Now, under a gauge transformation $\alpha \Rightarrow \alpha + \theta$, the shift symmetry, and $eA_\mu \Rightarrow eA_\mu - \partial_\mu\theta$. The term $\rho^2(\partial_\mu\alpha - eA_\mu)^2$ is gauge invariant! Defining $\mathcal{A}_\mu = -\frac{1}{e}\partial_\mu\alpha + A_\mu$ we can write the Lagrangian as:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\rho^2\mathcal{A}_\mu\mathcal{A}^\mu + \partial_\mu\rho\partial^\mu\rho + \mu^2\rho^2 - \lambda\rho^4 \tag{3.40}$$

Note that $F_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ also. Now, upon spontaneous symmetry breaking: $\rho = \frac{1}{\sqrt{2}}(v + h)$, where v is of course $\sqrt{\frac{\mu^2}{\lambda}}$, and the Lagrangian becomes:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2\mathcal{A}_\mu\mathcal{A}^\mu + e^2vh\mathcal{A}_\mu\mathcal{A}^\mu + \frac{1}{2}e^2h^2\mathcal{A}_\mu\mathcal{A}^\mu \\ &+ \frac{1}{2}\partial_\mu h\partial^\mu h - \mu^2h^2 - \lambda vh^3 - \frac{1}{4}\lambda h^4 + \frac{1}{4}v^4\end{aligned}$$

We now have a theory with a massive vector field $\mathcal{A}_\mu$ $(m = ev)$ while the NGB $\alpha(x)$ has disappeared from the theory. There is also a massive scalar particle h with a mass $\sqrt{2}\mu$. The massive vector field has three degrees of freedom. Sometimes people, rather poetically, refer to this as the fact that the gauge field has 'eaten' the NGB. In theories with greater symmetries there is a massive vector field for every broken generator.

For excellent pedagogical introductions to the Higgs mechanism we refer to [1] and [8].

Nambu-Goldstone bosons arising from the spontaneous breaking of an exact continuous symmetry are strictly massless to all orders in perturbation theory. If, however the symmetry is approximate or there are explicit symmetry breaking terms in the Lagrangian, the NGB's can acquire a mass through for example higher order loop diagrams. These NGB's are known as *pseudo-Nambu-Goldstone bosons*. This phenomenon can also occur in *collective symmetry breaking*. Collective symmetry breaking entails that various terms in the Lagrangian preserve a certain symmetry separatively, but joined together they break the symmetry or partially break it. This is a key ingredient in Little Higgs theories.

4

Little Higgs Models: General Introduction

Having gone through the necessary machinery in the previous chapters we can now begin at looking what Little Higgs models actually entail. Little Higgs models are actually successful implementations of an older idea. The idea was to stabilize the Higgs mass from large quantum corrections by constructing theories in which it was a pseudo-Nambu-Goldstone boson of a spontaneously broken (approximate) symmetry. The first attempts to construct theories based on this idea were not totally successful. Meaning that some of the large corrections to the Higgs mass could not be canceled or avoided. It appeared difficult to construct a model which fully stabilized the Higgs mass and thus the electro-weak scale, nevertheless the first successful model was realized in 2001. It was constructed by Arkani-Hamed, Cohen and Georgi [11] and based on a mechanism known as dimensional deconstruction and using unitary gauge symmetries to be broken down to smaller groups to produce Nambu-Goldstone bosons. It should be noted that in dimensional deconstruction the extra dimensions are employed purely as theoretical instrumentation. There is no higher dimensional interpretation of four dimensional theories constructed in this way.

Later on more elegant and simpler models based on the pseudo-Nambu-Goldstone idea were constructed. Usually the particular models are characterized by their global symmetry breaking pattern. A few examples are: $SU(5)/SO(5)$[13], $SU(6)/Sp(6)$[14], "the minimal moose" $[SU(3)]^2/SU(3)$[12] and general moose models $[SU(3)]^k/[SU(3)]^n$. Models with a discrete symmetry known as T-parity have also been constructed. In these models the heavy particles are odd under T-parity and the light standard model particles are even. This symmetry forbids effective operators (integrating out heavy fields) generated by tree-level exchange of heavy particles. This weakens electro-weak constraints, because only loop diagrams containing heavy particles can contribute. We will not discuss T-parity here, see for example [15]. In the next chapter we will discuss a particular simplified (toy) Little Higgs model, but first we shall try to uncover the basic mechanism known as collective symmetry breaking.

4.1 Collective Symmetry Breaking

Massless Nambu-Goldstone bosons always arise from the spontaneous breaking in the vacuum of a global symmetry. So this allows for a production mechanism of massless scalar fields. Consider for example the breaking of a global $SU(3)$ to $SU(2)$. There are five broken generators, so five massless NGB's appear. Four of these are contained in an $SU(2)$ (complex) doublet. However, this doublet does not have couplings to gauge bosons or fermions. In adding these couplings we have to add terms that explicitly do not respect the $SU(3)$ symmetry[4]. This also gives rise to non-derivative couplings turning the NGB's into *pseudo-NGB's*. If these terms are added with a very small coupling ϵ then all these non-derivative interactions are proportional to ϵ. This remains so including loop diagrams. These very small couplings thus allow us to control quantum corrections to the PNGB's. A problem is that standard model couplings to the Higgs doublet are not small ($\lambda_t \approx 1$). So even though (large!) quantum corrections to the Higgs mass are proportional to these couplings this does not adequately suppress these contributions to the Higgs mass. So simply coupling a NGB to standard model particles does not work.

Collective symmetry breaking involves two (or more) couplings in such a way that both couplings separately preserve a symmetry, but together they break (a part of) the symmetry. This is illustrated as follows. Consider the $SU(3)$ invariant Lagrangian $\mathcal{L}_0$, with two sets of interactions $\mathcal{L}_{1,2}$ added.

$$\mathcal{L} = \mathcal{L}_0 + \epsilon_1 \mathcal{L}_1 + \epsilon_2 \mathcal{L}_2 \tag{4.1}$$

Both ϵ terms separatively conserve an $SU(3)$ symmetry, so there are two $SU(3)$ symmetries, but both terms taken together this is broken down to the diagonal subgroup. This means that both terms transform in *precisely* the same way. This can visualized as tuning the ϵ_i's. Setting one of the two to zero, decouples the terms and restores the full two group symmetry. Setting both ϵ_i's non-zero aligns the two terms, thus the remaining symmetry is the diagonal subgroup.

Now with sufficient symmetry the NGB's are massless. This is sometimes expressed as the fact that they are protected by a symmetry. There exist no terms or diagrams that give rise to quantum corrections to the mass. So the NGB remains exactly massless. When however, the symmetry is reduced when both terms are present, diagrams and operators appear which do give rise to quantum corrections to the mass of the NGB's, making them pseudo-NGB's. These operators however, are proportional to both ϵ_i's and in well constructed theories only logarithmically divergent. These are acceptable and natural quantum corrections to stabilize the mass of the NGB's. In the following chapters we will present illustrations of the Little Higgs idea and collective symmetry-breaking.

5

An $SU(3)$ Simplified Little Higgs Model

5.1 Symmetry breaking and Nambu-Goldstone bosons

In this chapter we will discuss a particular Little Higgs model. This is a somewhat simplified model with a global $SU(3)$ symmetry [16]. Assume a global symmetry breaking pattern: $SU(3)/SU(2)$ by a scalar field ϕ. There are five broken generators and thus five NGB's. We parameterize these NGB's as follows:

$$\mathbf{\Pi} = \pi_i \, T_i = \begin{pmatrix} \eta/\sqrt{3} & 0 & h \\ 0 & \eta/\sqrt{3} & \\ h^\dagger & & -2\eta/\sqrt{3} \end{pmatrix}$$

Where the T_i's are the broken $SU(3)$ generators. The complex scalar field ϕ is parameterized with an exponential function.

$$\phi = e^{i\Pi/f}\phi_0 \qquad\qquad \phi_0 = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \tag{5.1}$$

The constant f is a symmetry breaking scale with the dimension of a mass. The Goldstone bosons transform linearly under unbroken transformations, but non-linearly (shift) under broken transformations. This non-linear, exponential form is the simplest of an infinite number of possible parameterizations, see [17]. It contains non-renormalizable terms, but that does not concern us, because we are only interested in its low energy, effective behavior. ϕ_0 is a vacuum expectation value (VEV) configuration, which breaks the $SU(3)$ symmetry. To obtain a low-energy effective Lagrangian we write all $SU(3)$ invariant terms constructed with ϕ with increasing powers of momentum (derivatives) of ϕ. The first (invariant) terms with zero derivatives can only be $\phi^\dagger\phi$ and $\epsilon^{abc}\phi_a\phi_b\phi_c$. This last term is zero because of the anti-symmetry of the ϵ symbol and the symmetry of the ϕ field. But we do not need constant terms. The next term is one with two derivatives, the most important one: $\partial_\mu\phi^\dagger\partial^\mu\phi = |\partial_\mu\phi|^2$. This term is usually normalized in such a way that the NGB's have canonical kinetic terms.

For convenience we will ignore the field η, which is an $SU(2)$ singlet, in much of the rest of the discussion. The field h is a complex $SU(2)$ doublet and we will attempt to make it the Higgs doublet as known from the standard model. To discover the interactions for h we expand the ϕ field:

$$\phi = \exp[\frac{i}{f}\begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}]\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} + i\begin{pmatrix} h \\ 0 \end{pmatrix} - \frac{1}{2f}\begin{pmatrix} 0 \\ 0 \\ h^\dagger h \end{pmatrix} + \ldots$$

Note that h has two components. One now finds for the kinetic term:

$$\partial_\mu\phi^\dagger\partial^\mu\phi = \partial_\mu h^\dagger\partial^\mu h + \frac{\partial_\mu(h^\dagger h)\partial^\mu(h^\dagger h)}{4f^2} + \ldots \tag{5.2}$$

There is a kinetic term for the Higgs doublet and interactions which are suppressed by the symmetry breaking scale f. Because the Lagrangian contains non-renormalizable interactions it is a low-energy effective field theory as anticipated. Let us have a look at what scale the theory becomes strongly coupled. A quadratically divergent diagram arises when h runs in a loop, as one can see by looking at the second term in equation (5.2). This term renormalizes the kinetic term. We might ask when does this term contribute to the same order as the first term. When the divergence is cut off at Λ one finds a value of:

$$\frac{1}{f^2}\frac{\Lambda^2}{16\pi^2} \sim 1 \tag{5.3}$$

We should only expect sensible results below the cut-off scale $\Lambda \sim 4\pi f$.

To summarize, we now have a theory with an $SU(2)$ (will be) Higgs doublet. It is a NGB resulting from the breaking of an exact global symmetry and thus exactly massless. There are also no diagrams that can generate a mass for h: it is massless to all orders in perturbation theory. The theory is a low-energy effective theory, with a cut-off scale $\Lambda \sim 4\pi f$.

This is still rather far from anything like the standard model Higgs doublet. One still requires gauge field interactions, Yukawa couplings to fermions and a quartic potential. All this should be taken care of without introducing dangerous one-loop quadratic divergences to the Higgs mass. We will now try to do just that.

5.2 Gauge Field Interactions

One would now like to construct gauge interactions with $SU(2)$ gauge bosons, this without introducing quadratically divergent contributions to the Higgs mass. We will turn our attention to the $U(1)$ group and hypercharge appearing in the standard model later. It appears we cannot introduce solely $SU(2)$ gauge interactions without generating quadratically divergent contributions to the Higgs mass: For example, the usual $|D_\mu h|^2$, with $SU(2)$ gauge fields

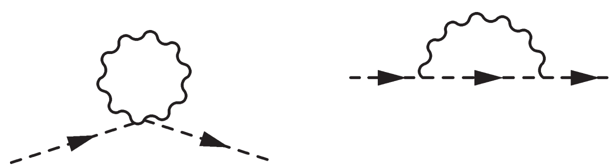


Fig. 5.1: Quadratically divergent $SU(2)$ gauge loops contributing to the Higgs mass

in the covariant derivative gives mass terms for h proportional to Λ^2. This is precisely the problem in the standard model we seek avoid. This should also come as no surprise to us since we have not taken any measures to avoid or cancel the divergences. A way to avoid the quadratic divergences to the mass is to gauge the $SU(3)$ symmetry. So let us add a covariant derivative for the ϕ field, $(D_\mu\phi)^\dagger D^\mu\phi = |D_\mu\phi|^2$, now with the eight generators and gauge fields associated with the $SU(3)$ group. There are quadratically divergent contributions, these come from the term:

$$g^2\phi^\dagger A^a_\mu T^a A^a_\mu T^a \phi \Rightarrow \frac{g^2}{2\pi^2}\Lambda^2\phi^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \phi = \frac{g^2}{2\pi^2}\Lambda^2 f^2 \tag{5.4}$$

We find a quadratically divergent constant contribution to the vacuum energy, though no mass term for the Higgs doublet. We appear to be on the right track, but there is a problem: we have lost the Higgs field. The NGB's are "eaten" by the gauge bosons corresponding to the broken $SU(3)$ generators, because we have broken the $SU(3)$ gauge symmetry. These gauge bosons have gained an extra longitudinal degree of freedom and are now massive, with a mass proportional to f. However a solution to this problem exists. We will use two scalar triplets instead of just one, ϕ_1 and ϕ_2, and introduce covariant derivatives for both:

$$\mathcal{L} = |D_\mu\phi_1|^2 + |D_\mu\phi_2|^2 \tag{5.5}$$

This is starting to get the form of the collective symmetry breaking Lagrangian of eq. (4.1). Let us have a look at this in an $SU(3)$ symmetric way. The scalar fields are parameterized as follows:

$$\phi_1 = e^{i\Pi_1/f} \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \qquad \phi_2 = e^{i\Pi_2/f} \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \tag{5.6}$$

Where we have chosen equal VEV's and symmetry breaking scales. This is done for simplicity. Later on we shall generalize this. The two terms again give rise to two sets of quadratical divergences, as we saw before, which give of course:

$$\frac{g^2}{2\pi^2}\Lambda^2(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2) = \frac{g^2}{2\pi^2}\Lambda^2(f^2 + f^2) \tag{5.7}$$

There is no mass or potential for the NGB's. There is only one set of massive gauge bosons, both fields have the same covariant derivative, meaning one set of linear combination of Π_1 and Π_2 is eaten and the other (orthogonal) combinations form the Higgs. Now that we have introduced two ϕ fields in the Lagrangian, we can draw a third possible diagram, shown in figure (5.2). This diagram involves both ϕ fields, coupled through gauge bosons.

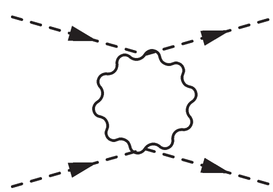


Fig. 5.2: Logarithmically divergent contribution to the Higgs mass

Integrating the loop, this diagram gives:

$$\frac{g^4}{16\pi^2}\log(\frac{\Lambda^2}{f^2})\ |\phi_1^\dagger\phi_2|^2 \tag{5.8}$$

This diagram is not quadratically divergent (!), but logarithmically and it contains a tree level mass term for the h doublet and also a quartic coupling. To see this, use the following parameterizations for the scalar fields $\phi_{1,2}$:

$$\phi_1 = \exp[i\begin{pmatrix} 0 & k \\ k^\dagger & 0 \end{pmatrix}] \cdot \exp[+i/f\begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}]\begin{pmatrix} 0 \\ f \end{pmatrix} \tag{5.9}$$

$$\phi_2 = \exp[i\begin{pmatrix} 0 & k \\ k^\dagger & 0 \end{pmatrix}] \cdot \exp[-i/f\begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}]\begin{pmatrix} 0 \\ f \end{pmatrix} \tag{5.10}$$

The logarithmically divergent diagram produces an operator which breaks the $[SU(3)]^2$ symmetry of the Lagrangian with two, *a priori*, independent ϕ fields. $[SU(3)]^2 \rightarrow SU(3)$. The ϕ fields cannot be rotated independently anymore. The previously massless NGB's now acquire masses. Only one $SU(3)$ group is gauged, the diagonal group (see chapter 7). This means five NGB's are eaten and five others remain and become massive. The k field can be rotated away by an $SU(3)$ gauge transformation known as the unitarity gauge and corresponds to the eaten NGB's. h Cannot be removed from both ϕ_1 and ϕ_2 and is a physical field. Working in the unitary gauge has the advantage of working with the physical spectrum of particles. When we rotate k away and calculate $\phi_1^\dagger \phi_2$ we find:

$$\phi_1^\dagger \phi_2 = \begin{pmatrix} 0 & 0 & f \end{pmatrix} \exp[-\frac{2i}{f} \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}] \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \tag{5.11}$$

$$= \begin{pmatrix} 0 & 0 & f \end{pmatrix} [1 - \frac{2i}{f} \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} \tag{5.12}$$

$$- \frac{2}{f^2} \begin{pmatrix} h^\dagger h & 0 \\ 0 & h^\dagger h \end{pmatrix} + \ldots] \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \tag{5.13}$$

$$= f^2 - 2h^\dagger h + \ldots \tag{5.14}$$

As one can see, a theory containing two scalar triplets can generate a mass-squared for h equal to:

$$\frac{g^4}{4\pi^2} \log(\frac{\Lambda^2}{f^2}) f^2 \tag{5.15}$$

Taking $\Lambda \sim 4\pi f$ and $f \sim 1$ TeV, this mass is approximately 100 GeV, which seems about right. Thus up until now, the theory of two complex scalar triplets, which both break $SU(3)$ to $SU(2)$, gives a Higgs doublet which does not receive quadratically divergent contributions to its mass. From this we find a Higgs mass of around 100 GeV, in agreement with current predictions and expectations. Let us closer inspect the mechanism which gives the Higgs a naturally light mass and one free of quadratic divergences.

