# A geometric notion of stationary black hole entropy

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ונמקו כל צבא השמים
ונגלו כספר השמים וכל צבאם יבול
כנבל עלה מגפן וכנבלת מתאנה

*And all the host of the heaven shall be dissolved, and the heavens shall be rolled together as a scroll, and all their host shall fall down like the leave falls from the vine tree, and as a falling fig from the fig-tree.*

Jesha'jahu 34:4

**Abstract**

In this thesis, the connection between the geometry of black holes and their entropy will be discussed. First, some mathematical tools will be overviewed which will be needed in this discussion. Then some elementary parts of general relativity will be discussed. From there, we will also make the step towards more general theories of gravitation by discussing the Lagrangian and Hamiltonian formalism. After that, all the needed information will be gathered to discuss a method due to Robert Wald to give an expression for the black hole entropy using the diffeomorphism invariant nature of a vast class of gravitational theories. When this method is understood, some examples of black hole solutions will be calculated. The so-called Gauss-Bonnet Lagrangian will give a rather surprising thermodynamical effect. The thesis will end with a discussion of Wald's method and interesting questions which could continue this research.

# Contents

# Notation and conventions

The following signature of the metric is used: $(-,+,+,+)$. The components of a general tensor $X$ will be notated $X^{\mu\ldots\nu}_{\alpha\ldots\beta}$. The determinant of the metric $g_{\mu\nu}$ is written $g$, and this is also used to denote the metric tensor itself; context always makes clear what is meant. For other determinants we use $|\ |$. For dual vectors we will use greek letters, and for vectors we will use latin capital letters. Greek indices indicate the coordinates on the space-time manifold and start with 0; Latin indices will indicate hypersurfaces and start with 1. Sometimes, in arguments or matrices, we will omit the index and just write $x$ for our coordinates. Ofcourse, throughout the thesis we will use Einstein's summation convention:

$$X^\mu \omega_\mu \equiv \sum_\mu X^\mu \omega_\mu$$

where $\mu$ runs over the space-time manifold. If basis vectors are written, brackets will be placed with the indices: $e_{(\mu)}$ and $e^{(\mu)}$. If indices are not written, it is clear from the context what is meant. If the coordinate basis is used, these brackets are dropped: $\partial_\mu$ and $dx^\mu$. Most of the time we will use $G = c = 1$, but sometimes these constants pop up. We will assume that space-time has four dimensions, but often the generalization is not that difficult. Equations and results which will be important later on will be boxed:

$$\boxed{1 \neq 2}$$

We will both use the notation $X_\mu X^\mu$ and $X^2$, and $F_{\mu\nu}F^{\mu\nu}$ and $\mathbf{F}^2$. Some useful geometric identities are listed in the appendix for reference. Forms like the Lagrangian $\mathbf{L}$ and the volume form $\epsilon$ will be notated in a bold style, except for zero-forms. Space-time is assumed to be equipped with a symmetric connection contructed from the metric and the cosmological constant $\Lambda$ will be put to zero. Most of the time vector fields will be denoted by $X$ or $\xi$. For fields and pullbacks we often will use the same symbol $\phi$, but the context will make clear what is meant; otherwise the latter will be denoted by $f$.

# Chapter 1

# Introduction and motivation

The idea of an object, which has an escape velocity such that even *light* can't escape is not very new; Laplace postulated already such an object in the 18th century. He based his idea upon Newtons theory of gravitation. When Einstein found his field equations of general relativity, his freshly found theory also included black hole solutions. A firm theoretical basis for these objects was created in this way.

For years the idea of a black hole being some sort of one-way membrane was generally accepted by most of the physicists. That's not strange; classically everything that passes the Schwarzschild radius is forced to stay inside the black hole. A curious fact which was discovered in the late sixties is that the equations of motion of a black hole have the same form as the thermodynamic equations. For instance, the surface of a black hole behaves as if it were an entropy. Bekenstein postulated that this is not a coincidence; he stated that this surface actually gives us a measure of the black hole entropy. He noted that the second law of thermodynamics could easily be violated by throwing in material in a black hole, if this black hole wouldn't possess some sort of entropy. So Bekenstein made the bold conjecture that black holes carry an intrinsic entropy, given by the surface times some constant, in the framework of general relativity. But according to thermodynamics, an entropy implies a temperature. It came as quite a shock when Stephen Hawking found in 1974 the possibility that black holes radiate thermally as black bodies and lose mass. This gave the needed temperature for a black hole entropy, but it took quite some time before the scientific community could accept such a new idea. With this a macroscopic notion of the black hole entropy was developed.

But what about the *microscopic* notion? A microscopic notion of entropy means that one should be able to count the microstates of the system. And for that one needs a quantum mechanical description. So here quantum mechanics and general relativity meet each-other. It is sometimes said that the black hole will be the object of understanding of a theory of quantum gravity just like the Hydrogen-atom gave enormous insight in the theory of quantum mechanics. So this gives us a very good reason of doing research on how to calculate such black hole entropies. Here we will focus on the macroscopic formulation.

Physicists expect that a theory of quantum gravity possesses a low energy effective action which describes the space-time for weak curvatures and long distances. "Weak" and "long" are with respect to the usual Planck quantities. This effective action should be the Hilbert action plus higher curvature terms induced by quantum effects. Ofcourse, here one assumes that those higher curvature terms are not of the same order as the quantum fluctuations themselves; otherwise it would be quite useless to study modifications to the classical case. In this framework black hole thermodynamics is considered here.

The author will assume from the reader some knowledge of differential geometry and general relativity, although most of the used tools will be discussed here. The goal of this thesis was to be as complete as possible and to find a compromise between physical relevance and mathematical rigor. Hopefully this goal is achieved in some way.

# Chapter 2

# Crucial differential geometry

Physicists and mathematicians often differ in their explanation of what a tensor essentially is. Here we will follow the mathematician's point of view. In this chapter the ideas of tensors, forms, coordinate transformations etc. will be shortly repeated.

## 2.1 General tensors

Let's assume we have a vector space $V$ and its dual $V^*$. A tensor $T$ of type (r,s) is defined as a multilinear mapping in the following way:

$$T : \underbrace{V^* \times V^* \times V^* \ldots \times V^*}_{r\ times} \underbrace{\times V \times V \ldots \times V}_{s\ times} \rightarrow \mathbb{R} \tag{2.1}$$

So we have the following construction[1]: with a vector space one is able to construct a dual vector space, and a general tensor then is defined via its action on these objects. A tensor of type (0,0) is a scalar, a tensor of type (1,0) is called a contravariant vector and a tensor of type (0,1) is called a covariant vector, a dual vector or a 1-form. Most of the time, our tensors will live on a 4-dimensional manifold called space-time. More specifically, a vector is defined in the tangent space $T_xM$ at a point $x$ on the manifold $M$. A covector is defined in the so-called cotangent space $T_x^*M$, which is the dual space of the tangent space. According to our definition, a covector is nothing more than a linear map, which maps vectors to the real line. So we have with $a, b \in \mathbb{R}$

$$\omega(aX + bY) = a\omega(X) + b\omega(Y) \tag{2.2}$$

Physicists sometimes tend to forget that there's also a basis in the game. We introduce bases on the tangent space and cotangent space, and with that we can write for $\omega \in V^*$ and $X \in V$ :

$$\begin{aligned} \omega &= \omega_\mu e^{(\mu)} \\ X &= X^\mu e_{(\mu)} \end{aligned} \tag{2.3}$$

[1] Formally we should introduce functions on manifolds first to relate the tangent vectors of those functions to vectors.

Note the braces around $\mu$ in our basisvectors; it emphasizes that $(\mu)$ indicates a whole vector, and not merely a component! It is a good thing to notice that the vector or dual vector itself is base independent,whereas the components ofcourse are not. This is obvious from the definition of a vector being the tangent vector of some curve on the manifold. The whole idea of a vector space and it's dual is that if we act with an element from the one space on the element of the other, we get a real number. We define this action via the basisvectors of the vector space and it's dual. For if we know what an action does to the (dual) basis, we also know what an action does to the (dual) vector. We define the duality between the two spaces as an orthonormality relation:

$$e^{(\mu)}(e_{(\nu)}) = \delta^{\mu}_{\nu} \tag{2.4}$$

With this, we can write the action of a vector on a dual vector in the following way:

$$X(\omega) = X^{\mu}\omega_{\nu}e^{(\nu)}e_{(\mu)} = X^{\mu}\omega_{\mu} \in \mathbb{R} \tag{2.5}$$

and we see that it does not depend on the choice of basis. If we know the (dual) vector, and we also know the basis, then we have a way to find our components. For instance, if we have a vector $X \in V$ and a basis $\{e_{(\mu)}\}$ then, according to our earlier discussion, we have

$$X(e^{(\mu)}) = X^{\nu}e_{(\nu)}(e^{(\mu)}) = X^{\mu} \tag{2.6}$$

The same goes for a dual vector $\omega$.

That raises a question: what are those basis vectors for our vector space and dual vector space? Vectors are defined in the tangent space, and it turns out we can take the differential operators as our basis: $e_{(\mu)} = \frac{\partial}{\partial x^{\mu}}$. This views the vector as an operator, and this gives us an intrinsic notion of the idea of a vector: it maps functions on the manifold to the real line. The set of all differential operators is called the coordinate basis. The basis of our dual tangent space can be taken as the set of differentials of $x^{\mu} : e^{(\mu)} = dx^{\mu}$.

In physics, we're often interested in how our components change if we turn to another observer. A point on a manifold can be described by different coordinate systems, and a manifold is defined in such a way that we can jump from one coordinate system to another via a coordinate transformation. For instance, in special relativity we want to be able to jump back and forth between two observers in a continuous way with Lorentz transformations. We already know how differentials and derivative operators change under a coordinate transformation $x \to x^{'}(x)$, namely

$$\begin{aligned} e^{('\mu)} &= dx^{'\mu} = \frac{\partial x^{'\mu}}{\partial x^{\nu}} dx^{\nu} \\ e_{('\mu)} &= \frac{\partial}{\partial x^{'\mu}} = \frac{\partial x^{\nu}}{\partial x^{'\mu}} \frac{\partial}{\partial x^{\nu}} \end{aligned} \tag{2.7}$$

We see that they transform in an opposite way. Because (dual) vectors are independent of the coordinates by construction, we have that for a vector $X$

$$X^{\mu}e_{(\mu)} = X^{'\mu}e_{('\mu)} = X^{'\mu}\frac{\partial x^{\nu}}{\partial x^{\mu}}e_{(\nu)} \tag{2.8}$$

This means that our component $X^\mu$ transforms via[2]:

$$X'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} X^\nu \tag{2.9}$$

In the same way we can derive that for a dual vector $\omega$:

$$\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\omega_\nu \tag{2.10}$$

If we work with gravitational theories, and more specifically general relativity, we work in a metric space. The metric is our symmetric and bilinear mapping which enables us to define lengths, angles and makes it possible to switch between covariant components and contravariant components. First space-time was just a topological space, but with this metric we add an additional piece of structure to it. For two vectors $X, Y \in V$ we define the metric via the inner product. This inner product should also be base independent:

$$X \cdot Y = Y \cdot X = g(X,Y) = X^\mu Y^\nu g(e_{(\mu)}, e_{(\nu)}) = g_{\mu\nu} X^\mu Y^\nu \tag{2.11}$$

where $g_{\mu\nu} = g(e_{(\mu)}, e_{(\nu)})$. So here $g$ is not the determinant of the metric, but the tensor acting on the vectors $(X,Y)$. Later on the context always will make it clear what $g$ means. The connection between contravariant and covariant components becomes clear now, if we consider the foregoing discussion:

$$X_\mu = X(e_{(\mu)}) = X^\nu g(e_{(\nu)}, e_{(\mu)}) = X^\mu g_{\mu\nu} \tag{2.12}$$

So with the metric we map vectors on dual vectors, and in fact we can lower any contravariant index on a tensor with the metric. Physically the metric is very important, because in general relativity it replaces the scalar potential of Newton. We raise indices via $g^{\mu\nu}$.

Ofcourse, with this we are eager to find out how a general tensor can be described by its components. For this, we have to introduce the concept of the tensor product $\otimes$. Take for example two tensors T and S of type $(m,0)$ and $(n,0)$ respectively. The tensor product of T and S, denoted $T \otimes S$, is defined by its action on dual vectors:

$$T \otimes S(\omega_{(1)}, \ldots, \omega_{(m)}, \theta_{(1)}, \ldots \theta_{(n)}) \equiv T(\omega_{(1)}, \ldots, \omega_{(m)}) S(\theta_{(1)}, \ldots \theta_{(n)}) \tag{2.13}$$

In words: first act with $T$ on the first $m$ dual vectors, and then act with $S$ on the remaining $n$ dual vectors, and then multiply the answers. This can be generalized easily for mixed tensors of all kinds. Now we can build a basis for arbitrary tensors with this tensor product. For instance, a tensor $T$ of type (2,1) can be written as

$$T = T^{\mu\nu}{}_\lambda e_{(\mu)} \otimes e_{(\nu)} \otimes e^{(\lambda)} \tag{2.14}$$

The behaviour of such objects under coordinate transformations is determined by eq.(2.1) and the transformation laws for vectors and dual vectors. The components of a mixed tensor will transform as

$$\boxed{X'^{\alpha\cdots'\beta}_{\ '\mu\cdots'\nu}(x') = \frac{\partial x'^\alpha}{\partial x^\lambda} \cdots \frac{\partial x'^\beta}{\partial x^\gamma} \frac{\partial x^\sigma}{\partial x'^\mu} \cdots \frac{\partial x^\rho}{\partial x'^\nu} X^{\lambda\cdots\gamma}_{\sigma\cdots\rho}(x)} \tag{2.15}$$

[2]We are always free to interchange the primes.

