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Master Thesis

The vacuum structure of $SO(p, q)$ gauged maximal supergravity in four dimensions

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Contents

Chapter 1

Introduction

In this thesis we will investigate the vacuum structure of the one-parameter family of $SO(p,q)$ $(p+q=8)$ gaugings of the maximal supergravity theory in four dimensions [1]. But first let us swiftly introduce the terms and concepts needed to understand this particular problem and also try to motivate the reason why we are looking at it.

The first thing one needs to know about is supersymmetry [2][3]. Supersymmetries are symmetries not described by Lie groups, but rather by Lie supergroups. They were discovered in an effort to combine space-time symmetries with internal symmetries. The interesting thing about supersymmetry is that it is a symmetry relating bosons to fermions, something that cannot be done via conventional symmetries. Another feature is that it usually enhances the UV behavior of quantum theories due to very specific cancelling between bosonic and fermionic terms, which are now related via supersymmetry. The first constructed supersymmetric theories all had global supersymmetry. However, none of these theories include gravity. Physicist, in their pursuite of a quantum theory of gravity, then tried to combine supersymmetry and gravity to see whether such theories could be constructed and possibly even finite. Now, if one were to add gravity to a global supersymmetric theory one would come to the conclusion that one has to promote the global supersymmetries to local ones. Vice-versa any locally supersymmetric theory would have to include gravity. This has to do with the fact that local supersymmetry means local space-time symmetries, which should be viewed as diffeomorphism invariance. Theories with local supersymmetry are therefor usually referred to as supergravity theories [4][6]. It was hoped that supergravity theories would be renormalizable due to their enhanced symmetries and thus constitute renormalizable theories of gravity. This turned out to be somewhat optimistic. However, the maximally supersymmetric theory in four dimensions still stands as a possible UV finite theory [7]. This is the theory we will be looking at in this thesis.

A general property of maximal supergravity theories in D dimensions, is that they can be written down in different inequivalent ways [5]. Amongst the fields present in such theories are different types of p-forms, depending on the theory at hand. These fields enter the Lagrangian via their $(p+1)$-form field strengths. One must then supplement the equations of motion with Bianchi identities for the field strengths. What one can now do is exchange the roles of the equations of motion and the Bianchi identities. One can view the Bianchi identities as equations of motion following from some theory with $(D-p-2)$-forms, and view the equations of motion as the Bianchi identities corresponding to the $(D-p-1)$-field strengths. Thus one can describe a theory using p-forms, or their duals, $(D-p-2)$-forms. Although the off-shell formulations are fundamentally different, on-shell they are equivalent. The maximal supergravity theory in four dimensions contains, amongst other fields, vector fields. Now, in four dimensions vector fields are dual to vector fields. The ones entering the Lagrangian are called the electric fields, whilst their duals are dubbed magnetic, and the concept of duality is in this case called electric-magnetic duality. Now, the thing is, that instead of merely exchanging the roles of the equations of motion and Bianchi identities, one can in this case actually rotate them amongst eachother. This basically means that instead of two formulations one can write down infinitely many Lagrangians, each with different combinations of electric and magnetic fields as the fundamental fields entering the Lagrangian [6]. These different formulations are said to describe the theory in different symplectic frames. Their equations of motion are all equivalent and exhibit a large duality group rotating the equation of motions and Bianchi identities. Different Lagrangians in general have different subgroups of this duality group as their symmetry group (which always rotate the electric fields amongst themselves).

Altough rather complicated, supergravities do lack some interesting features. For instance, they do not have a scalar potential. This means that the vacuum structure is trivial and hence one is severely constrained in generating interesting dynamics. In order to remedy this one can gauge supergravity theories with respect to some of their global symmetries. Possible gaugings can be conveniently described in a group theoretic manner via what is called the embedding tensor, which encodes how the different vector fields couple to the generators of the group one wishes to gauge. When one tries to gauge the maximal supergravity theory one encounters some complications. Firstly, different symplectic frames have different symmetry groups with thus different possible subgroups one can gauge. One would like to have some frame independent formulation. This can be done by also allowing the dual magnetic fields to couple to the gauge group. It is then in principle possible to, regardless of the chosen frame, gauge any subgroup of the larger duality group of the theory. However, one then stumbles upon questions like: Are two gaugings of the same subgroup but done in different frames equivalent? Are two gaugings of the same subgroup, but using different sets of electric and magnetic vector fields, equivalent? In an effort address these questions, the concept of so called symplectic deformations has recently been introduced, which tries to give a purely group theoretical answer.

To examine the dynamics of supergravity theories a lot of research has been done towards their vacuum structure. Also the maximal theory in four dimensions has been examined, and then especially $SO(p,q)$ gaugings [27]. These gaugings were originally performed purely electrically in some standard frame in the 80s. Many different vacua with varying properties have been found. Most turned out to be AdS vacua. De Sitter vacua, phenomologically the most interesting, were also found. However, they were all unstable. In fact many so called no-go theorems have been proven that various circumstances under which stable dS vacua are shown not to exist in supergravity theories [8]. Certainly when theories have higher dimensional origins, i.e. they can be viewed as certain geometric compacitifcation of higher dimensional supergravity theories lack such dS vacua. The original gaugings are known to have higher dimensional origins. Only recently it was discovered that there is in fact a one parameter family of inequivalent $SO(p,q)$ gaugings that can be obtained by either choosing a non standard frame or also using magnetic fields in the gauge process. The parameter of these gaugings does not seem to have a geometrical higher dimensional origin. Therefor it could be possible that stable dS vacua can be obtained in these theories. As of yet no stable dS have been found, but new vacua with interesting properties that have no counterpart in the original theories have. In this thesis we will try to find novel vacua in sectors that have not yet been examined.

1.1 Outline

The structure of this thesis is as follows. In Part 1 we will introduce the basic concepts, starting with global supersymmetry in chapter 2, followed by the descritption of (half-)maximal supergravity theories in chapter 3. The last chapter of the first part is devoted to the description of gauged supergravities. In Part II we will focus on the maximal supergravity in four space-time dimensions, starting off with the ungauged theory in chapter 5. We then discuss the gauging of the theory in chapter 6. In the final part, part III, we will specifically look at the $SO(p, q)$ gaugings, introducing them in chapter 7, and describe their vacuum structure in chapter 8.

Part I

Introducing Supergravity theories

Chapter 2

Global Supersymmetry

Theorists have been constructing field theories with all kinds of symmetries. The types of symmetries basically fell into two classes: space-time symmetries pertaining to the structure of space-time, and internal symmetries rotating different fields amongst eachother in some abstract field space. The total symmetry group of a theory was always the direct product of the space-time part and the internal part; i.e. they were truly unrelated to eachother. Eeventually people tried to construct theories in which the symmetry group is not such a direct product, but rather of a more complicated form encompassing both types of symmetries. In such cases the product of two internal symmetries could f.e. combine into a pure space-time symmetry and vice-versa. However, Coleman and Mandula proved that such a construction could not be done. They showed that in a relativistic quantum field theory space-time and internal symmetries can only be combined in a trivial way (i.e. in a direct product fashion). If this is not the case the S-matrix of the theory is trivial rendering the theory trivial. Now, of course, as all theorems, it is merely as strong as its assumptions. One such assumption is that the theory is described by an S-matrix, which need not be the case. In theories with conformal symmetry the S-matrix is ill-defined. We will not take this route. Rather we focus on the assumption that the symmetries of the theory are to be described by Lie groups, and their corresponding Lie algebras. However, it was soon realised that one can also look at symmetries described by what are called Lie supergroups and their corresponding Lie superalgebras. Such symmetries are, not surprisingly, called supersymmetries and do allow for a nontrivial combination of internal and space-time symmetries.

In this chapter we shall review the basics concerning such (global) supersymmetries: we introduce Lie superalgebras, their particle representations and finally comment on their field representations. The content of this chapter is based largely on [2] and [3], and can be found in any book on supersymmetry.

2.1 The algebras

Supersymmetries are described by Lie superalgebras. These are algebras in which the basic objects come in two flavors, which are often dubbed odd and even or, in physics, often bosonic and fermionic. Depending on whether an object is bosonic or fermionic it either satisfies commutation or anticommutation relations. We capture this as follows by introducing the bracket:

$$[O_a, O_b\} = O_a O_b - (-1)^{\sigma_a \sigma_b} O_b O_a = i f_{ab}^c O_c \tag{2.1}$$

Where σ_i is 0 for bosonic and 1 for fermionic objects. In addition there is a generalized Jacobi identity which has to be satisfied:

$$[O_a, [O_b, O_c\}\} = [[O_a, O_b\}, O_c\} + (-1)^{\sigma_a \sigma_b} [O_b, [O_a, O_c\}\} \tag{2.2}$$

By using the properties of the bracket and the generalized Jacobi identity one finds that there is always the following structure:

$$[B, B\} = [B, B] \propto B \tag{2.3}$$
$$[B, F\} = [B, F\} \propto F \tag{2.4}$$
$$[F, F\} = \{F, F\} \propto B \tag{2.5}$$

Where we let B denote generic bosonic operators and F fermionic.

What we would now like to do is find some superalgebra that contains the Poincare algebra in its bosonic part. I.e. we would like to obtain a supersymmetric extension of the Poincare algebra; we thus look at theories on four dimensional Minkowski space. One could also look at the higher dimensional Minkowski spaces, or entirely other background spaces with their corresponding isometry groups and supersymmetrically extend these to obtain supersymmetric theories in these background spaces. However, we will not do so in this thesis as the main theory we will look at is four dimensional.

2.1.1 Poincare

Let us first swiftly review the structure of the Poincare algebra. Te Poincare group is the isometry group of Minkowski space. The corresponding algebra is the Poincare algebra which is generated by the generators P^μ corresponding to translations, and the generators $M^{\mu\nu}$ generating boosts and rotations. Here $\mu, \nu = 0, ..., 3$ are space-time indices. The generators satisfy the following commutation relations:

$$[P_\mu, P_\nu] = 0 \tag{2.6}$$
$$[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu) \tag{2.7}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\mu\rho}\eta_{\nu\sigma} - M_{\nu\sigma}\eta_{\mu\rho}) \tag{2.8}$$

2.1.2 Simple Supersymmetry

A natural starting point would be to add a single fermionic generator Q_α to the Poincare algebra. Here $\alpha = 1, 2$ is a space-time spinor index labeling the components of the generator. (One immediatly realises that the structure of supersymmetry could depend very much on the dimension of Minkowski space-time, since spinor content varies. However, as noted earlier, we will restrict ourselves to four dimensions.) The only way such a fermionic generator can be added in a consistent manner is if it satisfies the following (anti-)commutation relations:

$$\{Q_\alpha, \bar{Q}_{\ddot{\alpha}}\} = 2\sigma^\mu_{\alpha\ddot{\alpha}} P_\mu \tag{2.9}$$

$$[Q_\alpha, P_\mu] = [\bar{Q}^{\ddot{\alpha}}, P_\mu] = 0 \tag{2.10}$$

$$\{Q_\alpha, Q_\beta\} = 0 \tag{2.11}$$

$$[Q_\alpha, M_{\mu\nu}] = i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta \tag{2.12}$$

$$[\bar{Q}^{\dot{\alpha}}, M_{\mu\nu}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} \tag{2.13}$$

Here σ^i are the familiar Pauli matrices, $\sigma^0 = \mathbb{1}$, and $\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$.

We have already noted that supersymmetry can in princple intertwine non trivially with internal symmetries. Let us therefor also consider some internal symmetries generated by T_a. The commutator $[Q_\alpha, T_a]$ generically vanishes though. Only in the case that the T_a generate an automorphism of the supersymmetric part of the algebra are they non trivial. The automorphism group varies with the amount of supersymmetry and also with the dimensions of space-time due to the different nature of spinors. In the case of four dimensions and one fermionic generator the automorphism group is $U(1)$:

$$Q_\alpha \mapsto e^{i\theta} Q_\alpha \tag{2.14}$$

$$\bar{Q}_{\ddot{\beta}} \mapsto e^{-i\theta} \bar{Q}_{\ddot{\beta}} \tag{2.15}$$

Thus if one of the generators of the internal symmetries generates this automorphism, lets call it R, it must satisfy:

$$[Q_\alpha, R] = Q_\alpha \tag{2.16}$$

$$[\bar{Q}_{\ddot{\alpha}}, R] = -\bar{Q}_{\ddot{\alpha}} \tag{2.17}$$

In all other cases the commutator must vanish between internal and supersymmetry generators.

2.1.3 Extended Supersymmetry

Instead of adding one fermionic generator one can add multiple. Let us denote them by Q^I_α. Here $I = 1, ..., N$ simply labels the number of such generators. One can then show that the most general structure one can obtain is the following:

$$\{Q^I_\alpha, \bar{Q}^J_{\ddot{\alpha}}\} = 2\sigma^\mu_{\alpha\ddot{\alpha}} P_\mu \delta^{IJ} \tag{2.18}$$

$$[Q^I_\alpha, P_\mu] = [\bar{Q}^{I\ddot{\alpha}}, P_\mu] = 0 \tag{2.19}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 0 \tag{2.20}$$

$$[Q^I_\alpha, M_{\mu\nu}] = i(\sigma_{\mu\nu})_\alpha^\beta Q^I_\beta \tag{2.21}$$

$$[\bar{Q}^{I\dot{\alpha}}, M_{\mu\nu}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}} \tag{2.22}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \tag{2.23}$$

$$\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^* \tag{2.24}$$

Here Z^{IJ} is antisymmetric and commutes with all the other generators present. They are called the central charges. Again, one can consider the effect of internal symmetries on the supersymmetry generators. One finds that again only internal symmetries that generate automorphisms of the supersymmetric part of the algebra can have non-vanishing commutators. In four dimensions, if there are no central charges, i.e. $Z^{IJ} = 0$, then the automorphism group is $U(N)$. If there are central charges it is reduced to a subgroup of $U(N)$ depending on the nature of the central charges.

2.2 Particle Representations

Having described the supersymmetry algebra we now turn to its simplest representations, the particle representations. The ones most easily obtained that are relevant to us are the single particle representations. Before discussing representations of the supersymmetry algebras, we swiftly review how one can obtain those of the Poincare algebra.

2.2.1 Poincare

Identifying the representations starts by constructing the Casimir operators, which are operators which commute with all elements of the Poincare algebra. Since they commute with all elements, all states in a representation have the same eigenvalues under the Casimir operators. We can thus classify the different representations according to their eigenvalues under these operators. The Poincare algebra has two Casimir operators: $C_1 = P^\mu P_\mu$ which labels the mass m, and $C_2 = W^\mu W_\mu$ (where $W^\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ is the Pauli Ljubanski vector) which labels the spin s. So for the Poincare algebra representations are labeled by mass and spin.

Next one wishes to see what the state content of the representations is, i.e. what are the quantities labeling the different states within a representation. Since in this thesis we only deal with massless theories, we will restrict ourselves to the massless representations.

One label amongst states is of course momentum, i.e. the eigenvalue under P_μ. The massless representations are most easily analyzed by choosing a frame in the direction of the momentum, such that the eigenvalue of the state under P_μ is $p_\mu = (E, 0, 0, E)$. One sees that an $SO(2)$ subgroup, called the little group, rotating the orthogonal spatial dimensions leaves it invariant. One then finds that $W^\mu = \lambda P^\mu$. Eigenstates of P_μ are thus also eigenstates of W_μ and we can use λ as an additional label. It is called the helicity. We thus find that the states are labelled as $|p^\mu, \lambda>$. Acting with CPT one obtains:

$$e^{2\pi i\lambda}|p^\mu, \lambda >= \pm|p^\mu, \lambda > \tag{2.25}$$

One concludes that λ should be integer or half integer (as is familiar). Thus every massless representation consists of states labelled by momentum (satisfying $p_\mu p^\mu = 0$) and (half-)integer helicity.