Without gauge interactions the theory would comprise two (non-linear sigma) scalar fields ϕ. Both fields break an $SU(3)$ to $SU(2)$. This means there are ten spontaneously broken generators and thus ten NGB's, which are exactly massless. If we now introduce $SU(3)$ gauge interactions for both fields (same diagonal group) ϕ_1 and ϕ_2, the two previously independent $SU(3)$ symmetries are broken down to one diagonal $SU(3)$ which is gauged. $[SU(3)]^2 \rightarrow SU(3)$. This can be seen by looking at the two boson-two scalar couplings in the Lagrangian.

$$\mathcal{L} \sim g_{(1)}^2 |A_\mu \phi_1|^2 + g_{(2)}^2 |A_\mu \phi_2|^2 \tag{5.16}$$

Now one set of NGB's is eaten, which are the k fields in our parametrization, and provides a set of massive gauge bosons corresponding to the broken generators. The other set gains a potential through loop diagrams as we just saw with the logarithmically divergent diagram. Suppose we now were to set the gauge coupling of ϕ_2 to zero. Then the Higgs could no longer gain a potential from loops and would be an exact, massless NGB again. Similarly if we were to set the gauge coupling to ϕ_1 to zero. When one of the gauge couplings is set to zero we restore the symmetry of two independent $SU(3)$ symmetries. The Higgs mass is protected by the approximate $[SU(3)]^2$ symmetry. So only when both ϕ fields have non-zero gauge couplings can h receive a potential from loops. Only when the two $SU(3)$ symmetries are broken explicitly to the diagonal $SU(3)$ can h become massive. We can now understand the absence of quadratically divergent loop contributions to the Higgs mass: Any diagram contributing to the h mass must involve both gauge couplings, however there are no quadratically divergent diagrams involving both gauge couplings.

This mechanism, as just described above, is used in the "Little Higgs" theories. The Little Higgs is pseudo-NGB of a spontaneously broken global symmetry. The symmetry type is also broken explicitly, though collectively, meaning that two or more (gauge) couplings are non-zero, such that it becomes a pseudo-NGB. If the symmetry (which was explicitly broken)is restored by setting the appropriate gauge coupling(s) to zero, the Higgs becomes an exact, massless NGB again. The Higgs mass is protected by an approximate symmetry. This mechanism is the key in every Little Higgs theory in all sectors. We have now discussed the gauge sector. In the next section we will try to do the exact same thing in the fermion (quark) sector.

5.3 Yukawa Couplings to Quarks

Let us continue with the Yukawa couplings of the scalar field to fermions. The most dangerous quadratic divergence to the Higgs mass is due to the Top quark. So the goal is to cancel this divergence. The quark doublets in the standard model are replaced by triplets: $\Psi = (t, b, T)$ which transform under the fundamental representation of the $SU(3)$ gauge symmetry. The singlets are t^c, b^c and the new T^c. Let us have a look at the $SU(2)$ Higgs doublet. The standard model Yukawa couplings are of the form: $h^\dagger Q t^c$ where

Q is the quark doublet. T^c and t^c as they stand now will mix, so it will be more convenient to change notation and call them t_1^c and t_2^c. Later on we will then call the appropriate linear combinations of t_1^c and t_2^c, t^c and T^c again. The Yukawa Lagrangian has the following form:

$$\mathcal{L}_{Yukawa} = \lambda_1 t_1^c \phi_1^\dagger \Psi + \lambda_2 t_2^c \phi_2^\dagger \Psi + h.c. \tag{5.17}$$

Note that there are $SU(3)$ symmetries in the Lagrangian which are broken collectively. We do not include couplings like $t_2^c \phi_1^\dagger \Psi$ and $t_1^c \phi_2^\dagger \Psi$, because they would lead to the quadratic divergences we want to avoid.

Now let us closer inspect the interactions for h. We again choose the parametrization of (5.9) and (5.10), though we will work in the unitary gauge, meaning that the k fields are rotated away. To see the interactions one substitutes the expansion for the ϕ fields.

$$\mathcal{L}_{Yukawa} = \lambda_1 t_1^c [\begin{pmatrix} 0 & 0 & f \end{pmatrix} - i \begin{pmatrix} h^\dagger & 0 \end{pmatrix} - \frac{1}{2f} \begin{pmatrix} 0 & 0 & h^\dagger h \end{pmatrix}] \begin{pmatrix} t \\ b \\ T \end{pmatrix} \tag{5.18}$$

$$+ \lambda_2 t_2^c [\begin{pmatrix} 0 & 0 & f \end{pmatrix} + i \begin{pmatrix} h^\dagger & 0 \end{pmatrix} - \frac{1}{2f} \begin{pmatrix} 0 & 0 & h^\dagger h \end{pmatrix}] \begin{pmatrix} t \\ b \\ T \end{pmatrix} \tag{5.19}$$

To simplify things somewhat let us equalize the couplings λ_1 and λ_2. This is, however not strictly necessary.

$$\lambda_1 \equiv \lambda_2 \equiv \frac{\lambda}{\sqrt{2}} \tag{5.20}$$

This is done merely to have a better overview of things, the considerations made here are not affected by this choice. Evaluating the Lagrangian gives us:

$$\mathcal{L}_{Yukawa} = \frac{\lambda}{\sqrt{2}} [fT(t_1^c + t_2^c) + ih^\dagger Q(t_2^c - t_1^c) - \frac{1}{2f} h^\dagger h T(t_1^c + t_2^c)] \tag{5.21}$$

$$= \lambda f T T^c + \lambda h^\dagger Q t^c - \frac{\lambda}{2f} h^\dagger h T T^c \tag{5.22}$$

Where we have now substituted T^c and t^c for the appropriate linear combinations:

$$T^c = \frac{t_1^c + t_2^c}{\sqrt{2}} \qquad t^c = \frac{i(t_1^c - t_2^c)}{\sqrt{2}} \tag{5.23}$$

So we find a Yukawa coupling for the Top quark and the Higgs boson, this means that λ=λ_t. There is also a mass for the heavy T fermion equal to

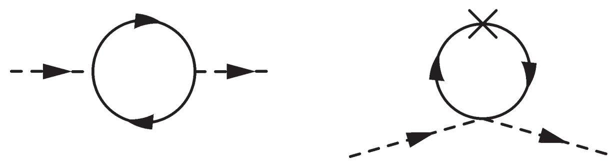


Fig. 5.3: Quadratically divergent fermion loops contributing to the Higgs mass cancel one another

$\lambda_t f$. The T fermion couples to the Higgs doublet with a coupling constant of $\lambda_t/2f$. We can now draw the Feynman diagrams, see figure (5.3), and see that indeed the quadratic divergent contributions to the Higgs mass cancel. The two diagrams in figure (5.3) give:

$$M_1 = \frac{\lambda_t^2}{16\pi^2}\Lambda^2 h^\dagger h \qquad M_2 = \lambda_t f \frac{-\lambda_t}{16\pi^2 f}\Lambda^2 h^\dagger h \tag{5.24}$$

The quadratically divergent contribution from the Top quark and the new T quark to the Higgs mass successfully cancel!

The absence of quadratic divergences can again be understood by inspecting the symmetries involved, similar to the case of the gauge bosons, which we did in the previous section. The Yukawa couplings preserve the $SU(3)$ gauge symmetry. When both couplings are non-zero there is only one $SU(3)$ symmetry, the diagonal gauge symmetry. Namely the λ_i couplings force the symmetries in both terms in (5.17) to be aligned. The fact that we equalized them is irrelevant here. Suppose we set one of the couplings λ_i to zero. Now there are two independent, global, $SU(3)$ symmetries. When broken to $SU(2)$, two sets of five NGB's arise, one linear combination is eaten and contributes in heavy gauge bosons, the other (orthogonal) combination is the Little Higgs, which is an exact NGB and thus massless. If we now set both couplings λ_i non-zero, only one $SU(3)$ symmetry, the diagonal, remains. Only one set of exact NGB's arises and is eaten, and the Higgs receives a potential from loops and becomes a pseudo-NGB. So again, any contribution to the Higgs potential must come from diagrams involving both couplings λ_i. But there are no quadratically divergent diagrams involving both couplings.

The couplings for the other up-type quarks must, and can be, added in precisely the same way. In the standard model, the down-type couplings are generated using a contraction of the Higgs doublet and the quark doublets with an epsilon tensor (or with the conjugate Higgs doublet $h^c = i\sigma_2 h$):

$$\lambda_b \epsilon_{ij} h_i Q_j b^c \tag{5.25}$$

The $SU(3)$ invariant epsilon contraction in our case shall be:

$$\frac{\lambda_b}{f}\epsilon_{ijk}\phi_1^i\phi_2^j\Psi_j b^c \tag{5.26}$$

This operator however, immediately breaks both $SU(3)$ symmetries to the diagonal group. This does give rise to quadratic divergences, but they are not problematic because of the relative small couplings involved. For the bottom quark this is approximately (30 GeV)^2. These divergences do not lead to fine-tuning with the cut-off of order 10 TeV that we use. This is a less elegant part of the $SU(3)$ model, though not a serious problem, but there are models which are completely free of quadratic divergences.

5.4 The Quartic Higgs Coupling

The next goal is to produce a quartic Higgs coupling large enough to induce electro-weak symmetry breaking. So we require a potential $V(\phi_1, \phi_2)$ satisfying that it does not have a mass term for the Higgs (at tree level), it contains a quartic coupling for the Higgs doublet and preserves the mechanism of collective symmetry breaking of the $SU(3)$'s, meaning that the quartic coupling must be generated when at least two couplings are non-zero. This property, which we use at all times, guarantees radiative stability and a naturally, relatively small mass for the Higgs boson. Constructing such a potential that obeys all these foregoing constraints does not seem possible for the $SU(3)/SU(2)$ model, which is another shortcoming, next to the inability to cancel down type quadratic divergences. Namely, the only non-trivial $SU(3)$, gauge invariant, operator is $\phi_1^\dagger\phi_2$. But this operator immediately breaks the two $SU(3)$'s to the diagonal group. It contains a mass for the Higgs as well as a quartic coupling.

$$\phi_1^\dagger\phi_2 \approx f^2 - 2h^\dagger h + \frac{2}{3f^2}h^\dagger h h^\dagger h + \dots \tag{5.27}$$

Functions of $\phi_1^\dagger\phi_2$ always contain a mass as well as a quartic coupling. Note that this operator contains only even powers of $h^\dagger h$. We might try using this operator and tune the coefficient in the following way:

$$\frac{C}{f^{2n-4}}(\phi_1^\dagger\phi_2)^n \approx c_1 f^4 - c_2 f^2 h^\dagger h + c_3 h^\dagger h h^\dagger h + \dots \tag{5.28}$$

Tuning the coefficient, one can generate a small enough mass or a quartic coupling large enough for EWSB, however not both. So this alone does not work. One could then try to use two such operators, with different powers, and tune both their coefficients to produce a cancelation of the mass and a large enough quartic coupling. This also does not work because large quantum corrections still arise from the quartic. There are good solutions to this problem, but all these require enlarging the symmetry and the extent of the model. These solutions however take away the relative simplicity and elegance of the $SU(3)$ model.

Another, slightly less elegant possibility then one would like, would be to just add a potential by hand with a rather small coefficient. This then automatically gives a small mass, which we want, but it also gives a small quartic coupling. But corrections to the quartic coupling from a loop diagram with a Top quark (four external Higgs fields on a top loop) together with the added tree level contribution can give the desired electro-weak symmetry breaking. This added tree level mass term however, contributes to the Higgs mass-squared resulting in a small degree of tuning, because the coefficient is small. This solution is not too elegant, but it is still better than some other models of electro weak symmetry breaking. So in the end there is a possibility for inducing electro-weak symmetry breaking, but its solution is not very elegant.

5.5 Color, Hypercharge and Phenomenology

To construct a model that resembles and, of course, extends the standard model requires some additional work. To construct the standard model including the $SU(3)$ Little Higgs, and the mechanism of collective symmetry breaking we have been discussing, we still need the (local) symmetry of color for the quarks and hypercharge. Color can simply be added, by adding the $SU(3)_{color}$ group. It does not change any of the arguments we discussed in the foregoing. The weak interactions were embedded in an $SU(3)$ gauge group, which is spontaneously broken down to $SU(2)$. The symmetry group of the standard model is $SU(2)_L \times U(1)_Y$. For this reason we gauge an extra $U(1)_X$ group. Now the full symmetry group of the model will be: $SU(3)_{color} \times SU(3)_{weak} \times U(1)_X$. In the conventions used here we have a VEV for the Higgs doublet as follows:

$$< h >= \begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{5.29}$$

Then Higgs doublet has hypercharge -1/2.

$$Q = Y + T^3$$

$$Y = -\frac{1}{\sqrt{3}}T^8 + T^x$$

Where the generators are:

$$T^8 = \frac{1}{2\sqrt{3}} \text{ diag } (1, 1, -2) \tag{5.30}$$

$$T^x = -\frac{1}{3} \text{ diag } (1, 1, 1) \tag{5.31}$$

So the ϕ_i fields are assigned $SU(3)_{weak} \times U(1)_X$ quantum numbers:

$$\phi_i = 3_{-\frac{1}{3}} \tag{5.32}$$

The combination of generators, Y, is unbroken by the VEV configuration

$$\phi_i \sim (0, 0, 1) \tag{5.33}$$

When the $SU(3)$ transformation properties are chosen, this specifies all the $U(1)_X$ charges for the particles. The $(SU(3)_{color} \times SU(3)_{weak})_{U(1)_X}$ quantum numbers for the quark sector are:

$$\Psi_Q = (3, 3)_{\frac{1}{3}} \tag{5.34}$$

$$d^c = (\bar{3}, 1)_{\frac{1}{3}} \tag{5.35}$$

$$u^c, U^c = (\bar{3}, 1)_{-\frac{2}{3}} \tag{5.36}$$

The first equation describes the quark triplet and the last two the down type quark and the two up type $SU(3)$ singlets respectively. For the lepton sector one finds:

$$\Psi_L = (1, 3)_{-\frac{1}{3}} \tag{5.37}$$

$$e^c = (1, 1)_1 \tag{5.38}$$

$$n^c = (1, 1)_0 \tag{5.39}$$

The lepton sector has only one additional singlet. The lepton sector, especially the neutrino part, is further described in chapter 6.

In this Little Higgs theory we expect extra gauge bosons in addition to the known $SU(2)$ and $U(1)$ gauge fields, because the symmetry has been enlarged. Let us briefly have a look at this. The more general and realistic form of this model is discussed in chapter 7, so we shall be somewhat brief and continue this line of discussion in chapters 7 and 8.

From the covariant derivative on the ϕ_i fields one can find the particular spectrum of the gauge fields and their masses. Let us do this in a particular useful and insightful way. The covariant derivative term is a real number. However, because we use more than one scalar field, it proofs to be very useful to use the trace and the cyclic property of the trace.

$$\sum_{i=1}^{2}(D_\mu\phi_i)^\dagger(D^\mu\phi_i) = \sum_{i=1}^{2}|(\partial_\mu + igA_\mu^a T^a - \frac{1}{3}ig_x A_\mu^x)\phi_i|^2 \tag{5.40}$$

$$= \text{Trace}[\ \sum_{i=1}^{2}|(\partial_\mu + igA_\mu^a T^a - \frac{1}{3}ig_x A_\mu^x)\phi_i|^2\ ] \tag{5.41}$$

The relevant part for finding the gauge boson masses is then:

$$\text{Trace}[\ \sum_{i=1}^{2}|(gA_\mu^a T^a - \frac{1}{3}g_x A_\mu^x)\phi_i|^2\ ] = \text{Trace}[\ (gA_\mu^a T^a - \frac{1}{3}g_x A_\mu^x)^2 \sum_{i=1}^{2}\phi_i\phi_i^\dagger\ ] \tag{5.42}$$

The trace is a linear operator. In principal the only thing we have to do now is to compute the matrix $\sum_{i=1}^{2}\phi_i\phi_i^\dagger$. Doing so one finds to first order:

$$\phi_1\phi_1^\dagger + \phi_2\phi_2^\dagger = \begin{pmatrix} hh^\dagger & 0 \\ 0 & f^2 - h^\dagger h \end{pmatrix} = \begin{pmatrix} \frac{v^2}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f^2 - \frac{v^2}{2} \end{pmatrix} \tag{5.43}$$

Here we already see a glimpse of the cancellation mechanism in the opposing signs for the terms that are quadratic in the Higgs doublet. In the last equality a vacuum expectation value has been assumed for h.

$$h = \begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{5.44}$$

In principal one can compute the masses to any order using the expansions for the ϕ fields. Now that the matrix $\sum_{i=1}^{2}\phi_i\phi_i^\dagger$ is known, the trace in equation (5.42) can be evaluated to obtain the mass eigenstates and the masses of the gauge bosons. When one does not yet assume a VEV for the Higgs, the mass eigenstates (and masses) that are present before electro-weak symmetry breaking can be found. These are the fields $A_\mu^{3'}$, B_μ and Z'_μ.

$$A_\mu^{3'} = A_\mu^3 \tag{5.45}$$

$$B_\mu = \frac{\sqrt{3}gA_\mu^x - g_x A_\mu^8}{\sqrt{3g^2 + g_x^2}} \tag{5.46}$$

$$Z'_\mu = \frac{g_x A_\mu^x + \sqrt{3}gA_\mu^8}{\sqrt{3g^2 + g_x^2}} \tag{5.47}$$

There appear also two combinations of A_μ^4 and A_μ^5 which form $W^{0\bar{0}'}$ and two combinations of A_μ^6 and A_μ^7 which form $W^{\pm'}$. Together these form an $SU(2)$ doublet of new heavy gauge bosons ($m^2 = 1/2g^2f^2$). The standard model $W^\pm$ are of course still massless at this level.