## 2.2 Forms

In physics we often encounter tensors which are antisymmetric. An antisymmetric covariant tensor $T$ in two indices is defined via

$$T(\cdots X, \cdots Y, \cdots) = -T(\cdots Y, \cdots X, \cdots) \tag{2.16}$$

for $X, Y \in V$. If we fill in our basis vectors for $X$ and $Y$, we see that the tensor components are antisymmetric if 2 arguments are changed. A same definition goes for a contravariant tensor. With this we define p-forms. A p-form is a tensor of type (0,p) which is totally antisymmetric. We need the so-called exterior product to define this p-form. In mathematical texts people often define the exterior product by determinants and the action of such a p-form on vectors. We will define it via the tensor components. To describe an antisymmetric tensor, we can antisymmetrize the basis $\{e^{(\mu)}\}$. So we define an antisymmetric tensorproduct of p basisvectors via the tensorproduct:

$$e^{[\mu_1} \otimes \cdots \otimes e^{\mu_p]} = \sum_{j=1}^{p!} (-1)^{\pi(j)} e^{\mu_1} \otimes \cdots \otimes e^{\mu_p} \tag{2.17}$$

The function $\pi(j)$ is 0 if the permutation of the indices is odd, and $\pi(j)$ is 1 if the permutation is even. With this we define the socalled wedge product $\wedge$ via the components:

$$e^{[\mu_1} \otimes \cdots \otimes e^{\mu_p]} = e^{\mu_1} \wedge e^{\mu_2} \wedge \ldots \wedge e^{\mu_p} \tag{2.18}$$

A general p-form can now be written as

$$\omega = \omega_{\mu_1 \cdots \mu_p} e^{\mu_1} \wedge \ldots \wedge e^{\mu_p} \tag{2.19}$$

The space of all p-forms on a manifold $M$ will be denoted as $\Lambda^p(M)$.We will encounter forms when we want, among other things, to do integration. They turn out to be the natural objects to integrate. This will be explored in the next chapter. A famous example of a 2-form is the Maxwell field tensor $\mathbf{F} = F_{\mu\nu} dx^\mu \wedge dx^\nu$.

To conclude this section, we remark that for an n-dimensional manifold there are only forms up to type $(0, n)$. That's most easily seen by the basis of our forms. If it contains 2 linear dependent vectors, the antisymmetrized product becomes 0. So in 4 dimensions there are zero,one,two,three and four-forms. A p-form on an n-dimensional manifold has $\binom{n}{p}$ distinct coefficients, due to the antisymmetric property of the form. And as a result, an n-form on an n-dimensional manifold has just one independent component, and can be written as $f(x) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$, with $f(x)$ a scalar function.

## 2.3 Symmetry and antisymmetry

A contravariant or covariant tensor always can be decomposed in an antisymmetric part and a symmetric part. Take for instance a tensor of type (0,p). The antisymmetric part is defined by

$$X_{[\mu_1 \cdots \mu_p]} \equiv \frac{1}{p!} \sum_{j=1}^{p!} (-1)^{\pi(j)} X_{perm(\mu_j)} \tag{2.20}$$

and the symmetric part is defined by

$$X_{(\mu_1 \cdots \mu_p)} \equiv \frac{1}{p!} \sum_{j=1}^{p!} X_{perm(\mu_j)} \tag{2.21}$$

where $perm(\mu_j)$ indicates the permutation of the indices. For instance, for a tensor of type (0,2) we get

$$X_{[\mu\nu]} \equiv \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu}) \tag{2.22}$$

and

$$X_{(\mu\nu)} \equiv \frac{1}{2}(X_{\mu\nu} + X_{\nu\mu}) \tag{2.23}$$

so that we have $X_{\mu\nu} = X_{(\mu\nu)} + X_{[\mu\nu]}$. A useful fact is that if we contract a (anti)symmetric tensor with a general tensor, we only have to consider the (anti)symmetric part of it. After all, the contraction between a symmetric tensor $X^{\mu\nu} = X^{(\mu\nu)}$ and an antisymmetric tensor $\omega_{\mu\nu} = \omega_{[\mu\nu]}$gives us 0:

$$\begin{aligned} X^{\mu\nu}\omega_{\mu\nu} &= \frac{1}{2}(X^{\mu\nu} + X^{\nu\mu})\omega_{\mu\nu} \\ &= \frac{1}{2}(X^{\mu\nu}\omega_{\mu\nu} - X^{\mu\nu}\omega_{\mu\nu}) = 0 \end{aligned} \tag{2.24}$$

It is easily proven that this is also true for the contraction between a total antisymmetric tensor and a total symmetric tensor of arbitrary type.

## 2.4 Pullbacks, pushforwards and diffeomorphisms

Coordinate transformations can be looked upon in different manners. Imagine we have an m-dimensional manifold $M$ with coordinate functions $x^\mu : M \to \mathbb{R}^m$. If we want to change coordinates, we can do two things. First, we could simply pick new coordinate functions $y^\mu : M \to \mathbb{R}^m$. An example of this is to change from Cartesian coordinates to spherical coordinates. We could also actually move the points on the manifold and evaluate the coordinates of the new point. This would be accomplished by a diffeomorphism[3] $\phi : \mathbb{R}^3 \to S^3$. This idea will be outlined here, and in the end we will obtain a nice and more general way of looking at coordinate transformations.

Take a look at figure (2.1). A question we can ask ourselves is: if we have a vector $X \in T_xM$ or a covector $\omega \in T_y^*N$, is there a natural way for these objects to induce a vector or covector on another manifold if we have a diffeomorphism between $M$ and $N$? Consider two arbitrary manifolds $M$ and $N$ with coordinate systems $x^\mu$ and $y^\alpha$ respectively. Imagine there exists a diffeomorphism $\phi : M \to N$ and a function $f : N \to \mathbb{R}$. We can construct a new function from $M$ to $\mathbb{R}$ by simple composition:

$$\phi_* f = f \circ \phi = f(\phi(x^\mu)) : M \to \mathbb{R} \tag{2.25}$$

[3]A diffeomporhism is a smooth map with a smooth inverse, and people sometimes say that two manifolds can be smoothly deformed into each other if there exists a diffeomorphism between them. In this way one can construct equivalence classes of manifolds on basis of the existence of such diffeomorphisms between them.

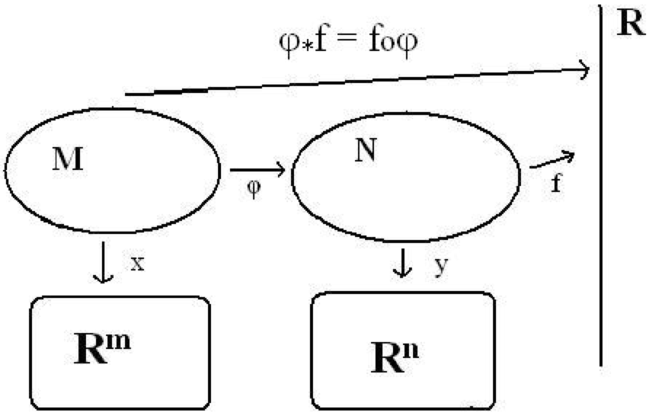


Figure 2.1: *The pullback*

Here the pullback $\phi_* f$ of $f$ is defined: the function $f$ is pulled back from $N$ to $M$. This can also be done for vectors and covectors, but there are some limitations on this. We can't pull back vectors from $N$ to $M$ with $\phi$, but we can push them forward! Remember we can look at a vector as a derivative operator $X : F(M) \to \mathbb{R}$, where $F(M)$ denotes the space of all smooth functions on $M$. So we can define the pushforward of $X$ via its action on functions on $N$. If $X \in T_x M$, the pushforward vector $\phi^* X \in T_y N$ is defined by

$$\boxed{(\phi^* X) f = X(\phi_* f)} \tag{2.26}$$

So what does this mean? $X$ is a vector on $M$ which has a coordinate base $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and thus can be expressed as $X^\mu \partial_\mu$. The pushforward of X is a vector on $N$ which has a coordinate base $\partial_\alpha = \frac{\partial}{\partial y^\alpha}$ and thus can be expressed as $(\phi^* V)^\alpha \partial_\alpha$. So the relation between those components can be discovered by looking at the action on a function:

$$\begin{aligned} (\phi^* V)^\alpha \partial_\alpha f &= V^\mu \partial_\mu [\phi_* f] \\ &= V^\mu \partial_\mu [f \circ \phi] \\ &= V^\mu \partial_\mu [f(y^\alpha(x^\mu))] \\ &= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f \end{aligned} \tag{2.27}$$

via the chain rule. So the pushforward operator $\phi^*$ can be seen as a matrix:

$$[\phi^*]^\alpha{}_\mu = \frac{\partial y^\alpha}{\partial x^\mu} \tag{2.28}$$

Now the nice thing about this discussion becomes clear: we obtained again the transformation law for vectors, only this is a generalization; $M$ and $N$ can be two different manifolds with even a different dimension.

Now we pick a covector $\omega \in T^*_y N$. With a diffeomorphism $\phi : M \to N$ we can't push it forward, but we *can* pull it back. This is done by looking at the action of $\omega$ on $\phi^* X \in T_y N$:

$$(\phi_* \omega) X = \omega(\phi^* X) \tag{2.29}$$

In this case the pullback operator is given by

$$[\phi_*]_\mu^{\ \alpha} = \frac{\partial y^\alpha}{\partial x^\mu} \tag{2.30}$$

Ofcourse we want to extend these ideas to more general tensors. That's no problem, since tensors are multilinear functions. A tensor of type $(0, l)$ is a linear map from the direct product of $l$ vectors to $\mathbb{R}$. So we define the pullback of a tensor $T = T_{\mu_1 \cdots \mu_l} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_l}$ via its action on the pushed-forward vectors $X^{(i)}$:

$$(\phi_* T)(X^{(1)}, \ldots, X^{(l)}) = T(\phi^* X^{(1)}, \ldots, \phi^* X^{(l)}) \tag{2.31}$$

with components

$$(\phi_* T)_{\mu_1 \cdots \mu_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_l}}{\partial x^{\mu_l}} T_{\alpha_1 \cdots \mu_l} \tag{2.32}$$

The same goes for a tensor of type $(k, 0)$. In that case we have for the components of $S$

$$(\phi^* S)^{\alpha_1 \cdots \alpha_k} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} S^{\mu_1 \cdots \mu_k} \tag{2.33}$$

Notice that the matrices of the two operations are the same, and that the difference lies in the contraction. Now a little example would be nice. The metric of the unit sphere in $\mathbb{R}^3$ induces a metric on $S^2 \subset \mathbb{R}^2$ by its surface. So we have the following case:

- $N = \mathbb{R}^3, y \in N$ with $y^i = (x, y, z)$
- $M = S^2, x \in S^2$ with $x^k = (\theta, \phi)$
- $\phi : M \to N$ with $\phi(\theta, \varphi) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$

The metric on $S^2$ now can be calculated by pulling the metric on $\mathbb{R}^3$ back to $S^2$. The metric on $\mathbb{R}^3$ is simply $g_{ij} = \delta_i^j$, so we get for the induced metric on $S^2$

$$\begin{aligned} (\phi^* g)_{kl} &= \frac{\partial y^i}{\partial x^k} \frac{\partial y^j}{\partial x^l} g_{ij} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \end{aligned} \tag{2.34}$$

The invertibality of the diffeomorphism $\phi$ allows us to use both $\phi$ and its inverse $\phi^{-1}$ to move tensors from $M$ to $N$. With this we can define the pushforward on a tensor T of type $(k, l)$:

$$\begin{aligned} &(\phi^* T)(\omega^{(1)}, \ldots, \omega^{(k)}, X^{(1)}, \ldots, X^{(l)}) = \\ &T(\phi_* \omega^{(1)}, \ldots, \phi_* \omega^{(k)}, [\phi^{-1}]^* X^{(1)}, \ldots, [\phi^{-1}]^* X^{(l)}) \end{aligned} \tag{2.35}$$

Physicists demand that any theory of gravitation is diffeomorphism invariant. This is just another way of saying that your theory should be invariant under coordinate transformations. If we have a metric $g_{\mu\nu}$ describing the geometry of the space-time $M$ and a matter field $\psi$, then $(g_{\mu\nu}, \psi)$ and $(\phi_* g_{\mu\nu}, \phi_* \psi)$ describe the same physics for any diffeomorphism $\phi : M \to M$.