2.2.2 Supersymmetry

For the supersymmetry algebras one follows a similar procedure. C_1 is still a Casimir operator, but C_2 is not. Spin is thus no longer a label for representations; i.e. representations contain states with different spins. One can define a new second Casimir, whose specific form is not relevant for our discussion. Since C_1 is still a Casimir operator all states within a multiplet have the same mass.

Again we will look at massless representations. We will consider the algebra with N generators Q_I and no central charges. Consider a massless state with P_μ-eigenvalue $p_\mu = (E, 0, 0, E)$. Plugging this in the defining commutation relation yields:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix} \delta^{IJ} \tag{2.26}$$

Implying $Q^I_1 = \bar{Q}^I_{\dot{1}} = 0$. So only half of the spinor components are non-trivially realized. One can now define creation and annihilation operators:

$$a_I = \frac{Q^I_2}{2\sqrt{E}}, \qquad a^\dagger_I = \frac{\bar{Q}^I_{\dot{2}}}{2\sqrt{E}} \tag{2.27}$$

These satisfy the standard commutation relations:

$$\{a_I, a^\dagger_J\} = \delta_{IJ}, \qquad \{a_I, a_J\} = \{a^\dagger_I, a^\dagger_J\} = 0 \tag{2.28}$$

From the commutators,

$$[M_{12}, Q^I_2] = -\frac{1}{2} Q^I_2, \qquad [M_{12}, \bar{Q}^I_{\dot{2}}] = \frac{1}{2} \bar{Q}^I_{\dot{2}} \tag{2.29}$$

we see that the supersymmetry generators raise and lower the helicity by one half.

One can now build up a representation by starting with some state which we define to have the lowest helicity in the representation. Let us denote this state by $|\lambda>$ (again it must be integer or half-integer). If it is to have the lowest helicity it must thus get annihilated by all a_I, i.e:

$$a_I |\lambda >= 0 \tag{2.30}$$

By repeatedly acting on $|\lambda>$ with the creation operators we obtain new states with higher helicities:

$$|\lambda + \frac{1}{2} >_I = a^\dagger_I |\lambda > \tag{2.31}$$

$$|\lambda + 1 >_{IJ} = a^\dagger_I a^\dagger_J |\lambda > \tag{2.32}$$

$$\dots \tag{2.33}$$

$$|\lambda + \frac{N}{2} >_{1\dots N} = a^\dagger_1 \dots a^\dagger_N |\lambda > \tag{2.34}$$

By taking into account that the operators anticommute we observe that there are $\binom{N}{k}$ states with helicity $\lambda + \frac{k}{2}$. In total we thus find that the total number of states is:

$$\sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} = 2^N \tag{2.35}$$

One also observes that the number of fermionic states is equal to the number of bosonic states, i.e:

$$n_B = n_F = 2^{N-1} \tag{2.36}$$

This is a general property of any representation. One can see this by introducing the operator $(-)^F$ which acts as:

$$(-)^F|B> = |B> \tag{2.37}$$

$$(-)^F|F> = -|F> \tag{2.38}$$

One can show that its trace vanishes. However one also swiftly obtains:

$$\mathrm{Tr}\{(-)^F\} = \sum_{bosons} < B|(-)^F|B> + \sum_{fermions} < F|(-)^F|F> \tag{2.39}$$

$$= \sum_{bosons} < B|B> - \sum_{fermions} < F|F> = n_B - n_F \tag{2.40}$$

Therefor the number of bosonic states is equal to the number of fermionic states.

Now, one can also immediately see that if one looks at a theory without gravity, the maximal amount of supersymmetry is $N = 4$, for else all representations contain states with spin 3/2 or spin 2. However, upon adding gravity one could extend it to $N = 8$. Going beyond $N = 8$ would imply states with spins larger than 2, for which we do not know how to write down suitable theories. Let us note that all these specifics about the amount of supersymmetry depends on the number of space-time dimensions one considers. This because the spinorial representations get larger as one goes up in dimensions which alters the way one counts the amount of supersymmetry. However, analogous statements can of course be made in other dimensions.

2.3 Supersymmetric Field Theories

Having constructed the particle representations is one thing, but of course we would like to describe our particles via a quantum field theory. We thus need to construct field representations of supersymmetry, such that after quantizing the fields, they give rise to states transforming under the particle representations as described above. After finding actual field representations one is faced with the problem of writing down an action which is invariant under supersymmetry transformations. We will not go into the different techniques. Let us simply show how one can construct field representations.

2.3.1 Field representations

Constructing field representations begins in a manner similar to that of particle representations. What one does is pick a field, which in principle could have an arbitrary space-time index structure (i.e. any spin), and demand that it is the lowest spin field of the representation. Then by repeatedly acting on the field with the supersymmetry generators, one generates more and more fields. Then eventually this process terminates due to the finite amount of supersymmetry.

To illustrate this consider a scalar field $\phi(x)$. The fact that it has lowest spin implies that it should get annihilated by Q, i.e. $[\bar{Q}_{\ddot{\alpha}}, \phi(x)] = 0$. Now, one can immediately see that $\phi(x)$ should be complex. If it were not we would get:

$$0 = [P_\mu, \phi(x)] \propto \partial_\mu \phi(x) \tag{2.41}$$

I.e. in this case $\phi(x)$ would actually be a constant and not so much a space-time dependent field. Now, the most general thing we can then write down is:

$$[Q_\alpha, \phi(x)] = \psi_\alpha(x) \tag{2.42}$$

This basically defines our new field $\psi_\alpha(x)$ which has a spinor index. Now, we continue and try to generate more fields:

$$\{Q_\alpha, \psi_\beta(x)\} = F_{\alpha\beta}(x) \tag{2.43}$$
$$\{\bar{Q}_{\dot{\alpha}}, \psi_\beta(x)\} = X_{\dot{\alpha}\beta}(x) \tag{2.44}$$

Using the generalized Jacobi identity on $(\phi, Q, \bar{Q})$ leads one to conclude that:

$$X_{\dot{\alpha}\beta}(x) \propto \partial_\mu \phi(x) \tag{2.45}$$

I.e. X is not independent. Now using Jacobi on (ϕ, Q, Q) yields:

$$F_{\alpha\beta}(x) = \varepsilon_{\alpha\beta} F(x) \tag{2.46}$$

I.e. we find another scalar. Now we can act on this new scalar with the supersymmetry generators and obtain:

$$[Q_\alpha, F] = \lambda_\alpha \tag{2.47}$$
$$[\bar{Q}_{\dot{\alpha}}, F] = \bar{\chi}_{\dot{\alpha}} \tag{2.48}$$

Again, using generalized Jacobi identities one finds that λ_α vanishes and $\bar{\chi}$ is proportional to derivatives of ψ. Therefor they are not new fields. Thus we find that the total field content of the representation is (ϕ, ψ, F).

This procedure can be extended to extended supersymmetry and one can of course start with a different lowest spin field. In this way one can build up all kinds of field multiplets.

Chapter 3

Supergravity

In the previous chapter we have discussed some aspects of theories with global supersymmetry. Apart from global supersymmetric theories, one can also construct theories that have local supersymmetry. One finds that one must introduce several spin-3/2 fields ψ_μ (called gravitinos) as gauge fields for the supersymmetries. Also from the anti-commutator of two supertransformations $\{Q, Q\} \propto P$, one finds that also the translation is local. Such local translations should be viewed as diffeomorphisms. One thus finds that a theory with local supersymmetry, necessarily also has diffeomorphism invariance. I.e. gravity is always included, hence one speaks of local supersymmetric theories as supergravities. The converse is also true: in any theory with gravity and supersymmetry, the supersymmetry should be realized locally.

Given our interest in the maximal supergravity theory in four dimensions, we will give an overview of some key aspects of a special class of supergravity theories, namely the (half-)maximal ones. These are the theories that exhibit the maximal amount (or half that) of supersymmetry in a given space-time dimension. These theories have a lot in common, and we can discuss their general structure simultaneously. We will focus on the structures describing the different fields present in theories. First, we will comment on the Lagrangians describing these theories. We will then go into the different sectors in detail: scalars and coset manifold models, vectors and p-forms and duality, and finally something about fermions.

For a general introduction to supergravity theories in general, we refer to the excellent book [4].

3.1 Field content and Lagrangian

The field content of a supergravity theory of course depends on the case at hand, but generally we could say the following. The bosonic field content consists of the metric $g_{\mu\nu}$, a collection of scalar fields ϕ^i, vector fields A_μ^M and possibly also higher-rank antisymmetric p-forms $B^I_{\nu_1 \ldots \nu_p}$. Of course these bosonic fields come with a bunch of fermionic fields as demanded by supersymmetry which in general are spin-1/2 fields χ and spin-3/2 Rarita-Schwinger fields ψ_μ (where we suppressed spinor indices). All fields are massless.

In general the Lagriangian will have the following structure: [5]

$$\mathcal{L} = -\frac{1}{2}R - \frac{1}{2}G_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4}\mathcal{M}_{MN}(\phi)F^M_{\mu\nu}F^{\mu\nu N} - \mathcal{I}(\phi,\chi,\psi)^{\mu\nu}_M F^M_{\mu\nu} + ... \tag{3.1}$$

Here the dots stand for kinetic terms for the higher-rank p-forms and also possible topological terms (i.e. terms that do not involve the metric). The matrices G_{ij}, $\mathcal{M}_{MN}$ and $\mathcal{I}^{\mu\nu}_M$ depend on the scalars and the last also includes fermion bilinears. This general form essentially follows from demanding diffeomorphism invariance. In general, supersymmetry highly constraints the objects G_{ij}, $\mathcal{M}_{MN}$, $\mathcal{I}^{\mu\nu}_M$ (and also the ones corresponding to the higher rank p-forms) and their specific structure varies among theories.

What we have said so far holds for general supergravity theories, but from now on we will restrict our attention to (half)-maximal supergravity theories. In these theories the coupling matrices are further restricted by the existence of a global symmetry group G under which the different fields transform. In general the p-forms transform under linear representations of G, whilst the scalar fields transform non-linearly. The fermions transform under representations of a local maximal compact subgroup $H \subset G$. We will start off by reviewing how a non-linear realization of G on the scalars is achieved.

3.2 Scalar sector

The scalars of all (half-)maximal supergravity theories are described by non-linear sigma models with coset manifolds as target spaces. In this section we will review general properties of such models, starting of with non-linear sigma models in general. This section is mostly based on [6]. Another usefull reference is [9].

3.2.1 Non-linear sigma models

Consider a collection of massless scalar fields, i.e. $\phi = (\phi^i)$. Then the simplest non-interacting theory one can write down simply contains the standard kinetic terms, i.e:

$$\mathcal{L} = \frac{1}{2}\delta_{ij}\partial_\mu\phi^i\partial^\mu\phi^j \tag{3.2}$$

However, one can also write down more general kinetic terms such that different fields get coupled which give the scalar sector a lot more interesting structure. To introduce this we take the viewpoint that the scalar fields are actually coordinates on some abstract space T, called the target space. The dimension of the target space is equal to the number of scalars. Applied to the just described simplest theory one observes that indeed one can view the scalars as coordinates in some euclidean space $T = \mathbb{R}^n$, with the standard euclidean metric $G_{ij}(\phi) = \delta_{ij}$. Now, one can generalize this rather straightforwardly by also consider a more complicated target space T, with a more complicated metric $G_{ij}(\phi)$. The Lagrangian describing such a theory is given by:

$$\mathcal{L}(\phi) = \frac{1}{2}G_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \tag{3.3}$$

I.e. the action is then given by:

$$S = \frac{1}{2}\int_M dx\sqrt{g}g^{\mu\nu}(x)G_{ij}(\phi(x))\partial_\mu\phi^i\partial_\nu\phi^j \tag{3.4}$$

Where $g_{\mu\nu}$ is the metric of the background space-time. As one can see such models are in general non-linear due to the fact that one does not have standard kinetic terms of the scalars, but kinetic terms with scalar dependent coefficients as captured by the metric of the target space. Such models are called a non-linear sigma models.

3.2.2 Coset groups

We will now turn our attention to a subset of non-linear sigma models, namely the coset space models. In these models the target space is of a very specific form: it is a coset manifold. Let us review some basics about coset groups.

Given a group G and a subgroup H one can partition the group G into a number of left cosets w.r.t. H. (One could also introduce right cosets but it is customary to use left cosets.) A left coset is a set S of group elements of G, for which there exists some $g \in G$ such that $S = gH = \{gh : h \in H\}$. One can easily see that $g_1H = g_2H$ iff $g_1^{-1}g_2 \in H$, otherwise $g_1H \cap g_2H = \emptyset$. Thus G is the union of a number of such disjunct cosets, i.e. if $g \in G$ then g is an element of precisely one coset. The set of all distinct cosets, denoted G/H, comes with a natural group structure defined via $(g_1H)(g_2H) = g_1g_2H$. One can define a natural left action of G on G/H via $g*(g_1H) = gg_1H$. One can also freely act with H from the right as $h*(g_1H) = (g_1H)h = g_1H$, mapping the coset to itself (since $Hh = H$).

Now, there are two ways in which one can describe coset groups. Firstly one can simply view it as an abstract group with a certain structure, without making reference to the fact that it is a coset group. It is not surprising that this is not the most convenient description. Rather, one would like to make full use of the fact that it is a coset group. One can do this by representing the cosets, i.e. the elements of G/H, by elements of G. We can thus represent an element from G/H by a corresponding member of that coset $g \in G$. One then usually denotes the coset as $[g]$, where g is then called a representative of the coset.

The thing is that we are free to choose our coset representatives; we can choose any element of the coset to represent it. Let us now represent the elements of G/H by elements of G. The (linear) action of G on G/H can be written as:

$$g*[g_1] = [gg_1] = [g_2] \tag{3.5}$$

where in general $[g_2] \neq [g_1]$. The H-action is given by:

$$h*[g_1] = [g_1]h = [g_1h] = [g_1] \tag{3.6}$$

In principle one should choose a specific set of coset representatives, but for the moment let us act as if we do not care. Then we can perform H transformations at will since they map any coset to itself. In fact for any coset we can take a different h acting on it, since if $g_1 \in [g_1]$ then also $g_1h \in [g_1]$. I.e. we have a 'local' action of H on G/H.

Now let us fix the representatives we wish to use. If one acts with G on the cosets, then in general it does not respect the representatives of g_1H and g_2H respectively, and act with some $g \in G$ which maps $[g_1]$ to $[g_2]$. Then $g[g_1] = [gg_1] = [g_2]$, but in general $gg_1 \neq g_2$. However, one can always perform an H transformation to put it in the right form, i.e. we can find an $h \in H$ such that $gg_1h = g_2$. Thus, when one is interested in preserving the chosen representatives upon acting with $g \in G$ on the coset, one must generally accompany it by an H transformation h (which in principle depends on g and g_1). In this way one obtains a non-linear realization of G on G/H which preserves the representatives, given by:

$$g[g_1] = [gg_1h(g, g_1)] \tag{3.7}$$

One can view the fixing of the representatives as 'gauge' fixing the 'local' H.