Evaluating the trace, assuming a VEV for the Higgs doublet, one finds the charged weak vector bosons W^+ and W^- of the standard model with a mass of $\frac{1}{4}g^2v^2$. The masses of the additional $SU(2)$ doublet of heavy gauge bosons ($W^{'\pm}, W^{0,\bar{0}}$) are split upon EWSB. We also find neutral gauge bosons. The gauge fields which correspond to the generators T^3, T^8 and T^x mix to form mass eigenstates, which are now the photon, which is massless, the neutral Z^0 boson and a heavy $Z^{'}$. We will also pay attention to this in chapter 7. For completeness, the explicit trace after EWSB is:

$$\begin{aligned}
\text{Trace}[M] &= \frac{1}{2}[((\frac{A_3}{2}+\frac{A_8}{2\sqrt{3}})g-\frac{A_x}{3}g_x)^2+g^2(\frac{A_1}{2}-i\frac{A_2}{2})(\frac{A_1}{2}+i\frac{A_2}{2}) \\
&+ g^2(\frac{A_4}{2}-i\frac{A_5}{2})(\frac{A_4}{2}+i\frac{A_5}{2})]v^2 \\
&+ [(\frac{A_8}{\sqrt{3}}g-\frac{A_x}{3}g_x)^2+g^2(\frac{A_4}{2}-i\frac{A_5}{2})(\frac{A_4}{2}+i\frac{A_5}{2}) \\
&+ g^2(\frac{A_6}{2}-i\frac{A_7}{2})(\frac{A_6}{2}+i\frac{A_7}{2})]\ (f^2-\frac{1}{2}v^2)
\end{aligned} \tag{5.48}$$

After performing an orthogonal transformation among the fields A_μ^3, A_μ^8 and A_μ^x to obtain the mass eigenstates one can read of the masses (squared) of all the massive combinations.

$$\begin{pmatrix} A_\mu^3 & A_\mu^8 & A_\mu^x \end{pmatrix} M \begin{pmatrix} A_\mu^3 \\ A_\mu^8 \\ A_\mu^x \end{pmatrix} = \begin{pmatrix} A_\mu^3 & A_\mu^8 & A_\mu^x \end{pmatrix} P M_{diag} P^T \begin{pmatrix} A_\mu^3 \\ A_\mu^8 \\ A_\mu^x \end{pmatrix} \tag{5.49}$$

$$= \begin{pmatrix} A_\mu & Z^0 & Z^{0'} \end{pmatrix} M_{diag} \begin{pmatrix} A_\mu \\ Z^0 \\ Z^{0'} \end{pmatrix} \tag{5.50}$$

Orthogonal transformations are outlined in the appendix on mathematical results. Keeping in mind the correct form in which mass terms should appear in a Lagrangian for the various charged and neutral fields one finds explicitly:

$$m^2\ W^\pm = \frac{1}{4}g^2v^2 \tag{5.51}$$

$$m^2\ W^{'\pm} = \frac{1}{2}g^2f^2 - \frac{1}{4}g^2v^2 \tag{5.52}$$

$$m^2\ W^{'0,\bar{0}} = \frac{1}{2}g^2f^2 \tag{5.53}$$

$$m^2\ Z^0 = \frac{1}{4} g^2 v^2 (1+t^2) \tag{5.54}$$

$$m^2\ Z'^0 = g^2 f^2 \frac{2}{3-t^2} - \frac{1}{4} g^2 v^2 (1+t^2) \tag{5.55}$$

Where $t = \frac{g'}{g} \equiv \tan\theta_W$, where θ_W is the weak mixing angle or Weinberg angle. $g^{'}$ is the hypercharge coupling as in the standard model.

$$g^{'} = \frac{g_x}{\sqrt{1+\frac{g_x^2}{3g^2}}} \tag{5.56}$$

We will show this in chapter 7.

6

Massive Neutrinos

6.1 75 Years of Neutrino Physics

With the new subatomic particle the neutron barely known, a new particle, the neutrino, was postulated in 1931 by Pauli for the first time in order to solve the problem of energy conservation in β-decay processes. Radioactive (β-)decays appeared to be violating conservation of energy and momentum. Pauli suggested that a neutral particle existed, which had escaped detection up until then. The neutrino had to be neutral, because a charged particle would easily have been detected. Conservation of spin angular momentum implied that the neutrino would have to be a spin 1/2 particle, making it a fermion. The neutrino was thought to be massless or having an extremely small mass. In 1934 Fermi developed a theory of radioactive decays (weak interactions), including Pauli's hypothesized particle. He calls it the neutrino (Italian for "little neutral one"). Fermi's theory explains many of the by then observed results. In 1959 the neutrino is finally discovered and in time understood to be the light partner of the electron. The question concerning the mass remains unanswered. In 1962 it is observed that neutrino's originating from muon experiments have quite different properties from the known electron neutrino. The new type of neutrino is called the muon neutrino. A third type of neutrino, the tau neutrino, is hypothesized when in 1978 the tau particle is discovered and recognized as the heavier partner of the electron and the muon. The tau neutrino had to await direct observation until the summer of 2000. In the late 1960's, experiments are conducted attempting to detect electron neutrino's coming from the sun. In these experiments however, only about half of the expected amount of neutrinos is detected. Also in later experiments neutrino deficits are reported. At LEP, using decay properties of the Z^0 boson, it is determined that no more than three neutrinos exist unless the undiscovered particles have quite different properties from the known neutrinos. To explain the apparent missing of neutrinos, the idea of neutrino mixing is proposed. During their flight neutrinos (from the sun) can metamorphose into other neutrinos, thus oscillating on their path of flight. This would explain the lack of electron neutrinos coming from the sun. However this also meant that neutrinos must have mass. In the 1990's, experiments are done to test this mixing phenomenon among neutrinos. In 1998 the Super-Kamiokande team announces evidence of neutrino masses. Nowadays the majority of the physics community has agreed upon the fact that neutrino's have a small, however non-vanishing, mass. However, the question of why these neutrino masses are so much smaller than the electroweak scale remains. There are some attractive scenario's such as the see-saw mechanism, but not yet any firm experimental verifications.

6.2 The Price of non-zero Neutrino Masses

6.2.1 Quark Mixing

Before attempting to introduce masses for the neutrinos, let's first have a look at the quark sector of the standard model which has a similar structure. In the quark sector, mixing is a rather well known phenomenon these days (speaking pretentiously). In the strong and electromagnetic interaction quark flavor is conserved. The weak interaction, however may change quark flavor. In a great deal of weak decays the flavor of a quark changes within a generation, thus a d changes to a u, or an s changes to a c, but weak decays like:

$$K^- \to \pi^0 + e^- + \bar{\nu}_e \tag{6.1}$$

change an s quark into a u quark. Cabibbo explained this by posing that the eigenstates of the weak interaction, the gauge eigenstates, are different from the mass eigenstates, which are the physical u, d, c, s quark fields:

$$\begin{pmatrix} d_\theta \\ s_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} \tag{6.2}$$

The matrix is known as the Cabibbo matrix for two generations. θ_c is the Cabibbo angle and has a value of approximately 15 degrees. The presence of the Cabibbo angle shows that one must distinguish between gauge eigenstates and mass eigenstates. This becomes clear by looking at how masses arise. One cannot just write a Dirac mass term for fermions, because left and right handed particles transform under different representations of $SU(2)$ and have different $U(1)$ charges. Instead one has to use the Higgs mechanism. Fermions are massless before spontaneous symmetry breaking. The fermion masses and eigenstates are determined by the Yukawa couplings when the Higgs field has a assumed a VEV. Generalizing to three generations the lefthanded lepton and quark doublets are:

$$l_{A,L} = \begin{bmatrix} \nu'_A \\ e'_A \end{bmatrix}_L \tag{6.3}$$

$$q_{A,L} = \begin{bmatrix} u'_A \\ d'_A \end{bmatrix}_L \tag{6.4}$$

The primed fields are gauge eigenstates and the A index indicates the generation. The Yukawa terms now have the following form:

$$\mathcal{L}_{Yuk} = f^{(e)}_{AB}\ \bar{l}_{A,L}\, \phi\, e'_{BR} \ +\ f^{(u)}_{AB}\ \bar{q}_{A,L}\, \tilde{\phi}\, u'_{BR} \ +\ f^{(d)}_{AB}\ \bar{q}_{A,L}\, \phi\, d'_{BR} \ +\ h.c. \tag{6.5}$$

After spontaneous symmetry breaking we find interaction terms and mass terms:

$$\begin{aligned} \mathcal{L} = & \frac{h}{\sqrt{2}}(f^{(e)}_{AB}\ \ \bar{e}'_{A,L}e'_{BR} \ \ +\ \ f^{(u)}_{AB}\ \ \bar{u}'_{A,L}u'_{BR} \ \ +\ \ f^{(d)}_{AB}\ \ \bar{d}'_{A,L}d'_{BR}) \\ & + \frac{v}{\sqrt{2}}(f^{(e)}_{AB}\ \ \bar{e}'_{A,L}e'_{BR} \ \ +\ \ f^{(u)}_{AB}\ \ \bar{u}'_{A,L}u'_{BR} \ \ +\ \ f^{(d)}_{AB}\ \ \bar{d}'_{A,L}d'_{BR}) \end{aligned} \tag{6.6}$$

In the basis of gauge eigenstates the mass matrices are:

$$M^{(i)}_{AB} = \frac{-v}{\sqrt{2}} f^{(i)}_{AB} \qquad\qquad i = e, u, d \tag{6.7}$$

These matrices mix generations and there is no *a priori* reason to assume that these mass matrices are diagonal and in fact they are not. However they can be diagonalized by socalled biunitary transformations [18], meaning that given a matrix M_{AB} there exist unitary matrices S and T such that:

$$S^\dagger M_{AB} T = M_{diag} \tag{6.8}$$

This is explained in the appendix on mathematical results. The relation between gauge eigenstates and mass eigenstates is now:

$$\bar{\psi}'_L M \psi'_R = (\bar{\psi}'_L S) S^\dagger M T (T^\dagger \psi'_R) = \bar{\psi}_L M_{diag} \psi_R \tag{6.9}$$

With:

$$\psi'_L = S\psi_L, \qquad \psi'_R = T\psi_R \tag{6.10}$$

Note that γ-matrices work on different indices than the transformation matrices. When we diagonalize all these mass matrices with appropriate unitary matrices we obtain matrices $S_e, S_u, S_d, T_e, T_u, T_d$. The fields 'observed in nature' are the mass eigenstates. So interaction terms and currents should be expressed in terms of mass eigenstates. Looking at the charged weak currents for the quarks one finds interesting implications:

$$j^\mu = \bar{u}'_{AL}\gamma^\mu d'_{AL} \tag{6.11}$$
$$= \bar{u}_{AL}\gamma^\mu (S^\dagger_{(u)} S_{(d)}) d_{AL} \tag{6.12}$$
$$= \bar{u}_{AL}\gamma^\mu (U)_{AB} d_{BL} \tag{6.13}$$

$$U = S^\dagger_{(u)} S_{(d)} \tag{6.14}$$

Thus the doublets expressed in terms of mass eigenstates are:

$$\begin{pmatrix} u \\ d'' \end{pmatrix} \quad \begin{pmatrix} c \\ s'' \end{pmatrix} \quad \begin{pmatrix} t \\ b'' \end{pmatrix}$$

Where

$$\begin{pmatrix} d'' \\ s'' \\ b'' \end{pmatrix} = U \begin{pmatrix} d \\ s \\ u \end{pmatrix} \tag{6.15}$$

With the doubly primed fields, the mass eigenstates. For three generations U is a unitary matrix, known as the Kobayashi-Maskawa (KM) matrix. For two generations one finds the Cabibbo matrix from the foregoing discussion. The KM matrix determines which type of transitions are possible between generations. If the KM matrix were to be diagonal, such transitions would not occur. However the elements of the KM matrix have to be determined experimentally. Another interesting consequence of the KM $(3\times\ 3)$ matrix is that an additional parameter appears which gives rise to a violation of the symmetry of combined charge conjugation and a parity transformation.

6.2.2 Neutrino Mixing

One can proceed analogous in the lepton sector. The charged weak current for leptons is:

$$j^\mu = \bar{\nu}'_{AL}\gamma^\mu e'_{AL} \tag{6.16}$$
$$= \bar{\nu}_{AL}(S^\dagger_{(\nu)} S_{(e)})_{AB}\gamma^\mu e_{BL} \tag{6.17}$$

This results in:

$$\begin{pmatrix} \nu_e'' \\ e \end{pmatrix} \quad \begin{pmatrix} \nu_\mu'' \\ \mu \end{pmatrix} \quad \begin{pmatrix} \nu_\tau'' \\ \tau \end{pmatrix}$$

Where

$$\begin{pmatrix} \nu_e'' \\ \nu_\mu'' \\ \nu_\tau'' \end{pmatrix} = V \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{6.18}$$

Where $V = S^\dagger_{(\nu)} S_{(e)}$. However, when one assumes that neutrinos are massless and do not (yet) have Yukawa couplings similar to the quarks and the charged leptons we can make any unitary redefinition of the neutrino fields without punishment. The fields are degenerate, so any unitary linear transformation would work just as well. In particular one can make a redefinition with V^{-1}, thus getting rid of the matrix V. This would imply that the lepton generations are completely decoupled, because V is in essence the unit matrix. The absence of mixing in the lepton sector results in the conservation laws for the lepton flavors, known as *conservation of electron number*, *muon number* and *tau number* respectively. This means that, for example, an electron cannot decay into a muon and a photon.

If neutrinos are not massless, but have at small mass difference one would not be able to get rid of the matrix V. In this case, next to lepton number violating processes, there should be the phenomenon of neutrino oscillations. This means that neutrinos change flavor as they travel through space and time. V is a matrix similar to the KM matrix, however there is no reason to assume that the entries will be in any way related to the parameters in the KM matrix. Also in this case one has to distinguish between *weak interaction eigenstates* and mass eigenstates. The relation is usually parameterized in the following manner:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 s_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{6.19}$$

Where the first column vector contains the weak interaction eigenstates and the last column vector contains the mass eigenstates. $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$. If at time $t = 0$ there is a source emitting only electron neutrinos, then these neutrino states are a superposition of the mass eigenstates. They evolve in time according to:

$$|\nu_e(t)> \;=\; c_1 e^{-iE_1 t}\,|\nu_1> \;+\; s_1 c_3 e^{-iE_2 t}\,|\nu_2> \;+\; s_1 s_3 e^{-iE_3 t}\,|\nu_3> \tag{6.20}$$

Now the probability of finding a muon neutrino at time t equals:

$$| < \nu_\mu(t)|\nu_e(t) > |^2 \tag{6.21}$$

The term "neutrino oscillations" comes from the fact that the expressions for these probabilities have an oscillating behavior with respect to position. An electron neutrino can change into a muon neutrino, however the reverse process occurs as well. Concluding: for neutrino oscillations to occur one must have non-zero and non-degenerate masses and mixing angles.

One might ask: "do charged leptons oscillate"? The answers would be *no*, because we detect and identify charged leptons using their mass. We do not detect neutrino's using their mass. We say: "the electron has a mass of 0.51 MeV". We identify flavor by mass, so flavor states and mass states coincide. Oscillations occur when mass states and flavor states do not coincide.

In the past few years neutrinos have in fact been found to oscillate and thus the majority of the physics community is convinced that neutrinos have a small, though non-vanishing mass. The question is now how to incorporate neutrino masses into gauge theories. A possibility would be to just simply couple the neutrino fields in the appropriate way to the Higgs field along with right handed partners. But taking into account the extreme smallness of neutrino masses, one would require extremely suppressed Yukawa couplings: $\lambda_\nu \sim 10^{-12}$. This number requires an explanation and raises questions concerning naturalness.

6.3 Neutrino Masses in Little Higgs Models

In the standard model problems arise concerning the introduction of neutrino masses. The Yukawa couplings appear to be suppressed to an extreme extent, which apparently has some kind of reason behind it. As the model we will discuss now [19], some *Little Higgs theories* can provide a possible solution to this problem. Consider the particular $SU(3)$ invariant Little Higgs model we have been discussing in the previous chapter. In this model quark masses can be generated without introducing quadratic divergences into the Higgs mass. It includes an $SU(3)$ triplet $\Psi_Q = (Q, T)$ and two singlets, t_1^c and t_2^c, per generation. The singlets mix to form the linear combinations t^c and T^c. The Yukawa Lagrangian has the following form:

$$\mathcal{L}_{Yukawa} = \lambda_1 \phi_1^\dagger \Psi t_1^c + \lambda_2 \phi_2^\dagger \Psi t_2^c + h.c. \tag{6.22}$$

$$= \lambda f T T^c + \lambda h^\dagger Q t^c - \frac{\lambda}{2f} h^\dagger h T T^c + h.c. \tag{6.23}$$

Where the couplings were equalized for clarity. Note that the formulation is in terms of lefthanded (Weyl) spinors. The c superscript on the singlets denotes charge conjugation:

$$\psi^c = C\gamma^0\psi^* \tag{6.24}$$

Working in a lefthanded context, note that righthanded singlets are indicated by a conjugated field which is lefthanded. Note that:

$$(\psi^c)_L = (\psi_R)^c \tag{6.25}$$

Where the subscripts L and R denote left- and right handed respectively. A Dirac mass term now has the form:

$$m_{Dirac} = -m\ [(\psi^c)_L^T C\psi_L + h.c.] \tag{6.26}$$

In the following the matrix C will be absorbed into the lefthanded components of the Ψ field. In one of the previous chapters one could see that the Lagrangian (6.22) was free of one loop quadratic divergent corrections to the Higgs mass. Little Higgs theories are also capable of accommodating the observed small neutrino masses. A possibility for introducing neutrino masses would be to proceed precisely in the lepton sector as in the quark sector, which was treated before, but then there would again be the problem of the Yukawa couplings to the neutrinos, needing to be very suppressed. A better solution is to proceed in a similar way, however slightly different from the quark sector. A particular model for the neutrino sector includes an $SU(3)$ triplet, containing the known lefthanded leptons of the standard model and a heavy N: Ψ_L and one singlet n^c per generation. So the difference with the quark sector is that now there is only one $SU(3)$ singlet, instead of two.

$$\Psi_L = \begin{pmatrix} -iL \\ N \end{pmatrix} \tag{6.27}$$

The phase in L is used to simplify the Lagrangian. The Lagrangian for the neutrino part is to be the following:

$$\begin{aligned} \mathcal{L}_\nu &= \lambda_\nu \phi_1^\dagger \Psi_L n^c + h.c. && (6.28)\\ &= \lambda_\nu f N n^c - \lambda_\nu h^\dagger L n^c - \frac{\lambda_\nu}{2f} h^\dagger h N n^c + h.c. && (6.29) \end{aligned}$$

Note that this Lagrangian does not contain a Dirac mass term for the neutrino since we have not included a righthanded partner. When the Higgs field also acquires a vacuum expectation value v, one can write the following coupling/mass matrix to first order.

$$\frac{1}{2}\begin{pmatrix} \nu & N & n^c \end{pmatrix} \begin{pmatrix} 0 & 0 & -\lambda_\nu v \\ 0 & 0 & \lambda_\nu f \\ -\lambda_\nu v & \lambda_\nu f & 0 \end{pmatrix} \begin{pmatrix} \nu \\ N \\ n^c \end{pmatrix} \tag{6.30}$$

If this matrix is diagonalized there will in total be three exact Dirac fields, one for every generation, with a mass of approximately $m = \lambda_\nu f$ plus $\mathcal{O}(v/f)$ terms, but no mass for the (standard model) Majorana neutrinos. N and n^c pair up to form this Dirac fermion. The eigenstates are:

$$\begin{aligned} \nu' &\approx \nu + \frac{v}{f} N \\ N' &\approx \frac{N - n^c - \frac{v}{f}\nu}{\sqrt{2}} \\ n^{c'} &\approx \frac{n^c + N - \frac{v}{f}\nu}{\sqrt{2}} \end{aligned}$$

The neutrinos are still exactly massless in this scenario. To generate masses for the neutrinos one would have to introduce additional singlets, but then the suppressed coupling problem is encountered again. If one maintains only one singlet, but also couples it to the ϕ_2 field, one loop quadratically divergent contributions to the Higgs mass will arise. Another solution would be to allow for a breaking of the global symmetry of lepton number. This is done by introducing a Majorana mass term for the singlet. This is allowed, because it does not transform under a non-trivial representation of the gauge group.