But there is more; the line integral can also be properly defined with this machinery. Take a form $\omega \in \Lambda^n(M)$, and a smooth curve $c : [a, b] \subset \mathbb{R} \to M$. Then we can pull back $\omega$ from $M$ to $[a, b] \subset \mathbb{R}$ and as a result $c_* \omega \in \Lambda^1(\mathbb{R})$. So we define

$$\int_c \omega = \int_a^b c_* \omega \tag{2.36}$$

This object is actually independent of the parametrization of $c$. Is this definition sensible? Well, if we write $c_* \omega(t) = h(t)dt$, then we have that

$$\begin{aligned} h(t) &= h(t)dt(e_1) \\ &= (c_* \omega(t))(e_1) \\ &= \omega(c(t))(c^* e_1) \\ &= (\omega(c(t)))\dot{c}(t) \end{aligned} \tag{2.37}$$

and we recognize our well-know expression $\int_c \omega_\mu dx^\mu = \int_a^b \omega_\mu(c(t)) \frac{dx^\mu}{dt} dt$. Later on we will encounter the same idea in looking at surface integrals.

# Chapter 3

# Differentiation and integration

In differential geometry, we can define a lot of different differential operators, because there are quite some choices of "taking differences of tensors" on a manifold. We will especially take a look at Lie-derivatives, but also covariant differentiation, exterior differentiation and functional differentiation will pass by briefly.

## 3.1 Covariant differentiation

On our manifold we can define several different types of derivative operators. The partial derivative is the most simple one, but it isn't tensorial, and to write down equations which are covariant we need a tensorial derivative. The one that should be familiar is the covariant derivative. In general one constructs a covariant derivative by the demand that the equations of motion should be invariant under certain symmetry transformations. In quantum electrodynamics for instance these transformations are $U(1)$ transformations. In general relativity and other theories of gravity, these transformations are general coordinate transformations. Here, the covariant derivative is an operator from (k,p) to (k,p+1) types of tensors. For this derivative one needs an extra piece of structure on the manifold, called the connection $\Gamma$. The connection tells you how to compare basisvectors at different points on your manifold. We will only consider the case in which the connection is made from the metric. The connection coefficients are defined mathematically as the components of the directional derivative of the basis vectors. This expresses the idea that $\nabla_X e_{(\mu)}$ should be a linear combination of the basisvectors $\{e_{(\mu)}\}$:

$$\nabla_\mu e_{(\nu)} = \Gamma^\lambda_{\nu\mu} e_{(\lambda)} \tag{3.1}$$

This is also denoted as $\nabla_X e_{(\mu)} = e_{(\nu)} \omega^\nu{}_\mu(X)$, where $\omega$ is called the connection form. With some very basic assumptions about the connection[1][2], like $\nabla_{fX} Y = f \nabla_X Y$ and some familiar rules for differentiation about products, one can derive the covariant derivative of an arbitrary tensor $T$. For instance, one demands that for a scalar function $f$ the covariant derivative with respect to a vector field $X$ turns out to be

$$\nabla_X f = X(f) = X^\mu \partial_\mu f \tag{3.2}$$

The covariant derivative of a vector $Y$ with respect to $X = X^\nu \partial_\nu$ in a coordinate base is

$$\nabla_X Y = X^\nu (\partial_\nu Y^\mu + Y^\alpha \Gamma^\mu_{\alpha\nu})\partial_\mu \tag{3.3}$$

What a covariant derivative does to a form $\omega$ can be derived by the fact that $\omega(Y) \in \mathbb{R}$ and the product rule. One gets

$$\nabla_X[\omega(Y)] = (\nabla_X \omega)Y + \omega(\nabla_X Y) \tag{3.4}$$

If one fills in the basis vectors for $X = X^\mu \partial_\mu$ and $\omega = \omega_\nu dx^\nu$ and uses $dx^\mu(e_\nu) = \delta^\mu_\nu$ the familiar result for covariant vectors is obtained:

$$\nabla_X \omega = X^\lambda(\partial_\lambda \omega_\nu - \Gamma^\mu_{\nu\lambda}\omega_\mu)dx^\nu \tag{3.5}$$

With this the expression for the covariant derivative acting on any general tensor can be obtained:

$$\nabla_\lambda T^{\mu\cdots}_{\nu\cdots} = \partial_\lambda T^{\mu\cdots}_{\nu\cdots} + \Gamma^\mu_{\rho\lambda} T^{\rho\cdots}_{\nu\cdots} + \cdots - \Gamma^\rho_{\nu\lambda} T^{\mu\cdots}_{\rho\cdots} + \cdots \tag{3.6}$$

Most of the time we only differentiate with respect to the basis vectors $e_{(\mu)} = \partial_\mu$ instead of a general vector field $X$, and then we can forget about the components $X^\mu$. We shall demand that the covariant derivative of the metric is zero. So, raising or lowering indices and taking covariant derivatives commute. With this demand, the connection becomes

$$\boxed{\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})} \tag{3.7}$$

This can also be derived by a variational argument. Throughout this thesis, we will only work with this so called metric connection, which is symmetric in $\mu$ and $\nu$. This implies that in a coordinate base our space-time is torsion free.

## 3.2 Lie-differentiation

Now to another kind of derivative operator, called the Lie-derivative $\pounds_X$. This takes (k,p) types to (k,p) types. First we will approach it via the tensor components and after that we will show the connection with pullbacks. According to the theory of differential equations a vector field induces a certain flow; for a given vector field on a manifold, every vector can be viewed as the tangent vector of a curve. So, we can set up a congruence of curves with a given vectorfield and vice versa. These curves can be used to compare tensors at different points on the manifold. That is the essence of Lie-differentiating a general tensor $T(x)$ with respect to a vector field $X(x)$: Take $T(x)$, drag it along the curves set up by $X$ to a point $y$, and take the difference between the two tensors in the limit $x \to y$. So we want to find an expression for $T(x^{'}) - T^{'}(x^{'})$. What we essentially do here is to push the tensor forward, and then pull it back to compare it with the tensor already there.

For an infinitesimal transformation of a coordinate $x^\mu$ we can write

$$\begin{aligned} x^{'\mu} &= x^\mu + \frac{dx^\mu}{ds}\delta s \\ &= x^\mu + X^\mu(x)\delta s \end{aligned} \tag{3.8}$$

where we have defined the congruence of curves created by the vector field $X$. To get an expression for $T(x^{'})$, we use Taylor's theorem:

$$T(x^{'}) = T(x + X^\mu \delta s) = T(x) + \delta s X^\lambda \partial_\lambda T(x) + \dots \tag{3.9}$$

Higher orders won't be necessary, because we will take the limit $\delta s \to 0$. Now we want an expression for $T^{'}(x^{'})$. We will take a tensor with components $T^{\mu\nu}$; the idea can be easily generalized.

$$\begin{aligned} T^{'\mu\nu}(x^{'}) &= \frac{\partial x^{'\mu}}{\partial x^\alpha}\frac{\partial x^{'\nu}}{\partial x^\beta}T^{\alpha\beta}(x) \\ &= [\delta^\mu_\alpha + \delta s\ \partial_\alpha X^\mu][\delta^\nu_\beta + \delta s\ \partial_\beta X^\nu]T^{\mu\nu}(x) \\ &= T^{\mu\nu}(x) + [\partial_\alpha X^\mu(x)T^{\alpha\nu}(x) + \partial_\beta X^\nu T^{\mu\beta}(x)]\delta s + \dots \end{aligned} \tag{3.10}$$

again only up to first order. If one wants to generalize the idea for covariant tensors, the relation $x^{'}(x)$ should be inverted to $x(x^{'})$ in order to write down the explicit coordinate transformations. Now we can define our Lie-derivative:

$$\pounds_X T = \lim_{\delta s \to 0} \frac{T(x^{'}) - T^{'}(x^{'})}{\delta s} \tag{3.11}$$

In the specific case that $T$ has components $T^{\mu\nu}$, we see that

$$\boxed{\pounds_X T^{\mu\nu} = X^\lambda \partial_\lambda T^{\mu\nu} - T^{\mu\lambda}\partial_\lambda X^\nu - T^{\lambda\nu}\partial_\lambda X^\mu} \tag{3.12}$$

Note that we used an active point of view here for the coordinate transformations, something we will come back to in chapter four. Important properties of the Lie-derivative are that it is linear and Leibniz. We will see that this derivative becomes important when Killing vector fields and inducing coordinate transformations come into play. Now one could complain about the fact that the expression for the Lie-derivative contains partial derivatives. Fortunately, we can replace all partial derivatives by covariant derivatives because of the symmetry of the connection: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$!

Now all this "comparing tensors" business should raise the desire to use the formalism of pullbacks to define Lie-derivatives. For this we need one-parameter groups of diffeomorphisms. Remember that a vector field $X = X^\mu \partial_\mu$ induces a congruence of curves. These curves will be our diffeomorphisms:

$$\frac{\partial \phi}{\partial t} = X_{\phi(P,t)}, \qquad \phi(P, t = 0) = P \tag{3.13}$$

for an arbitrary point $P$ on our manifold. Note that we wrote the parameter as $t$, and not as $\delta s$, because this is an arbitrary parameter and not just an infinitesimal quantity. But in the case of an infinitesimal $t$, the differential equation (3.13) is again our coordinate transformation

$$x'^{\mu} = (\phi_t(x))^{\mu} = x^{\mu} + tX^{\mu} \tag{3.14}$$

where $x^{\mu}$ is a component of $P$ and $x'^{\mu}$ a component of $\phi_t(P)$. Now we want to define again the Lie derivative of a covariant tensor $T$. What we do is to evaluate $T$ in a point $\phi_t(x)$ and pull it back with $\phi_t^*$ to the original point $x$. Then we can compare it with $T$ already at $x$. Figure (3.1) should make this idea clear for a vector $X$. Then the Lie derivative of a covariant tensor is simply

$$\boxed{\pounds_X T = \lim_{t\to 0} \frac{\phi_t^* T_{\phi_t(x)} - T_x}{t}} \tag{3.15}$$

where $T_x$ simply means "$T$ evaluated in $x$." With some calculating one can see that the two given definitions are the same. Sometimes it is said that Lie-

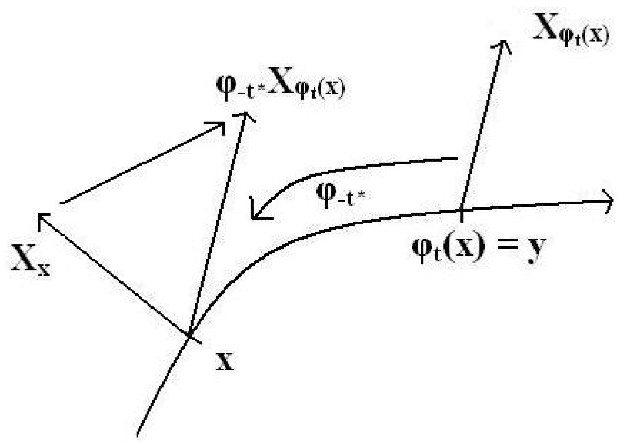


Figure 3.1: *The Lie-derivative*

differentiation is a more primitive notion of differentiation than the covariant one because there is no need for a connection. This is also true for exterior differentiation.

## 3.3 Exterior differentiation

Last but not least we look at the so-called exterior derivative $d$. We are going to make heavily use of it, so familiarity with them is important. This derivative operator is only used for forms[1], and it takes a p-form to a (p+1)-form. If we have a p-form $\omega = \omega_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$, we define the exterior derivative $d\omega$

[1]It is also used for so-called vector-valued forms, and this results in the existence of the spin connection.

as

$$\boxed{d\omega = \frac{1}{p!}\frac{\partial \omega_{\mu_1\cdots\mu_p}}{\partial x^\lambda} dx^\lambda \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}} \tag{3.16}$$

Because it is written in an antisymmetric basis with (p+1) elements, we see indeed that this is a (p+1) form. For example, if we have a function $f(x)$, we have that

$$df(x) = \frac{\partial f}{\partial x^\mu} dx^\mu \tag{3.17}$$

This expression should be familiar, but in this context we see that we explicitly wrote down the gradient of $f(x)$ in its basis $\{dx^\mu\}$. Because this is a 1-form, $f(x)$ can be seen as a 0-form. Especially in writing down Maxwell's equations and Stoke's theorem this derivative is very convenient. Properties of this operator are that it is linear, Leibniz and that $d^2 = 0$ due to the fact that partial derivatives commute. To see that explicitly, one can take a p-form $\omega_{\rho_1\cdots\rho_p} dx^{\rho_1} \ldots dx^{\rho_p}$ and consider $d^2\omega$:

$$\begin{aligned} d^2\omega &= \frac{1}{p!(p+1)!}\frac{\partial^2 \omega_{\mu\nu\rho_1\cdots\rho_p}}{\partial x^\mu \partial x^\nu} dx^\mu \wedge dx^\nu \wedge dx^{\rho_1} \wedge \ldots dx^{\rho_p} \\ &= 0 \end{aligned} \tag{3.18}$$

because the first part is linear in $\mu$ and $\nu$ and the second part is antilinear in $\mu$ and $\nu$. Due to the antilinear character of the exterior product, all of the partial derivatives in it's expression can easily be replaced by covariant derivatives if the connection is symmetric. Let's see this explicitly. For instance, we know that the electromagnetic field strength tensor $\mathbf{F}$ is a two-form. Locally, it can be written as $d\mathbf{A}$, where $\mathbf{A} = A_\nu dx^\nu$ is the familiar vector potential[2]. This becomes

$$\mathbf{F} = d\mathbf{A} = \frac{\partial A_\nu}{\partial x^\mu} dx^\mu \wedge dx^\nu \tag{3.19}$$

and in component form we see that

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ \mathbf{F} &= \nabla_\mu A_\nu dx^\mu \wedge dx^\nu \end{aligned} \tag{3.20}$$

In general, the relation between $\nabla$ and $d$ can be stated as

$$d\omega = \nabla \wedge \omega \tag{3.21}$$

or in components $(d\omega)_{\mu\nu\lambda\cdots} = \nabla_{[\mu}\omega_{\nu\lambda\cdots]}$. So the field tensor $\mathbf{F}$ is not altered when we introduce a symmetric connection on our space-time. Gauge invariance is expressed by the fact that $\mathbf{A}$ and $\mathbf{A} + df$ give the same $\mathbf{F}$. It's also useful to define the concepts of a p-form $\omega$ being closed or exact:

- $\omega$ is closed if $d\omega = 0$.
- $\omega$ is exact if $\omega = d\alpha$ for a (p-1)-form $\alpha$.