Let us summarize. If one does not specify the representatives, one has a linear global G-action and a linear local H-action on G/H. If one does fix the representatives then the local H-action is lost and one ends up with a non-linear realization of G on G/H.

3.2.3 Coset Manifolds

We now turn to the case where G and H are both Lie groups. The resulting coset group G/H is then also a Lie group, and thus in particular a manifold, and the group is called a coset manifold. These are the spaces that arise in the non-linear sigma models describing the scalar sector of the (half-)maximal theories. In fact, in these theories H is a maximal compact subgroup of G (i.e. H is maximal among compact subgroups) and from now on we assume that H is indeed such.

The dimension of a coset manifold is given by $D = \text{dim}G$ - $\text{dim}H$ and can thus, locally, be described by D coordinates $\phi = (\phi^1, ..., \phi^D)$. Each value of ϕ thus corresponds to a certain coset. One can then locally calculate everything of interest, such as the metric $G_{ij}(\phi)$ in terms of these D local coordinates. However, we will use the description in terms of coset representatives as described in the previous section. Thus, we will use elements of G to represent elements of G/H. Now, elements of G locally parametrized by dim $G =$ dim $G/H+$ dim $H = D + n$ coordinates, $\bar{\phi} = \{\bar{\phi}^1, ..., \bar{\phi}^{D+n}\}$. As long as we do not specify the coset representatives we thus have a redundant description in terms of more coordinates then necessary. Also we thus have a linear action of G on G/H and a 'local' action of H. One can then describe the cosets via representatives $L(\bar{\phi}) \in G$. Under left multiplication by $g \in G$, the coset with representative $L(\phi)$ is in general carried into another coset, with representative element $L(\phi')$. That is, $gL(\phi)$ lies in the coset characterized by ϕ'. However, as we have seen, to put $gL(\phi)$ in the right representative form, say $L(\phi')$ one must act from the right with some $h \in H$, i.e:

$$gL(\phi)h = L(\phi'), \quad h = h(g, \phi) \tag{3.8}$$

Upon fixing the coset representatives one specifies the form of $L(\bar{\phi})$ effectively reducing to a description in terms of D coordinates, which are the true coordinates on G/H.

G and H being Lie groups, they come with corresponding Lie algebras g and h. We can decompose (as vector spaces) the Lie algebra of G w.r.t. to that of H, i.e. $g = k \oplus h$. Here k is not necessarily a Lie algebra, however it does contain the generators of g which do not lie in h. When considering the coset manifold G/H the relevant generators are those of k, now dubbed coset generators, as those in h are in fact redundant. When g is a semi-simple Lie algebra and H is a maximal compact subgroup one can show that:

$$[h, h] \subset h \tag{3.9}$$
$$[h, k] \subset k \tag{3.10}$$
$$[k, k] = h \tag{3.11}$$

The basic object that enters the Lagrangian is the left-invariant one-form $L(\phi)^{-1}dL(\phi)$ with $L(\phi) \in G$. This one-form takes values in the Lie algebra of G and it is easily seen that indeed it is invariant under a global (left-)action $g' : G \to G$:

$$(g'L(\phi))^{-1}d(g'L(\phi)) = L(\phi)^{-1}g'^{-1}g'dL(\phi) \tag{3.12}$$
$$= L(\phi)^{-1}dL(\phi) \tag{3.13}$$

Since the one-form takes values in g, we can decompose it as:

$$L(\phi)^{-1}dL(\phi) = \omega(\phi) + e(\phi) \tag{3.14}$$
$$= \omega_i(\phi)d\phi^i + e_i(\phi)d\phi^i \tag{3.15}$$

where ω and e are one-forms taking values in h and k respectively. The local H-transformations act as:

$$L(\phi)^{-1}dL(\phi) \to h(\phi)^{-1}L(\phi)^{-1}(dL(\phi)h(\phi) + L(\phi)dh(\phi)) \tag{3.16}$$
$$= h(\phi)^{-1}L(\phi)^{-1}dL(\phi)h(\phi) + h(\phi)^{-1}dh(\phi) \tag{3.17}$$

Or in terms of ω and e:

$$\omega + e \to h^{-1}(\omega + e)h + h^{-1}dh \tag{3.18}$$

I.e:

$$\omega_i \to h^{-1}\omega_i h + h^{-1}\partial_i h \tag{3.19}$$
$$e_i \to h^{-1}e_i h \tag{3.20}$$

Where we used that $[h, h] \subset h$ and $[h, k] \subset k$. We thus see that the ω_i transform as gauge fields corresponding to the local H, and the e_i transform in a linear representation.

3.2.4 Coset manifold models

As already noted, the scalar sector of (half-)maximal supergravity theories are described by non-linear sigmal models with coset manifolds as target spaces. We will now use the theory of the previous sections to describe such models. We introduce the scalars via a group element $L(\phi) \in G$ and for the moment do not fix the coset representatives. We thus make use of the redundant description and are adding non-physical scalars to our theory. However, as discussed, upon gauge fixing w.r.t. the local H we can get rid of those extra fields. (Since we have a global G working from the left, and a local H working from the right, the coset representative comes with two indices: one global G index M which is down, and one local

H index $\bar{M}$ which is up. Thus writing it with indices we get $L(\phi)_M^{\bar{M}}$.)

In a field theory, the scalar fields of course depend on space-time coordinates and in that sense give a map from space-time M to our coset space G/H. One can then can pull back the differential form earlier on G/H to obtain a differential form on M:

$$L(\phi(x))^{-1}dL(\phi(x)) = L(\phi(x))^{-1}\partial_\mu L(\phi(x))dx^\mu \tag{3.21}$$

Again we can decompose it accordingly:

$$L(\phi(x))^{-1}\partial_\mu L(\phi(x)) = P_\mu(\phi(x)) + Q_\mu(\phi(x)) \tag{3.22}$$

With $P_\mu \in k$ and $Q_\mu \in h$. Relating them to ω and e:

$$Q_\mu(\phi(x)) = \omega_i(\phi(x))\partial_\mu\phi(x)^i \tag{3.23}$$

$$P_\mu(\phi(x)) = e_i(\phi(x))\partial_\mu\phi(x)^i \tag{3.24}$$

The H transformations as discussed in the previous section depend on ϕ, which in turn depend on space-time coordinates. What one can now do is lift these transformations $h(\phi(x))$ to transformation $h(x)$ depending arbitrarily on x. We thus have under such a transformation:

$$L(\phi(x)) \to L(\phi(x))h(x) \tag{3.25}$$

This is just for convenience. Upon doing so, we in fact lift all objects to depend on x instead of $\phi(x)$; we thus consider an unrestricted space-time dependent $L(x) \in G$ (and correspondingly $Q(x)$ and $P(x)$). However, upon fixing the representatives we obtain the correct description in terms of the physical fields $\phi(x)$. The behavior of $P(x)$ and $Q(x)$ under the local H-transformations is then:

$$Q_\mu(x) \to h^{-1}(x)Q_\mu(x)h(x) + h^{-1}(x)\partial_\mu h(x) \tag{3.26}$$

$$P_\mu(x) \to h^{-1}(x)P_\mu(x)h(x) \tag{3.27}$$

Not surprisingly we see that $Q_\mu(x)$ acts as a gauge field associated with the local H transformations, and P transforms in a linear representation of H. Again, when one fixes the representatives one gets a non-linear realization of G on G/H as described earlier.

We can now use the one-form to construct the kinetic part of the Lagrangian. Since the one-form $L(\phi(x))^{-1}\partial_\mu L(\phi(x))$ is invariant under the rigid G transformations so are Q and P. The fact that Q acts as a gauge field compels us to introduce a corresponding H-covariant derivative:

$$D_\mu L = \partial_\mu L - VQ_\mu \tag{3.28}$$

And we get:

$$L^{-1}D_\mu L = P_\mu \tag{3.29}$$

One then finds that we can write down the following Lagrangian which is G and local H-invariant:

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}D_\mu L^{-1}D^\mu L \tag{3.30}$$

An alternative way to write this down is to introduce $\mathcal{M} = LL^T$ and obtain:

$$\mathcal{L} = \frac{1}{8}\mathrm{tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}) \tag{3.31}$$

If one explicitly works out the Lagrangian one of course obtains one of the general form of a non-linear sigma model.

3.3 Vector fields and antisymmetric p-forms

We now turn to the vector fields, and more generally possible anti-symmetric p-forms [5]. These fields all transform linearly under the global group G, with the vectors in the fundamental representation. They do not transform under the local H. The fields enter the Lagrangian via their corresponding field strengths:

$$F^I_{\nu_1 \ldots \nu_{p+1}} = (p+1)\partial_{[\nu_1} B^I_{\nu_2 \ldots v_{p+1}]} \tag{3.32}$$

Observe that they are thus all Abelian fields, i.e. each field comes with its own $U(1)$ group of gauge transformations. The bosonic part of the Lagrangian is then:

$$\mathcal{L} = -\frac{1}{2(p+1)!}\mathcal{M}_{IJ} F^I_{\nu_1 \ldots \nu_{p+1}} F^{J\nu_1 \ldots \nu_{p+1}} \tag{3.33}$$

Here $\mathcal{M}_{IJ}$ is the scalar matrix as introduced in the previous section, which has to be evaluated in the corresponding representation of G under which the given p-form transforms.

One very interesting property of theories with p-forms is that of the on-shell duality between massless p-forms and $(D-p-2)$-forms in D space-time dimensions. To see this we calculate the equations of motion:

$$\partial^\mu (\mathcal{M}_{IJ} F^I_{\mu\nu_1 \ldots \nu_{p+1}}) = 0 \tag{3.34}$$

In addition one has the Bianchi identity which on must impose to guarantee that indeed the field strengths follow from a p-form potential:

$$\partial_{[\nu_1} F^I_{\nu_2 \ldots \nu_{p+2}]} = 0 \tag{3.35}$$

One can also choose to describe these equations in terms of the dual field strenght:

$$G_{\mu_1 \ldots \mu_{D-p-1} I} = \frac{e}{(p+1)!}\varepsilon_{\mu 1 \ldots \mu_{D-p-1}\nu_1 \ldots \nu_{p+1}} \mathcal{M}_{IJ} F^{J\nu_1 \ldots \nu_{p+1}} \tag{3.36}$$

The equations of motion and the Bianchi identity then become:

$$\partial_{[\mu_1} G_{I\mu_2 \ldots \mu_{D-p}]} = 0 \tag{3.37}$$

$$\partial^\mu (\mathcal{M}^{IJ} G_{I\mu\nu_1 \ldots \nu_{D-p-2}}) = 0 \tag{3.38}$$

One thus sees that the roles of the equation of motion and the Bianchi identity are exchanged w.r.t. the dual field strength. The Bianchi identity for the dual field strength ensures that it can be written in terms of a $(D-p-2)$-form C_I:

$$G_{\mu_1 \ldots \mu_{D-p-1} I} = (D-p-1)\partial_{[\mu_1} C_{I\mu_2 \ldots \mu_{D-p-1}} \tag{3.39}$$

One can now choose to describe the theory in terms of these dual $(D-p-2)$-forms C_I instead of the original p-forms B^I. The theory can thus be described in different off-shell formulations i.e. one with p-forms and the other with $(D-p-2)$-forms, which after proper dualization (i.e. imposing the relation relating G to F) are on-shell equivalent. Such a relation between two seemingly different theories is called a duality.

Now, one can see that if the space-time dimension is even, i.e. $D = 2n$, then $(n-1)$-forms are dual to $(n-1)$-forms. This leads to extra possible transformations: one can rotate the original field strength F and its dual G into each other as they are both n-form field strengths. Especially the case where $D =$ is particularly interesting. In this case the vector fields are dual to eachother and one finds that there is an infinite set of inequivalent Lagrangians, which at the level of the equations of motion are equivalent. In particular when one considers gauging these theories one obtains very interesting structures. We will discuss this in detail in the upcoming chapters.

Let us also note that although the dualities above were shown with only standard kinetic terms, it in principle extends to the full theory, including fermions.

3.4 Fermions

The fermionic fields present in supergravities transform under the local H group coming from the G/H coset manifold, i.e. they come with local indices $\bar{N}$. Now, Q_μ can be used in covariant derivatives for the fermions since it plays the role of a gauge connection w.r.t the local H under which the fermions transform. For example for the gravitinos one gets:

$$D_\mu \psi_\nu^{\bar{N}} = \partial_\mu \psi_\nu^{\bar{N}} - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\psi_\nu^{\bar{N}} - (Q_\mu)_{\bar{M}}^{\bar{N}}\psi_\nu^{\bar{M}} \tag{3.40}$$

(Here the term containing the spin connection ω is there due to gravity.) Since P_μ transforms in a linear representation of H, it can be used to construct H-invariant fermionic interaction terms involving scalars. The vector fields do not transform under H, and the only way to couple them to fermions is by again using the scalars. One can schematically write:

$$F^M L_M^{\bar{N}}(\bar{\psi}\psi)_{\bar{N}} \tag{3.41}$$

The description of the scalars in terms of coset representatives $L(\phi)$ transforming under global G transformations and local H transformations is thus essential for the possibility to couple the different fields to each other. There is of course plenty more one can say about the fermionic terms, but we will not do so. In this thesis the role of the fermions is not really essential; the prominent role is reserved for the scalars and vectors.

Chapter 4

Gauged Supergravity

Although supergravity theories are already quite complex theories, they generically lack some properties one would like to see in a theory. For example, they do not contain a scalar potential and the vacuum structure is thus trivial. Without a scalar potential a theory can harbour far less interesting dynamics. A scalar potential could for example allow for the existence of a variety of vacua with different properties. Spontaneous symmetry breaking by ending up in one of those vacua would lead to theories with different residual symmetry groups, cosmological constants, generated masses, etc. Therefor a theory with a non trivial vacuum structure could potentially be phenomologically interesting. Ungauged supergravity theories however, thus do not allow for such effects. Another point is that the internal symmetries are not gauged, whilst our successfull field theories are all theories with gauge symmetries. A natural step would be to try to gauge some of the global symmetries present. This can indeed be done and one finds that upon gauging the theory a scalar potential emerges, thus leading to a more interesting theory.

In this chapter we will do precisely that to obtain gauged supergravities. Let us swiftly recall the standard gauging procedure usually applied in field theories. The starting point is some theory, without vector fields, with a corresponding Lagrangian which has some global symmetry group G. One then demands that the Lagrangian should in fact be invariant under local G transformations. Working things out one concludes that vector fields, the gauge fields which transform in the adjoint representation of G, have to be added to the theory to be able to meet the requirement of local invariance.

When trying to apply this procedure to supergravities we immediately encounter a difference. Namely, we don't start with a Lagrangian which does not contain vector fields. They are already present and demanded by supersymmetry, and in addition also have particular couplings to the other fields. One must thus specify which vectors are used for the gauging and modify the existing terms accordingly. Also in principle we would like to preserve supersymmetry. We haven't come all this way with the supersymmetry to lose it while gauging the theory. It is not hard to imagine that only a specific subset of possible gaugings will respect the supersymmetry of a given theory.

It turns out that a very convenient formulation of the gauging procedure exists, called the embedding tensor formalism, which we will review in this chapter. Afterwards we will comment on how one, given a scalar potential, can obtain vacua and also some of their properties.