Of course in the lepton sector the right handed electron singlet is also maintained. The mass term for the electron[3], which is a Dirac mass term, is generated by a contraction of the two scalar fields and the lefthanded lepton triplet with an epsilon tensor.

$$\mathcal{L}_e = \frac{\lambda_e}{2f} \epsilon_{ijk} e^c \phi_1^i \phi_2^j \Psi_Q^k \tag{6.31}$$

This term does lead to quadratic divergent contributions to the Higgs mass, but they are harmless due to the smallness of the electron mass. This is similar to the quark sector. Upon expansion of the scalar fields the mass term for the electron will then in correct form come out to be:

$$\frac{\lambda_e v}{\sqrt{2}} \left[(e^c)_L^T C e_L + h.c. \right] \tag{6.32}$$

Note that the epsilon tensor has only non-zero elements when all indices are unequal.

Let us consider the scenario in which lepton number symmetry is broken. One can add to the Lagrangian (6.28) a Majorana mass term for the singlet n^c:

$$\mathcal{L}_\nu = \lambda_\nu \phi_1^\dagger \Psi_L n^c - \frac{1}{2} M (n^c)^T C n^c + h.c. \tag{6.33}$$

The non-diagonal Majorana mass matrix now has an additional element.

$$\frac{1}{2}\begin{pmatrix}\nu & N & n^c\end{pmatrix}\begin{pmatrix}0 & 0 & -\lambda_\nu v\\ 0 & 0 & \lambda_\nu f\\ -\lambda_\nu v & \lambda_\nu f & M\end{pmatrix}\begin{pmatrix}\nu\\ N\\ n^c\end{pmatrix} \tag{6.34}$$

Diagonalizing this matrix still results in a zero mass of the neutrino at tree level. However, it gets a mass at the loop level. The diagram in figure(6.1) generates operators which give rise to a radiatively induced Majorana mass for the neutrinos.

$$\mathcal{L} = \frac{1}{2\Lambda_\nu}(\phi_2^\dagger\Psi_L)(\phi_2^\dagger\Psi_L) + h.c. \tag{6.35}$$

$$\approx \frac{1}{2\Lambda_\nu}(< h > \nu + fN)(< h > \nu + fN) + h.c. \tag{6.36}$$

The external Φ_2 legs contribute VEV's.

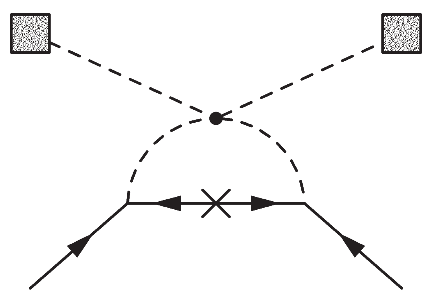


Fig. 6.1: One loop diagram inducing neutrino masses

This diagram then effectively becomes a correction to the neutrino propagator and gives a radiatively induced mass correction to the neutrino. Calculating the diagram in figure (6.1) (see appendix B) gives:

$$\frac{1}{\Lambda_\nu} \approx \frac{\lambda M}{16\pi^2 f^2}\frac{x-1-\log(x)}{(x-1)^2} \tag{6.37}$$

Here λ is the $|\phi_1^\dagger\phi_2|^2$ coupling and $x = m_h^2/(\lambda_\nu f)^2$. Completing the mass matrix gives:

$$\frac{1}{2}\begin{pmatrix}\nu & N & n^c\end{pmatrix}\begin{pmatrix}\frac{v^2}{\Lambda_\nu} & \frac{vf}{\Lambda_\nu} & -\lambda_\nu v\\ \frac{vf}{\Lambda_\nu} & \frac{f^2}{\Lambda_\nu} & \lambda_\nu f\\ -\lambda_\nu v & \lambda_\nu f & M\end{pmatrix}\begin{pmatrix}\nu\\ N\\ n^c\end{pmatrix} \tag{6.38}$$

To find the correct masses from this matrix we need to diagonalize it. This neutrino mass matrix is real and symmetric. Such a matrix can be diagonalized using orthogonal transformations. This is explained in the appendix on mathematical results.

$$A = PA_{diag}P^T \tag{6.39}$$

Where the columns of P are the eigenvectors of A and the diagonal entries of A_{diag} are the eigenvalues of A.

The eigenvalues are of course obtained by solving the characteristic polynomial of A. We know that among the eigenvalues we will find two relatively large masses for the Dirac field proportional to f. And a small mass for the neutrinos. Let us write the mass matrix in the following way, pulling out an overall factor of f, such that the matrix only contains dimensionless parameters which we can then consider to be large or small and in which we may expand.

$$M = f\begin{pmatrix} a^2bL_\nu & abL_\nu & -\lambda a \\ abL_\nu & bL_\nu & \lambda \\ \lambda a & \lambda & b \end{pmatrix} \tag{6.40}$$

Where the new parameters represent:

$$a = \frac{v}{f} \tag{6.41}$$

$$b = \frac{M}{f} \tag{6.42}$$

$$L_\nu = \frac{\lambda}{16\pi^2}\frac{x-1-\log(x)}{(x-1)^2} \tag{6.43}$$

The equation to be solved is then:

$$-x^3 + b(1 + L_\nu + a^2L_\nu)x^2 + (1 + a^2)(\lambda_\nu^2 - b^2L_\nu)x - 4a^2b\lambda_\nu^2L_\nu = 0 \tag{6.44}$$

Looking for the small neutrino mass, we can drop the higher order terms and keep only terms linear in x. One then finds an eigenvalue of approximately.

$$m \approx 4a^2bL_\nu \tag{6.45}$$

We thus find a mass for the neutrino which has a value of approximately:

$$m_\nu \approx \frac{4v^2}{\Lambda_\nu} \tag{6.46}$$

For the mass of the Dirac fermion, which is large we drop the terms proportional to a^2 and b^2. One then indeed finds that the mass is:

$$M_{Nn} \approx \lambda_\nu f \tag{6.47}$$

It is now a quasi Dirac fermion however, because the mass of the N and the $SU(3)$ singlet n^c are split due to the introduction of the Majorana mass for the singlet. The masses are split by an amount of $\mathcal{O}(M/f)$. We can restore it again to an exact Dirac fermion if M is taken to zero. This will also make the neutrinos massless again, which is consistent.

If one makes the approximation $m_h^2 \ll (\lambda_\nu f)^2$, thus $x \ll 1$, in the expression for the neutrino mass the mass of the neutrino is approximately:

$$m_\nu \approx \frac{\lambda M v^2}{2\pi^2 f^2} \log(\frac{\lambda_\nu f}{m_h}) \tag{6.48}$$

To obtain a neutrino mass of approximately 1 eV, the Majorana mass M should be approximately 1 keV. This is actually a rather small mass, which one would not expect from a term that is not really constrained by any symmetry, such as Majorana mass terms. This is a somewhat less elegant point of the model.

As can be seen the mass of the neutrinos depends only logarithmically on the coupling. So large changes in the coupling are suppressed by the logarithm.

6.4 Summary

Little Higgs models can provide an explanation of small neutrino masses. This, without introducing large scale masses or additional (to the model) dynamics. The lepton sector is similar to the quark sector, though slightly different. There is only one gauge singlet, which has a Majorana mass term proportional to M, which is relatively small. The Majorana mass term for n^c breaks conservation of lepton number. There is no need for suppressed Yukawa couplings. The model contains two heavy leptons, n^c and N', which form a quasi Dirac fermion, of approximately mass $\lambda_\nu f$ in addition to the standard model and a very light Majorana neutrino, which is interpreted as the standard model neutrino. The heavy masses of the Dirac partners are split due to the introduction of the Majorana mass for n^c. The neutrino is massless at tree level, but gets a radiatively induced Majorana mass from loop diagrams stemming from the operator $(\phi_2^\dagger \Psi_L)(\phi_2^\dagger \Psi_L)$. In only the standard model this approach would not work, because one would again come across the need for suppressed couplings. The mass of the light neutrino is proportional to M. In the limit of M going to zero, lepton number symmetry is restored and the neutrino mass goes to zero. In this particular model the neutrinos are Majorana particles.

7

The Simplest Little Higgs

In this chapter we will again discuss the model we have discussed before but now in a somewhat more general, technical and summarizing way [20]. We will also explicitly show the desired cancellation among quadratic divergences which are to be avoided if a stable electroweak scale is to be a result.

We will start with a theory of two fundamental scalars Φ_1 and Φ_2 which are both triplets of $SU(3)$. The model contains a global $[SU(3) \times U(1)]^2$ symmetry. The diagonal subgroup is gauged so the gauge symmetry is $[SU(3) \times U(1)]$. What this means is the following: In group theory, given a group G, the diagonal subgroup of the n-fold direct product is the subgroup $(g, g, \ldots, g)$, with g an element of G. In our case $n = 2$ and the group G is $[SU(3) \times U(1)]$.

The symmetry is spontaneously broken in the vacuum in that the scalar fields assume vacuum expectation values. These vacuum expectation values are now not assumed to be equal and designated f_1 and f_2. So schematically:

$$[SU(3) \times U(1)]^2 \rightarrow [SU(2) \times U(1)]^2 \tag{7.1}$$

In the above, the gauge symmetry is also broken: $[SU(3) \times U(1)] \rightarrow [SU(2) \times U(1)]$. Where the latter is the gauge symmetry of the standard model. From equation (7.1) one sees that 10 Nambu-Goldstone bosons (NGB) emerge, namely 10 group generators are broken. However, in breaking the gauge symmetry 5 NGB's are 'eaten'. They are in a sense unphysical and form the longitudinal degrees of freedom of the massive gauge fields. These fields can be transformed away from the theory. So, after the symmetry-breaking we are left with 5 physical NGB's. 4 Of these form the (complex) Higgs doublet (h) and the last is a real scalar field called η. Now we can parameterize the scalar fields Φ_1 and Φ_2 as follows:

$$\Phi_1 = e^{i\frac{f_2}{f_1 f}\Theta_1} \begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix}_{31}$$
$$\Phi_2 = e^{i\frac{f_1}{f_2 f}\Theta_2} \begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix}_{13}$$

Where $f = \sqrt{f_1^2 + f_2^2}$. The subscripts indicate how the fields transform under $SU(3)_1 \times SU(3)_2$. As mentioned earlier, 5 linear combinations of NGB's are eaten. The 5 linear combinations that are orthogonal to the eaten ones are physical. Under $SU(3)$ transformations this remains valid. Define $\tan(\phi) = f_2/f_1 = t_\phi$. The eaten combination has the form[23]:

$$\Theta_{eaten} = \cos(\phi)\Theta_1 + \sin(\phi)\Theta_2 \tag{7.2}$$

The physical linear combination is then:

$$\Theta_{physical} = -\sin(\phi)\Theta_1 + \cos(\phi)\Theta_2 = \begin{pmatrix} \frac{\eta}{\sqrt{2}} & 0 & h^0 \\ 0 & \frac{\eta}{\sqrt{2}} & h^- \\ (h^0)^* & (h^-)^* & \frac{\eta}{\sqrt{2}} \end{pmatrix} \tag{7.3}$$

We now parameterize the Φ fields as follows:

$$\Phi_1' = e^{i\Theta_{eaten}} e^{i\frac{f_2}{f_1 f}\Theta_{physical}}$$
$$\Phi_2' = e^{i\Theta_{eaten}} e^{-i\frac{f_1}{f_2 f}\Theta_{physical}}$$

This parameterization has the form of an $SU(3)$ (broken) transformation. The unphysical, eaten fields must be rotated away. These become the longitudinal degrees of freedom of the massive gauge bosons. Rotating away these unphysical eaten fields thereby fixes the $SU(3)$ gauge: The $SU(3)$ symmetry is broken in the vacuum. This gauge is known as the unitary gauge and gives the physical particle spectrum. The physical fields cannot be simultaneously rotated away because of the sign difference. What remains still is an $SU(2)$ symmetry. So we now find the following:

$$\Phi_1 = e^{i\frac{f_2}{f_1 f}\Theta} \begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix}$$
$$\Phi_2 = e^{-i\frac{f_1}{f_2 f}\Theta} \begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix}$$

Where

$$\Theta = \begin{pmatrix} \frac{\eta}{\sqrt{2}} & 0 & h^0 \\ 0 & \frac{\eta}{\sqrt{2}} & h^- \\ (h^0)^* & (h^-)^* & \frac{\eta}{\sqrt{2}} \end{pmatrix} \tag{7.4}$$

h is a doublet of $SU(2)$ becoming the standard model Higgs doublet and η is a real $SU(2)$ singlet. The normalizations are chosen to produce canonical kinetic terms in the effective Lagrangian. Note that all fields are still massless at this level.

Upon expanding one finds:

$$\Phi_1 = c_\phi f \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{it_\phi}{f} \begin{pmatrix} h \\ \eta/\sqrt{2} \end{pmatrix} - \frac{t_\phi^2}{2f^2} \begin{pmatrix} \sqrt{2}h\eta \\ h^\dagger h + \eta^2/2 \end{pmatrix} + \ldots \right] \tag{7.5}$$

$$\Phi_2 = s_\phi f \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{i}{t_\phi f} \begin{pmatrix} h \\ \eta/\sqrt{2} \end{pmatrix} - \frac{1}{2t_\phi^2 f^2} \begin{pmatrix} \sqrt{2}h\eta \\ h^\dagger h + \eta^2/2 \end{pmatrix} + \ldots \right] \tag{7.6}$$

Where $t_\phi = tan(\phi) = f_2/f_1$ and c_ϕ and s_ϕ the cosine and sine respectively.

Now let us have a look at the gauge sector of the model. Mass terms for, and scalar field interactions with gauge bosons result from the covariant derivative part of the Lagrangian:

$$\mathcal{L} = |D_\mu \Phi_1|^2 + |D_\mu \Phi_2|^2 \tag{7.7}$$

Where the covariant derivative for the ϕ fields in our case has the form:

$$D_\mu = \partial_\mu + igA_\mu^a T^a - \frac{1}{3} i g_x A_\mu^x \tag{7.8}$$

Where the T^a's are the generators of $SU(3)$ and T^x (diag(-1/3)) the $U(1)$ generator. In chapter 5 we found the following field combinations upon the breaking of $SU(3)$:

$$\begin{aligned} W_\mu^{\prime \bar{0}0} &= \frac{1}{\sqrt{2}}(A_\mu^4 \pm iA_\mu^5) \\ W_\mu^{\prime \pm} &= \frac{1}{\sqrt{2}}(A_\mu^6 \pm iA_\mu^7) \\ Z_\mu^{\prime 0} &= \frac{\sqrt{3}gA_\mu^8 + g_x A_\mu^x}{\sqrt{3g^2 + g_x^2}} \end{aligned}$$

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(A^1_{\mu} \mp iA^2_{\mu})$$
$$B_{\mu} = \frac{-g_x A^8_{\mu} + \sqrt{3} g A^x_{\mu}}{\sqrt{3g^2 + g_x^2}}$$
$$A^3_{\mu}$$

The last three combinations are massless. They are the $W^{\pm}_{\mu}$ of the standard model, the hypercharge boson which is the orthogonal combination to Z'_{μ} and the field A^3_{μ}. The masses after $SU(3)$ breaking are:

$$m^2 W'^{0\bar{0}}_{\mu} = \frac{1}{2} g^2 f^2$$
$$m^2 W'^{\pm}_{\mu} = \frac{1}{2} g^2 f^2$$
$$m^2 Z'^{0}_{\mu} = g^2 f^2 \frac{2}{3 - t^2}$$

Which can be found by looking at the terms quadratic in the gauge fields. One sees that there are indeed precisely five massive combinations, corresponding to the five broken generators. After $SU(2)$ breaking the fields B_{μ} and A^3_{μ} form orthogonal linear combinations which are the Z^0_{μ} and the photon. The covariant derivative in terms of mass eigenstates (before the breaking of $SU(2)$) becomes:

$$\begin{aligned} D_{\mu} = \partial_{\mu} &+ \frac{ig}{\sqrt{2}}[W^{+}_{\mu}(T^1 + iT^2) + W^{-}_{\mu}(T^1 - iT^2)] \\ &+ \frac{ig}{\sqrt{2}}[W'^{0}_{\mu}(T^4 + iT^5) + W'^{\bar{0}}_{\mu}(T^4 - iT^5)] \\ &+ \frac{ig}{\sqrt{2}}[W'^{-}_{\mu}(T^6 + iT^7) + W'^{+}_{\mu}(T^6 - iT^7)] \\ &+ igA^3_{\mu}T^3 \\ &+ i\frac{1}{\sqrt{3g^2 + g_x^2}} Z'_{\mu}(\sqrt{3}g^2 T^8 + g_x^2 T^x) \\ &+ i\frac{\sqrt{3} g g_x}{\sqrt{3g^2 + g_x^2}} B_{\mu}(\frac{-1}{\sqrt{3}} T^8 + T^x) \end{aligned}$$

From this we find that the hypercharge coupling, that is the coupling to B_{μ}, equals:

$$g' = \frac{\sqrt{3} g g_x}{\sqrt{3g^2 + g_x^2}} = \frac{g_x}{\sqrt{1 + \frac{g_x^2}{3g^2}}} \tag{7.9}$$

The weak mixing angle is then:

$$\frac{g'}{g} = \tan(\theta_w) = t_w = \frac{\sqrt{3}g_x}{\sqrt{3g^2 + g_x^2}} = \frac{\sqrt{3}}{\sqrt{1 + \frac{3g^2}{g_x^2}}} \tag{7.10}$$