[2] This is a consequence of Poincaré's lemma.

Exact forms are always closed, but closed forms are not always exact. The exterior product also obeys a product rule; for a p-form $\omega$ and a q-form $\alpha$ we have

$$d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^p \omega \wedge d\alpha \tag{3.22}$$

To conclude this section, we note a very nice and useful relation between the Lie-derivative and the exterior derivative if we are dealing with vectors and forms. If we have a vector $X$ and a p-form $\omega$ and we denote with $(X \cdot \omega)$ the contraction between $X$ and the first index of $\omega$, then the following can be proven[3]:

$$\boxed{\pounds_X \omega = d(X \cdot \omega) + X \cdot d\omega} \tag{3.23}$$

With the identity $d^2 = 0$ it can be shown that the Lie-derivative and the exterior derivative commute:

$$\begin{aligned} d(\pounds_X \omega) &= d(X \cdot d\omega) \\ d(X \cdot d\omega) &= \pounds_X d\omega - X \cdot d^2\omega \\ &= \pounds_X d\omega \end{aligned} \tag{3.24}$$

A convenient application of eq.( 3.23) is for n-forms: if $\omega \in \Lambda^n(M)$ then $\pounds_X \omega = d(X \cdot \omega)$.

## 3.4 Killing vector fields

In discussing black holes, we will encounter so called Killing vector fields. These fields tell us something about the present symmetries. In this section we will again work with infinitesimal vector fields. This is justified by the fact that every transformation can be constructed from these infinitesimal transformations due to the continuous character of them. Imagine we are changing coordinates $x \to x^{'}(x)$, and we take a look at how this affects our metric components $g_{\mu\nu}$. We call this transformation an isometry if we have

$$g^{'}_{\mu\nu}(x) = g_{\mu\nu}(x) \tag{3.25}$$

On the left hand side the metric is transformed, and then the argument is pulled back to the original coordinate. We could also express this isometry as

$$\phi_t^*(g_{\mu\nu} dx^\mu \otimes dx^\nu) = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{3.26}$$

If we change primes on the expression for the transformed metric, we see that $x \to x^{'}(x)$ is an isometry if

$$g_{\mu\nu}(x) = \frac{\partial x^{'\alpha}}{\partial x^\mu} \frac{\partial x^{'\beta}}{\partial x^\nu} g_{\alpha\beta}(x^{'}) \tag{3.27}$$

So if we again apply an infinitesimal coordinate transformation we have

$$x^\mu \to x^{'\mu} = x^\mu + \delta s X^\mu(x) \qquad \frac{\partial x^{'\mu}}{\partial x^\alpha} = \delta^\mu_\alpha + \delta s \; \partial_\alpha X^\mu \tag{3.28}$$

[3]In some texts the notation $\iota_X \omega = \omega \cdot X$ is encountered for this contraction.

Substituting this in (3.9) and using again Taylor's theorem up to first order, we find that we have an isometry if the Lie-derivative of the metric vanishes:

$$\pounds_X g_{\mu\nu} = X^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu X^\lambda + g_{\nu\lambda}\partial_\mu X^\lambda \tag{3.29}$$

Considering the metric compatibility $\nabla_\lambda g_{\mu\nu} = 0$ and the fact that we can replace partial derivatives by covariant derivatives, we conclude that $X$ induces an infinitesimal symmetry if

$$\begin{aligned} \pounds_X g_{\mu\nu} &= \nabla_\nu X_\mu + \nabla_\mu X_\nu \\ &= 0 \end{aligned} \tag{3.30}$$

The vector field $X$ is then called a Killing vector field: the flow of this vector field leaves the metric invariant. A useful identity for Killing-vectors comes from the identity

$$\nabla_{[\mu}\nabla_{\nu]}X^\alpha = \frac{1}{2}R^\alpha{}_{\beta\mu\nu}X^\beta \tag{3.31}$$

We see that for a Killing vector $X$ this implies that

$$\nabla_\mu\nabla_\nu X^\alpha = R^\alpha{}_{\nu\mu\lambda}X^\lambda \tag{3.32}$$

and this enables us to rewrite higher derivatives of a Killing vector field in terms of the Riemann-tensor.

### 3.4.1 An example: 3-dimensional Minkowksi space-time

Let's calculate a little example. We take a 3-dimensional flat space-time with coordinates $x^i = (t, x, y)$ and line-element

$$ds^2 = -dt^2 + dx^2 + dy^2 \tag{3.33}$$

in Cartesian coordinates. We can rewrite Killing's equations for the Killing vector field $X$ in a coordinate basis as

$$\partial_i X_j + \partial_j X_i = 2X_k\Gamma^k_{ij} \;\rightarrow\; \partial_i X_j + \partial_j X_i = 0 \tag{3.34}$$

This gives us 6 differential equations:

$$\begin{aligned} 2\partial_i X_i &= 0, \quad i = 1,2,3 \\ \frac{\partial X^2}{\partial t} + \frac{\partial X^1}{\partial x} &= 0 \\ \frac{\partial X^3}{\partial x} + \frac{\partial X^2}{\partial y} &= 0 \\ \frac{\partial X^3}{\partial t} + \frac{\partial X^1}{\partial y} &= 0 \end{aligned} \tag{3.35}$$

This on it's turn gives us six solutions:

$$\begin{aligned} X^{(1)} &= \frac{\partial}{\partial t}, \qquad X^{(4)} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \\ X^{(2)} &= \frac{\partial}{\partial x}, \qquad X^{(5)} = x\frac{\partial}{\partial t} - t\frac{\partial}{\partial x} \\ X^{(3)} &= \frac{\partial}{\partial y}, \qquad X^{(6)} = y\frac{\partial}{\partial t} - t\frac{\partial}{\partial y} \end{aligned} \tag{3.36}$$

Note that this doesn't explicitly depend on the metric signs; we would had obtained a similar result if we would had taken the 3-dimensional Cartesian space with line element $ds^2 = dx^2 + dy^2 + dz^2$. The fact that these 3-dimensional spaces have 6 Killing vectors gives them the name "maximally symmetric". One can show that an n-dimensional space can have at most $\frac{n}{2}(n+1)$ linear independent Killing vectors.

## 3.5 Tensor densities

It appears to be necessary to define something called a tensor density. It transforms just like a tensor, except that there is an additional Jacobian $J$ to some power $W$ involved. The $W$ is the weight, and so we have for a tensor density that

$$\begin{aligned} Z'^{\alpha\cdots}_{\ \beta\cdots} &= M^W \frac{\partial x'^\alpha}{\partial x^\mu} \cdots \frac{\partial x^\nu}{\partial x'^\beta} Z^{\mu\cdots}_{\nu\cdots} \\ M &\equiv |\frac{\partial x}{\partial x'}| \end{aligned} \tag{3.37}$$

From the definition it follows that the tensorproduct of 2 tensor densities of weight V and W produces a tensor density with weight V+W. The covariant derivative of a tensor density is defined as the following:

$$\nabla_\lambda Z^{\alpha\cdots}_{\beta\cdots} = \partial_\lambda Z^{\alpha\cdots}_{\beta\cdots} + \Gamma^\alpha_{\rho\lambda} Z^{\rho\cdots}_{\beta\cdots} + \cdots - \Gamma^\rho_{\beta\lambda} Z^{\alpha\cdots}_{\rho\cdots} - \cdots - W\Gamma^\rho_{\rho\lambda} Z^{\alpha\cdots}_{\beta\cdots} \tag{3.38}$$

This is just the ordinary transformation rule with an extra term accounting for the Jacobian at the end. This definition is not fixed, but often chosen as to satisfy metric compatibility which implies $\nabla_\mu \sqrt{g} = 0$. What is important for us, is the case of $W = 1$ and $\lambda = \alpha$:

$$\boxed{\nabla_\alpha Z^\alpha = \partial_\alpha Z^\alpha} \tag{3.39}$$

We see that the covariant divergence of a vector density is equal to its ordinary divergence.

Now for two examples which will be important if we want to integrate fields on manifolds. The first one is already familiar, namely the volume element $d^n x$. Under a coordinate-transformation the volume-element $d^n x$ transforms via

$$\begin{aligned} d^n x' &= |\frac{\partial x'}{\partial x}| d^n x \\ &= M^{-1} d^n x \end{aligned} \tag{3.40}$$

The volume element is a tensor density with weight $W = -1$. This results in the ordinary rule for how integrals transform under a change of coordinates. Now the second example. The metric transforms as

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}(x) \\ &= M^\lambda{}_{'\mu} M^\rho{}_{'\nu} g_{\lambda\rho}(x) \end{aligned} \tag{3.41}$$

Taking determinants and square roots on both sides, we obtain

$$\sqrt{|g'(x')|)} = M\sqrt{|g(x)|} \tag{3.42}$$

which is a tensor density of $W = 1$. According to eq.(3.38), $\nabla_\mu\sqrt{|g(x)|} = 0$ implies

$$\partial_\mu\sqrt{g} - \Gamma^\lambda_{\lambda\mu}\sqrt{g} = 0 \tag{3.43}$$

which is exactly what we want due to the identity $\Gamma^\lambda_{\lambda\mu} = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}$. So we see that the definition (3.38) makes sense. Also, the combination $d^4x\sqrt{|g(x)|}$ is a tensor density of weight $W = 0$: a scalar.

## 3.6 Integration

Now we look at integration on manifolds. An example of such an integral is the action, which we want to define on the space-time manifold. As said before, forms are often encountered if you want to integrate. The nice thing about forms is that they come pre-equipped with a notion of a measure. Let's see what that means. If we consider ordinary integrals, we encounter the volume element $d^nx$, which is something like "an infinitesimal n-dimensional volume". In the language of forms however, this expression has an exact meaning. In general, the volume $V(X, \ldots, Z)$ of a set vectors $\{X, \ldots, Z\}$ is given by the determinant of the matrix constructed from these vectors[4]. We also know that these operations on these vectors are oriented and multilinear. So that should ring a bell; there is a close connection between forms, which are multilinear and antisymmetric, and determinants! In mathematical texts, a form is often defined via its action on vectors via the determinant and then the connection is clear at once. Now we put this more mathematically. The determinant of an $n \times n$ matrix $A$ with elements $A^{\mu\nu}$ is given by

$$|A| = \varepsilon_{\mu_1\cdots\mu_n}A^{1\mu_1}\ldots A^{n\mu_n} \tag{3.44}$$

Here $\varepsilon$ is the antisymmetric alternating Levi-Civita symbol. In fact this is a tensor density with $W = -1$, and in Minkowski space-time it is a Lorentz tensor. The generalization of this tensor becomes

$$\epsilon^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{|g|}}\varepsilon^{\mu\nu\alpha\beta}, \qquad \epsilon_{\mu\nu\alpha\beta} = \sqrt{|g|}\varepsilon_{\mu\nu\alpha\beta} \tag{3.45}$$

with

$$-\varepsilon_{\mu\nu\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta} = \begin{cases} 1 & \text{if } \mu\nu\alpha\beta \text{ is an even permutation of 0123,} \\ -1 & \text{if } \mu\nu\alpha\beta \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise} \end{cases} \tag{3.46}$$

Also, the manifold should be orientable[5], or else the symbol cannot be globally defined. Being warned for this, we set up a tetrad basis of one-forms $\{\omega^{\hat\mu}\}$, and define the volume form $\epsilon$ by

$$\epsilon = \omega^{\hat 1} \wedge \ldots \wedge \omega^{\hat n} \tag{3.47}$$

[4]In this matrix, the n'th row consists of the components of the n'th vector.

[5]The most familiar example of a non-orientable manifold is the famous Möbius strip.