An introduction to the embedding tensor formalism can for instance be found in [5] and [10] on which our first three sections are based.

4.1 The embedding tensor formalism

Let us consider the following situation. We have a Lagrangian with a global symmetry group G, and we wish to gauge some subgroup $H \subset G$. The theory already contains a number of vector fields A^M. Let t_α be the generators of G and let t_r be the generators of $H \subset G$.

Now, certain linear combinations of the vectors fields are going to be used to perform the gauging. However, not necessarily all the vectors fields are used, since the gauge group might have less generators than there are vector fields. (Conversely, one can at best gauge subgroups with the same number of generators as vector fields present.) In general we can introduce the covariant derivative:

$$D_\mu = \partial_\mu + igA_\mu^M X_M \tag{4.1}$$

Note that the X_M are thus in general not the generators of the gauge group. However they are of course (sometimes vanishing) linear combinations of them, i.e:

$$X_M = \vartheta^r_M t_r \tag{4.2}$$

Here ϑ^r_M thus encodes how the different vectors fields couple to the generators, that is, which linear combinations of the vector fields couple to specific generators of the gauge group. Note that the rank of ϑ is equal to the number of generators of H, as only that many linear combinations of the vector fields are actually used in the gauging.

We could also rewrite this in terms of the generators of G instead of H, since every generator of H can be expressed in terms of generators of G:

$$X_M = \Theta^\alpha_M t_\alpha \tag{4.3}$$

Here the rank of Θ is of course also the same as the number of generators of H. The object Θ^α_M is called the embedding tensor and forms the basis of the embedding tensor formalism which we will discuss in a short while.

Thus upon fixing a subgroup H, one is naturally lead to these three objects X_M, ϑ^r_M and Θ^α_M. In fact, the whole gauging can be formulated by simply using any of the three objects (we will give the details in the next section). However we will see that the embedding tensor is by far the most usefull. One can see this because it makes no explicit reference to the subgroup H and is in that sense gives a formulation independent of the gauge group. One can now take the following viewpoint: instead of fixing a subgroup H and explicitly perform the gauging, one could start off with a general embedding tensor Θ^α_M and see what happens. A priori we do not know that indeed a given embedding tensor will give rise to a consistent gauging of some subgroup $H \subset G$. However one can rather easily obtain constraints on the embedding tensor which ensure that indeed it gives rise to a consistent gauging. Let us note that if one keeps the components unspecified, one really views the embedding tensor as a true tensor transforming according to its index structure (i.e. $\Theta \in V^* \otimes adj(G)$, with $V*$ the representation dual to the fundamental of G). In this way one can develop a lot of theory by simply analyzing the group theoretic properties of the embedding tensor.

Now, there are basically two constraints that one would like to impose on the embedding tensor. Firstly, the embedding tensor should give rise to X_M that close into a subalgebra, i.e:

$$[X_M, X_N] = a^P_{MN} X_P \tag{4.4}$$

Calculating the bracket gives:

$$[X_M, X_N] = \Theta^\alpha_M \Theta^\beta_N [t_\alpha, t_\beta] \tag{4.5}$$

$$= \Theta^\alpha_M \Theta^\beta_N f^\gamma_{\alpha\beta} t_\gamma \tag{4.6}$$

Equating the two expressions we find:

$$\Theta^\alpha_M \Theta^\beta_N f^\gamma_{\alpha\beta} = a^P_{MN} \Theta^\gamma_P \tag{4.7}$$

Thus, closure results in a quadratic contraint on the embedding tensor.

Secondly, the embedding tensor should be invariant under gauge transformations. To see this we note that components of the embedding tensor are simply the gauge charges, which you of course want to be invariant under specific gauge choices. Upon introducing the generator for a gauge transformation, $\delta_M = \Theta^\alpha_M \delta_\alpha$, this leads to the following condition:

$$0 = \delta_P \Theta^\alpha_M = \Theta^\beta_P \delta_\beta \Theta^\alpha_M \tag{4.8}$$

$$= \Theta^\beta_P (t_\beta)^N_M \Theta^\alpha_N + \Theta^\beta_P f^\alpha_{\beta\gamma} \Theta^\gamma_N \tag{4.9}$$

Here we used that the α index lives in the adjoint representation for which of course $(t_\alpha)^\gamma_\beta = f^\gamma_{\alpha\beta}$. This condition is called the quadratic constraint. One can actually easily show that closure follows from it. Contracting with t_α yields:

$$\begin{aligned} 0 &= (\Theta^\beta_P (t_\beta)^N_M \Theta^\alpha_N + \Theta^\beta_P f^\alpha_{\beta\gamma} \Theta^\gamma_M) t_\alpha \\ &= \Theta^\beta_P (t_\beta)^N_M \Theta^\alpha_N t_\alpha + \Theta^\beta_P f^\alpha_{\beta\gamma} \Theta^\gamma_M t_\alpha \\ &= (X_P)^N_M X_N + \Theta^\beta_P \Theta^\gamma_M [t_\beta, t_\gamma] \\ &= (X_P)^N_M X_N + [X_P, X_M] \end{aligned} \tag{4.10}$$

I.e:

$$[X_M, X_N] = -X^P_{MN} X_P \tag{4.11}$$

Here we introduced $X^N_{PM} = \Theta^\alpha_P (t_\alpha)^N_M$. We note that in general the quadratic constraint is stronger then mere closure.

We thus find that the quadratic constraint on the embedding tensor ensures that it corresponds to a consistent gauging. Note also that every consistent gauging naturally gives rise to an embedding tensor satisfying the quadratic constraint. The quadratic constraint is thus a necessary and sufficient condition on the embedding tensor. Thus, if we want to find all possible gaugings, one 'simply' has to solve for all possible solutions of the quadratic constraint.

4.2 Modifying the Lagrangian

Apart from introducing the covariant derivatives, one must modify the field strength of the vector fields used in the gauging, to capture their new non-abelian nature. I.e:

$$F_{\mu\nu} \to F_{\mu\nu} + i[A_\mu, A_\nu] \tag{4.12}$$

Due to the specific couplings already present between the vector fields and the other fields, introducing the covariant derivatives and modifying the field strenght is not enough to make the Lagrangian invariant w.r.t. the local transformations. In addition one must add new terms to the Lagrangian such that indeed the entire action is invariant under the gauge group. One can find these by the standard Noether procedure. This basically goes as follows. One calculates the variations of the original terms in the Lagrangian and identifies the nonvanishing parts. Then, one adds terms to the Lagrangian which can cancel these terms with their corresponding variations. These new terms though, can in addition also come with new nonvanishing variations under the gauge group which would then also have to be cancelled by additional terms. Eventually this procedure should stop and the end result is a Lagrangian which is invariant under the gauge group.

Of course the specific terms that are needed will depend on the theory one is looking at. However, generically one finds that one has to add fermionic mass terms and a scalar potential. It turns out that one can specify all the terms, in terms of what is called the T-tensor, which is basically the embedding tensor dressed up with scalars:

$$T_{\bar{N}}^{\bar{\beta}}(\phi) = \Theta_M^\alpha L_{\bar{N}}^M(\phi) L_\alpha^{\bar{\beta}}(\phi) \tag{4.13}$$

Here $L_\alpha^{\bar{\beta}}(\phi)$ is the scalar matrix evaluated in the adjoint representation instead of the fundamental of G.
The generic form of the fermionic mass terms is as follows:

$$g(\bar{\psi}^i A_{ij} \psi^j + \bar{\chi}^A B_{Ai} \psi^i + \bar{\chi}^A C_{AB} \chi^B) + h.c. \tag{4.14}$$

Here the i and A label the H-representations under which the gravitinos and spin-1/2 particles transform respectively. The tensors A_{ij}, B_{Ai} and C_{AB} can be extracted from the T-tensor. They are those parts of the T-tensor that transform under those representations of H that make the fermionic mass terms invariant under H.
The scalar potential can be expressed as:

$$V(\phi) = g^2 V_{\alpha\beta}^{MN} \Theta_M^\alpha \Theta_N^\beta \tag{4.15}$$

Here $V_{\alpha\beta}^{MN}$ is a scalar dependent matrix to be extracted from the T-tensor. The detailed form depends on the theory.

Let us stress that the specific terms, which can all be written in terms of the T-tensor, are completely determined by the embedding tensor, since the T-tensor directly follows from it. I.e. the entire gauging can be fully described in a gauge group covariant way with the embedding tensor.

4.3 Supersymmetry

As we have already noted, generically a certain consistent gauging, i.e. one satisfying the quadratic constraints, does not preserve supersymmetry. However, it is not desireable to already lose the supersymmetry of the theory at the level of gauging. Therefor one would like to find some extra condition on the embedding tensor, such that compatibility with supersymmetry is ensured. It turns out that there is no general expression one can write down that works for all theories, as is the case for consistency. Although the particular constraint on the embedding tensor depends on the theory at hand, it is always a linear constraint. If we write the embedding tensor in terms of its irreducible components, i.e.

$$\Theta = \theta_1 \oplus ... \oplus \theta_n \tag{4.16}$$

then the linear constraint projects out those irreducible components θ_i that are incompatible with supersymmetry. Schematically we write this as:

$$P\Theta = 0 \tag{4.17}$$

Such a projection always translates into a linear constraint on the embedding tensor. One can see what irreducible components should be projected out by looking at the representation content of the fermionic mass terms which are added in the gauging procedure. Only the parts of the variations coming from those representations in the embedding tensor that are also contained in the fermionic mass terms can be cancelled. The others must thus be projected out.

4.4 Vacua

Recall the followings facts about vacua. One is interested in finding maximally symmetric (w.r.t. space-time symmetries) constant backgrounds. I.e. we look for constant solutions to the field equations, around which to perform the usual perturbation theory. Maximal symmetry forces the background space-time to be Minkowski, De Sitter or Anti De Sitter. Also, one has to put to zero all background values of fields transforming non-trivially under space-time symmetries. Therefor, only the scalars can acquire (constant) nonzero background values. The scalar potential determines which background values scalar fields can acquire: they are precisely the stationary points of the scalar potential. Possible vacua are thus fully classified by the stationary points of the scalar potential. A constant scalar configuration $\phi = \phi_0$ thus corresponds to a vacuum if and only if

$$\frac{\partial V(\phi)}{\partial \phi_i}|_{\phi_0} = 0, \quad i = 1, ..., n \tag{4.18}$$

Where n is the number of scalars in the theory. The scalar potential also determines whether the corresponding vacuum is Minkowski, dS or AdS. Let us see how one can determine the various properties of vacua from the dynamics of the scalars. Since we set all fields to zero except the scalars, the only relevant part of the full Lagrangian is the scalar plus gravity sector [11]:

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}G_{ij}(\phi)\partial_\mu \phi^i \partial \mu \phi^j - g^2 V(\phi) \tag{4.19}$$

The Einstein equations following from it are (after setting the fields equal to their vevs, $\phi = \phi_0$):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} g^2 V(\phi_0) \tag{4.20}$$

I.e. the metric is a solution to the Einstein equations with non-zero cosmological constant, $\Lambda = g^2 V(\phi_0)$. Depending on whether it is positive, zero or negative, one finds a dS, Minkowski or AdS solution respectively. Other properties of a vacuum can be obtained by expanding the Lagrangian around ϕ_0. I.e. we introduce $\bar{\phi} = \phi - \phi_0$ and rewrite everything in terms of these deviations around the vev. We then find:

$$\mathcal{L} = \frac{1}{2} R - \frac{1}{2} G_{ij}(\phi_0) \partial_\mu \bar{\phi}^i \partial \mu \bar{\phi}^j - g^2 (V(\phi_0) + \frac{1}{2} \nabla_i \partial_j V(\phi)|_{\phi_0} \bar{\phi}^i \bar{\phi}^j + ...) \tag{4.21}$$

One can now define the masses of the perturbations of the scalar fields $\bar{\phi}$ as:

$$[m^2]^i_j = G^{ik}(\phi_0) \nabla_k \partial_j V(\phi)|_{\phi_0} \tag{4.22}$$

If the background is Minkowskian or dS, negative masses signal instabilities, whereas in AdS negative masses up to some lowest value are allowed (called the Breitenlohner-Freedman bound). In principle one would like to avoid such instable vacua as they are unphysical.

(Note that one must of course also consider perturbations of the other fields around their (vanishing) vevs. Upon doing so one can also calculate the masses of these fields and examine possible residual symmetries.)

4.4.1 Warners Trick

The scalar potential that emerges upon gauging a supergravity theory is in general highly non trivial. On the one hand this is desireable since then it could potentially harbour very interesting dynamics. However, it also makes analyzing its vacuum structure very difficult. For example, the scalar potential of maximal supergravity in four dimensions depends nontrivially on 70 scalars, which makes an analytic determination of its vacua practically impossible. However, luckily there is a nice trick due to Warner which greatly aids us in finding vacua of the scalar potential.

First some facts. Given a compact subgroup H of the gauge group G, one can consistently truncate the theory to the scalars which are singlets under H. In other words, one sets all fields which transform non trivially under H to zero, and only keeps those which are invariant. Upon doing so one ends up with a simpler scalar potential in terms of these invariant scalars, which one might be able to analyse. Now, here comes the trick [22]: all stationary points of this simpler potential correspond to stationary points of the full potential. Let us be a bit more precise. Consider n scalar fields ϕ describing the full theory, with corresponding scalar potential $V(\phi)$. Now consider some compact subgroup H of the gauge group, and assume that there are k scalars invariant under H. Also assume that we have chosen our basis so that the first k scalars are precisely the invariant ones. Now if then $(\phi_1, ..., \phi_k)$ is a vacuum of the reduced scalar potential, then $(\phi_1, ..., \phi_k, 0, ..., 0)$ is a vacuum of the full scalar potential.

In general, the larger the subgroup H, the less singlets there are and thus the simpler the potential becomes. One could thus first take the maximal subgroups of the gauge group and examine the resulting potentials, subsequently one takes the maximal subgroups of those subgroups and analyses their corresponding potentials, etc. Proceeding in this fashion one obtains more and more singlets and thus more and more complicated potentials and in principle one finds more and more vacua. Eventually one reaches $H = 1$ which thus corresponds to the full theory. Note also that if $H_1 \subset H_2$ then the set of all vacua with H_1 symmetry contains the set of all vacua with H_1 symmetry.

Although Warners trick indeed gives a convenient method to analytically analyse part of the vacuum structure, in practice scalar potentials with more than three scalars are already very difficult to manage. We are thus still rather limited, and we do not come even remotely close to finding the full vacuum structure of the theories, which can have up to 70 scalars. Nevertheless it is only known method by which one can in practice find analytic expressions for vacua and it lies at the basis of what we will do in this thesis.

Part II

Maximal Supergravity in four dimensions

Chapter 5

Ungauged Maximal Supergravity

Let us now finally turn to the theory which we will be investigating in detail, namely maximal supergravity in four dimensions. As we have already noted the maximal amount of supersymmetry in four dimensions is $N = 8$ and the field content is fixed as being the corresponding massless gravity multiplet:

$$\big(g_{\mu\nu}(\times 1) \quad \psi_\mu(\times 8) \quad A_\mu(\times 28) \quad \chi(\times 56) \quad \phi(\times 70)\big) \tag{5.1}$$

As it turns out, the physics of the theory is unique. That is, there is only one maximally supersymmetric theory in four dimensions. One cannot add more multiplets or add for instance superpotentials. This is of course due to the large amount of supersymmetry which, not surprisingly, severely constrains the possible couplings in the theory. However, there is a certain freedom in the way one can describe the theory which has to do with electric-magnetic duality. We will discuss the concept of electric-magnetic duality in detail in the first section. In section 2 we give an overview of the most important properties of the theory and how this duality plays a role.