The hypercharge generator Y comes out to be:

$$Y = \frac{-1}{\sqrt{3}}T^8 + T^x \tag{7.11}$$

Noting that

$$g_x = g\frac{t_w}{\sqrt{1 - t_w^2/3}} \tag{7.12}$$

we can write the covariant derivative slightly different as:

$$\begin{aligned} D_\mu = \partial_\mu &+ \frac{ig}{\sqrt{2}}[W_\mu^+(T^1 + iT^2) + W_\mu^-(T^1 - iT^2)] \\ &+ \frac{ig}{\sqrt{2}}[W_\mu^{'0}(T^4 + iT^5) + W_\mu^{'\bar{0}}(T^4 - iT^5)] \\ &+ \frac{ig}{\sqrt{2}}[W_\mu^{'-}(T^6 + iT^7) + W_\mu^{'+}(T^6 - iT^7)] \\ &+ igA_\mu^3 T^3 \\ &+ i\frac{g}{\sqrt{3 - t_w^2}}\ Z'_\mu\ (t_w^2 Y + \sqrt{3}T^8) \\ &+ igt_w\ B_\mu\ Y \end{aligned}$$

This expression contains only the parameters g and t_w. Using this form of the covariant derivative we can look for masses and mass eigenstates upon electro-weak ($SU(2)$) symmetry breaking. This goes in the same way as the breaking of $SU(3)$ but now h also assumes a vacuum expectation value. Doing so we find the masses of the charged gauge bosons relatively easily.

$$\begin{aligned} m^2 W^{'\bar{0}0} &= \frac{1}{2}g^2 f^2 \\ m^2 W^{'\pm} &= \frac{1}{2}g^2 f^2 - \frac{1}{4}g^2 v^2 \\ m^2 W^{\pm} &= \frac{1}{4}g^2 v^2 \end{aligned}$$

Note that the masses of the $W^{'\bar{0}0}$ and $W^{'\pm}$ are split upon EW symmetry breaking. The neutral gauge bosons are slightly more complicated because the heavy Z' mixes with the Z^0. This leads to deviations form the standard model in the couplings of the Z^0, but we will come back to this later on when we construct the covariant derivative. After EWSB the relevant part of the lagrangian has the following form:

$$\mathcal{L}_{neut} = \frac{g^2(f^2 - \frac{v^2}{2})}{3 - t_w^2}\,(Z'^0_\mu)^2 + \frac{1}{2}g^2v^2(\frac{1}{2}A^3_\mu - \frac{t_w}{2}B_\mu + \frac{1 - t_w^2}{2\sqrt{3 - t_w^2}}\,Z'^0_\mu)^2 \quad (7.13)$$

To find the correct mass eigenstates this needs to be diagonalized:

$$\begin{pmatrix} A^3_\mu & B_\mu & Z'_\mu \end{pmatrix} M \begin{pmatrix} A^3_\mu \\ B_\mu \\ Z'_\mu \end{pmatrix} = \begin{pmatrix} A^3_\mu & B_\mu & Z'_\mu \end{pmatrix} P M_{diag} P^T \begin{pmatrix} A^3_\mu \\ B_\mu \\ Z'_\mu \end{pmatrix} \quad (7.14)$$

$$= \begin{pmatrix} A_\mu & Z^0 & Z'^0 \end{pmatrix} M_{diag} \begin{pmatrix} A_\mu \\ Z^0 \\ Z'^0 \end{pmatrix} \quad (7.15)$$

We now distinguish between the Z'^0 and Z'. These indicate respectively the mass eigenstate after and before EWSB. Upon diagonalizing we find a massless linear combination which is the photon and two massive linear combinations. One heavy the other light, which are the Z'^0 and Z^0 respectively.

$$\begin{aligned} m^2 A_\mu &= 0 \\ m^2 Z^0_\mu &= \frac{1}{4}g^2v^2(1 + t_w^2) \\ m^2 Z'^0_\mu &= g^2f^2\frac{2}{3 - t_w^2} - \frac{1}{4}g^2v^2(1 + t_w^2) \end{aligned}$$

The mass eigenstates are the following linear combinations:

$$\begin{aligned} A_\mu &= (t_w A^3_\mu + B_\mu)/N_1 \\ Z^0_\mu &= (aA^3_\mu - at_w B_\mu + Z'_\mu)/N_2 \\ Z'^0_\mu &= (bA^3_\mu - bt_w B_\mu + Z'_\mu)/N_3 \end{aligned}$$

Where the N_i are normalization factors and a and b are:

$$a = -\frac{8}{\sqrt{3 - t_w^2}\,(1 + t_w^2)(1 - t_w^2)(v/f)^2} \qquad b = -\frac{1}{(1 + t_w^2)a} \quad (7.16)$$

Writing these mass eigenstates in a more recognizable form one finds:

$$\begin{aligned} A_\mu &= s_w A^3_\mu + c_w B_\mu \\ Z^0_\mu &= \frac{c_w A^3_\mu - s_w B_\mu + \frac{c_w}{a} Z'_\mu}{\sqrt{1 + c_w^2/a^2}} \\ Z'^0_\mu &= \frac{\frac{c_w}{a} A^3_\mu - \frac{s_w}{a} B_\mu - c_w(1 + t_w^2) Z'_\mu}{\sqrt{\frac{1}{a^2} + c_w^2(1 + t_w^2)^2}} \end{aligned}$$

Having determined the mass eigenstates after EWSB we can now express the covariant derivative in these mass eigenstates:

$$
\begin{aligned}
D_\mu = \partial_\mu &+ \frac{ig}{\sqrt{2}}[W^+_\mu(T^1+iT^2)+W^-_\mu(T^1-iT^2)] \\
&+ \frac{ig}{\sqrt{2}}[W'^0_\mu(T^4+iT^5)+W'^{\bar{0}}_\mu(T^4-iT^5)] \\
&+ \frac{ig}{\sqrt{2}}[W'^-_\mu(T^6+iT^7)+W'^+_\mu(T^6-iT^7)] \\
&+ igs_wA_\mu(T^3+Y) \\
&+ iZ^0_\mu(gc_wT^3-gs_wt_wY+\frac{g}{ac_w(1+t_w^2)\sqrt{3-t_w^2}}T^z) \\
&+ iZ'^0_\mu(\frac{gc_w}{a\sqrt{1+t_w^2}}T^3-\frac{gs_wt_w}{a\sqrt{1+t_w^2}}Y-\frac{g}{c_w\sqrt{1+t_w^2}\sqrt{3-t_w^2}}T^z)
\end{aligned}
$$

We can write this slightly different using the expression for a:

$$
\begin{aligned}
D_\mu = \partial_\mu &+ \frac{ig}{\sqrt{2}}[W^+_\mu(T^1+iT^2)+W^-_\mu(T^1-iT^2)] \\
&+ \frac{ig}{\sqrt{2}}[W'^0_\mu(T^4+iT^5)+W'^{\bar{0}}_\mu(T^4-iT^5)] \\
&+ \frac{ig}{\sqrt{2}}[W'^-_\mu(T^6+iT^7)+W'^+_\mu(T^6-iT^7)] \\
&+ ieA_\mu\, Q \\
&+ i\frac{g}{c_w}Z^0_\mu(T^3-s_w^2Q-\frac{v^2}{8f^2}(1-t_w^2)T^z) \\
&+ i\frac{g}{c_w}Z'^0_\mu(-\frac{v^2}{8f^2}(1-t_w^2)\sqrt{1+t_w^2}\sqrt{3-t_w^2}(T^3-s_w^2Q)-\frac{1}{\sqrt{1+t_w^2}\sqrt{3-t_w^2}}T^z)
\end{aligned}
$$

Where the charge generator Q and the generator T^z are:

$$Q = T^3 + Y \qquad T^z = \sqrt{3}T^8 + t_w^2 Y \tag{7.17}$$

We find that $e = gs_w$ and that the Z^0 has some additional interactions, proportional to $(v/f)^2$ which lead to deviations from the standard model.

Now that we have found a covariant derivative to work with let us have a look at the supposed cancellation of quadratical divergences to the Higgs mass in the gauge sector. These divergences arise from couplings which are quadratic in h with two gauge bosons. As we saw in chapter 5:

$$\mathcal{L} = |D_\mu\Phi_1|^2 + |D_\mu\Phi_2|^2 \to \ \mathrm{Tr}[g^2(A^a_\mu T^a)^2(\Phi_1\Phi_1^\dagger+\Phi_2\Phi_2^\dagger)] \tag{7.18}$$

$$\Phi_1\Phi_1^\dagger + \Phi_2\Phi_2^\dagger = \begin{pmatrix} hh^\dagger & 0 \\ 0 & f^2 - h^\dagger h \end{pmatrix} \tag{7.19}$$

The matrix only contains the terms that are quadratic in h or less. For the charged gauge bosons we found:

$$A_\mu^a T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+ & W_\mu^0 \\ W_\mu^- & 0 & W_\mu^{\prime -} \\ W_\mu^{\bar{0}} & W_\mu^{\prime +} & 0 \end{pmatrix} \tag{7.20}$$

So one finds:

$$\frac{g^2}{2}\mathrm{Tr}[\begin{pmatrix} 0 & W_\mu^+ & W_\mu^0 \\ W_\mu^- & 0 & W_\mu^{\prime -} \\ W_\mu^{\bar{0}} & W_\mu^{\prime +} & 0 \end{pmatrix}\begin{pmatrix} 0 & W_\mu^+ & W_\mu^0 \\ W_\mu^- & 0 & W_\mu^{\prime -} \\ W_\mu^{\bar{0}} & W_\mu^{\prime +} & 0 \end{pmatrix}\begin{pmatrix} hh^\dagger & 0 \\ 0 & f^2 - h^\dagger h \end{pmatrix}]$$

$$= \frac{g^2}{2}((W_\mu^{\prime +}W^{\mu\prime -} + W_\mu^0 W^{\mu\bar{0}})(f^2 - h^\dagger h) + W_\mu^+ W^{\mu -} h^\dagger h + \mathrm{Tr}[\begin{pmatrix} W_\mu^0 \\ W_\mu^- \end{pmatrix}\begin{pmatrix} W^{\mu\bar{0}} & W^{\mu +} \end{pmatrix} hh^\dagger])$$

$$\Rightarrow \frac{1}{2}g^2 f^2 (W_\mu^{\prime +}W^{\mu\prime -} + W_\mu^0 W^{\mu\bar{0}}) + \frac{1}{2}g^2 < h >^2 (W_\mu^+ W^{\mu -} - W_\mu^{\prime +}W^{\mu\prime -})$$

Upon assuming a VEV for the Higgs doublet the quadratic divergences caused by the W bosons in the standard model are precisely cancelled by the heavy gauge bosons W'. Note that they couple equally, but with opposite sign. This is due to the symmetry of of the model.

In a similar manner we can show that the quadratic divergences due to the neutral gauge bosons cancel. This goes as follows.

$$\frac{g^2}{c_w^2}\mathrm{Tr}[(A_\mu^a T^a)^2_{neut}(\Phi_1\Phi_1^\dagger + \Phi_2\Phi_2^\dagger)] = \frac{g^2}{c_w^2}\mathrm{Tr}((Z_\mu^0 Z^{0\mu}\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & (-\frac{1}{2}+sw^2)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ Z_\mu^{\prime 0} Z^{\prime 0\mu}\begin{pmatrix} \frac{(\frac{1}{2}-\frac{t_w^2}{2})^2}{(3-t_w^2)(1+t_w^2)} & 0 & 0 \\ 0 & \frac{(\frac{1}{2}-\frac{t_w^2}{2})^2}{(3-t_w^2)(1+t_w^2)} & 0 \\ 0 & 0 & \frac{1}{(3-t_w^2)(1+t_w^2)} \end{pmatrix}$$

$$+ Z_\mu^{\prime 0} Z^{0\mu}\begin{pmatrix} -\frac{\frac{1}{2}-\frac{t_w^2}{2}}{\sqrt{3-t_w^2}\sqrt{1+t_w^2}} & 0 & 0 \\ 0 & -\frac{2(-\frac{1}{2}+s_w^2)(\frac{1}{2}-\frac{t_w^2}{2})}{\sqrt{3-t_w^2}\sqrt{1+t_w^2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}).$$

$$\begin{pmatrix} hh^\dagger & 0 \\ 0 & f^2 - h^\dagger h \end{pmatrix})$$

$$\Rightarrow \frac{g^2}{4c_w^2} < h >^2 Z_\mu^0 Z^{0\mu} - \frac{g^2}{4c_w^2} < h >^2 Z'^0_\mu Z'^{0\mu}$$
$$+ \; g^2 f^2 \frac{1}{3-t_w^2} Z'^0_\mu Z'^{0\mu} - \frac{g^2(1-t_w^2)}{2c_w\sqrt{3-t_w^2}} < h >^2 Z_\mu^0 Z'^{0\mu}$$

The last term, which connects the Z^0 and Z'^0, does not give rise to a quadratic divergence. Upon VEV the Z^0 and Z'^0 couple equally but with opposite sign to the Higgs boson, precisely cancelling quadratic divergences. For completeness the couplings of the various particles to the Higgs boson are given in table (7.1). Having considered the the gauge sector in some

vertex	coupling
hhW^+W^-	$+\frac{g^2}{4}$
$hhW'^+W'^-$	$-\frac{g^2}{4}$
hhZ^0Z^0	$+\frac{g^2}{8c_w^2}$
$hhZ'^0Z'^0$	$-\frac{g^2}{8c_w^2}$
$hhZ^0Z'^0$	$-\frac{g^2(1-t_w^2)}{4c_w\sqrt{3-t_w^2}}$

Tab. 7.1: Vertices and corresponding couplings

detail we will now move on to the fermion sector of the model, in particular to the quark part, because the lepton sector was already discussed in some detail in chapter 6 and the couplings of gauge bosons to leptons are similar to those in the quark sector. The Yukawa Lagrangian for the quarks has the following form:

$$\mathcal{L} = i\lambda_{1n}\Phi_1^\dagger \Psi_{Qn} u_{1n}^c + i\lambda_{2n}\Phi_2^\dagger \Psi_{Qn} u_{2n}^c + i\frac{\lambda_{mn}}{\Lambda}\epsilon_{ijk}\Phi_1^i \Phi_2^j \Psi_{Qn}^k d_m^c + h.c. \quad (7.21)$$

The last term contains the couplings to down type quarks. Since the top quark divergence is troublesome in the standard model we will focus our attention on the up type quarks in the Lagrangian, this means on the first two terms.

The subscript $n = 1, 2, 3$ indicates the generation. Ψ_Q is an $SU(3)$ triplet and contains the SM quark doublet and the heavy partner U,C and T along with a phase factor. u_1^c and u_2^c are the singlets belonging to the first and third member of the triplet. Remember that we are working in the gauge eigenstate basis. Substituting the expansions (7.5) and (7.6) for the Φ fields results in the following terms (where d_n stands for the SM quark doublet and

η was ignored):

$$\mathcal{L} = \lambda_{1n} c_\phi f[-U_n u_{1n}^c + \frac{t_\phi}{f} h^\dagger d_n u_1^c + \frac{t_\phi^2}{2f^2} h^\dagger h U_n u_{1n}^c] \tag{7.22}$$

$$+ \lambda_{2n} s_\phi f[-U_n u_{2n}^c - \frac{1}{t_\phi f} h^\dagger d_n u_2^c + \frac{1}{2t_\phi^2 f^2} h^\dagger h U_n u_{2n}^c] + h.c. \tag{7.23}$$

One sees that the heavy third member U_n of the triplet Ψ_{Qn} couples to the (normalized) linear combination of the heavy singlet U_n^c:

$$U_n^c = \frac{\lambda_{1n} c_\phi \quad u_{1n}^c + \lambda_{2n} s_\phi \quad u_{2n}^c}{\sqrt{(\lambda_{1n} c_\phi)^2 + (\lambda_{2n} s_\phi)^2}} \tag{7.24}$$

Together with the U_n's these linear combinations form mass terms:

$$-M U_n U_n^c = -f\sqrt{(\lambda_{1n} c_\phi)^2 + (\lambda_{2n} s_\phi)^2} U_n U_n^c \tag{7.25}$$

Having found the linear combination forming the heavy singlet, the light singlet is formed by the orthogonal combination:

$$u_n^c = \frac{-\lambda_{2n} s_\phi \quad u_{1n}^c + \lambda_{1n} c_\phi \quad u_{2n}^c}{\sqrt{(\lambda_{1n} c_\phi)^2 + (\lambda_{2n} s_\phi)^2}} \tag{7.26}$$

This linear combination is massless, but it will however become massive after EWSB to which we shall proceed now. Assuming a VEV for h in (7.22) and (7.23) gives:

$$\mathcal{L} = \lambda_{1n} c_\phi f[-U_n u_{1n}^c + \frac{t_\phi}{f} \frac{v}{\sqrt{2}} u_n u_1^c + \frac{t_\phi^2}{2f^2} \frac{v^2}{2} U_n u_{1n}^c] \tag{7.27}$$

$$+ \lambda_{2n} s_\phi f[-U_n u_{2n}^c - \frac{1}{t_\phi f} \frac{v}{\sqrt{2}} u_n u_2^c + \frac{1}{2t_\phi^2 f^2} \frac{v^2}{2} U_n u_{2n}^c] + h.c. \tag{7.28}$$

Expressing this Lagrangian as:

$$\mathcal{L} = a U_n U_n^c + b U_n u_n^c + c u_n u_n^c + d u_n U_n^c + h.c. \tag{7.29}$$

Solving the equations for a,b,c and d gives to leading order in (v/f):

$$a = -f\sqrt{\lambda_{1n}^2 c_\phi^2 + \lambda_{2n}^2 s_\phi^2}$$
$$b = 0$$
$$c = -\frac{v}{\sqrt{2}} \frac{\lambda_{1n}\lambda_{2n}}{\sqrt{\lambda_{1n}^2 c_\phi^2 + \lambda_{2n}^2 s_\phi^2}}$$
$$d = \frac{v}{\sqrt{2}} \frac{(\lambda_{1n}^2 - \lambda_{2n}^2) c_\phi s_\phi}{\sqrt{\lambda_{1n}^2 c_\phi^2 + \lambda_{2n}^2 s_\phi^2}}$$