This volume form represents a volume of unity, and the accompanying metric with components $g_{\hat{\mu}\hat{\nu}}$ has determinant $\hat{g} = \pm 1$. Now we transform this tetrad basis to an arbitrary basis $\omega^{\mu} = M^{\mu}{}_{\hat{\mu}}\omega^{\hat{\mu}}$ and express $\epsilon$ in this new basis:

$$\begin{aligned} \epsilon &= M^{\hat{1}}{}_{\mu_1} \dots M^{\hat{n}}{}_{\mu_n} \omega^{\mu_1} \wedge \dots \wedge \omega^{\mu_n} \\ &= M^{\hat{1}}{}_{\mu_1} \dots M^{\hat{n}}{}_{\mu_n} \varepsilon^{\mu_1 \cdots \mu_n} \omega^1 \wedge \dots \wedge \omega^n \\ &= M \omega^1 \wedge \dots \wedge \omega^n \end{aligned} \tag{3.48}$$

Here $M$ is the determinant of the matrices $M^{\mu}{}_{\hat{\mu}}$ which pops up in the transformation. We see from the transformation rule for the metric that

$$g = M^2 g' \ \rightarrow \ g = M^2 \hat{g} = \pm M^2 \ \rightarrow \ M = \sqrt{|g|} \tag{3.49}$$

And with this we obtain an expression for the volume form in a general basis:

$$\boxed{\epsilon = \sqrt{|g|}\omega^1 \wedge \dots \wedge \omega^n} \tag{3.50}$$

Note that this is an n-form, and so every n-form on an n-dimensional manifold will be proportional to $\epsilon$. With introducing the hodge * operator for $\omega = \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \in \Lambda^p(M)$ as

$$*\omega \equiv \frac{1}{(n-p)!} \omega^{\mu_1 \cdots \mu_p} \epsilon_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_n} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_n} \tag{3.51}$$

such that $*\omega \in \Lambda^{n-p}$. We can also write

$$\epsilon = *1 \tag{3.52}$$

Now we want to give a link between $d^n x$ and n-forms. This will be made clear by a two-dimensional example. Imagine that we want to integrate the function $F(x, y)$:

$$\int F(x, y) dx dy \tag{3.53}$$

In ordinary calculus it is learned that there really isn't a difference between $dxdy$ and $dydx$; they represent the same area. Now pick 2 other variables, like $\theta(x, y)$ and $\phi(x, y)$. We know that the integral transforms with a Jacobian. But according to the calculus of one-forms we have

$$\begin{aligned} dx &= \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi \\ dy &= \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi \end{aligned} \tag{3.54}$$

If we plug this transformation into our integral and expect it to equal the Jacobian, we see that these rules are consistent if the differentials obey antisymmetric product rules:

$$\begin{aligned} d\theta d\theta &= d\phi d\phi = 0 \\ d\theta d\phi &= -d\phi d\theta \end{aligned} \tag{3.55}$$

So instead of writing $dxdy$ we should write $dx \wedge dy$, and the order *does* matter! Explicitly we find that

$$\begin{aligned} dx \wedge dy &= dx(\theta,\phi) \wedge dy(\theta,\phi) \\ &= \Big(\frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi\Big) \wedge \Big(\frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi\Big) \\ &= \Big(\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi}\Big) d\phi \wedge d\theta \end{aligned} \tag{3.56}$$

That Jacobian should look familiar. We can also write that under a pullback $\phi$ we have $\phi_*(dx \wedge dy) = M dx \wedge dy$, where $M$ is the Jacobian. At first sight it can be confusing to see a wedge product transforming like a density, but remember that $dx$ is the differential of a coordinatefunction, not a scalar. This can be made clear by the identification

$$dx^0 \wedge \cdots \wedge dx^{n-1} = \frac{1}{n!} \varepsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \tag{3.57}$$

On the right-hand side, $\varepsilon$ is a tensor density, and the wedgeproduct is an n-form, so the left-hand side should also be a tensor density. Now, the argument for our identification of volume-elements with forms can easily be extended to arbitrary dimension, and with this the connection between the volume-form and the integration measure is clear.

Now we define the integral of an n-form $\omega$ on an n-dimensional manifold. Such an object can always be written as $\omega = f(x)\epsilon = \sqrt{|g|} f dx^1 \wedge \ldots \wedge dx^n$, and we define

$$\int \omega = \int \cdots \int f(x) dx^1 \ldots dx^n \tag{3.58}$$

We can now make the identification

$$d^n x \rightarrow dx^1 \wedge \ldots \wedge dx^n \tag{3.59}$$

if we choose a coordinate basis. A scalar function $f(x) \in \Lambda^0(M)$ is integrated as

$$\boxed{\int \sqrt{|g|} f(x) d^n x = \int f(x)\epsilon} \tag{3.60}$$

We see that this integral is a scalar quantity, and thus properly defined. From this discussion it should be clear that we can view every integrand as an n-form. In a strict sense, we can construct an equivalence class between tensor densities and n-forms.

## 3.7 Stokes' theorem

We are now in a position to turn to Stokes' theorem. Imagine we have an n-dimensional manifold $M$ available, which has $\Omega$ as n-dimensional submanifold. The border $\partial\Omega$ is (n-1)-dimensional and can be defined via chains. Stokes tells us that

$$\boxed{\int_\Omega d\omega = \int_{\partial\Omega} \omega} \tag{3.61}$$

where $d\omega \in \Lambda^n(M)$ is the exterior derivative of $\omega \in \Lambda^{n-1}(M)$. Here it is understood that $M$ is orientable and more of such subtleties. With the identification of integrands as being forms, we also see that this is true for the divergence of vector densities. This should be familiar from basic vector analysis. In many applications, the exact details of the boundary and the region are not needed; one can simply see that one gets an integral over the boundary, and often the boundary conditions let this integral vanish. The theorem then states about vector densities

$$\int_\Omega \sqrt{|g|}\nabla_\mu X^\mu d^4x = \int_{\partial\Omega} \sqrt{|g|} X^\mu d^3x_\mu \tag{3.62}$$

which actually denotes

$$\int_\Omega \nabla_\lambda X^\lambda \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \int_{\partial\Omega} X^\lambda \epsilon_{\lambda\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{3.63}$$

This makes people make the following identification, when concerning integration:

$$\int_{\partial\Omega} d(X\cdot\epsilon) = \int_{\partial\Omega} \nabla_\mu (X^\mu \epsilon) \tag{3.64}$$

If the manifold $\Omega$ itself is compact[6], then $\partial\Omega = \emptyset$ and as a result

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega = 0 \tag{3.65}$$

Maybe it is nice to say something more about integration on surfaces in the context of pullbacks. We saw how the line integral could be defined with pullbacks, and now we are going to use the pullback to define integration on surfaces. For this we take $N \subset M$ being an (n-1) dimensional submanifold of $M$, and $U$ being an open (n-1) dimensional manifold. Let $S : U \to M$ be a local parametrization of $S(U) \subset M$. If $\omega \in \Lambda^{n-1}(M)$, then $S_*\omega \in \Lambda^{n-1}(U)$ and with this we can define the integral of $\omega$ over $S(U)$ as

$$\int_{S(U)} \omega = \int_U S_*\omega \tag{3.66}$$

Just like in the case of line integrals, this object is invariant under a change of parametrization, as it should be.

[6]A compact set is closed and bounded, and a compact manifold often can be read as "manifold without boundary". For instance, the circle $S^1$ is a compact one-dimensional manifold, but the real line $\mathbb{R}$ is not.

# Chapter 4

# Lagrangians, symmetries and variations

Many physical theories are derivable from the action principle, and so is the theory of general relativity. In fact, the famous mathematician Hilbert obtained the field equations via this principle before Einstein did! The symmetries which are present also become clear from the Lagrangian of the system. Symmetries play a prominent role in modern physics. In the standard model forces are introduced by promoting global symmetries of the Lagrangian to local symmetries. For this we need covariant derivatives, in which the correction terms are the gauge-fields of the particular force. In the theory of general relativity we work on space-time itself; the relevant field here is the metric. By demanding the invariance under diffeomorphisms defined on that space-time the equations get a tensorial character. This introduces a certain connection[1] which gives rise to the gravitational field. So it is not an exaggeration to state that symmetries play a crucial role in modern physics. The invariance under diffeomorphisms is also used here to define the Noether current, from which a possible candidate for the black hole entropy is obtained. In what follows, we will review the action principle, how to define variations, and the Noether method.

## 4.1 The action principle

Imagine that we have an n-dimensional manifold $M$ and a so called target manifold $T$. In our case $M$ is the space-time manifold and $T$ is the set of all possible values of the fields. For example, in the case when we only have a metric field in 4 dimensions, we can identify $T$ with $\mathbb{R}^{16}$ and in the case of only one real scalar field we have $T = \mathbb{R}$. The configuration space of smooth functions from $M$ to $T$ is denoted by $\Phi$, so $\Phi : M \to T$. It is the space of all kinematically allowed field configurations. The action $S : \Phi \to \mathbb{R}$ then gives us the equations of motion when it is extremized under certain boundary conditions. The action can be written as an integral over a space-time region of a certain function called

[1] The specific connection has to be obtained by specific assumptions.

the Lagrangian $\mathscr{L}$, and so the equations of motion are given by

$$\boxed{\delta S[\phi] = \delta \int_M \mathscr{L}[\phi] d^4x = 0} \tag{4.1}$$

which ofcourse is coordinate-independent. We use squared brackets to emphasize that $\mathscr{L}$ depends on the field and its derivatives, and here it is considered as a scalar density. There are some assumptions for this method:

- The variation of the field $\phi \in \Phi$ is zero on the boundary of our space-time volume, $\delta\phi|_{\delta M} = 0$, so $M$ is considered as being a compact region.
- $\mathscr{L}$ depends only on the fields and its derivatives, *not* on the coordinates $x$.
- The functional derivative of $\mathscr{L}$ is evaluated as if $\phi$ and its derivatives are independent functions.

The first condition means that the fields involved are local, so that $\lim_{x\to\infty} \phi(x) = 0$. This and other properties of the fields, like smoothness and causal restrictions, would specify the configuration space $\Phi$, but those considerations are not very important to us. The boundary conditions are imposed, because the action principle should give us a description of the evolution of the system; the beginning and endpoint are assumed to be given, which means that the fields are fixed there. If $\phi$ satisfies the given boundary conditions and it satisfies the constraint $\frac{\delta S}{\delta \phi} = 0$, then it is said that $\phi$ lies in the subspace of on-shell solutions. Most of the time we will do the variations explicitly, but we can use the Euler-Lagrange equations to exhibit symmetries of our Lagrangian density. We will discuss the case that $\mathscr{L}[\phi] = \mathscr{L}(\phi, \partial_\mu \phi)$ and derive how this functional derivative looks like[2]; the extension to higher order derivatives of $\phi$ is simply a matter of doing more partial integrations and imposing more boundary conditions. The variation of $S$ can be written as

$$\begin{aligned} \delta S &= \int (\frac{\partial \mathscr{L}}{\partial \phi}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi))d^4x \\ &= \int \Big(\partial_\mu \Big[\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big] - \partial_\mu \Big[\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\Big]\delta\phi\Big) d^4x \end{aligned} \tag{4.2}$$

Note that we have put $\delta(d^4x) = 0$. When we write down explicitly the Jacobian for a coordinate transformation $x \to x'^\mu = x^\mu + \delta x^\mu$, we get up to first order

$$J = |\frac{\partial x'}{\partial x}| = 1 + \partial_\mu(\delta x^\mu) \tag{4.3}$$

so that

$$\delta(d^4x) = d^4x' - d^4x = \partial_\mu(\delta x^\mu)d^4x \tag{4.4}$$

The variation of the coordinates is zero by hypothesis, so we can only consider the variation in $\mathscr{L}$. Then we can also use that the operations of varying the field and taking derivates of the field commute:

$$\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi) \tag{4.5}$$

[2]Examples of this case are $\phi$ begin a scalar field or the metric tensor due to metric compatability.

This will not be true anymore if we consider covariant derivatives $\nabla$; this will be shown later on. The total derivative in eq.(4.39) can be converted into a surface integral via Stokes' theorem, and this term becomes 0. This boundary term will become important to us in the next chapters. We could a priori add an extra boundary term to the action which cancels the boundary term obtained by the partial integration, but we won't. Alltogether we find that the integrand vanishes for arbitrary $\delta\phi$, and the equations of motion are thus given by the celebrated Euler Lagrange equations:

$$\frac{\delta \mathscr{L}}{\delta \phi} \equiv \frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = 0 \tag{4.6}$$

and this gives us the functional derivative of $\mathscr{L}$. In most of the theories which are derived according to (4.1), the Lagrangian density functional $\mathscr{L}$ depends on the field and the first derivative of the field. In a theory of gravity, we take as our fundamental field the metric $g_{\mu\nu}(x)$. One can show that at any point on the manifold, the metric can be put into its canonical form and the first derivatives can be set to 0 ( see for instance [3] or [2] ). So a proper, non-trivial Lagrangian density should contain up to second order derivatives of the metric: $\mathscr{L} = \mathscr{L}(g_{\mu\nu}, \partial_\lambda g_{\mu\nu}, \partial_\lambda \partial_\rho g_{\mu\nu})$. A same derivation as above defines the functional derivative of $\mathscr{L}$ with respect to the metric field $g_{\mu\nu}$:

$$\frac{\delta \mathscr{L}}{\delta g_{\mu\nu}} \equiv \frac{\partial \mathscr{L}}{\partial g_{\mu\nu}} - \partial_\lambda \Big( \frac{\partial \mathscr{L}}{\partial(\partial_\lambda g_{\mu\nu})} \Big) + \partial_\lambda \partial_\rho \Big( \frac{\partial \mathscr{L}}{\partial(\partial_\lambda \partial_\rho g_{\mu\nu})} \Big) \tag{4.7}$$

This looks a little intimidating, and when you start doing calculations with it, you will discover it often actually *is*. When we introduce higher order derivatives in our Lagrangian, we have two options: we can impose boundary conditions for those derivatives of the variations of the fields involved, but we could also add terms to the boundary terms of the action which don't change the equations of motion. These terms then can help to get rid of the $\partial_\mu \delta\phi$ in the total derivative. When doing variations, if often is much easier to just do the variations explicitly instead of using the functional derivative.