5.1 Electric-Magnetic Duality

To get a feeling for electric-magnetic duality we first discuss the simplest and familiar example of pure Maxwell theory. Subsequently we will look at a more complicated scenario. We follow [10]. For a full discussion of electric-magnetic duality we refer to [12] and [13].

5.1.1 Pure Maxwell

Let us consider pure Maxwell theory with only n vector fields A^M_μ, and let us describe the theory in terms of the field strengths $F^M_{\mu\nu} = \partial_\mu A^M_\nu - \partial_\nu A^M_\mu$. The Lagrangian is then given by:

$$\mathcal{L} = -\frac{1}{4}\delta_{MN}F^M_{\mu\nu}F^{N\,\mu\nu} \tag{5.2}$$

The equations of motion are found to be:

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial F^M_{\nu\rho}} \tag{5.3}$$

$$= \partial_{[\mu} G_{\nu\rho] M} \tag{5.4}$$

Where we defined:

$$G_{M\,\rho\sigma} = 2\varepsilon_{\rho\sigma\mu\nu} \frac{\partial \mathcal{L}}{\partial F^{M\,\mu\nu}} \tag{5.5}$$

$$= -\varepsilon_{\rho\sigma\mu\nu} F^M_{\mu\nu} \tag{5.6}$$

$$= -(*F^M_{\rho\sigma}) \tag{5.7}$$

In addition we must impose the Bianchi identities

$$\partial_{[\mu} F^M_{\nu\rho]} = 0 \tag{5.8}$$

to demand that the field strengths can indeed be written in terms of a vector potential. Now, the thing about four dimensions is that the G^M are 2-form field strengths just as F^M. Thus in principle we could rotate F^M and G^M into each other while leaving the equations of motion plus the Bianchi identities invariant. Since these equations fully specify the theory, such a rotation would leave the underlying physics invariant. However, this transformation will in general not leave the Lagrangian invariant and such a transformation is called a duality. Now, we call the field strengths F^M electric, and their dual field strengths G_M magnetic; hence the group of all such transformations is called the electric-magnetic duality group.

To examine it in more detail, let us perform such a rotation, i.e:

$$\begin{pmatrix} F'^N \\ G'^N \end{pmatrix} = \begin{pmatrix} A^N_M & B^N_M \\ C^N_M & D^N_M \end{pmatrix} \begin{pmatrix} F^M \\ G^M \end{pmatrix} \tag{5.9}$$

Note that this transformation is formally only defined on the level of the equations of motion, i.e. on-shell. The transformation does not work at the level of action/Lagrangian. Now, these symmetries of the equations of motion are nice, but still we would like to impose the condition that the rotated equations of motion again follow from an action. This action could in principle be different in its functional dependence on the fields than the original action. This amounts to demanding that the defining relation between F^M and G^M remains true. I.e. we demand that:

$$G'_M = \frac{\partial \mathcal{L}'(F')}{\partial F'^M} \tag{5.10}$$

This will guarantee that the rotated equations of motion indeed follow from a new Lagrangian $\mathcal{L}'(F')$. We must thus demand that $G'^M = - * F'^M$. Combining this with the expressions for $F^{'M}$ and G'^M in terms of the original field strenghts we find:

$$C^N_M F^M - D^N_M * F^M = -A^N_M * F^M - B^N_M F^M \tag{5.11}$$

I.e. $A = D$ and $C = -B$. So only the following transformations are consistent:

$$\begin{pmatrix} A^N_M & B^N_M \\ -B^N_M & A^N_M \end{pmatrix} \tag{5.12}$$

All these transformations (ignoring rescalings) make up the group $U(n)$ (one can see this by going to a complex basis for the field strengths). The duality group of pure Maxwell theory with n vector fields is thus $U(n)$. Let us note that the new Lagrangian $\mathcal{L}'$ is given by:

$$\mathcal{L}' = -\frac{1}{4}\delta_{MN}F'^{M}_{\mu\nu}F'^{N\mu\nu} \tag{5.13}$$

And is thus the same functional as the original Lagrangian. This is because pure Maxwel theory is a bit dull. We will see that if one adds scalars to the theory, things get more interesting and inequivalent Lagrangians desribing the same physics can be obtained.

5.1.2 General considerations

From the easiest case just considered, we will now make a leap and consider a generic theory whose scalar plus vector sector is as one typically encounters in supergravities:

$$\mathcal{L} = \frac{1}{4}e\mathcal{I}_{MN}(\phi)F^{M}_{\mu\nu}F^{N\mu\nu} - \frac{1}{8}\mathcal{R}_{MN}(\phi)\varepsilon^{\mu\nu\rho\sigma}F^{M}_{\mu\nu}F^{N}_{\rho\sigma} + \frac{1}{2}G_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \tag{5.14}$$

(We ignore terms involving fermions. It has been shown that these play no essential role; they are always inert under electric-magnetic duality transformations.) However, it is more convenient to describe the theory in terms of self-dual and anti-self-dual fieldstrengths. These are defined as:

$$F^{\pm}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}) \tag{5.15}$$

With $\tilde{F}_{\mu\nu} = -\frac{1}{2}ie\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The Lagrangian is then given by:

$$\mathcal{L} = \frac{1}{2}e\mathrm{Im}(F^{+}_{\mu\nu}{}^{M}G^{+}_{M}{}^{\mu\nu}) \tag{5.16}$$

With the magnetic dual field strengths:

$$G^{+}_{M}{}^{\mu\nu} = 2ie^{-1}\frac{\partial\mathcal{L}}{\partial F^{+}_{\mu\nu}{}^{M}} = \mathcal{N}_{MN}(\phi)F^{N+\mu\nu} \tag{5.17}$$

And $\mathcal{N} = \mathcal{R} + i\mathcal{I}$. The equations of motion (for the vector fields) and Bianchi identities are given by:

$$\partial_{[\mu}\mathrm{Im}G^{+}_{\nu\rho]N} = 0 \tag{5.18}$$
$$\partial_{[\mu}\mathrm{Im}F^{N+}_{\nu\rho]} = 0 \tag{5.19}$$

Lets now rotate them amongst eachoter via some matrix S:

$$\begin{pmatrix} F'^{+} \\ G'^{+} \end{pmatrix} = S\begin{pmatrix} F^{+} \\ G^{+} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F^{+} \\ G^{+} \end{pmatrix} \tag{5.20}$$

Where A, B, C, D are real invertable $n \times n$ matrices, with n the number of vector fields. This transformation leave the combined set of field equations invariant. However, we must additionally impose, in order to ensure that the dynamics follow from a new Lagrangian, that the relation between F'^{+} and G'^{+} is of the same form as that of the original fields. I.e. we impose:

$$G'^{+\mu\nu}_{M} = 2ie^{-1}\frac{\partial \mathcal{L}'}{\partial F'^{+M}_{\mu\nu}} = \mathcal{N}'_{MN}(\phi)F'^{N+\mu\nu} \tag{5.21}$$

Now, in order for this to hold we find, by using the relation between the original fields:

$$\mathcal{N}'(\phi) = (C + D\mathcal{N}(\phi))(A + B\mathcal{N}(\phi))^{-1} \tag{5.22}$$

Consistency then requires that $\mathcal{N}'(\phi)$ is a symmetric matrix (as is $\mathcal{N}(\phi)$, which is precisely the case when:

$$A^TC - C^TA = B^TD - D^TB = 0, \qquad A^TD - C^TD = \mathbb{1}, \qquad B^T = B, C^T = C \tag{5.23}$$

These are the defining relations such that $S \in Sp(2n, \mathbb{R})$. However, we cannot conclude that this symplectic group is indeed the group which leaves all the equations of motion invariant, since we also have the equations of motion for the scalars. In addition, the transformation $\mathcal{N} \to \mathcal{N}'$ must be obtainable by suitable transformations on the scalars. That is, we must demand that corresponding to each transformation S on the field strengths, there exists some transformation $\phi \to \phi'$ of the scalar fields, such that:

$$\mathcal{N}(\phi') = \mathcal{N}'(\phi) \tag{5.24}$$

Such a transformation is in general non-linear. Also in order for all the equations of motion to be invariant we must have that this corresponding transformation must leave the equations of motion of the scalars invariant. In general one then finds that only a subgroup $G \subset Sp(2n, \mathbb{R})$ can be realized non-linearly on the scalars and in addition leave its equations of motion invariant. One can in addition show that the metric and fermions must always be inert under such transformations. It is interesting to note that without scalars the duality group can be at most the compact subgroup $U(n) \subset Sp(2n, \mathbb{R})$. Adding scalars can thus enlarge the duality group of the theory.

Concluding, it is possible for a theory to have a duality group $G \subset Sp(2n, \mathbb{R})$ which leaves the combined equations of motion and Bianchi identities invariant for all fields. It acts linearly on the field strenghts and their duals, non-linearly on the scalars and trivially on the metric and fermions. In general it does not leave the action invariant: only the the so called electric subgroup $G_e \subset G$ consisting of lower block triangular matrices could leave it invariant. Those are precisely the transformations which rotate the electric field strengths amongst eachother. In addition G_e must leave the scalar sector invariant. Theories with such duality groups can be described by multiple inequivalent Lagrangians which differ in their functional dependence on the scalars. This because after a duality transformation the scalar matrix changes its form as discussed above. As a result the symmetry group G_e might vary amongst the different Lagrangians. Different Lagrangians are said to describe the theory in different symplectic frames.

5.2 An overview

Although it might be surprising due the large amount of supersymmetry in the maximal theory, it turns out that it has a very large duality group. In fact, this possibility was exploited in the early construction of the maximal supergravity theory [14][15]. It has been shown that its duality group is $E_{7(7)} \subset Sp(56,\mathbb{R})$. This group thus leaves the equations of motion plus Bianchi identities invariant, but not the Lagrangian. As a results one can choose to describe the theory in different symplectic frames. Each with its own Lagrangian with its corresponding symmetry group G_e. We shall discuss how the inequivalent Lagrangians are characterized later in this section. This choice of symplectic frame is the only freedom one has in describing the unique underlying physics.

Let us give an overview of the different transformation properties of the different fields; both on-shell as well as off-shell [6].

5.2.1 Fields and groups

The scalars are found to parametrize the coset manifold $E_{7(7)}/SU(8)$. The equations of motion are invariant under the global duality group $E_{7(7)}$ which acts non-trivially on the scalars and field strengths, but trivially on the metric and the fermions. As discussed, the duality group only acts on-shell. The scalars then live in the **133** of $E_{7(7)}$ and the vectors, along with their on-shell magnetic duals, live in the fundamental, **56**.

Off-shell, i.e. at the level of the Lagrangian, the fields transform under representations of the symmetry group of the Lagrangian one chooses to describe it.
The scalars are introduced via the familiar redundant description in terms of 133 scalars living in the **133** of $E_{7(7)}$, which under different symmetry groups undergoes different branchings via:

$$E_{7(7)} \to G_e \tag{5.25}$$

In terms of $SU(8)$ representations we find $\mathbf{133} \to \mathbf{70} + \mathbf{63}$. The ones living in the **70** are the physical scalars and correspond to the non compact directions of $E_{7(7)}$, while the ones in the **63** are the unphysical compact directions which can be gauged away.

The fermions always transform in different linear representations of $SU(8)$:

$$\psi_\mu \in \mathbf{8} \tag{5.26}$$

$$\chi \in \mathbf{56} \tag{5.27}$$

The vector fields are Abelian, so we have an $U(1)^{28}$-gauge group. They transform in a 28-dimensional representation of the symmetry group G_e. Off-shell they thus do not transform under $E_{7(7)}$.

The general structure of the Lagrangian is as described in chapter 3. The details are not essential for our purposes.

5.2.2 Symplectic frames

The different Lagrangians can be parametrized by a symplectic matrix [6]. The way this is done is by modifying the coset representative via:

$$L(\phi)_M^{\bar{N}} \to S_M^P L(\phi)_P^{\bar{N}} \tag{5.28}$$

I.e. anywhere in the Lagrangian where the coset representative enters, one modifies it as above. This changes the way the $E_{7(7)}$ acts on the scalars; it corresponds to changing the embedding of $E_{7(7)} \subset Sp(56, \mathbb{R})$. By changing this embedding, one also in general changes changing the symmetry group of the theory which corresponds to subset of lower triangular matrices. I.e. those matrices that do not mix the magnetic vectors with the electric vectors. Thus to switch from one Lagrangian to the other, one simply modifies the coset representative as above. However, not all symplectic matrices will lead to truly inequivalent Lagrangians. If the transformation is an element of $E_{7(7)}$ it can be absorbed in a redefinition of the scalars so no inequivalent Lagrangian is obtained. Also if the transformation is an element of the the diagonal $GL(28, \mathbb{R})$, then it can be reabsorbed in a field redefinition of the electric vector fields (and correspondingly their magnetic duals). Thus, the inequivalent Lagrangians are parametrized by matrices from the double coset $E_{7(7)} \backslash Sp(56, \mathbb{R})/GL(28, \mathbb{R})$. Any choice for such a matrix determines an inequivalent symplectic frame to describe the theory in.

Chapter 6

Gauged Maximal Supergravity

We now turn the the gauging procedure of our maximal supergravity theory in four dimensions. Due to the fact that the theory can be written in different symplectic frames, gauging the theory comes with some complications. The main reason is that given the theory in a certain frame, one can only gauge subgroups of the duality group $E_{7(7)}$ that are also symmetries of the action. I.e. one can only gauge subgroups of the electric group G_e which varies amongst different symplectic frames. (Note that these are precisely the groups that rotate the 28 electric vectors of a given frame amongst each other.)

We thus see that the possible gaugings one can perform differs depending on the symplectic frame one picks. Thus in order to see what gaugings are possible one must go through all the infinitely many inequivalent frames. Clearly, this might not be the most convenient way to go. Ideally one would like to formulate a gauging procedure which is covariant w.r.t. the frame. This can be achieved by not only allowing electric gaugings, i.e. gaugings in which the electric vector fields act as gauge fields, but also allowing their dual magnetic vector fields to act as such. Gaugings in which one uses linear combinations of electric and magnetic vectors are called dyonic gaugings. By allowing this one can now gauge any subgroup of the duality group $E_{7(7)}$. (I.e. one can also gauge groups that rotate electric and magnetic vectors into eachother.) Thus by selecting different linear combinations of the electric and magnetic vector fields one can gauge all the different subgroups (whose rank does not exceed 28) of $E_{7(7)}$.

In the first section we will discuss the different aspects of dyonic gaugings in general and how the embedding tensor formalism can be extended to such gaugings. We will then focus on the specific case of gauging maximal supergravity in four dimensions and the concept of symplectic deformations.

6.1 Dyonic gaugings

Before tackling the general case, we start of with the simplest possible example. This will give us a feeling about the different aspects of such gaugings.