So we find:

$$\mathcal{L} = -f\sqrt{\lambda_{1n}^2 c_\phi^2 + \lambda_{2n}^2 s_\phi^2} U_n U_n^c - \frac{v}{\sqrt{2}} \frac{\lambda_{1n}\lambda_{2n}}{\sqrt{\lambda_{1n}^2 c_\phi^2 + \lambda_{2n}^2 s_\phi^2}} u_n u_n^c$$
$$+ \frac{v}{\sqrt{2}} \frac{(\lambda_{1n}^2 - \lambda_{2n}^2) c_\phi s_\phi}{\sqrt{\lambda_{1n}^2 c_\phi^2 + \lambda_{2n}^2 s_\phi^2}} u_n U_n^c + h.c.$$

This lagrangian leads to mixing between heavy quarks and SM up quarks. It is more difficult to handle than the simplified cases with equal couplings or aligned VEV's. The mass eigenstates after EWSB are found by diagonalizing the last equation. It is not immediately obvious how this should be done but it can be achieved using biunitary transformations:

$$S^\dagger M T = M_{diag} \tag{7.30}$$

Where is our case:

$$M = \begin{pmatrix} -M & M\,\delta \\ 0 & -m \end{pmatrix} \tag{7.31}$$

The idea is outlined in the appendix on mathematical results. We shall not give the EW mass eigenstates another name, but indicate the SU(3) states with a prime:

$$U_n = U_n' - \delta u_n' \tag{7.32}$$
$$u_n = \delta U_n' + (1 - \frac{1}{2}\delta^2) u_n' \tag{7.33}$$

Where δ equals:

$$\delta = \frac{v}{\sqrt{2} f} \frac{(\lambda_{1n}^2 - \lambda_{2n}^2) c_\phi s_\phi}{(\lambda_{1n}^2 c_\phi^2 + \lambda_{2n}^2 s_\phi^2)} \tag{7.34}$$

Inverting these equations gives:

$$U_n' = U_n + \delta u_n \tag{7.35}$$
$$u_n' = (1 - \frac{1}{2}\delta^2) u_n - \delta U_n \tag{7.36}$$

The δ^2 term in the first equation is omitted because we do not really need it here and it only gives precision corrections. In the second equation it is maintained because it appears in the couplings to known weak interaction gauge bosons. The conjugate fields do not mix to $\mathcal{O}$(v/f). Having determined the EW mass eigenstates let us have closer look at the cancellation of quadratic divergences in the quark sector. The Lagrangian had the following form:

$$\mathcal{L} = i\lambda_{1n} u^c_{1n} \Phi_1^\dagger \Psi_{Qn} + i\lambda_{2n} u^c_{2n} \Phi_2^\dagger \Psi_{Qn} + h.c. \tag{7.37}$$

We can write the mass matrix squared as:

$$MM^\dagger = \begin{pmatrix} \lambda_1 \Phi_1^\dagger \\ \lambda_2 \Phi_2^\dagger \end{pmatrix} \begin{pmatrix} \lambda_1 \Phi_1 & \lambda_2 \Phi_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 \Phi_1^\dagger \Phi_1 & \lambda_1 \lambda_2 \Phi_1^\dagger \Phi_2 \\ \lambda_1 \lambda_2 \Phi_2^\dagger \Phi_1 & \lambda_2^2 \Phi_2^\dagger \Phi_2 \end{pmatrix} \tag{7.38}$$

With the singlets in a row vector to the left and the triplet to the right of M. The mass matrix squared is hermitian, so it can be diagonalized using unitary transformations:

$$U^\dagger M M^\dagger U = M^2_{diag} \tag{7.39}$$

Where U has eigenvectors of $MM^\dagger$ as columns and M_{diag} has the absolute value of the eigenvalues on the diagonal. We already determined the EW eigenstates, so we shall concentrate on the masses of the particles and the cancellation of quadratic divergences. When we use the expansions for the Φ fields we can find an expression for $MM^\dagger$:

$$MM^\dagger = f^2 \begin{pmatrix} \lambda_1^2 c_\phi^2 & (1/f^2)\lambda_1\lambda_2 \Phi_1^\dagger \Phi_2 \\ (1/f^2)\lambda_1\lambda_2 \Phi_2^\dagger \Phi_1 & \lambda_2^2 s_\phi^2 \end{pmatrix} \tag{7.40}$$

Calculating $\Phi_1^\dagger \Phi_2$ and $\Phi_2^\dagger \Phi_1$ to fourth order gives:

$$\Phi_1^\dagger \Phi_2 = c_\phi s_\phi f^2 [1 - \frac{1}{f^2}(\frac{t_\phi^2}{2} + 1 + \frac{1}{2t_\phi^2})h^\dagger h + \frac{1}{4f^4}(h^\dagger h)^2] = \Phi_2^\dagger \Phi_1 \tag{7.41}$$

Inserting this to $\mathcal{O}(v^2/f^2)$ in the mass matrix squared and diagonalizing gives us the following expressions for masses and couplings to the Higgs boson:

$$M_U^2 = f^2(\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2) - \frac{\lambda_1^2 \lambda_2^2}{(\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2)} < h^\dagger h > \tag{7.42}$$

$$m_u^2 = \frac{\lambda_1^2 \lambda_2^2}{(\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2)} < h^\dagger h > \tag{7.43}$$

So one finds to $\mathcal{O}(v^2/f^2)$:

$$M_U = f\sqrt{\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2} - \frac{\lambda_1^2 \lambda_2^2}{2(\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2) f \sqrt{\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2}} < h^\dagger h >$$

$$m_u = \frac{\lambda_1 \lambda_2}{\sqrt{\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2}} < h >$$

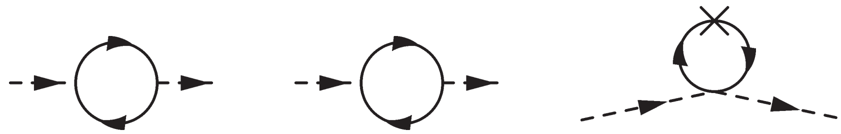


Fig. 7.1: Quadratically divergent fermion loops contributing to the Higgs mass squared

The quadratically divergent diagrams appearing in the quark sector are shown in figure (7.1). The quadratically divergent parts of these diagrams cancel among each other. This is of course due to the symmetry of the model. Remembering the mixing expressed in equations (7.32-7.36) we can express this cancellation with an equation for the coupling constants, similar to the gauge sector, though somewhat more complicated. The subscripts refer to the top quark sector.

$$\lambda_{tt}^2 + \lambda_{Tt}^2 = M_T \lambda_{TT} \tag{7.44}$$

Where the couplings are:

λ_{tt}	$\frac{\lambda_1 \lambda_2}{\sqrt{\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2}}$
λ_{Tt}	$\frac{\lambda_1 \lambda_2 \delta}{\sqrt{\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2}}$
λ_{TT}	$\frac{\lambda_1^2 \lambda_2^2 (1+\delta^2)}{2(\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2) M_T}$

Experimental verification of this particular model should include testing equation (7.44). When we return to the quark masses a bit more one sees that one of the couplings $\lambda_{1,2}$ should be small because of the light masses of the SM u and c quarks. Let us choose the first coupling. The resulting masses and mixing angles are shown in table (7.2).

M_U	$f\lambda_U s_\phi$
M_C	$f\lambda_C s_\phi$
M_T	$f\sqrt{\lambda_1^2 c_\phi^2 + \lambda_2^2 s_\phi^2}$
M_u	$\frac{1}{\sqrt{2}}\lambda_u v$
M_c	$\frac{1}{\sqrt{2}}\lambda_c v$
M_t	$\frac{1}{\sqrt{2}}\lambda_t v$
δ_u	$\frac{-v}{\sqrt{2}ft_\phi}$
δ_c	$\frac{-v}{\sqrt{2}ft_\phi}$
δ_t	$\frac{v}{\sqrt{2}f}\frac{(\lambda_{1n}^2-\lambda_{2n}^2)c_\phi s_\phi}{(\lambda_{1n}^2 c_\phi^2+\lambda_{2n}^2 s_\phi^2)}$

Tab. 7.2: Quark masses and mixing angles.

The new coupling constants are of course found by taking λ_1^u small.

We can now also determine the interactions of quarks with gauge bosons. After assigning quantum numbers to the various quarks the interaction currents are obtained. We already gave these quantum numbers in chapter 5.

$$\Psi_Q = (3,3)_{\frac{1}{3}} \tag{7.45}$$

$$d^c = (\bar{3},1)_{\frac{1}{3}} \tag{7.46}$$

$$u^c, U^c = (\bar{3},1)_{-\frac{2}{3}} \tag{7.47}$$

The interactions follow from the covariant derivative term in the Lagrangian:

$$\mathcal{L} = i\bar{\Psi}_Q \not{D}_\mu \Psi_Q \tag{7.48}$$

The gauge bosons then couple to their respective currents. For example the W_μ^+ couplings to quarks, which we will use later on, are:

$$\mathcal{L}_{W^+} = -gW_\mu^+ J^{\mu+} = -gW_\mu^+ \frac{1}{\sqrt{2}}[(1-\frac{1}{2}\delta_u^2)\bar{u}_L\gamma^\mu d_L - \delta_u \bar{U}\gamma^\mu d_L] \tag{7.49}$$

In the next chapter we will discuss some phenomenology of this particular Little Higgs model known as the Simplest Little Higgs.

8

Phenomenology

8.1 Little Higgs: Phenomenological Implications

Little Higgs models contain some spectacular new physics at the TeV scale. With the CERN Large Hadron Collider (LHC), due to begin operation in 2007, and the TeVatron as new generation particle accelerators, we are sure to get more than a glimpse of this fascinating, new, beyond the standard model, physics. Be it Little Higgs or Super Symmetry or other possibilities. Here we shall focus on experimental consequences of Little Higgs models, in particular on the $SU(3)$ 'simple' group model.

Although these models all predict many new particles and phenomena there appear to be as many varieties of the idea as newly predicted particles. For this reason it is important that one tries to find and predict features that distinguish Little Higgs models from other high energy scenarios and further more that distinguish between different Little Higgs models. Of course all models have to reproduce the gauge group of the electro-weak interactions $SU(2)_L \times U(1)_Y$ at low energy. From this one can divide the Little Higgs models in two classes. One class is the models where the extended symmetry group is a product group. By this we mean that the EW group arises from the diagonal breaking of two or more gauge groups. The second class of models possess a single symmetry group or 'simple' group. The $SU(3) \times U(1)$ model is an example of a simple group model. In this model the gauge symmetry is already the diagonal $SU(3)$ to begin with, which breaks to $SU(2)$. An example of a product group model is the 'Littlest Higgs', which is based on the gauge group $[SU(2) \times U(1)]^2$, breaking to $SU(2)_L \times U(1)_Y$.

An example of a difference between these two classes of models is the number of additional fermions. In the product group models it usually sufficient to introduce only one new heavy quark, to cancel the top divergence to the Higgs mass. This is done by writing group symmetric couplings of fermions to the scalar fields, and add only for the top quark, in the manner of collective symmetry breaking, extra terms which involve a mass and interactions, upon symmetry-breaking, for the additional heavy T, which ensure the quadratic divergence cancelation. Similar terms are then also written for the first two generations of quark doublets, but then there is no need for the introduction of heavy additional fermions and extra terms, because of the insignificant contribution to the quadratic divergence of the Higgs mass. It can however be done. For the simple group models there is no such freedom. In the simple group models all the fermions, including the TeV scale ones, transform under the fundamental representation of an $SU(N)$ group, meaning that all three generations require an additional TeV fermion. In the product group models SM fermions and TeV ($SU(2)_L$ singlet) fermions transform quite differently. Also, in the case of a 'simple' group model, $SU(N) \times U(1)$ gauge couplings, are fixed in terms of SM couplings. This is because the 'simple' group, is broken down to the SM EW group.

Another important goal is of course determining wether or not a Little Higgs mechanism is realized in nature. This means that it is of crucial importance to experimentally verify the relations among couplings in the scalar-, gauge- and fermion sector which express the Little Higgs mechanism and the all important quadratic divergence cancellations.

All Little Higgs models contain particles in addition to the SM. These particles have TeV scale masses and many of them stand a good chance of being detected at the LHC, if they exist. Production of these new particles can give strong indications as to dismiss or confirm particular models. Also, model parameters can be extracted from these collider processes to test a (particular) Little Higgs scenario even further. In the following we shall discuss some of these features in somewhat more detail. In doing this we will focus on the $SU(3)$ model.

8.2 TeV Scale Quarks

In the Simplest Little Higgs, every generation contains a third quark (U,C,T) in addition to the SM u and d type quarks. It has a charge $+2/3$ and a TeV scale mass. It is charged under the $SU(3)_c$ group of color. The heavy quarks are $SU(2)_L$ singlets up to mixing of $\mathcal{O}(v/f)$ with the standard model up-type quarks as we saw in chapter 7. These additional quarks for one have the role of cancelling the quadratic divergences to the Higgs mass coming from standard model up type quark loops, especially the one from the top quark.

Suppose in future experiments a heavy unknown quark-like particle is to be discovered, how can one distinguish between a Little Higgs partner and for example a fourth generation quark? An attempt to distinguish the heavy T fermion from a fourth generation top-like quark can be made by detecting the flavor changing neutral current (FCNC) decays $T \rightarrow Zt$ and $T \rightarrow Ht$[21]. These currents must be highly suppressed if T were to be a fourth generation top quark, because then it would have to be charged under $SU(2)_W$, highly suppressing FCNC (CKM transformations cancel in the neutral current), where as in Little Higgs models it is (virtually) an $SU(2)$ singlet and then these currents are perfectly possible.

A note: There FCNC processes in the standard model, however they occur only at the loop level for example in the $K^0 \rightarrow \bar{K}^0$ system.

Looking at the mass of the T quark, one can understand that it can be small in order to avoid fine tuning, that is avoiding large, highly accurate cancellations between different contributions. The lower the T mass, the lesser the amount of tuning. To avoid fine tuning of more than 10% the T mass should not be much higher than circa 2 TeV.

$$M_T = \sqrt{\lambda_1^2 f_1^2 + \lambda_2^2 f_2^2} \tag{8.1}$$

As can be seen, the mass of the T quark can be a lot lower than the largest of the two f's, if we take the particular coupling down. Next to avoiding fine tuning, this is also convenient, because it allows one to give the heavy gauge bosons a relatively large mass, which are all proportional to $(f_1^2 + f_2^2)$, which appears to be favored by EW constraints, and at the same time keep the T mass low. This is an elegant feature of the $SU(3)$ model.

After Electro-Weak symmetry breaking (EWSB) the mass eigenstates of the up type quarks mix with with their heavy partner, by an amount of $\mathcal{O}(v/f) = \delta$. This means that the heavy partner obtains couplings to SM model fermions and gauge bosons and therefore might be produced at LHC through these mixing angles. The Feynman rules for these vertices are the same as the SM Feynman rules for the up type quark coupling to dW_μ^+ except for the extra factor which takes into account the mixing δ. From chapter 7 we know that these Feynman rules are:

$$W^{\mu +}\bar{U}d :\rightarrow \tfrac{ig\delta}{\sqrt{2}}\gamma^\mu \tfrac{1-\gamma^5}{2} \qquad Z^{\mu 0}\bar{U}u :\rightarrow \tfrac{ig\delta}{2c_w}\gamma^\mu \tfrac{1-\gamma^5}{2}$$

Single heavy quark production is possible by socalled Wd fusion. It proceeds through the mixing with the SM up-type quark, which leads to a $W_\mu^+\bar{Q}d$ coupling. They can also be pair produced, if m_T is not too large, but with a smaller cross-section through QCD processes. The matrix elements squared for producing the heavy partner U of the SM up and down quarks through Wd and Zu fusion are:

$$|\mathcal{M}|^2_{dW_\mu^+\to U} = \frac{g^2\delta_U^2}{12}\,(\frac{m_U^4}{m_W^2})\,(1-\frac{m_W^2}{m_U^2})\,(1+2\frac{m_W^2}{m_U^2}) \tag{8.2}$$

$$|\mathcal{M}|^2_{uZ_\mu^0\to U} = \frac{g^2\delta_U^2}{24c_w^2}\,(\frac{m_U^4}{m_Z^2})\,(1-\frac{m_Z^2}{m_U^2})\,(1+2\frac{m_Z^2}{m_U^2}) \tag{8.3}$$

The derivation of these expressions can be found in the appendix on matrix elements and decay rates. The cross-section for U production should be rather large compared to C or T production, because the last two require the generation of initial state s and b quarks, while U production can be accomplished from the u and d quarks in the proton at LHC, which is a proton-proton collider.

Let us have a look at the decay processes of the heavy quarks. The dominant decay modes in most Little Higgs models are to standard model particles: $Q \to qZ$, $Q \to q'W$ and $Q \to qH$. The partial widths to these final state particles for example for the U quark is:

$$\Gamma_{U\to dW_\mu^+} = \frac{g^2\delta_U^2}{64\pi}\,(\frac{m_U^3}{m_W^2})\,(1-\frac{m_W^2}{m_U^2})^2\,(1+2\frac{m_W^2}{m_U^2}) \tag{8.4}$$

Furthermore:

$$\frac{1}{2}\Gamma_{U\to dW_\mu^+} \approx \Gamma_{U\to uZ_\mu^0} \approx \Gamma_{U\to uH} \approx \frac{\lambda_U^2}{32\pi}m_U \approx 9.9\cdot 10^{-3}\lambda_U^2\,m_U \tag{8.5}$$

Where the second line is obtained by considering the Goldstone boson equivalence theorem (see the appendix of decay rates). The last expression is obtained by substituting the expressions for δ_U and the masses of the U-quark and the W boson. These results are calculated in the mentioned appendix. These might be the only dominant decay modes, depending on the mass spectrum. If the Q mass is large enough it also decays into qW'^0 and $q'W'^+$. There is also a rather small coupling to $q\eta$. Let us focus a bit on the T quark.

T can decay into various final states. A branching fraction is the fraction of decays into a particular state and is equal to the quotient of the partial width and the total decay width. The mass of the T quark can be reconstructed from each of the three SM decay modes[21]. The partial widths to tZ, bW and tH are then just proportional to λ_T^2.

The partial widths to tW'^0 and bW'^+ can be expressed in terms of the gauge couplings, which are related to the standard model couplings, and the W'^0 and W'^+ masses. These masses can be expressed in the mass of the heavy Z'^0 boson, which in turn can be determined from its decays into dileptons.

The partial width to $t\eta$ can also be calculated and can after an η mass measurement be determined.