Now we have to take a closer look at derivatives of $\mathscr{L}$ with respect to tensor fields. We neglected an important detail in differentiating objects with respect to tensor fields. These tensor fields can have certain symmetries, for instance $g_{\mu\nu} = g_{(\mu\nu)}$ or $F_{\mu\nu} = F_{[\mu\nu]}$. The components of these derivatives are not independent anymore and we have to account for that. We impose that the partial derivative is fixed by the variation of $\mathscr{L}$ via

$$\delta \mathscr{L} = \frac{\partial \mathscr{L}}{\partial T_{\mu \cdots \nu}} \delta T_{\mu \cdots \nu} + \dots \tag{4.8}$$

So we impose that the derivative with respect to a tensor field has the same symmetry properties as the tensor field itself.

In this thesis $\mathscr{L}$ will be defined in different ways. Most of the time we will define it without the factor $\sqrt{|g|}$ in it, so that it is a scalar. Then the action is written as

$$S = \int \sqrt{|g|} \mathscr{L} d^4 x \tag{4.9}$$

If we do include this factor, it is explicitly mentioned and then $\mathscr{L}$ is actually a tensor density. Later on, we will view $\mathscr{L}$ as an n-form and write $\mathbf{L}$ for it, which is justified by our discussion on integration. In that case, the action becomes simply

$$S = \int \mathbf{L} = \int \mathscr{L}\epsilon \tag{4.10}$$

and $\mathbf{L} \in \Lambda^n(M)$. It turns out that in some cases this last point of view is very convenient, but be aware of this differences.

## 4.2 A note on variations

Because we are dealing a lot with variations, it's no luxury to take a good look at them. What do we actually mean by varying a tensor field $\psi$ with $\delta\psi$? The subtle point here is that in a relativistic field theory coordinates have no physical meaning. First of all, we used variations in the action to derive the equations of motion. These variations were arbitrary variations and didn't involve variations of the coordinates; we were merely interested in what equations the field configurations obey, and this is independent of the coordinates.

If we do involve coordinate variations in the game, we can define two kinds of variations of the field. Recall how we pushed forward a point on our manifold via

$$\begin{aligned} \delta x^\mu &= x'^\mu - x^\mu \\ &= \lim_{t\to 0}(\phi_t(x))^\mu \end{aligned} \tag{4.11}$$

which is in fact a member of the one-parameter group diffeomorphisms with infinitesimal $t$.

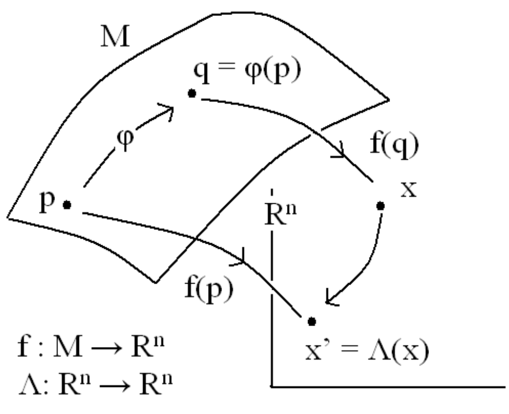


Figure 4.1: *The action of a diffeomorphism*

The first variation of the field is written as

$$\begin{aligned}\delta\psi &= \psi^{'}(x) - \psi(x) \\ &= \lim_{t\to 0}\frac{1}{t}[\phi_t^{*}\psi(\phi_t(x)) - \psi(x)]\end{aligned} \quad (4.12)$$

What we essentialy do here, is to transform the field $\psi$ via a coordinate transformation, and after that we pull the coordinates back to compare the two fields at one and the same point on the manifold. That's nothing more than our Lie-derivative, in which we used a vector flow $\xi$ which was associated with the diffeomorphism $\phi$. So from now on we will denote this variation as $\delta_{\xi}$.

We could also define another kind of variation:

$$\tilde{\delta}\psi = \psi^{'}(x^{'}) - \psi(x) \quad (4.13)$$

Here we perform a coordinate transformation on the field, and by this the coordinate transforms along with the field. It refers to the value of the field $\psi$ again at the same point in two different coordinate systems. Note that this

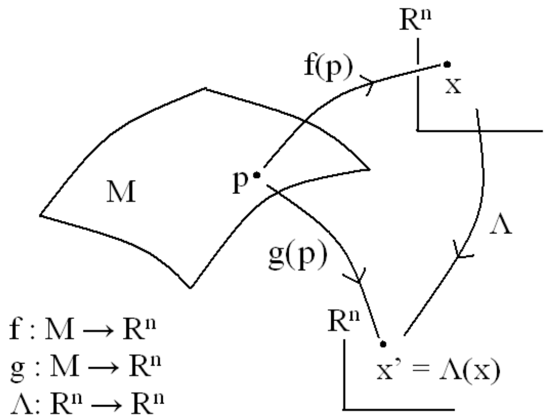


Figure 4.2: *The action of a general coordinate transformation*

doesn't involve a change of the geometric point on the manifold, and as such it doesn't involve a diffeomorphism $\phi : M \to M$ on the manifold itself. But we do have to know how the fields transform under coordinate transformations; these are diffeomorphisms in the coordinate space. Physically this is precisely what a Lorentz transformation does; two observers look at the same event ( point on the manifold ) from two different frames of reference, or coordinate systems, but here we consider also general coordinate transformations. Now these two variations are connected via the relation

$$\delta\psi = \tilde{\delta}\psi - \delta x^{\mu}\partial_{\mu}\psi \quad (4.14)$$

The last term is called the transport term. It accounts for the fact that in $\delta\psi$ the coordinate has been pulled back after the coordinate transformation. [3]

[3] See the appendix for a derivation of this result.

Let's take an example by looking at $\psi$ as a scalar function. Then we have by definition that

$$\tilde{\delta}\psi = 0 \rightarrow \delta\psi = -\delta x^\mu \partial_\mu \psi \tag{4.15}$$

which indeed is the Lie-derivative of a scalar field. The vector $\delta x^\mu$ is often given an own name like $\xi^\mu$. The fact that a vector field $\xi$ induces a variation $-\xi^\mu \partial_\mu \psi$ in a scalar field should be familiar. As a next example, consider the metric $g_{\alpha\beta}$. Under a coordinate transformation $x \rightarrow x - \xi$ we obtain, up to order $O(\partial\xi)$,

$$\tilde{\delta} g_{\alpha\beta} = g_{\alpha\mu}\partial_\beta \xi^\mu + g_{\mu\beta}\partial_\alpha \xi^\mu \tag{4.16}$$

This implies that

$$\delta_\xi g_{\alpha\beta} = \xi^\mu \partial_\mu g_{\alpha\beta} + g_{\alpha\mu}\partial_\beta \xi^\mu + g_{\mu\beta}\partial_\alpha \xi^\mu \tag{4.17}$$

which is what we want. To summarize all this potentially confusing material: the combination of a push-forward and a coordinate transformation is a diffeomorphism $\phi$ on the manifold $M$ which can be induced by a Lie-derivative infinitesimally. Under $\phi$, the transformed tensor components are evaluated at the same numerical values of the coordinates as the original tensor components in the original coordinates. If we define an action $S$ we have to keep in mind the fact that $\tilde{\delta}(\sqrt{g}d^4x) = 0$ because this is a scalar quantity, but that $\pounds_\xi(\sqrt{g}d^4x) \neq 0$ in general. So general coordinate transformations can be seen as diffeomorphisms in the coordinate space (for example, $\mathbb{R}^n$) which don't change the geometrical point, whereas push-forwards can be seen as diffeomorphisms on the manifold $M$, which *do* change the geometrical point. If we explore further this coordinate business, we will discover that the passive point of view and the active point of view of coordinate transformations are interchangeble; in each region of the manifold there is a one-to-one correspondence between a certain coordinate transformation and an active diffeomorphism. We won't go deeper into this, but this discussion leads to the correspondence between diffeomorphism invariance and general covariance, and the so-called "hole argument" of general relativity which troubled Einstein for quite some time.

It should be noted that in discussing the variational principle, some authors look at the field $\psi$ as a one-parameter family $\psi(\lambda; x)$ of field configurations on space-time, where $\lambda \in \mathbb{R}$. If the coordinate $x$ is held fixed, $c(\lambda) = \psi(\lambda; x)$ describes a curve in configuration space. The variation $\delta\psi$ is then defined as

$$\delta\psi(x) = \frac{d\psi(\lambda; x)}{d\lambda}|_{\lambda=0} \tag{4.18}$$

This can be considered as a tangent vector of the curve $c(\lambda)$ at the point $\psi(\lambda = 0; x)$ in configuration space. The variation of the Lagrangian density $\mathscr{L}$ is then

$$\delta\mathscr{L} = \frac{d\mathscr{L}}{d\lambda}|_{\lambda=0} \tag{4.19}$$

In theories of gravity the metric field is coupled to the matter fields via covariant derivatives and Lorentz invariance. Both fields are taken to be dynamical. Let's make a clear distinction between the collection of matter fields $\psi$ and the metric field $g$. The total collection of dynamical fields is called $\phi^a = (\psi, g)$. The Lagrangian $\mathbf{L}$ is written as

$$\mathbf{L} = \mathbf{L}(\psi, \nabla\psi, g) \tag{4.20}$$

in which indices are surpressed, $\psi$ is taken to be a covariant vector field for simplicity and so $\nabla\psi = \partial\psi - \Gamma\psi$. The generalization to higher order derivatives and arbitrary matter fields is quite straight forward. The total variation $\delta_t$ concerns the variation $\delta_m$ which concerns the matter fields and the variation $\delta_g$ which concerns the metric and the connection. The matter field is independent of the metric and vice versa. For instance, for the vector potential $\mathbf{A}$ being a one-form we write $\delta_g \mathbf{A} = 0$ and $\delta_A g = 0$. Note that this isn't true anymore for vectors with contravariant components as $A = A^\mu \partial_\mu$. The following identities are made:

$$\begin{aligned} \delta_m \psi &\equiv \delta\psi \\ \delta_m g &= \delta_m \Gamma = \delta_m R = 0 \\ \delta_g g^{\mu\nu} &\equiv \delta g^{\mu\nu} \\ \delta_g \psi &= 0 \\ \delta_t &= \delta_g + \delta_m \end{aligned} \tag{4.21}$$

These on their turn give rise to the following identities:

$$\begin{aligned} \delta_g \nabla\psi &= \delta_g(\partial\psi - \Gamma\psi) = -\psi\delta_g\Gamma \\ \delta_t \psi &= \delta\psi \\ \delta_t g &= \delta g \\ \delta_t \nabla\psi &= \nabla\delta_m\psi - \psi\delta_g\Gamma \end{aligned} \tag{4.22}$$

Our interest lies in the total variation of the Lagrangian, $\delta_t \mathbf{L}$:

$$\begin{aligned} \delta_t \mathbf{L} &= \frac{\partial \mathbf{L}}{\partial\psi}\delta_t\psi + \frac{\partial \mathbf{L}}{\partial\nabla\psi}\delta_t\nabla\psi + \frac{\partial \mathbf{L}}{\partial g^{\mu\nu}}\delta_t g^{\mu\nu} \\ &= \frac{\partial \mathbf{L}}{\partial\psi}\delta_m\psi + \frac{\partial \mathbf{L}}{\partial\nabla\psi}(\nabla\delta_m\psi - \psi\delta_g\Gamma) + \frac{\partial \mathbf{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} \\ &= \Big[\frac{\partial L}{\partial\psi}\delta_m\psi + \frac{\partial \mathbf{L}}{\partial\nabla\psi}\nabla\delta_m\psi\Big] + \Big[\frac{\partial \mathbf{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} - \frac{\partial \mathbf{L}}{\partial\nabla\psi}\psi\delta_g\Gamma\Big] \\ &= \delta_m \mathbf{L} + \delta_g \mathbf{L} \end{aligned} \tag{4.23}$$

Here the variations can be rewritten as

$$\begin{aligned} \delta_m \mathbf{L} &= \Big[\frac{\partial \mathbf{L}}{\partial\psi} - \nabla\frac{\partial \mathbf{L}}{\partial\nabla\psi}\Big]\delta\psi + \nabla\Big[\frac{\partial \mathbf{L}}{\partial\nabla\psi}\delta\psi\Big] \\ &\equiv E_\psi\delta\psi + \nabla\cdot\Theta \\ \delta_g \mathbf{L} &= E_g^{\mu\nu}\delta g_{\mu\nu} \end{aligned} \tag{4.24}$$

where the symplectic potential $\Theta$ and the equations of motion $E(\psi)$ are defined. This will be encountered again in chapter seven.