6.1.1 Simplest example

Let us consider a theory involving one scalar field ϕ and one vector field A_μ (with corresponding field strength $F_{\mu\nu}$), described by the following Lagrangian [10]:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.1}$$

Apart from the $U(1)$ symmetry of the vector field, one see that the theory has an additional global symmetry:

$$\delta_\Sigma\phi = \Sigma \qquad \delta_\theta A_\mu = \partial_\mu\theta \tag{6.2}$$

Now, the global symmetry of the scalar can be gauged via the standard gauging procedure, i.e by introducing a covariant derivative:

$$D_\mu\phi = \partial_\mu\phi - A_\mu \tag{6.3}$$

Upon inserting this in the Lagrangian, we end up with the gauged theory:

$$\mathcal{L} = -\frac{1}{2}D_\mu\phi D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.4}$$

With now the local symmetry:

$$\delta\phi = \theta(x) \qquad \delta A_\mu = \partial_\mu\theta(x) \tag{6.5}$$

So far nothing new has been done. However, let us now try something radically different by using the dual vector field, $\bar{A}_\mu$ to gauge the global symmetry of the scalar, i.e. let us introduce a covariant derivative with respect to $\bar{A}_\mu$:

$$\bar{D}_\mu\phi = \partial_\mu\phi - \bar{A}_\mu \tag{6.6}$$

Upon changing the derivative to this new covariant derivative in the Lagrangian one hasn't quite yet obtained a consistent gauging of the theory. To see this we perform a general transformation of the magnetic field, i.e. we consider $\delta\bar{A}_\mu$. We then see that:

$$\delta\mathcal{L} = (-\bar{D}_\mu\phi)\delta\bar{A}^\mu \tag{6.7}$$

We can cancel this by realizing that scalars can be dualized to 2-forms in four dimensions, and that the duality relation between a scalar and a 2-form $B_{\mu\nu}$ is:

$$\bar{D}^\mu\phi = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} \tag{6.8}$$

One is thus compelled to add a term to the Lagrangian which precisely induces such a term upon varying the magnetic field. It is easily seen that the topological term $\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{A}_\mu\partial_\nu B_{\rho\sigma}$ does the job, resulting in:

$$\delta\mathcal{L} = (-\bar{D}^\mu\phi + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma})\delta\bar{A}_\mu = 0 \tag{6.9}$$

We thus end up with:

$$\mathcal{L} = -\frac{1}{2}\bar{D}_\mu\phi\bar{D}^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{A}_\mu\partial_\nu B_{\rho\sigma} \tag{6.10}$$

With the following symmetries:

$$\delta\phi = \bar{\theta}(x), \quad \delta A_\mu = \partial_\mu \theta(x) \tag{6.11}$$
$$\delta \bar{A}_\mu = \partial_\mu \bar{\theta}(x), \quad \delta B_{\mu\nu} = -\Pi_{\mu\nu} \tag{6.12}$$

It seems we are done, but we aren't quite there yet. From the equation of motion of the two form, $0 = \varepsilon_{\mu\nu\rho\sigma}\partial^\rho \bar{A}^\sigma$, one sees that $\bar{A}_\mu$ is purely gauge, i.e. it has no propagating degrees of freedom. To remedy this, one must transfer the degree of freedom of the electric vector field to it. This can be done by modifying the field strength to incorporate the two-form: $F_{\mu\nu} \to H_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$. One can then introduce simultaneous gauge transformations for $B_{\mu\nu}$ and the electric field A_μ:

$$\delta A_\mu = \partial_\mu \theta(x) - \Sigma_\mu(x), \quad \delta B_{\mu\nu} = 2\partial_{[\mu}\Sigma_{\nu]}(x) \tag{6.13}$$

As can be seen, the electric vector field gets intertwined with the two-form via the simultaneous gauge transformations. In principle one could also introduce extra gauge transformations for the two-form which would then demand the introduction of a three form also transforming under that transformation, etc. etc. This is called the tensor hierarchy. However, this is of no relevance to us as we only need to go as far as to introduce the two-forms.

Summarizing we find that the magnetically gauged theory is given by:

$$\mathcal{L} = -\frac{1}{2}\bar{D}_\mu\phi\bar{D}^\mu\phi - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{A}_\mu\partial_\nu B_{\rho\sigma} \tag{6.14}$$

With local symmetries:

$$\delta\phi = \bar{\theta}(x), \quad \delta A_\mu = \partial_\mu\theta(x) - \Sigma_\mu(x) \tag{6.15}$$
$$\delta\bar{A}_\mu = \partial_\mu\bar{\theta}(x), \quad \delta B_{\mu\nu} = 2\partial_{[\mu}\Sigma_{\nu]}(x) \tag{6.16}$$

Unfortunately though, in this example the magnetic gauging is actually equivalent to the electric gauging. This can be seen by simply integrating out $B_{\mu\nu}$ via its equation of motion $B_{\mu\nu} = -F_{\mu\nu} + \varepsilon_{\mu\nu\rho\sigma}\partial^\rho\bar{A}^\sigma$. The resulting Lagrangian is:

$$\mathcal{L} = -\frac{1}{2}D_\mu\phi D^\mu\phi - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \tag{6.17}$$

(Where $G_{\mu\nu}$ is the field strength of $\bar{A}_\mu$.) I.e. upon renaming $\bar{A}_\mu \to A_\mu$ one obtains the electrically gauged theory. This leads us to the following question: when are gaugings with of the same group but done via different vectors equivalent? We will discuss this later on in this chapter. However, the next example already shows that in general they can be inequivalent.

6.1.2 Less simple example

Let us spice things up a little bit by consider a theory involving a complex scalar field z coupled to the field strength [10]:

$$\mathcal{L} = -\frac{1}{4(Im(z))^2}\partial_\mu z\partial^\mu z - \frac{1}{4}Im(z)F_{\mu\nu}F^{\mu\nu} \tag{6.18}$$

The following global symmetry is associated with the scalar field:

$$\delta z = \theta \tag{6.19}$$

Where θ is real, i.e. the symmetry is a shift of the real part of z. Traditional gauging via A_μ leads to:

$$\mathcal{L} = -\frac{1}{4(Im(z))^2} D_\mu z D^\mu z - \frac{1}{4} Im(z) F_{\mu\nu} F^{\mu\nu} \tag{6.20}$$

Before going to the magnetic gauging let us summarize what we have done in the simple example. We introduced covariant derivatives w.r.t. the magnetic field. Then we saw that we had to introduce a two-form (dual to the scalar) and adding a topological term. Then we had to transfer the degrees of freedom from the electric field to the magnetic field by introducing new gauge transformation simultaneously working on the electric field and the two-form. We also had to modify the field strength to include the two-form. Following exactly the same procedure, one obtains:

$$\mathcal{L} = -\frac{1}{4(Im(z))^2} \bar{D}_\mu z \bar{D}^\mu z - \frac{1}{4} Im(z) H_{\mu\nu} H^{\mu\nu} + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{A}_\mu \partial_\nu B_{\rho\sigma} \tag{6.21}$$

With local symmetries:

$$\delta z = \bar{\theta}(x), \quad \delta A_\mu = \partial_\mu \theta(x) - \Sigma_\mu(x) \tag{6.22}$$

$$\delta \bar{A}_\mu = \partial_\mu \bar{\theta}(x), \quad \delta B_{\mu\nu} = 2\partial_{[\mu} \Sigma_{\nu]} \tag{6.23}$$

Integrating out $B_{\mu\nu}$ yields:

$$\mathcal{L} = -\frac{1}{4(Im(z))^2} \bar{D}_\mu z \bar{D}^\mu z - \frac{1}{4Im(z)} G_{\mu\nu} G^{\mu\nu} \tag{6.24}$$

We now see that the magnetic gauging is truly different from the electric gauging since the coupling of z to the vector field has fundamentally changed. However it does correspond to an electrically gauged theory in another frame where the Lagrangian has the form:

$$\mathcal{L} = -\frac{1}{4(Im(z))^2} \partial_\mu z \partial^\mu z - \frac{1}{4Im(z)} F_{\mu\nu} F^{\mu\nu} \tag{6.25}$$

This is in fact a general statement: every dyonic gauging in one frame can be formulated purely electrically in some other frame.

6.1.3 General case

The general procedure as illustrated in the previous two examples also works for more complicated scenarios in which more fields (of any type) are present in the theory or where one uses arbitrary dyonic combinations for the gauging. The bottom line is that one always has to transfer the degrees of freedom from the electric vectors to the linear combinations of electric and magnetic vectors used in the gauging. This is done by introducing a set of two forms and intertwining it with the vectors via the new gauge transformation. The details of course highly depend on the theory at hand.

There are two drawbacks of this procedure. First of all it clearly depends on which subgroup one wishes to gauge. I.e. it is not gauge covariant. Also it is not clear whether two gaugings of the same group, but using different vectors for the gauging are inequivalent. The first drawback can be rather easily by extending the embedding tensor formalism to dyonic gaugings. We will review how this is done in the next section. The question of equivalence of gaugings has not been fully answered yet, but recently significant progress has been made by introducing so called symplectic deformations.

6.2 Dyonic gaugings and the embedding tensor

Consider a subgroup H of the duality G that we want to gauge (which is thus not necessarily a subgroup of the symmetry group). Let us use an arbitrary linear combination of electric and magnetic vectors to perform a gauging. To this end we introduce a new vector containing an electric part and a magnetic part [4][16]:

$$A_M = (A^\Lambda, A_\Lambda) \tag{6.26}$$

Where A^Λ denote electric vectors and A_Λ their magnetic duals. We then introduce the covariant derivative:

$$\begin{aligned} D &= \partial + igA^M X_M & (6.27)\\ &= \partial + igA^\Lambda X_\Lambda + igA_\Lambda X^\Lambda & (6.28)\end{aligned}$$

Where now:

$$\begin{aligned} X_M &= \Theta_M^\alpha t_\alpha = \Theta_\Lambda^\alpha t_\alpha + \Theta^{\Lambda\alpha} t_\alpha & (6.29)\\ &= \vartheta_M^r t_r & (6.30)\end{aligned}$$

(Here the t_α are now generators of the duality group, and t_r again of the subgroup H) I.e. the new embedding tensor splits into an electric and a magnetic part:

$$\Theta_M^\alpha = (\Theta_\Lambda^\alpha, \Theta^{\Lambda\alpha}) \tag{6.31}$$

Again, for this gauging to be consistent, the embedding tensor should be invariant under the gauge group, i.e:

$$0 = (\Theta_P^\beta (t_\beta)_M^N \Theta_N^\alpha + \Theta_P^\beta f_{\beta\gamma}^\alpha \Theta_M^\gamma) t_\alpha \tag{6.32}$$

We thus again obtain a quadratic constraint. Any embedding tensor satisfying the quadratic constraint again leads to a consistent gauging of some subgroup H of the duality group. If one wants to preserve supersymmetry, one must impose an additional linear constraint, which again projects out any unwanted representations contained in the embedding tensor.

One must of course also modify the Lagrangian, which in this case is done as discussed in the previous section. I.e. one must add two forms to the theory and introduce a new field strength for the vectors, etc. There are many subtleties that one encounters, but for our purposes they are not essential. For generic theories, one of course also obtains fermionic mass terms and a scalar potential to make the full Lagrangian gauge invariant. These can again be fully expressed in terms of the embedding tensor, or equivalently the T-tensor.

6.3 Gauged maximal supergravity in four dimensions

Let us apply the theory of the previous sections to the maximal four dimensional supergravity. That is, we start with the Lagrangian of the ungauged theory, and gauge the theory via the embedding tensor Θ_M^α.

6.3.1 The embedding tensor

Since our duality group is $E_{7(7)}$, and M is a fundamental index and α an adjoint index, we find that:

$$\Theta_M^\alpha \in \mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480} \tag{6.33}$$

We already noted that only those representations in the embedding tensor are also present in possible fermionic mass terms are consistent with supersymmetry. Now the possible ($SU(8)$) representations in the fermionic mass terms are (with their hermitian conjugates):

$$(\bar{\psi}\psi) :(8 \otimes 8)_{sym} = \mathbf{36} \tag{6.34}$$

$$(\bar{\psi}\chi) :(8 \otimes \bar{\mathbf{56}}) = \mathbf{28} + \mathbf{4\bar{2}0} \tag{6.35}$$

$$(\bar{\chi}\chi) :(\bar{\mathbf{56}} \otimes \bar{\mathbf{56}}) = \mathbf{420} + \mathbf{11\bar{7}6} \tag{6.36}$$

The $E_{7(7)}$ representations contained in the embedding tensor branch under $SU(8)$ as follows:

$$\mathbf{56} \to \mathbf{28} + h.c. \tag{6.37}$$

$$\mathbf{912} \to \mathbf{36} + \mathbf{420} + h.c. \tag{6.38}$$

$$\mathbf{6480} \to \mathbf{28} + \mathbf{420} + \mathbf{1280} + \mathbf{1512} + h.c. \tag{6.39}$$

Therefor one sees that only the **912** is consistent with supersymmetry and we must thus impose a linear constraint that precisely kills the other representations. Thus any embedding tensor satisfying the quadratic constraint which lives in the **912** gives a consistent gauging respecting supersymmetry.

The gauge process is as we have already described. We introduce covariant derivatives, i.e. we let $\partial_\mu \to D_\mu$. In addition we modify the field strengths to incorporate the newly introduced two-forms, i.e. $F_{\mu\nu} \to H_{\mu\nu}$. One must also add fermionic mass terms and a scalar potential. In addition one must introduce topological terms involving the magnetic fields and two-forms, and interwine the fields not used in the gauging with the two-forms via introducing new gauge transformations. This in order to transfer the degrees of freedom to those linear combinations which are used in the gauging.

Now, the detailed form of the actual terms are not essential for our purposes. Except of course that of the scalar potential. One finds that it can be expressed as:

$$V = \frac{g^2}{672}(X_{MN}^R X_{PQ}^S \mathcal{M}^{MP}\mathcal{M}^{NQ}\mathcal{M}_{RS} + 7X_{MN}^Q X_{PQ}^N \mathcal{M}^{NP}) \tag{6.40}$$

Where $\mathcal{M} = LL^T$. The theory in its full glory can be found in [16].

6.3.2 Symplectic Deformations

To address the question of inequivalent gaugings [17], it is most convenient to describe the gauging not via the embedding tensor but via:

$$X_M = \vartheta^r_M t_r \tag{6.41}$$

Where t_r are the generators of the gauge group. As we know, a consistent gauging requires some contraints on the objects ϑ^r_M. This object describes the entire gauging, which can be expressed in terms of the T-tensor. Note that since it is not the bare embedding tensor, but rather the T tensor that enters in the fermionic mass terms and scalar potential, theories with the same T-tensor correspond to equivalent gauged theories. This is regardless of the chosen symplectic frame.

Now, consider a gauging as defined by some ${X^0_{MN}}^P$. We can then always find a frame such that X^0 is electric, i.e. $X_M^{NP} = 0$ if M is larger than $\dim(G_{gauge})$. Also, by choosing a suitable basis for the generators of the gauge group, we can set:

$${X^0_{MN}}^P = \delta^r_M (t_r)^P_N \tag{6.42}$$

This will be our starting point: the gauging via X^0 as given above, with corresponding T-tensor ${T(\phi)^0_{MN}}^P$. We will now show that we can obtain possibly inequivalent gaugings by acting on our gauging with an element of the normalizer of the gauge group with respect to the symplectic group. I.e. we consider:

$$N \in \mathcal{N}_{Sp(56,\mathbb{R})}(G_{gauge}) = \{A \in Sp(56,\mathbb{R}) | AG_{gauge} = G_{gauge}A\} \tag{6.43}$$

In fact, we will also show that any possibly inequivalent gauging can be obtained from such a transformation.