The only free parameter still is λ_T. Measuring the T decay rate into any final state, allows this free parameter λ_T to be determined. Having determined λ_T, a prediction for the pattern of branching fractions can be made. Individually measuring all the branching fractions can then test the validity of the model. If the standard model decay modes are the only ones then these branching ratios will simply be:

$$BR(T \to tH) : BR(T \to tZ) : BR(T \to tW) \Rightarrow \frac{1}{4} : \frac{1}{4} : \frac{1}{2} \tag{8.6}$$

By the Goldstone equivalence theorem.

Kong[22] indicated that the embedding of the fermions in triplets all transforming as $\mathbf{3}$ of $SU(3)$ leaves the theory with a gauge group $SU(3)_W \times U(1)_Y$ anomalous. This does not dismiss the theory immediately since it is an effective theory and the anomalies can still be cancelled by fermions appearing at the scale of 10 TeV. An intriguing thing is that the model can only be constructed anomaly free if there are a multiple of three generations. It possible to construct an anomaly free theory[22].

By looking at the charge asymmetry in the final state of $U(+\bar{U})$ production[1], one can distinguish between the socalled universal- (anomalous) and anomaly-free embedding of the fermion sector. In the anomaly free theory the first two generations transform as $\bar{\mathbf{3}}$ of $SU(3)$, and the third as $\mathbf{3}$.

$$\begin{array}{lll} Q^T_{1,2,3} = (u, d, iU) & & \textit{universal} \\ Q^T_{1,2} = (d, -u, iD) & Q^T_3 = (t, b, iT) & \textit{anomaly free} \end{array}$$

In the anomaly free embedding the heavy partner of the first two generations has a charge $-1/3$, whereas in the universal embedding all heavy partners have charge $+2/3$. The final state of the heavy quark production to $W^\pm$ gauge bosons, in the universal embedding will for the greater part contain positively charged leptons, while in the anomaly free theory the final state will contain mostly negatively charged leptons. This feature distinguishes between an anomalous (effective) theory and one which is anomaly free.

[1] $\bar{U}$ production makes up only a small part of the total production cross-section

8.3 TeV Scale Gauge Bosons

Extension of the gauge symmetry in Little Higgs models implies that new gauge bosons should manifest themselves at the TeV scale. In chapter seven we determined the expression for the masses of these newly predicted gauge bosons:

$$
\begin{aligned}
m^2 W'^{\bar{0}0} &= \frac{1}{2} g^2 f^2 \\
m^2 W'^{\pm} &= \frac{1}{2} g^2 f^2 - \frac{1}{4} g^2 v^2 \\
m^2 Z'^0_\mu &= g^2 f^2 \frac{2}{3 - t_w^2} - \frac{1}{4} g^2 v^2 (1 + t_w^2)
\end{aligned}
$$

To avoid fine tuning of more than 10 percent these gauge boson masses should not be much higher than ca. 5 TeV. Lower bounds on the Z' mass are around 1 TeV, see for example [24]. Electro-Weak constraints on f are rather tight. However, in the $SU(3)$ model the additional parameter $t_\phi = f_2/f_1$ can relax these constraints somewhat, making the fermion sector more independent from the gauge sector. Taking $f \geq 3.9$ TeV and $t_\phi = 3$ [25] gives a Z' mass $M_{Z'} \geq 2.2$ TeV. These values give a T mass of 2.3 TeV. The $SU(2)$ doublet W' has a lower limit on its mass of $M_{W'} \geq 1.8$ TeV. After EWSB the heavy charged boson has a slightly lower mass.

8.4 Testing the Cancelation Mechanism

Naturally the all important test would be to determine wether or not nature is based on a Little Higgs mechanism. This implies testing the quadratic cancelation, that is, experimentally verifying the relation between the couplings that expresses this cancelation. In the fermionic sector this is:

$$
M_T \lambda_{TT} = \lambda_{tt}^2 + \lambda_{Tt}^2 \tag{8.7}
$$

One would hope to measure the couplings directly and model independent, however that does not seem possible by current means nor the LHC. It would require an experimental determination of λ_{TT} from a measurement of the production cross-section of TH, which too small to be observable at the LHC. However model dependent relations do stand a good chance of being possible to test. In the $SU(3)$ model it requires the independent experimental determination of five parameters. These are the four parameters: λ_t, λ_T, m_T and f. In the previous section we discussed (independent) ways of obtaining λ_T and m_T. f can be obtained from measurements in the gauge boson sector. In the most general case one also requires a fifth parameter, namely the ratio $\frac{f_2}{f_1} = t_\phi$ of the vacuum condensates. Here, the similarity of the gauge structure of the three generations is a useful fact. As already noted, the single heavy U production cross section is much larger then that of the T quark. By measuring M_U and the total cross section for heavy U production and knowing f one can determine λ_U and t_ϕ independently from the T sector. The relations in the anomaly free embedding are similar.

$$m_Q^{1,2} = s_\phi \lambda_Q^{1,2} f \tag{8.8}$$

$$\delta_Q = \frac{v}{\sqrt{2} f t_\beta} \tag{8.9}$$

So, having selected a particular model, testing the Little Higgs mechanism seems experimentally within reach at the LHC.

In the gauge boson sector the cancellations occur between the heavy Z' and the SM Z^0 and between the $W'^{\pm}$ and the SM $W^{\pm}$. These two couplings should sum to zero in both cases, which in the model they of course do.

Concluding: if Little Higgs models exist it should be possible to identify them at the LHC. Many of the parameters in these models can be measured and with phenomenological analysis one should be able to distinguish between models and scenarios.

APPENDIX

A

Mathematical Results

A.1 The Baker-Campbell-Hausdorff Result

For the exponents of two square matrices A and B it is in general not true that their product is equal to the exponent of the sum the two matrices. However there exists a matrix C such that [9]:

$$e^A \cdot e^B = e^C \tag{A.1}$$

Consider the expression:

$$e^{\lambda A} \cdot e^{\lambda B} = e^{(\sum_{k=1}^{k=\infty} \lambda^k P_k)} \tag{A.2}$$

For $\lambda = 1$ the matrix C equals:

$$C = P_1 + P_2 + P_3 + \dots \tag{A.3}$$

Expanding the left hand side of (A.2) gives:

$$\begin{aligned} e^{\lambda A} \cdot e^{\lambda B} = 1 + \lambda(A + B) &+ \lambda^2(\frac{1}{2!}A^2 + AB + \frac{1}{2!}B^2) \\ &+ \lambda^3(\frac{1}{3!}A^3 + \frac{1}{2!}A^2B + \frac{1}{2!}AB^2 + \frac{1}{3!}B^3) + \dots \end{aligned}$$

Expanding the right hand side results in:

$$\begin{aligned} e^{(\sum_{k=1}^{k=\infty} \lambda^k P_k)} &= 1 + (\sum_{k=1}^{k=\infty} \lambda^k P_k) + \frac{1}{2!}(\sum_{k=1}^{k=\infty} \lambda^k P_k)^2 + \frac{1}{3!}(\sum_{k=1}^{k=\infty} \lambda^k P_k)^3 + \dots \\ &= 1 + \lambda P_1 + \lambda^2(P_2 + \frac{1}{2!}P_1^2) + \lambda^3(P_3 + \frac{1}{2!}P_1P_2 + \frac{1}{2!}P_2P_1 + \frac{1}{3!}P_3^3) + \dots \end{aligned}$$

Equating the terms with the same power of the coefficient λ, which can in principal take any value, gives us:

$$\begin{aligned} P_1 &= A + B \\ P_2 &= \frac{1}{2}(AB - BA) = [A, B] \\ P_3 &= \frac{1}{12}(A^2B - 2ABA + BA^2 + AB^2 - 2BAB + B^2A) \\ &= \frac{1}{12}([A, [A, B]] + [[A, B], B]) \end{aligned}$$

So one can find an expression for the matrix C, which is the following:

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [[A, B], B]) + \dots \tag{A.4}$$

Here it can be seen that C is only equal to the sum of A and B to first order or exactly equal if A and B commute.

A.2 The Goldstone Theorem

Here we show a proof of the Goldstone theorem [8]. It is an elegant proof in the sense that is both rather simple and quite general. The Goldstone theorem states that for every spontaneously broken, continuous symmetry, a massless scalar field known as a Nambu-Goldstone boson (NGB) appears.

From the Noether theorem we know that for every continuous symmetry (generator) there exists a conserved charge Q. The time derivative of Q is zero. This can be expressed as:

$$[H, Q] = 0 \tag{A.5}$$

Where H is the Hamiltonian. Let the vacuum state be denoted by $|0>$. With an appropriate rescaling of the Hamiltonian we can always write:

$$H|0>= 0 \tag{A.6}$$

When the symmetries persist, the vacuum is invariant under the symmetry transformation:

$$e^{i\theta Q}|0>= |0> \tag{A.7}$$

Expanding this means that:

$$Q|0>= 0 \tag{A.8}$$

Now suppose that the symmetry is spontaneously broken in the vacuum, thus the vacuum is no longer invariant under the symmetry transformation. Thus: $Q|0> \neq 0$. The energy of that state $Q|0>$ can be found as follows.

$$HQ|0>=[H,Q]|0>=0 \tag{A.9}$$

Remember that H does annihilate the vacuum (eq.A.6). So the state $Q|0>$ has the same energy as the vacuum. We can also express Q as a space integral over a current.

$$Q=\int \mathrm{d}^D x J^0(x,t) \tag{A.10}$$

Now consider now the state $|s>$ in the theory:

$$|s>=\int \mathrm{d}^D x\ e^{ikx} J^0(x,t) \tag{A.11}$$

This state has a spatial momentum k ($P|s>=k|s>$). When k goes to zero this state becomes $Q|0>$, which has energy zero. So the state $|s>$ has zero energy when the momentum goes to zero. In a relativistic theory this means that it describes a massless particle. This is the NGB.

An alternative proof can be found in [2].

A.3 Diagonalizing Matrices

Consider an $n \times n$ matrix A that has n linearly independent eigenvectors v_i corresponding to eigenvalues λ_i. Let P the a matrix which has the v_i's as its columns. P is invertible ($\det(P) \neq 0$). Notice that $P^{-1}v_i$ is the i-th column of $P^{-1}P = 1$. This means that $P^{-1}v_i$ is the i-th column of the identity matrix e_i.

$$P^{-1}v_i = e_i \tag{A.12}$$

This implies:

$$\begin{aligned}
P^{-1}AP &= P^{-1}A[v_1, v_2, \ldots, v_n] \\
&= P^{-1}[Av_1, Av_2, \ldots, Av_n] \\
&= P^{-1}[\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n] \\
&= [\lambda_1 P^{-1}v_1, \lambda_2 P^{-1}v_2, \ldots, \lambda_n P^{-1}v_n] \\
&= [\lambda_1 e_1, \lambda_2 e_2, \ldots, \lambda_n e_n] \\
&= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}
\end{aligned}$$

So A can be diagonalized by a matrix P as in the foregoing such that:

$$P^{-1}AP = D \tag{A.13}$$

Where D is a diagonal matrix with the eigenvalues of A on the diagonal.

The eigenvectors are automatically linearly independent if the eigenvalues are distinct, meaning that they are all different. If A is a symmetric matrix ($A = A^T$) then the eigenvectors of different eigenvalues are orthogonal to each other. Namely, suppose that we have eigenvalues λ_1 and λ_2 corresponding to eigenvalues v and w:

$$\lambda_1 v \cdot w = Av \cdot w = v \cdot A^T w = v \cdot Aw = v \cdot \lambda_2 w = \lambda_2 v \cdot w \tag{A.14}$$

This can only be true if $v \cdot w = 0$, thus if v and w are orthogonal.

There are always n linearly independent eigenvectors if A is symmetric, so A will be diagonalizable. If the entries of A are all real, then there exist n mutually orthogonal eigenvectors. The matrix P then contains n orthogonal columns and also its inverse is equal to cP^T where c is a normalization factor. Matrices for which $P^{-1} = P^T$ are called orthogonal matrices. Thus real symmetric matrices can be diagonalized by orthogonal matrices P.

$$P^T AP = D \qquad\qquad (\text{if} \quad A = A^T) \tag{A.15}$$

A.4 Biunitary transformations

Suppose a matrix M is non-singular, but that it has no further special properties. Such a matrix M can be diagonalized by unitary matrices S and T. Namely, the matrix M can always be written as the product of a hermitian matrix and a unitary matrix: $M = HV$. And a general theorem is that hermitian matrices can be diagonalized by unitary matrices. The proof of diagonalizing M goes as follows [18]. The matrix $MM^\dagger$ is hermitian and also positive. So this matrix can be diagonalized by a unitary matrix S.

$$S^\dagger M M^\dagger S = D^2 \tag{A.16}$$

With

$$D^2 = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix} \tag{A.17}$$

The matrix S has the eigenvectors of $MM^\dagger$ as its columns. Hermitian matrices have real eigenvalues and different eigenvalues have corresponding orthogonal eigenvectors. It is possible to find an orthonormal basis of $\mathcal{C}^n$ consisting only of eigenvectors. A unitary matrix rotates an orthonormal basis into another orthonormal basis.

The matrix S is determined up to a diagonal phase matrix. Namely:

$$(SF)^\dagger(MM^\dagger)(SF) = D^2 \tag{A.18}$$

Now define H and V such that:

$$H = SDS^\dagger \qquad V = H^{-1}M \tag{A.19}$$

Where D has the absolute values of the eigenvalues of $MM^\dagger$ on the diagonal. V is a unitary matrix, namely:

$$\begin{aligned} VV^\dagger &= H^{-1}MM^\dagger H^{-1} \\ &= H^{-1}(SD^2S^\dagger)H^{-1} \\ &= H^{-1}(SDS^\dagger)(SDS^\dagger)H^{-1} \\ &= H^{-1}HHH^{-1} = 1 \end{aligned}$$

We can now write:

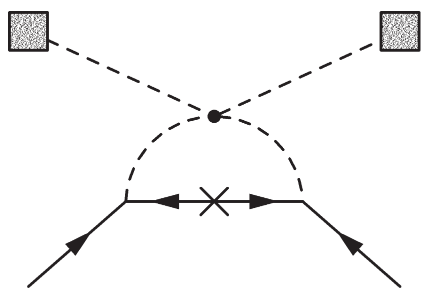
$$D = S^\dagger HS = S^\dagger MV^\dagger S = S^\dagger MT \tag{A.20}$$

Where $T = V^\dagger S$. So the matrix M can be diagonalized by unitary matrices $S^\dagger$ and T.

B

Neutrino Loop Diagram

Calculation of the diagram in figure (6.1):



$$
\begin{aligned}
i\frac{1}{\Lambda_\nu} &= (-i\lambda)(i\lambda_\nu)^2(-iM)\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p-k)^2-m^2}\frac{i}{(p-k)^2-m^2}\frac{i}{\not k - m_{n^c}}\frac{i}{(-\not k)-m_{n^c}} \\
&= -\lambda\lambda_\nu^2 M\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}\frac{1}{k^2-m^2}\frac{1}{k^2-(\lambda_\nu f)^2} \\
&= -\lambda\lambda_\nu^2 M\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)^2(k^2-(\lambda_\nu f)^2)}
\end{aligned}
$$

Where we have put the external momenta to zero, collected factors of i and put the terms in a common denominator. A standard method for evaluating this type of integral is by use of the following particular identity:

$$
\frac{1}{a_1^2 a_2\ldots a_n} = n!\int_0^1\frac{x_1\,\mathrm{d}x_1\,\mathrm{d}x_2\ldots\,\mathrm{d}x_n}{a_1x_1+a_2x_2\ldots a_nx^n}\;\delta[1-\sum_{i=1}^{n}x_i] \qquad \text{(B.1)}
$$

The x_i in this identity are known as Feynman parameters. More general forms of this identity exist but they are not needed here. Upon using the identity (B.1) we arrive at:

$$
\begin{aligned}
i\frac{1}{\Lambda_\nu} &= -\lambda\lambda_\nu^2 M \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)^2(k^2-(\lambda f)^2)} \\
&= -\lambda\lambda_\nu^2 M \int \frac{d^4k}{(2\pi)^4} \cdot 2\int_0^1 \frac{x\,\mathrm{d}x\,\mathrm{d}y}{[(k^2-m^2)x+(k^2-(\lambda f)^2)y]^3}\delta[1-(x+y)]
\end{aligned}
$$

Using the delta function results in:

$$
\begin{aligned}
i\frac{1}{\Lambda_\nu} &= -\lambda\lambda_\nu^2 M \int \frac{d^4k}{(2\pi)^4} \cdot 2\int_0^1 \frac{x\,\mathrm{d}x}{[(k^2-m^2)x+(1-x)(k^2-(\lambda f)^2)]^3} \\
&= -\lambda\lambda_\nu^2 M \int \frac{d^4k}{(2\pi)^4} \cdot 2\int_0^1 \frac{x\,\mathrm{d}x}{[k^2+x((\lambda f)^2-m^2)-(\lambda f)^2]^3} \\
&= -i\lambda\lambda_\nu^2 M \int \frac{d^4k_E}{(2\pi)^4} \cdot 2\int_0^1 \frac{x\,\mathrm{d}x}{[-k^2-x(m^2-(\lambda f)^2)-(\lambda f)^2]^3}
\end{aligned}
$$

Where in the last line we rotated to Euclidean space using a Wick rotation:

$$
k^0 \to ik_E^0 \tag{B.2}
$$

$$
\begin{aligned}
i\frac{1}{\Lambda_\nu} &= i\lambda\lambda_\nu^2 M \int \frac{d^4k_E}{(2\pi)^4} \cdot 2\int_0^1 \frac{x\,\mathrm{d}x}{[k^2+x(m^2-(\lambda f)^2)+(\lambda f)^2]^3} \\
&= \frac{i\lambda\lambda_\nu^2 M}{16\pi^2} \int_0^1 \frac{x\,\mathrm{d}x}{[x(m^2-(\lambda f)^2)+(\lambda f)^2]}
\end{aligned}
$$

In the last line the integration over k is performed using the integral representation formula for beta-functions in the following way:

$$
\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+a^2)^3} = \int_0^\infty \frac{dk}{(2\pi)^4} \frac{2\pi^2 k^3}{(k^2+a^2)^3} \tag{B.3}
$$

$$
= \int_0^\infty \frac{dk^2}{16\pi^2} \frac{k^2}{(k^2+a^2)^3} \tag{B.4}
$$

$$
\int_0^\infty \frac{t^{m-1}dt}{(t+a^2)^n} = \frac{1}{(a^2)^{n-m}} \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)} \tag{B.5}
$$