## 4.3 Diffeomorphism invariance

Having discussed the issue of coordinates in the beginning of this section, let us briefly come to the notion of diffeomorphism invariance. In the theory of special relativity, we had a set of preferred frames of reference in which the laws of physics were the same: inertial frames. So we had the notion of absolute acceleration. In special relativity, but also in quantum field theory, the metric is a fixed background so it makes sense to assign a physical meaning to points on this manifold. If we use a diffeomorphism and shift all the dynamical fields with respect to this fixed background metric we change the physical situation; conventional field theories are not invariant under these transformations. However, this changes in general relativity. Here the metric is part of those dynamical fields, and if we shift all the fields including the metric, nothing changes; only the relative distances, measured with the space-time interval, have physical meaning, because we shift the geometry along. Also, we can't solve the equations of motion for the gravitational field and the other dynamical fields independently; these are coupled according to Einstein's field equations and so we have to solve those equations 'simultaneously'. So a theory of gravity, based on the idea that the metric determines the space-time geometry, is diffeomorphism invariant. It's however not a feature of the gravitational field alone, but of every general relativistic field theory in general. This statement is expressed as

$$\mathbf{L}(f_*\phi) = f_*\mathbf{L}(\phi) \tag{4.25}$$

where $f_* : M \to M$ and $\phi$ are the dynamical fields including the metric. This means that instead of one space-time we should regard the equivalence class of sets $(M, \phi)$, where the equivalence class is determined by $(M, \phi) \sim (M, f_*\phi)$. This makes the space of field solutions very large. It also changes the way we derive symmetries in field theories; with a fixed background metric we could say that rotating or translating all dynamical fields should leave the physical system unchanged, but that doesn't make any sense with a dynamical background. In general relativity we therefore use Killing vectors to derive symmetries, and quantities like the total energy should be defined at asymptotical infinity where the metric is assumed to be flat.

## 4.4 The Noether method

Here we will briefly discuss the Noether method. The symmetries which are important to us are given by the diffeomorphism invariant character of our theories of gravity. So we consider the infinitesimal transformations of the coordinates:

$$\begin{aligned} \tilde{\delta}x^{\mu} &= x^{'\mu} - x^{\mu} \\ \tilde{\delta}\phi &= \phi^{'}(x^{'}) - \phi(x) \end{aligned} \tag{4.26}$$

Note that we wrote also a $\tilde{\delta}$ for the coordinate transformations; it's again the difference between the coordinates in two different coordinate systems, but the point is the same. Let's obtain an expression for $\tilde{\delta}S$:

$$\tilde{\delta}S = \int [\tilde{\delta}d^4x\mathscr{L} + \tilde{\delta}\mathscr{L}d^4x] = 0 \tag{4.27}$$

The following identities will be useful to us:

$$\begin{aligned}
\tilde{\delta} d^4x &= d^4x \partial_\mu \tilde{\delta} x^\mu \\
\tilde{\delta} \mathscr{L} &= \delta \mathscr{L} + \tilde{\delta} x^\mu \partial_\mu \mathscr{L} \\
\delta \mathscr{L} &= \frac{\partial \mathscr{L}}{\partial \phi} \delta \phi + \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi} \delta(\partial_\mu \phi) + \frac{\partial \mathscr{L}}{\partial \partial_\mu \partial_\nu \phi} \delta(\partial_\mu \partial_\nu \phi) + \cdots
\end{aligned} \tag{4.28}$$

We are at ease by putting $[\delta, \partial_\mu] = 0$ because $\delta$ doesn't involve any change of coordinates. So the variation becomes, if we pull out the total factor $d^4x$ and identify a total derivative:

$$\tilde{\delta} S = \int [\partial_\mu (\tilde{\delta} x^\mu \mathscr{L}) + \delta \mathscr{L}] d^4x \tag{4.29}$$

Now some integration by parts gives us the following form of $\tilde{\delta} S$:

$$\tilde{\delta} S = \int [\partial_\mu \Big[ \mathscr{L} \tilde{\delta} x^\mu + \Big( \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi} - \partial_\nu \frac{\partial \mathscr{L}}{\partial \partial_\mu \partial_\nu \phi} \Big) \delta \phi + \frac{\partial \mathscr{L}}{\partial \partial_\mu \partial_\nu \phi} \partial_\nu (\delta \phi) + \cdots \Big] + \frac{\delta \mathscr{L}}{\delta \phi} \delta \phi] d^4x \tag{4.30}$$

The scalar character of $S$ implies thus

$$\int [\partial_\mu j^\mu + \frac{\delta \mathscr{L}}{\delta \phi} \delta \phi] d^4x = 0 \tag{4.31}$$

where we identified the current $j$

$$j^\mu = T^\mu{}_\nu \tilde{\delta} x^\nu - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \phi)} \partial_\lambda \phi \partial_\nu \tilde{\delta} x^\rho + \Big[ \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \partial_\nu \Big( \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \phi)} \Big) \Big] \tilde{\delta} \phi + \cdots \tag{4.32}$$

Here $T^\mu{}_\nu$ is the canonical energy-momentum tensor, and the $\cdots$ stand for terms wich are proportional to derivatives of $\tilde{\delta} \phi$. If $\phi$ lies in the subspace of on-shell solutions, then we have the continuity equation

$$\partial_\mu j^\mu = 0 \tag{4.33}$$

Remember that $j$ is a vector density, so this expression is covariant. If we have a metric available, then the vector $J = \sqrt{|g|} j$ can be constructed and we obtain[4]

$$\nabla_\mu J^\mu = 0 \tag{4.34}$$

There is a clever way to find the Noether current which is associated with global symmetry transformations of $\delta \phi$. We can construct a general transformation out of the symmetry transformations. The action is invariant under a transformation $\pounds_\xi \phi$, so it should also be invariant under a transformation $\epsilon(x) \pounds_\xi \phi$ if $\epsilon(x)$ is constant. The variation of the action must then have the following form:

$$\delta_\xi S = \int [\partial_\mu \epsilon(x) j^\mu] d^4x \tag{4.35}$$

This function $\epsilon(x)$ satisfies certain boundary conditions, such that the variation of the Lagrangian can be written as $\partial_\mu j^\mu$ modulo the equations of motion. Then

[4] Later we will write for the Noether current as a form $\mathscr{J} \in \Lambda^{n-1}(M)$.

the variation of the action can also be written as

$$\begin{aligned}\delta_\xi S &= \int_\Omega \partial_\mu(\epsilon(x)j^\mu)d^4x - \int_\Omega \epsilon(x)\partial_\mu j^\mu d^4x \\ &= -\int_\Omega \epsilon(x)\partial_\mu j^\mu d^4x \end{aligned} \tag{4.36}$$

because $\epsilon(x) = \partial_\mu\epsilon(x) = 0$ for $x \in \partial\Omega$ with $\Omega \subset M$.

## 4.5 Symmetries and Killing vectors

To conclude this chapter, we show the connection between symmetries and Killing vectors via the action. Imagine we have the action from which the equations of motion for time-like geodesics can be deduced:

$$S = \frac{1}{2}\int d\lambda\, g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{4.37}$$

Now we perform a coordinate transformation:

$$x^\mu \to x^\mu - \alpha\xi^\mu, \qquad \dot{x}^\mu \to \dot{x}^\mu - \alpha\dot{\xi}^\mu \tag{4.38}$$

which we consider as active. The variation of the action becomes

$$\delta S = \frac{1}{2}\int [\delta g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + g_{\mu\nu}\delta\dot{x}^\mu\dot{x}^\nu + g_{\mu\nu}\dot{x}^\mu\delta\dot{x}^\nu]d\lambda \tag{4.39}$$

The variations of $g_{\mu\nu}$, $x^\mu$ and $\dot{x}^\mu$ with respect to the active coordinate transformation are

$$\begin{aligned}\delta x^\mu &= \alpha\xi^\mu \\ \delta\dot{x}^\mu &= \alpha\dot{\xi}^\mu = \alpha\frac{\partial\xi^\mu}{\partial x^\rho}\frac{dx^\rho}{d\alpha} = \alpha\ \partial_\rho\xi^\mu\dot{x}^\rho \\ \delta g_{\mu\nu} &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho = \alpha\partial_\rho g_{\mu\nu}\xi^\rho \end{aligned} \tag{4.40}$$

If this is plugged in expression (4.39) and some indices are relabeled, the variation becomes

$$\begin{aligned}\delta S &= \frac{\alpha}{2}\int d\lambda\dot{x}^\mu\dot{x}^\nu[\xi^\rho\partial_\rho g_{\mu\nu} + \partial_\mu\xi^\rho g_{\rho\nu} + \partial_\nu\xi^\rho g_{\rho\mu}] \\ &= \frac{\alpha}{2}\int d\lambda\dot{x}^\mu\dot{x}^\nu \pounds_\xi g_{\mu\nu} \end{aligned} \tag{4.41}$$

It becomes clear that the action is invariant to first order, if the Lie-derivative of $g_{\mu\nu}$ vanishes! The vector field $\xi$ induces a conserved charge $Q$:

$$Q = \frac{\partial\mathscr{L}}{\partial\dot{x}^\mu}\delta x^\mu = \frac{\partial\mathscr{L}}{\partial\dot{x}^\mu}\xi^\mu \equiv \xi^\mu p_\mu \tag{4.42}$$

Here the four-momentum $p^\mu$ is defined and the constant factor $\alpha$ is omitted from the charge. In the case of the present action,

$$p_\mu = m\frac{dx^\nu}{d\tau}g_{\mu\nu} \tag{4.43}$$

if the parametrization with the eigentime $\tau$ is used. In the case of black holes we now have a useful expression for energy conservation when we know the metric which describes the particular black hole. For static black holes, we have that $\partial_t g_{\mu\nu} = 0$ with t the coordinate time, and with this $\partial/\partial t$ is a Killing vector. In that case we see that the conserved charge is

$$mk^{\mu}\frac{dx^{\nu}}{d\tau}g_{\mu\nu} = mg_{00}\frac{dt}{d\tau} \tag{4.44}$$

These symmetries can also be found without the Lagrangian. If $x^{\mu}(\lambda)$ is a geodesic curve with tangent vector $X^{\mu} = \frac{dx^{\mu}}{d\lambda}$ and $\xi$ is a Killing vector, then

$$\begin{aligned} X^{\mu}\nabla_{\mu}(\xi_{\nu}X^{\nu}) &= \xi^{\mu}\xi^{\nu}\nabla_{\mu}X_{\nu} + \xi^{\mu}X_{\nu}\nabla_{\mu}\xi^{\nu} \\ &= 0 \end{aligned} \tag{4.45}$$

This is due to the Killing's equation for $\xi$ and the geodesic equation for $X$. Be aware that the action we used here describes the evolution of a particle on a metric background, but not the evolution of the metric itself. What we essentially do is shifting the metric field with the Lie-derivative acting as a diffeomorphism, while keeping the particle fixed. Or the other way around, just what one prefers.

## 4.6 Aspects of general relativity

Here some aspects of the theory of general relativity are reviewed which are relevant to us. General relativity considers space-time as a pseudo-Riemannian manifold of which the geometry of it is determined by the distribution of energy. Gravity then is solely induced by the metric. The scalar potential of Newton thus is replaced by the metric tensor. Physically the equations were derived by Einstein by the demand that the geometric tensor constructed from the Riemann tensor has zero covariant divergence because of energy and momentum conservation. In fact, he just looked for a tensorial form of the Poisson equation for the gravitational field in which the matter density is replaced by the energy momentum tensor. The field equations which describe the evolution of the metric are in component form

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{4.46}$$

So in a vacuum the Ricci tensor vanishes. This doesn't have to mean that the metric is flat, and this is the origin of the existence of gravitational waves. Metric compatibility enables us to introduce a term proportional to the metric on the left hand side called the cosmological constant. We will ignore this constant throughout this thesis. On first sight, one would think that there are ten independent equations, but the Bianchi identities put four constraints on the equations, so we're left with six equations from which the metric can be found. This is the origin of the theory being a gauge-theory, because the four abundant degrees of freedom correspond with our ability to choose our four coordinates freely. We already stressed that the theory of general relativity is more than just these field equations; the geometry of space-time is determined by it on which all the other fields live. So the field $g_{\mu\nu}$ is in this sense not "just another ordinary classical field". The motion of a particle in a gravitational field is determined by the equation

$$\begin{aligned} ma^{\mu} &= m(\ddot{x}^{\mu} + \Gamma^{\mu}_{\rho\lambda}\dot{x}^{\rho}\dot{x}^{\lambda}) \\ &= m\dot{x}^{\nu}\nabla_{\nu}\dot{x}^{\mu} \\ &= F^{\mu} \end{aligned} \tag{4.47}$$

where $\dot{x}^{\mu} = \frac{dx^{\mu}}{d\tau}$, and $F^{\mu}$ is any force[5]. In the absence of these forces eq.(4.47) reduces to the geodesic equation. So a particle always travels along a geodesic if there are no external forces.