Now, consider such a transformation N, then it acts on the as gauging:

$${X^0_{MN}}^P \to N^Q_M N^R_N {X^0_{QR}}^S (N^{-1})^P_S \tag{6.44}$$

We also know that since N normalizes the gauge group that when acting on the generators of the gauge group (t_r), that they get mapped onto themselves, i.e:

$$N^N_M (t_r)^P_N (N^{-1})^Q_P = g^s_r (t_s)^Q_M \tag{6.45}$$

Plugging this in we obtain the new gauge connection:

$${X_{MN}}^P = N^Q_M \delta^r_Q g^s_r (t_s)^P_N \tag{6.46}$$

Now, everything is done symplectically covariant, so the newly obtain gauging, $\vartheta^s_M = N^Q_M \delta^r_Q g^s_r$, also satisfies the necessary constraints and thus yields a consistent gauging.

Conversely, starting with some gauging ϑ^s_M coupling to the same generators as the standard one δ^s_M, one can always find a matrix N in the normalizer of the gauge group such that ϑ and δ are related as above. We thus conclude that every possible gauging of the same group are related by elements in the normalizer of that gauge group. However, in general many of them could give rise to equivalent theories. We will now show when this is the case.

To examine this we must thus look at the T-tensors of the two gaugings. Now consider the two gaugings ϑ and δ, with corresponding T-tensors:

$$X^0 \Rightarrow T^0_{MN}{}^P = L^{-1M}_{\bar{M}} L^{-1N}_{\bar{N}} X^0_{MN}{}^P L^{\bar{P}}_P \tag{6.47}$$

$$X^\vartheta \Rightarrow T^\vartheta_{MN}{}^P = L^{-1M}_{\bar{M}} L^{-1N}_{\bar{N}} X^\vartheta_{MN}{}^P L^{\bar{P}}_P \tag{6.48}$$

If we now change frames via N^{-1} (i.e. we let $L^{\bar{N}}_M \to N^{-1}_M{}^P L^{\bar{N}}_P$), then by definition the X^ϑ gauging in the original frame gets mapped to the X^0 in the N^{-1} frame. However, the T-tensor is invariant, so we find that the X^0 gauging in the N^{-1} frame has the same T-tensor as the X^ϑ gauging in the original frame. Therefor the X^ϑ gauging in the original frame describes the same physics as the X^0 gauging in the N^{-1} frame.

Alternatively, we could have applied N^{-1} only on the coset and not on the gaugings, obtaining:

$$T^0_{MN}{}^P \to L^{-1M}_{\bar{M}} L^{-1N}_{\bar{N}} N^Q_M N^R_N X^0_{MN}{}^P N^{-1}_S{}^P L^{\bar{P}}_P \tag{6.49}$$

$$= L^{-1M}_{\bar{M}} L^{-1N}_{\bar{N}} X^\vartheta_{MN}{}^P L^{\bar{P}}_P \tag{6.50}$$

$$= T^\vartheta_{MN}{}^P \tag{6.51}$$

I.e. in this case we map the X^0 gauging in the original frame to the X^0 gauging in the N^{-1} frame. In the two frames the two gaugings thus have different T tensors. Such a transformation working only on the coset and not on the gauging, is called a symplectic deformation of the gauged theory. The equations of motion and Bianchi identities are in general not invariant under such a transformation and such theories are thus (potentially) inequivalent.

However, not all such symplectic deformations are for our purposes truly inequivalent. In fact the meaning of 'equivalent' depends on your interests. For example two theories that give the same equations of motions at the classical level, could differ at the quantum level. Now, we will only be interested in the classical theory and a carefull analysis shows that the space characterizing inequivalent models is:

$$\mathcal{S}_{Sp(56,\mathbb{R})}(X^0) \backslash \mathcal{N}_{Sp(56,\mathbb{R})}(G_{\text{gauge}}) / \mathcal{N}_{\mathbb{Z}_2 \ltimes E_{7(7)}}(G_{\text{gauge}}) \tag{6.52}$$

Where $\mathcal{S}_{Sp(56,\mathbb{R})}(X^0)$ is the set of symplectic matrices that stabilize X^0, i.e. for which $A^Q_M A^R_N X^0_{QR}{}^S (A^{-1})^P_S = X^0_{MN}{}^P$. Thus, two transformations N^{-1} sitting in the same coset of this group give rise to equivalent models, while two transformations in different cosets give rise to inequivalent models.

Part III

The vacuum structure of SO(p,q) gaugings

Chapter 7

SO(p,q) gaugings

We are now ready to look at the specifics of the $SO(p,q)$-gaugings of maximal supergravity in four dimensions. We will first discuss some properties of the $SO(p,q)$ groups, or rather their corresponding Lie algebras $so(p,q)$. Then we will discuss the theory of symplectic deformations and comment on the fact that there is a one-parameter family of inequivalent gaugings. We then give explicit formulae regarding the $SO(p,q)$-gaugings which we need in order to calculate scalar potentials.

7.1 The $SO(p,q)$-groups

Since we are going to look at $SO(p,q)$ gaugings, we will first discuss some general properties of the $SO(p,q)$ groups [18]. Recall that the group $SO(p,q)$ is defined as the set of matrices with unit determinant that leave invariant the metric:

$$\eta_{p,q} = \begin{pmatrix} \mathbb{1}_p & O \\ O & -\mathbb{1}_q \end{pmatrix} \tag{7.1}$$

I.e. it is the group of all matrices for which:

$$A\eta_{p,q}A^T = \eta_{p,q} \tag{7.2}$$

Being Lie groups, they are locally all described by corresponding Lie algebras, $so(p,q)$, which are all different real sections of the complex Lie algebra $so(8,\mathbb{C})$.

For our purposes only the local properties of the groups are important, and we can therefor focus on describing their Lie algebras, which is simpler then dealing with the actual groups. Let $X \in so(p,q)$, then it has the following general form:

$$X = \begin{pmatrix} \Lambda_1 & \Sigma \\ \Sigma^T & \Lambda_2 \end{pmatrix} \tag{7.3}$$

Where Λ_1 and Λ_2 are anti-symmetric $p \times p$ and $q \times q$-matrices respectively. The off-diagonal Σ is an arbitrary $p \times q$-matrix. An algebra element thus consists of an antisymmetric part described by the diagonal blocks, and a symmetric part described by the off-diagonal blocks. A convenient basis for such matrices can be labelled by antisymmetric index pairs $[ij]$, $i, j = 1, ..., 8$ as follows:

$$(X_{[ij]}) = \begin{cases} e_{ij} - e_{ji} & 1 \leq i < j \leq p \\ e_{ij} - e_{ji} & p+1 \leq i < j \leq q+q \\ e_{ij} + e_{ji} & 1 \leq i \leq p, \quad p+1 \leq j \leq p+q \end{cases} \tag{7.4}$$

Where $(e_{ij})_{mn} = \delta_{im}\delta_{jn}$, i.e. the matrix with a one as (i, j) entry, and zeros elsewhere. Upon doing the counting one finds that the algebras are all 28 dimensional as vector spaces.

The elements of $so(p, q)$ can be divided into two disctinct parts, namely a compact part and a non-compact part. Whether an element X is compact or non-compact depends on the sign of $\kappa(X, X)$, where $\kappa(-,-) = Tr(-,-)$ is the Killing form of the Lie algebra. A group element corresponding to a compact generator is described by a compact parameter, one corresponding to a non-compact parameter by a non-compact parameter. One finds that the compact generators are precisely the anti-symmetric ones, whilst the non-compact generators are the symmetric ones. The number of compact generators of $so(p, q)$ is thus given by $\frac{1}{2}p(p-1) + \frac{1}{2}q(q-1)$, the remaining generators are non-compact. For example, the algebra $so(8)$ is fully compact, i.e. all 28 generators are compact, whilst for instance $so(4, 4)$ has 12 compact and 16 non-compact generators.

7.1.1 Compact subgroups of $SO(p, q)$

As already noted, Warners trick only works for subgroups that are compact. Therefor we are only interested in compact subgroups $H \subset SO(p, q)$. Now, any subgroup of $SO(8)$ is compact, since $SO(8)$ is itself compact. However, in general for $SO(p, q)$ we should thus only look at subgroups which are contained in its compact part. That is, they should be contained in the maximal compact subgroup $SO(p) \times SO(q) \subset SO(p, q)$. In terms of the algebras, we thus only look at subalgebras contained in the antisymmetric part of $so(p, q)$, i.e. only elements of block diagonal form. These are precisely elements contained in the block diagonal $so(p) \times so(q) \subset so(p, q)$.
Ideally, one would like to have a full description of all the different maximal subalgebras of $so(p) \times so(q)$ with in turn the maximal subgroups of those, etc. etc. I.e. one would like to get some network of chains of maximal subgroups fully depicting the subalgebra structure. However, especially when trying to identify whether two subgroups in different chains are equivalent, things become messy as each subgroup often has around 3 to 4 maximal subgroups.

We will therefor not give an exhaustive classification. However, let us do make a couple of remarks. First of all, most of the subgroups contained in $so(p) \times so(q)$ are products of the form $so(m) \times so(n)$ with $so(m) \subset so(p)$ and $so(n) \subset so(q)$. In addition one also has many diagonally embedded $so(n)$ algebras. These are obtained by identifying some $so(n) \times so(n)$ subalgebra and then taking the diagonal $so(n) \subset so(n) \times so(n)$.

7.1.2 Triality

In the case of $so(8)$ and $so(4, 4)$ there is the matter of triality, which greatly affects the subgroup structure. Let us consider $so(8)$. At the basis of triality lies the fact that it has three inequivalent eight dimensional irreps, which we dub $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$. The $\mathbf{8}_v$ is the familiar fundamental vector representation. In this representation the generators are given by:

$$(X_{[ij]})_{ab} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} \tag{7.5}$$

However, one also has the spinor and cospinor representations acting on spinor and cospinors respectively. These arise from the fact that the 16-dimensional spinor representation is reducible and splits in these two 8 dimensional representations. In terms of gamma matrices this is the fact that the 16×16 gamma matrices, Γ_i (satysifying the $so(8)$ clifford algebra), of $so(8)$ can be put into a block form as follows:

$$\Gamma_i = \begin{pmatrix} O & (\gamma_i)_{\alpha\dot{\alpha}} \\ (\gamma_i)_{\dot{\alpha}\alpha} & O \end{pmatrix} \tag{7.6}$$

Where $(\gamma_i)_{\dot{\alpha}\alpha}$ is the transpose of $(\gamma_i)_{\alpha\dot{\alpha}}$ and real (α are indices, $\dot{\alpha}$ cospinor indices). These again satisfy the $so(8)$ clifford algebra. From these gamma matrices one can construct the gamma-2 matrices:

$$(\gamma_{ij})_{\alpha\beta} = \frac{1}{2}(\gamma_{i\alpha\dot{\alpha}}\gamma_{j\beta\dot{\alpha}} - \gamma_{j\alpha\dot{\alpha}}\gamma_{i\beta\dot{\alpha}}) \tag{7.7}$$

$$(\gamma_{ij})_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(\gamma_{i\alpha\dot{\alpha}}\gamma_{j\alpha\dot{\beta}} - \gamma_{j\alpha\dot{\alpha}}\gamma_{i\alpha\dot{\beta}}) \tag{7.8}$$

$$(\gamma_{\alpha\beta})_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(\gamma_{i\alpha\dot{\alpha}}\gamma_{i\beta\dot{\beta}} - \gamma_{i\beta\dot{\alpha}}\gamma_{i\alpha\dot{\beta}}) \tag{7.9}$$

One can show that they satisfy the same commutation relations as the generators $(X_{[ij]})_{ab}$ and hence generate $so(8)$ algebras. The first two can be shown to be inequivalent to eachother and also to the standard $so(8)$ algebra. We thus find the three inequivalent 8-dimensional representations generated by [20]:

$$\mathbf{8}_v : \quad (X_{[ij]})_{ab} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} \tag{7.10}$$

$$\mathbf{8}_v : \quad (X_{[ij]})_{\alpha\beta} = \frac{1}{4}(\gamma_{ij})_{\alpha\beta} \tag{7.11}$$

$$\mathbf{8}_v : \quad (X_{[ij]})_{\dot{\alpha}\dot{\beta}} = \frac{1}{4}(\gamma_{ij})_{\dot{\alpha}\dot{\beta}} \tag{7.12}$$

Now the interesting thing is that there exists an outer automorphism (i.e. an automorphism not obtained by conjugation by some $SO(8)$ element) that permutes these representations. I.e. we can perform some outer automorphism on $so(8)$ such that the roles of the three irreps get interchanged.

That is one can also consider the spinor index as the fundamental one. The representation matrices are then given by:

$$(X_{[\alpha\beta]})_{\delta\gamma} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta} \tag{7.13}$$

$$(X_{[\alpha\beta]})_{\dot{\alpha}\dot{\beta}} = \frac{1}{4}(\gamma_{\alpha\beta})_{\dot{\alpha}\dot{\beta}} \tag{7.14}$$

$$(X_{[\alpha\beta]})_{ab} = \frac{1}{4}(\gamma_{\alpha\beta})_{ab} \tag{7.15}$$

Considering the cospinor index to be the fundamental gives similar results.

There are thus three different sets of real matrices acting on the fundamental representation, each satisfying the commutation relations of $so(8)$:

$$(X_{[ab]})_{ij} = \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi} \tag{7.16}$$

$$(X_{[\alpha\beta]})_{ij} = \frac{1}{4}(\gamma_{\alpha\beta})_{ij} \tag{7.17}$$

$$(X_{[\dot\alpha\dot\beta]})_{ij} = \frac{1}{4}(\gamma_{\dot\alpha\dot\beta})_{ij} \tag{7.18}$$

These different sets real anti-symmetric matrices satisfying the same $so(8)$ commutation relations, are inequivalent because the three 8-dimensional irreps are. Now consider some subalgebra $h_v \subset so(8)$. It is generated by some subset of the matrices $(X_{[ab]})_{ij}$. Now, the corresponding subsets of the matrices $(X_{[\alpha\beta]})_{ij}$ and $(X_{[\dot\alpha\dot\beta]})_{ij}$ generate isomorphic subalgebras, $h_s, h_c \subset so(8)$. These subalgebras are in general also inequivalent. We thus find that each subalgebra comes in three flavors, v, s, c which in general will be inequivalent and hence allow us to probe different parts of the vacuum structure.

For $so(4,4)$ things are similar since also in this case one can decompose the 16-dimensional spinor representation into two real spinor representations. For the other $so(p,q)$ algebras there is no such decomposition of the 16-dimensional spinor representation and the concept of triality does not apply. See [19].

7.2 The gauging

We will now turn to the details of the actual $SO(p,q)$ gauging. These can be found in [15] and also [21]

7.2.1 The $SL(8)$ frame

We will perform the $SO(p,q)$ gaugings in the standard $SL(8)$ frame. This is the frame whose Lagrangian has a diagonally embedded $SL(8) \subset E_{7(7)}$ as its symmetry group. In this basis an element of the Lie algebra $e_{7(7)}$ takes the following form:

$$t = \begin{pmatrix} \Lambda_{[AB]}^{[CD]} & \Sigma_{[AB][CD]} \\ *\Sigma^{[AB][CD]} & -\Lambda_{[CD]}^{[AB]} \end{pmatrix} \tag{7.19}$$

Here we used the splitting of a fundamental $E_{7(7)}$ index $M = 1, ..., 56$ into an electric and a magnetic part: $M \rightarrow^{[AB]}_{[AB]}$, where $A, B = 1, ..., 8$ such that $[AB] = 1, ..., 28$. We use these antisymmetric index pairs because it enables one to give a neat description of the algebra elements. One can then write $\Lambda_{AB}^{CD} = \lambda_{[A}^{[C}\delta_{B]}^{D]}$ where λ_A^B is a generator of $SL(8)$ in the fundamental representation (i.e. it is a traceles 8×8 matrix). We also have that $\Sigma_{[AB][CD]} = \sigma_{ABCD}$, where the four tensor is real and fully antisymmetric.