Where $\Gamma(x)$ is the gamma-function. Substituting the particular values relevant to us in the expression (B.5) then gives:

$$
\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+a^2)^3} = \frac{1}{32\pi^2 a^2} \tag{B.6}
$$

Continuing the calculation by also performing the integral over $x$results in:

$$\begin{aligned} i\frac{1}{\Lambda_\nu} &= \frac{i\lambda\lambda_\nu^2 M}{16\pi^2}[\frac{-(\lambda f)^2 + m^2 - (\lambda f)^2\log(\frac{m^2}{(\lambda f)^2})}{((\lambda f)^2 - m^2)^2}] \\ &= \frac{i\lambda\lambda_\nu^2 M}{16\pi^2}[\frac{-(\lambda f)^2 + m^2 - (\lambda f)^2\log(\frac{m^2}{\lambda^2 f^2})}{(\lambda f)^4(1 - \frac{m^2}{\lambda^2 f^2})^2}] \\ &= \frac{i\lambda M}{16\pi^2 f^2}[\frac{x - 1 - \log(x)}{(1-x)^2}] \end{aligned}$$

Where the quantity x is defined to be:

$$x = \frac{m_{Higgs}^2}{(\lambda_\nu f)^2} \tag{B.7}$$

Having gone through the calculation we find a final expression for $1/\Lambda_\nu$ which is:

$$\frac{1}{\Lambda_\nu} = \frac{\lambda M}{16\pi^2 f^2}[\frac{x - 1 - \log(x)}{(1-x)^2}] \tag{B.8}$$

C

Matrix Elements and Decay Rates

C.1 Matrix Element $d + W_\mu^+ \to U$

In this appendix we try to outline the calculation of the matrix element of the production of a heavy U quark through down-quark W_μ^+ fusion. The diagram for this process is shown in figure(C.1). The incoming d quark

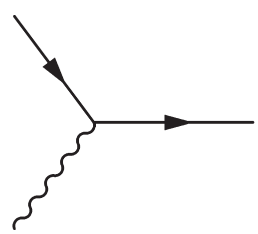


Fig. C.1: Diagram for U quark production

has momentum $\vec{q}$, the W boson has momentum $\vec{p}$ and the outgoing heavy U quark has momentum $\vec{k}$. Using the Feynman rules for this Little Higgs theory (ch.7) the amplitude for the process is:

$$i\mathcal{M} = \frac{ig\delta_U}{\sqrt{2}}\bar{u}(k)\gamma^\mu(\frac{1-\gamma^5}{2})u(q)\varepsilon_\mu(p) \tag{C.1}$$

What we want is the absolute square of this amplitude:

$$|\mathcal{M}|^2 = \frac{g^2\delta_U^2}{8}|\bar{u}(k)\gamma^\mu u(q) - \bar{u}(k)\gamma^\mu\gamma^5 u(q)|^2\varepsilon_\mu(p)\varepsilon_\nu^*(p) \tag{C.2}$$

Upon expanding the square we find four terms:

$$
\begin{aligned}
|\mathcal{M}|^2 = \frac{g^2\delta_U^2}{8} \ [\ & \bar{u}(k)\gamma^\mu u(q)\cdot\bar{u}(q)\gamma^\nu u(k) \\
& - \bar{u}(k)\gamma^\mu\gamma^5 u(q)\cdot\bar{u}(q)\gamma^\nu u(k) \\
& - \bar{u}(k)\gamma^\mu u(q)\cdot\bar{u}(q)\gamma^\nu\gamma^5 u(k) \\
& + \bar{u}(k)\gamma^\mu\gamma^5 u(q)\cdot\bar{u}(q)\gamma^\nu\gamma^5 u(k)] \ \varepsilon_\mu(p)\varepsilon^*_\nu(p)
\end{aligned} \tag{C.3}
$$

Note that complex conjugation of Dirac bilinears goes as follows:

$$
\begin{aligned}
(u(\bar{k})\gamma^\mu u(q))^* &= u^\dagger(q)\gamma^{\mu\dagger}\gamma^0 u(k) \\
&= u^\dagger(q)\gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0 u(k) \\
&= u^\dagger(q)\gamma^0\gamma^\mu u(k) \\
&= \bar{u}(q)\gamma^\mu u(k)
\end{aligned}
$$

The terms including γ^5 can be conjugated in the same way remembering that:

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{C.4}$$

If we assume that the initial state particles are not specifically polarized and that the final state particle polarization is irrelevant then we need to average over initial spin states and polarizations of the gauge boson and sum over final spin states. Consider the first term in (C.3). Summing over initial and final spin states gives:

$$
\begin{aligned}
\sum_{s,s'=1}^{2} \bar{u}_a^{s'}(k)\gamma^\mu_{ab}u_b^s(q)\cdot\bar{u}_c^s(q)\gamma^\nu_{cd}u_d^{s'}(k) &= \sum_{s,s'=1}^{2} u_d^{s'}(k)\bar{u}_a^{s'}(k) \ \gamma^\mu_{ab} \ u_b^s(q)\bar{u}_c^s(q) \ \gamma^\nu_{cd} \\
&= (\not{k} + m_U)_{da} \ \gamma^\mu_{ab} \ (\not{q} + m_d)_{bc} \ \gamma^\nu_{cd} \\
&= \mathrm{Tr}[(\not{k} + m_U)\gamma^\mu(\not{q} + m_d)\gamma^\nu]
\end{aligned}
$$

Where the spin sums satisfy the relation[2]:

$$\sum_{p=1}^{2} u^p(k)\bar{u}^p(k) = \not{k} + m \tag{C.5}$$

If we analyze the remaining three terms in the same way this gives for the second term:

$$\sum_{s,s'=1}^{2} \bar{u}_a^{s'}(k)(\gamma^\mu\gamma^5)_{ab}u_b^s(q)\cdot\bar{u}_c^s(q)\gamma^\nu_{cd}u_d^{s'}(k) = \mathrm{Tr}[(\not{k} + m_U)\gamma^\mu\gamma^5(\not{q} + m_d)\gamma^\nu] \tag{C.6}$$

Continuing along this line one finds:

$$
\begin{aligned}
|\mathcal{M}|^2 = \frac{g^2\delta_U^2}{8} \mathrm{Tr}\ [\ & (\not{k} + m_U)\gamma^\mu(\not{q} + m_d)\gamma^\nu \\
& - (\not{k} + m_U)\gamma^\mu\gamma^5(\not{q} + m_d)\gamma^\nu \\
& - (\not{k} + m_U)\gamma^\mu(\not{q} + m_d)\gamma^\nu\gamma^5 \\
& + (\not{k} + m_U)\gamma^\mu\gamma^5(\not{q} + m_d)\gamma^\nu\gamma^5]\ \varepsilon_\mu(p)\varepsilon_\nu^*(p)
\end{aligned}
\tag{C.7}
$$

In evaluating the resulting traces, some identities involving gamma matrices are useful. As a first, a trace involving an odd number of gamma matrices γ^μ equals zero. Further more:

$$
\begin{aligned}
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}) \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) &= 0 \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) &= -4i\epsilon^{\mu\nu\rho\sigma} \\
\{\gamma^5, \gamma^\mu\} &= 0
\end{aligned}
$$

Using these identities we can evaluate the traces. There are a total of sixteen terms, many of which are zero. There are terms involving γ^5 which are proportional to the totally anti-symmetric epsilon tensor. The sum over gauge boson polarization vectors however, is a symmetric quantity thus the result from these terms will be a zero contribution.

$$
\sum_{polarizations} \varepsilon_\mu(k)\varepsilon_\nu^*(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}
\tag{C.8}
$$

Evaluating the traces gives us an expression for $|M|^2$. Remembering to average over initial spin polarizations and gauge boson polarizations one finds:

$$
|\mathcal{M}|^2 = \frac{g^2\delta_U^2}{6}[k^\mu q^\nu + k^\nu q^\mu - g^{\mu\nu} k\cdot q]\cdot[-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_W^2}]
\tag{C.9}
$$

$$
= \frac{g^2\delta_U^2}{6}[k\cdot q + \frac{2(p\cdot k)(p\cdot q)}{m_W^2}]
\tag{C.10}
$$

In the las two equations we have (safely) neglected the rest mass of the d-quark ($\sim$ MeV), which is far lower than the masses of the W-boson and the heavy U-quark.

To find expressions for the dot products between the momentum vectors consider the following expressions:

$$
\begin{aligned}
(k - q)^2 &= k^2 + q^2 - 2k\cdot q \\
2k\cdot q &= -(k - q)^2 + k^2 + q^2 = m_U^2 + m_d^2 - m_W^2
\end{aligned}
$$

We then find for the dot products:

$$2k \cdot q \approx m_U^2 - m_W^2 \tag{C.11}$$

$$2p \cdot q \approx m_U^2 - m_W^2 \tag{C.12}$$

$$2p \cdot k \approx m_U^2 + m_W^2 \tag{C.13}$$

Where we have neglected the d mass. Inserting these expressions in (C.10) one obtains the squared matrix element for heavy U-quark production through d-quark W^μ fusion.

$$|\mathcal{M}|^2 = \frac{g^2\delta_U^2}{12}\;(m_U^2 - m_W^2 + \frac{(m_U^2 - m_W^2)(m_U^2 + m_W^2)}{m_W^2}) \tag{C.14}$$

We can express this in a clearer form:

$$|\mathcal{M}|^2_{dW_\mu^+\to U} = \frac{g^2\delta_U^2}{12}\;(\frac{m_U^4}{m_W^2})\;(1 - \frac{m_W^2}{m_U^2})\;(1 + 2\frac{m_W^2}{m_U^2}) \tag{C.15}$$

The squared matrix element for heavy U-quark production through u-quark Z_μ fusion is now easily determined to be:

$$|\mathcal{M}|^2_{uZ_\mu^0\to U} = \frac{g^2\delta_U^2}{24c_w^2}\;(\frac{m_U^4}{m_Z^2})\;(1 - \frac{m_Z^2}{m_U^2})\;(1 + 2\frac{m_Z^2}{m_U^2}) \tag{C.16}$$

C.2 U quark Decay Rate

The heavy U-quark predominantly decays into dW, uZ and uH, where H is the Higgs boson. Let us have a look at the decay rate for U into dW. The diagram for this process is shown in figure(C.2). This diagram looks very

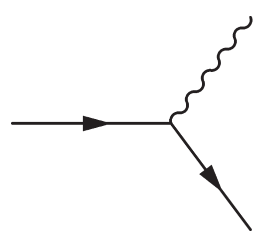


Fig. C.2: Diagram for U quark decay into Wu

much like the diagram for U production through dW fusion. The incoming

U quark has momentum $\vec{k}$, the d and W have momenta $\vec{q}$ and $\vec{p}$ respectively. The matrix element squared for the decay process is:

$$|\mathcal{M}|^2_{U\to dW^+_\mu} = \frac{g^2\delta_U^2}{4}\ (\frac{m_U^4}{m_W^2})\ (1-\frac{m_W^2}{m_U^2})\ (1+2\frac{m_W^2}{m_U^2}) \tag{C.17}$$

To find the decay width Γ we still have to multiply by phase space which, in the two particle final state case, has a simple form[2]:

$$\Gamma = \frac{1}{2m_U}\int \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{8\pi}(\frac{2|p|}{E_{cm}})\cdot|\mathcal{M}|^2_{U\to dW^+_\mu} \tag{C.18}$$

Where $|p|$ is the final state three-momentum magnitude of either particle in the center-of-mass frame, $p = -q$. Note that:

$$E_{cm} = m_U \tag{C.19}$$
$$2q\cdot p = m_U^2 - m_d^2 - m_W^2 \tag{C.20}$$
$$2k\cdot q = m_U^2 + m_d^2 - m_W^2 = 2E_{cm}E_d = 2m_U E_d \tag{C.21}$$
$$2k\cdot p = m_U^2 - m_d^2 + m_W^2 = 2E_{cm}E_W = 2m_U E_W \tag{C.22}$$

One can then find from (C.21), again neglecting the d-quark rest mass:

$$\frac{2|p|}{E_{cm}} = 1-\frac{m_W^2}{m_U^2} \tag{C.23}$$

So that the decay width becomes:

$$\Gamma_{U\to dW^+_\mu} = \frac{g^2\delta_U^2}{64\pi}\ (\frac{m_U^3}{m_W^2})\ (1-\frac{m_W^2}{m_U^2})^2\ (1+2\frac{m_W^2}{m_U^2}) \tag{C.24}$$

and

$$\Gamma_{U\to uZ^0_\mu} = \frac{g^2\delta_U^2}{128\pi c_w^2}\ (\frac{m_U^3}{m_Z^2})\ (1-\frac{m_Z^2}{m_U^2})^2\ (1+2\frac{m_Z^2}{m_U^2}) \tag{C.25}$$

The heavy U-quark can also decay into a u-quark and a Higgs boson.

The decay rate is approximately equal to the case of decay into a Z^0_μ boson, by the Goldstone boson equivalence theorem. This theorem states that[2], at high energy, meaning large M_U relative to the gauge boson mass in our case, the amplitude for emission or absorption of a (longitudinally polarized) massive gauge boson becomes equal to the amplitude for emission or absorption of the associated eaten goldstone boson. So there is an equal decay into the four degrees of freedom of the (complex) Higgs doublet. The longitudinal polarizations of the massive gauge bosons and the physical Higgs boson. The polarization vector for the gauge bosons is ϵ_μ which in the high

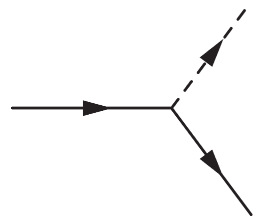


Fig. C.3: Diagram for U quark decay into Wh

energy limit becomes $k_\mu/M + \mathcal{O}(m/E_k)$ (longitudinal). To replace gauge bosons with their longitudinal mode, and thus the goldstone boson, they must be external, and they must be in the high energy limit. One cannot use this theorem for loop diagrams since then one integrates over the momentum, and in some region the gauge boson will have momentum less than its mass, and there the approximation is incorrect.

So we find:

$$\frac{1}{2}\Gamma_{U\to dW_\mu^+} \approx \Gamma_{U\to uZ_\mu^0} \approx \Gamma_{U\to uH} \approx \frac{\lambda_U^2}{32\pi} m_U \approx 9.9\cdot 10^{-3}\lambda_U^2\ m_U \qquad \text{(C.26)}$$

Where we have substituted in the $\Gamma_{U\to dW_\mu^+}$ expression expressions for δ_U, m_U and m_W:

$$\delta_U^2 = \frac{v^2}{2f^2 t_\phi^2} \qquad \text{(C.27)}$$

$$m_U^2 = s_\phi^2 \lambda_U^2 f^2 \qquad \text{(C.28)}$$

$$m_W^2 = \frac{1}{4} g^2 v^2 \qquad \text{(C.29)}$$

D

Quadratically divergent diagrams in the Standard Model

In this appendix we will present a calculation of the (most) troublesome quadratically divergent diagrams occurring in the standard model. Let us begin with the top quark loop.

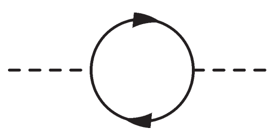


Fig. D.1: Quadr. div. due to the Top quark

$$
\begin{aligned}
-iM^2_{top} &= -\int \frac{\mathrm{d}^4\mathrm{p}}{(2\pi)^4}(\frac{-i\lambda_t}{\sqrt{2}})^2\,\mathrm{Tr}[\frac{i\ \not{p}}{p^2}\ \ \frac{i\ \not{p}}{p^2}]\cdot 3 \\
&= -\frac{3}{2}\int \frac{\mathrm{d}^4\mathrm{p}}{(2\pi)^4}\ \lambda_t^2\,\mathrm{Tr}[\frac{p_\mu p_\nu \gamma^\mu\gamma^\nu}{p^4}] \\
&= -\frac{3}{2}\int \frac{\mathrm{d}^4\mathrm{p}}{(2\pi)^4}\ \lambda_t^2\ \frac{4p^2}{p^4} \\
&= -\frac{3}{2}\int \frac{\mathrm{dp}}{(2\pi)^4}\ 2\pi^2 p^3\ \lambda_t^2\ \frac{4}{p^2}\cdot(-i) \\
&= -\frac{3}{8\pi^2}\lambda_t^2\Lambda^2\cdot(-i)
\end{aligned}
$$

The factor of three in the first line comes from the fact that the top quark comes in three colors. In the fourth line we have performed a Wick rotation to Euclidean spacetime and continued to evaluate the integral in spherical coordinates. We neglected the masses of the intermediate particles because they play no role in the dominant divergence of the diagram. We shall now proceed to the second diagram which involves the gauge bosons of the standard model. The charged W's are running in the loop.

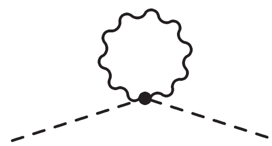


Fig. D.2: Quadr. div. due to gauge bosons

$$\begin{aligned} -iM^2 &= 2 \cdot \int \frac{\mathrm{d}^4 p}{(2\pi)^4} (ig_{\mu\nu}) \frac{1}{4} g^2 \frac{-i(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2})}{p^2} \\ &= 2 \cdot \frac{3}{4} g^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2} \\ &= 2 \cdot \frac{3}{4} g^2 \int \frac{\mathrm{d}p}{16\pi^4} \, 2\pi^2 p \, (-i) \\ &= 2 \cdot \frac{3}{64\pi^2} g^2 \Lambda^2 (-i) \end{aligned}$$

Where in the second line the factor of three comes from the trace of the gauge boson propagator in the Landau gauge. If we also include the contribution of the Z^0 boson we find a total contribution of:

$$-iM^2_{gauge} = +\frac{9}{64\pi^2} g^2 \Lambda^2 (-i) \tag{D.1}$$

The last contribution comes from the Higgs self-interaction:

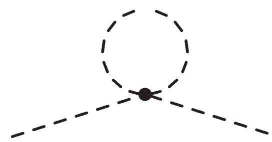


Fig. D.3: Quadr. div. due to the Higgs self-interaction

$$
\begin{aligned}
-iM^2_{self\ int.} &= -i\frac{\lambda}{4}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \frac{i}{p^2}\cdot 4!\cdot\frac{1}{2} \\
&= 3\lambda\int \mathrm{d}p\ \frac{2\pi^2}{16\pi^4}\ p\ (-i) \\
&= \frac{3}{16\pi^2}\lambda\ \Lambda^2(-i)
\end{aligned}
$$

E

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