There are thus 63 matrices $\Lambda_{[AB]}^{[CD]}$ and 70 matrices $\Sigma_{[AB][CD]}$. They reflects the splitting $\mathbf{133} \rightarrow \mathbf{63}+\mathbf{70}$ under $E_{7(7)} \rightarrow SL(8)$. We can conveniently describe the set of these matrices as follows. Firstly the 63 matrices Λ can be labelled by index pairs ${}_A^B$. We write:

$$(\Lambda_A^B)_{[CD]}^{[EF]} = (\lambda_A^B)_{[C}^{[E}\delta_{D]}^{F]} \tag{7.20}$$

Where $(\lambda_A^B)_C^D = -\delta_A^D\delta_C^B - \frac{1}{8}\delta_A^B\delta_C^D$. Note that this gives a slightly redundant description as we introduce 8 tracless diagonal matrices (λ_A^A) (no summation), whilst this set can be described by 7. This is simply to enable us to give a better interpretation to some results. The remaining 70 generators can be labelled by fully antisymmetric indices combinations $[ABCD]$ and their explicit expressions are:

$$(\Sigma_{ABCD})_{[EF][GH]} = \frac{1}{24}\varepsilon_{ABCDEFGH} \tag{7.21}$$

And thus:

$$(*\Sigma_{ABCD})^{[AB][CD]} = \delta_{ABCD}^{EFGH} \tag{7.22}$$

We thus have the following explicit expressions for the generators of $E_{7(7)}$ in the $SL(8)$ basis:

$$(t_A^B) = \begin{pmatrix} -\delta_{[C}^B\delta_{D]A}^{EF} - \frac{1}{8}\delta_A^B\delta_{CD}^{EF} & O \\ O & \delta_{[E}^B\delta_{F]A}^{CD} + \frac{1}{8}\delta_A^B\delta_{EF}^{CD} \end{pmatrix} \tag{7.23}$$

$$(t_{[ABCD]}) = \begin{pmatrix} O & \frac{1}{24}\varepsilon_{ABCDEFGH} \\ \delta_{ABCD}^{EFGH} & O \end{pmatrix} \tag{7.24}$$

Due to the convenient diagonal embedding $SL(8,\mathbb{R}) \subset E_{7(7)}$ we can easily identify the subgroups $SO(p,q) \subset SL(8,\mathbb{R})$. One simply looks at subsets of λ that generate them. F.e. the $SO(8)$ subgroup is obtain by restricting to antisymmetric λ.

To identify which generators correspond to the physical 70 scalars, we must identify the non compact generators. This can be easily done as follows. One can decompose Λ in terms of its symmetric and antisymmetric parts and one can decompose Σ in terms of a selfdual and anti-selfdual part. Upon calculating the Killing-form one finds that the symmetric part of Λ and the selfdual part of Σ are non-compact directions, and the antisymmetric part of Λ and anti-selfdual part of Σ are compact directions. Upon counting the numbers, one thus finds that 35 of the physical scalars live in the symmetric part of Λ and 35 in the selfdual part of Σ.

7.2.2 The embedding tensor

As discussed, in order the preserve supersymmetry, the embedding tensor must live in the **912** of $E_{7(7)}$. We also know that it has to be a singlet under the gauge group. One has the following branching under $E_{7(7)} \to SO(p,q)$:

$$\mathbf{912} \to 2(\mathbf{1} + ...) \tag{7.25}$$

One thus finds two singlets that could possibly give consistent gaugings [1]. Using the splitting of the fundamental index as desribed above we can write:

$$\Theta_M^\alpha = (\Theta_{[AB]}^\alpha, \Theta^{\alpha\,[AB]}) \tag{7.26}$$

Let us now also use the splitting of the adjoint index: $\alpha \to ({}^A_B, {}_{[ABCD]})$ and write:

$$\Theta_M^\alpha = ((\Theta_A^B)_M, (\Theta^{ABCD})_M) \tag{7.27}$$

Now, since $SO(p,q) \subset SL(8)$ the embedding tensor only has nonvanishing components for $\alpha =_A^B$, as $SL(8)$ is fully contained in the corresponding generators. The first singlet, which is purely electrical, is of the form:

$$(\Theta_D^C)_{[AB]} = \delta_{[A}^C \theta_{B]D} \tag{7.28}$$

By using $SL(8,R)$ invariance one can let $\theta_{AB} = (\eta^{p,q})_{AB}$ to obtain:

$$(\Theta_D^C)_{[AB]} = \delta_{[A}^C (\eta^{p,q})_{B]D} \tag{7.29}$$

It also satisfies the quadratic constraint and thus gives a consistent gauging. The second, purely magnetic singlet, is given by:

$$(\Theta_D^C)^{[AB]} = \delta_D^{[A} \xi^{B]C} \tag{7.30}$$

where $\xi = \theta^{-1}$ in order to satisfy the quadratic constraint. It can thus be chosen as:

$$(\Theta_D^C)^{[AB]} = \delta_D^{[A} (\eta^{p,q})^{B]C} \tag{7.31}$$

The most general embedding tensor one can now write down is a linear combination of the two singlets, i.e:

$$(\Theta_D^C)_M = (\delta_{[A}^C \delta_{B]D}, c\delta_D^{[A} \delta^{B]C}) \tag{7.32}$$

Where $c \in \mathbb{R}$. Alternatively one can introduce a different parameter to characterize the linear combination:

$$(\Theta_D^C)_M = (\cos\omega \delta_{[A}^C \delta_{B]D}, \sin\omega \delta_D^{[A} \delta^{B]C}) \tag{7.33}$$

With $\omega \in [0, 2\pi]$.

The gauge connection X_M thus takes the form:

$$X_M = (\Theta_B^A)_M t_A^B \tag{7.34}$$

These dyonic gaugings, characterized by the constant c (or ω), were long thought to give rise the same physics; for instance it can be shown that purely electric and magnetic gaugings do indeed result in equivalent physics. However, in general this turned out not to be the case. By using suitable tensor classifiers it was shown that in the case of $SO(8)$, inequivalent gaugings are obtained for $c \in [0, \sqrt{2}-1]$. Or in terms of ω for $\omega \in [0, \frac{\pi}{8}]$. The case $c = \omega = 0$ correspond to the original gauged theories.

However, a deeper understanding of the structure underlying these inequivalent gaugings was not known. It actually sparked the research leading to the theory about symplectic deformations as discussed in the previous chapter. Indeed, by applying that theory to the $SO(8)$ case one precisely obtains that inequivalent gaugings can be characterized by $\omega \in [0, \frac{\pi}{8}]$. One also obtains that result for $SO(4,4)$. However, the other $SO(p,q)$ gaugings have a larger domain of inequivalence, namely $[0, \frac{\pi}{4}]$. This has to do with the fact that for $SO(8)$ and $SO(4,4)$ the triality transformations can be embedded in $E_{7(7)}$ allowing for an additional identification between gaugings, leading to a smaller range of inequivalent gaugings than for the other $SO(p,q)$ groups for which there is no such triality relation.

Chapter 8

Vacuum structure

We will now discuss the vacuum structure of the $SO(p,q)$-gaugings. First we will review what results have already been obtained by others, then we turn to some results of our own.

8.1 Overview

Before discussing the recent developments regarding the one parameter family of inequivalent gaugings, let us first discuss some results already obtained for the original gaugings. Already in the 80s, the vacuum structure of the original $SO(p,q)$-gaugings was examined in detail.

In particular the $SO(8)$ gauging has been well examined. For instance in [22] all critical points that preserve at least an $SU(3)$ subgroup have been classified. These were all found to be AdS vacua. Other parts were discussed in [23]. These were all examined analytically. For the $SO(8)$ gauging many new critical points with no or a few $U(1)$ factors as residual symmetry have been obtained via numerical methods [25][26]. One might wonder whether the used method would be able to find most vacua, however it turns out that it is somehow better at finding vacua with little symmetry than with a lot. For instance it did not find certain critical points that were later analytically found. Also it considered only AdS vacua. The other $SO(p,q)$ gaugings have also been looked at. For example in [24]. However, few critical points have been obtained, apart from the maximmaly symmetric $SO(p) \times SO(q)$ vacua at the origin which they all exhibit. For example the $SO(5,3)$ gauging contains a $SO(5) \times SO(3)$ dS vacuum at the origin. It seems that the $SO(p,q)$ potentials simply harbour less critial points than the $SO(8)$ potentials.

After the discovery of the one-parameter family new results were swiftly obtain in a series of papers. The entire $SO(4)$-invariant sector was examined in [31]. The $SU(3)$ sector of the $SO(8)$ gaugings was examined in [29] and also in [30]. The G_2 were examined in [28]. In [33] it was found that in the $SO(4,4)$ gauging dS vacua can be obtained which get arbitrarily close to being stable. Truly stable dS vacua have still not been found however. In [31] and [32] the very interesting relation between triality and the periodicity of ω were discussed. Here it was observed that s and c embeddings would results in equivalent potentials for different values of ω. To be more precies, it was found that the potentials obtains via the s embedding at some ω are the same as those obtained via the c embedding only with $\omega + \frac{\pi}{4}$. Later this behavior was also discussed in the context of symplectic deformations.

All in all many vacua have been obtain, certainly with the recent developments concerning the family of gauging, with all kinds of residual symmetries and stability properties. Certainly, many subgroups have been picked to truncate to, but a systematic analysis has not really been made. This leaves room for the possibility of finding undiscovered vacua, by looking at other subgroups to truncate to.

8.2 Results

In order to be able to calculate the scalars invariant under some subgroup $SO(p,q)$ and calculating the scalar potential of the corresponding truncation, we had to resort to Mathematica to do the work for us. The general structure of the program code is as follows. One first constructs the basis as described in the previous chapter for the $E_{7(7)}$ generators. From these one picks the symmetric combinations in the set $\{t_A^B\}$ to construct a general scalar in the 35_v. Likewise one picks the combinations of the generators t_{ABCD} corresponding to self-dual forms Σ_{ABCD} to construct a general scalar in the 35_s.
One then chooses a subgroup of which one knows the specific embedding $H \subset SO(p,q) \subset SL(8)$ and therefor the corresponding embedding $H \subset E_{7(7)}$. One then calculates the commutators of the generators of that subgroup with the general scalars in the 35_v and the 35_s. The invariant scalars are precisely those subsets for which the commutator with the subgroup generators vanish. Demanding this results in a set of linear equations which can be solved to obtain the subset of the physical scalars are invariant.
One can then straightforwardly calculate the potential using only the invariant scalars, which can then be further analysed, either exact or numerically.

In order to test our code we have first tried to examine the cases considered in [31] and [33]. We obtained the same critical points as in those articles, indicating that our code is correct. Unfortunately it took us quite a while to get everything to the point of working correctly, which left us with little time to systematically examine other subgroups. Since the $SO(8)$ and $SO(4,4)$ case have been examined in more detail than say the $SO(5,3)$ gauging, we looked at this specific case.

We looked at the some large subgroups of the maximal compact subgroup $SO(5) \times SO(3)$. Of these only $SO(5)\times SO(2)$, $SO(4)\times SO(3)$ had a manageable number of invariant scalars, namely 2 and 3 respectively. Upon analyzing the corresponding scalar potentials we found, in both cases, only one critical point, regardless of the value of c. This is the maximally symmetric unstable dS vacuum at the origin, which is already known. Smaller subgroups were found to have too many invariant scalars and we were not able to analyze the corresponding potentials, at least not in the available time. Once could for instance try to find if one could mod out some of these scalars by a discrete group similar to what is done in [31] and [33], to obtain a manageable number of scalars.

Chapter 9

Conclusion

In this thesis we examined the vacuum structure of the one parameter family of $SO(p, q)$ gaugings of maximal supergravity in four dimensions. We started off in Chapter 2 by examining global supersymmetry, a symmetry relating bosons and fermions. In Chapter 3 we looked at field theories with local supersymmetry, and saw that such theories necessarily contain gravity. Hence the name supergravities. We especially focused on the class of supergravity theories with the maximal amount (or half that) of supersymmetry possible, which were all seen to have a similar structure. In Chapter 4 we turned to the problem of gauging these supergravity theories. This in order to generate a scalar potential, which ungauged theories lack, which allows for more interesting dynamics. We saw that a very convenient way to describe all possible gaugings is via the so called embedding tensor, which is an object that captures how the different vector fields of the theory couple to the generators of the group one wishes to gauge. We also noted that in order to examine the vacuum structure of such theories one can restrict oneself to scalars invariant under some compact subgroup of the gauge group. This allows for enormous simplifications.

In Part II we turned our attention to the maximally supersymmetric supergravity in four dimensions, which exhibits $N = 8$ local supersymmetry. We found that electric-magnetic duality plays a prominent role. Due to the existence of such a duality group one can describe the unique theory with different inequivalent Lagrangians, each with their own symmetry group. This amounts to choosing a certain symplectic frame that determines which vector fields are considered electric and enter the Lagrangian and which are their magnetic duals. In Chapter 6 we saw that due to this freedom one can obtain inequivalent gaugings of the same subgroup of the duality group. These symplectic deformations were only recently discovered and still under examination and amount to using different linear combinations of electric and magnetic vector fields for the gauging.

We then examined the $SO(p, q)$ gaugings in detail in Part II, and found that there is a family of inequivalent such gaugings. In Chapter 8 we discussed known results concerning the vacuum structure and saw that the discovery of the one-parameter family sparked renewed interest in it. New vacua were obtained , which have no counterpart in the previously only known standard gauging, with new interesting properties.

Finally we presented the results of our attempt to contribute to our knowledge of the vacuum structure by finding new vacua. Sadly we were unable to find previously unknown vacua. This due to limited time and the problem that many subgroups come with to many invariant scalars. This limits us in our ability to analyse the corresponding scalar potentials.

The fact that we did not find any new vacua does not mean none can be found. We were only able to look at very few subgroups and a more systematic approach might be in order. Also, the problem that most subgroups come with many invariant scalars could be tackled by further reducing this number by, for example, modding out by some discrete subgroup of automorphisms of the gauge group which in addition leave the subgroup one looks at invariant. In addition more computational power could be feasible in order to also be able to examine more complicated scalar potentials.

Let us conclude with some remarks which address some open problems concerning other topics of this thesis rather than the vacuum structure. For example, the theory on symplectic deformations and the cosets describing inequivalent gaugings has just recently been developed for the maximal supergravity. It would be interesting to see how these concepts could be applied to other four-dimensional theories with less supersymmetry.

Another interesting line of research would be about a possible higher dimensional origin of the c (or ω) parameter. It is long known that the originally constructed $SO(8)$ gauged theory has a clear higher dimensional origin as a compactification of 11 dimensional supergravity on S^7. It would be interesting to see whether something similar holds for the whole family of gaugings.

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