But now we look at the theory from an action- and gauge point of view. The vacuum field equations can be derived from the Hilbert-action,

$$\boxed{S = \frac{1}{16\pi}\int \sqrt{|g|}Rd^4x} \tag{4.48}$$

when we consider the metric as the fundamental field. When we want to couple matter to the gravitational field we introduce an action $S_M$ which describes this matter field. This defines the energy-momentum tensor $T^{\mu\nu}$:

$$\boxed{T^{\mu\nu} = -\frac{1}{\sqrt{|g|}}\frac{\delta S_M}{\delta g_{\mu\nu}}} \tag{4.49}$$

which is symmetric because $g_{\mu\nu}$ is. It is already said that the theory of general relativity can be regarded as a gauge-theory, where the symmetry group is the group of general coordinate transformations. These transformations are given by diffeomorphisms, or to be more specific: Lie-derivatives. A general action can be written as $S[\psi_i, g] = S_H[g] + S_M[\psi, g]$, where $S_H$ is the Hilbert action and $S_M$ is the matter action. A general variation of the action then reads

$$\begin{aligned} \delta S &= \int d^4x \Big(\frac{\delta S}{\delta \psi_i}\delta\psi_i + \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \partial_{\mu}B^{\mu}[\psi, g]\Big) \\ &= \int d^4x \Big(\frac{\delta S_H}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta S_M}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta S_M}{\delta \psi_i}\delta\psi_i + \partial_{\mu}B^{\mu}[\psi, g]\Big) \end{aligned} \tag{4.50}$$

Here $B^{\mu}[\psi, g]$ denotes the boundary terms of the metric and the matter fields. Now we consider the following case: the equations of motion hold, and the variation is generated by a vector field $\xi$. Being coordinate-independent, $\delta_{\xi}S = 0$. The action $S_H$ doesn't contain the matter field $\psi$, so the equations of motion for the matter field imply that the third term on the right is zero. We then end up only with the variation in the metric, and the total variation reads

$$\begin{aligned} \delta_{\xi}S &= \int d^4x \frac{\delta S}{\delta g_{\mu\nu}}\nabla_{\mu}\xi_{\nu} \\ &= -\int d^4x \sqrt{|g|}\xi_{\nu}\nabla_{\mu}\Big(\frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta g_{\mu\nu}}\Big) \\ &= 0 \end{aligned} \tag{4.51}$$

[5]Note that gravity is not seen as a force anymore, because the idea of acceleration due to a gravitational field breaks down due to the equivalence principe.

Boundary conditions on the vector field $\xi$ are imposed here[6]. The variation of the action results in the gauge identity

$$\boxed{\nabla_\mu\Big(\frac{1}{|g|}\frac{\delta S}{\delta g_{\mu\nu}}\Big) = 0} \tag{4.52}$$

for a general covariantly defined action. So what does this give us? If we insert the Hilbert action into this identity, we end up with the contracted Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$. If, on the other hand, we insert the matter action into it, we end up with

$$\begin{aligned}\nabla_\mu\Big(\frac{1}{|g|}\frac{\delta S_M}{\delta g_{\mu\nu}}\Big) &= \nabla_\mu T^{\mu\nu}_{(\psi)} \\ &= 0\end{aligned} \tag{4.53}$$

So diffeomorphism invariance implies that the energy momentum tensor is conserved in a covariant way. In the next section a closer look will be given on this sort of conservation laws. To make this identification of general relativity as a gauge theory more clear, we also give the situation in the case of electromagnetism with no electric sources. Here one considers the variation of the Maxwell action

$$S[\mathbf{A}] = -\frac{1}{4}\int\sqrt{|g|}\mathbf{F}^2 d^4x \tag{4.54}$$

under the gauge transformation $\delta A_\mu = \partial_\mu\Lambda(x)$. Varying the Lagrangian, we obtain

$$\begin{aligned}\delta(F^{\mu\nu}F_{\mu\nu}) &= 2F^{\mu\nu}\delta F_{\mu\nu} \\ &= 4F^{\mu\nu}\partial_\mu\delta A_\nu\end{aligned} \tag{4.55}$$

due to the antisymmetric contraction. We already know that under the gauge transformation $\delta S[\mathbf{A}] = 0$, because $\mathbf{F}$ is. The variation of $S[\mathbf{A}]$ becomes

$$\begin{aligned}\delta S[A] &= -\int\Big(\partial_\mu(F^{\mu\nu}\delta A_\nu) - \delta A_\nu\partial_\mu F^{\mu\nu}\Big)d^4x \\ &= -\int\Big(\partial_\nu(\partial_\mu F^{\mu\nu})\Lambda(x) - \partial_\mu(F^{\mu\nu}\partial_\nu\Lambda(x) - \partial_\nu F^{\nu\mu}\Lambda(x))\Big)d^4x \\ &= 0\end{aligned} \tag{4.56}$$

Here we have used the equations of motion $\partial_\mu F^{\mu\nu} = 0$. If we now also impose boundary conditions on $\Lambda(x)$ and $\partial_\mu\Lambda(x)$ we get the gauge identity

$$\partial_\mu\partial_\nu F^{\mu\nu} = 0 \tag{4.57}$$

which is, however, not very illuminating because this is trivially satisfied due to $[\partial_\mu, \partial_\nu] = 0$ which we have already used.

To conclude this section, some energy conditions will be considered, because they play an important role in the formulation of the laws of black hole mechanics. It is often assumed that $T_{\mu\nu}$ obeys the weak energy condition for timelike vectors $X$:

$$T_{\mu\nu}X^\mu X^\nu \geq 0 \tag{4.58}$$

[6] Don't think of this vector field as being a Killing vector field; it's arbitrary except for the boundary conditions.

which is a coordinate-independent expression of the demand that for the energy density $\rho = T_{00}$ we have $\rho \geq 0$. The dominant energy condition expresses the idea that the energy-momentum flow $j_\mu = T_{\mu\nu}X^\nu$ is always future-directed along timelike vectors or null vectors and $j_\mu j^\mu \geq 0$. A last energy condition, which Hawking used to establish the second law for black hole thermodynamics for Einstein's field equations, is implied by these field equations and the weak energy condition. If we take null vectors $X$ and we contract these with the field equations, we get

$$\begin{aligned} G_{\mu\nu}X^\mu X^\nu &= R_{\mu\nu}X^\mu X^\nu - \frac{1}{2}Rg_{\mu\nu}X^\mu X^\nu \\ &= R_{\mu\nu}X^\mu X^\nu \\ &= 8\pi T_{\mu\nu}X^\mu X^\nu \geq 0 \end{aligned} \tag{4.59}$$

So we see that for null vectors we always have

$$\boxed{R_{\mu\nu}X^\mu X^\nu \geq 0} \tag{4.60}$$

which is the so-called null energy condition.

## 4.7 Conserved charges and gravity

We saw in the previous section the definition of the energy momentum tensor of the matter field and noted that it should be conserved in a covariant way. The whole problem in defining the total energy locally is the equivalence principle: locally one can erase the existence of a gravitational field by transforming to an accelerating reference frame. This is the physical consequence of the idea that space-time can be described by a differentiable manifold; a differentiable manifold is locally Euclidian, and at every point in space-time the tangent space is isomorphic with Minkowski space-time. So if the local energy of a space-time region would be calculated by integrating the energy momentum tensor, it would depend on the coordinates. Because the equivalence principle doesn't hold globally, the total energy of a space-time region can be defined in a consistent way. This quantity *is* conserved.

The equation $\nabla_\mu T^{\mu\nu} = 0$ can be seen as the covariant generalization of the conservation equation $\partial_\mu T^{\mu\nu} = 0$ in the case of flat space-time. The covariant equation can be rewritten as

$$\partial_\mu T^{\mu\nu} = -\Big(\Gamma^\mu_{\sigma\mu}T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma}T^{\mu\sigma}\Big) \tag{4.61}$$

If we would like to see if the energy is conserved in some region $\Omega$, we have to consider the divergence of the integral over $T$. With covariant derivatives, an integrand with zero divergence equals an integral with zero divergence, but with partial derivatives this is not true. The integral can be rewritten using the symmetry of $T$:

$$\begin{aligned} \partial_\mu\Big(\int_\Omega T^{\mu\nu}\sqrt{|g|}d^4x\Big) &= \int_\Omega \Big(\Gamma^\lambda_{\mu\lambda}T^{\mu\nu} - \Gamma^\mu_{\sigma\mu}T^{\sigma\nu} - \Gamma^\nu_{\mu\sigma}T^{\mu\sigma}\Big)\sqrt{|g|}d^4x \\ &= -\int_\Omega \Gamma^\nu_{\mu\sigma}T^{\mu\sigma}\sqrt{|g|}d^4x \end{aligned} \tag{4.62}$$

and this is in general non-vanishing. This shouldn't give us trauma's because it is the *total* energy that should be conserved, and not only the energy in the matter field. Also, the equation $\partial_\mu T^{\mu\nu} = 0$ which we would like to be satisfied is only a tensorial equation if $T^{\mu\nu}$ would be a Lorentz tensor[7]. Symmetries like conservation of angular momentum or energy are naturally defined with respect to the vacuum. They are founded by performing a translation or rotation in Minkowski space-time, and by looking at the consequences of the invariance of the action under such transformations. This symmetry group is the Poincaré group. This suggests that if we work with asymptotically flat space-times, we should define the Hamiltonian as a boundary integral at infinity. It is ofcourse not trivial how to define such an integral and it depends on the behaviour of the dynamical fields at spatial infinity. It has to be understood as a limiting process, for spatial infinity is not part of our original space-time manifold.

If we would define the total energy momentum tensor of the gravitational field plus the matter field with the action $S$ we would get $T^{\mu\nu}_{(\psi+g)} = 0$ due to the equations of motion. One possible interpretation is that "the total energy and momentum flux of the matter fields always cancels that of the gravitational field", but it's not clear if this interpretation makes any sense. According to some physicists, the search for a local energy is looking for the wrong thing, but at least we now are au fait with the subtleties around this matter. In the next chapter some variations and derivatives are calculated which are needed for further analysis.

[7]A Lorentz tensor is a tensor transforming tensorially only under the group of Lorentz transformations $\Lambda^\mu{}_\nu$, with $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$ and $\eta$ the Minkowski metric.

# Chapter 5

# Some explicit variations and derivatives

To avoid too much "it's easy to show that"-sentences some explicit variations and derivatives will be shown here. Complicated or tedious variations can often be derived more easily if one thinks about the form of the result and the present symmetries first before starting a brute calculation. It can save quite some paper too, so it's also more environmental.

## 5.1 Variations concerning the metric

We will need some expression involving $\delta g_{\mu\nu}$, $\delta g^{\mu\nu}$, $\delta g$ and $\delta\sqrt{|g|}$. First we notice that

$$g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda \rightarrow \delta(g_{\mu\nu}g^{\nu\lambda}) = 0 \tag{5.1}$$

So, if we want to switch between $\delta g_{\mu\nu}$ and $\delta g^{\mu\nu}$ we use

$$\delta g^{\alpha\lambda} = -g^{\alpha\beta}g^{\rho\lambda}\delta g_{\beta\rho} \tag{5.2}$$

Be aware of the fact that $[\delta, g_{\mu\nu}] \neq 0$, so in contractions which concerns $\delta g_{\mu\nu}$ or $\delta g^{\mu\nu}$ we are *not* free to raise or lower indices without introducing a minus-sign; for example,

$$X^{\mu\nu}\delta g_{\mu\nu} = -X_{\mu\nu}\delta g^{\mu\nu} \tag{5.3}$$

If the metric is looked upon as a matrix and $G^{\mu\nu}$ is the cofactor of $g_{\mu\nu}$, the inverse is defined by

$$g^{\mu\nu} = \frac{1}{g}G^{\nu\mu} \tag{5.4}$$

where $G^{\nu\mu}$ is ofcourse the transpose of $G^{\mu\nu}$. For a fixed $\mu$ the determinant $g$ can be expanded along the row $\nu$:

$$g = \sum_{\nu=1}^{n} g_{\mu\nu}G^{\mu\nu} \tag{5.5}$$

where there is no summation over the index $\mu$. Because $g_{\mu\nu}$ doesn't appear in $G^{\mu\nu}$ for a fixed $\mu$ and arbitrary $\nu = 1 \dots n$ [1], it is clear that

$$\frac{\delta g}{\delta g_{\mu\nu}} = G^{\mu\nu} = g g^{\mu\nu} \quad \rightarrow \quad \delta g = g g^{\mu\nu} \delta g_{\mu\nu} \tag{5.6}$$

If we now want to vary $\delta\sqrt{|g|}$, we get

$$\begin{aligned} \delta\sqrt{|g|} &= \frac{\delta\sqrt{|g|}}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \\ &= \frac{1}{2\sqrt{|g|}} \frac{\delta g}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \\ &= \frac{1}{2}\sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu} \end{aligned} \tag{5.7}$$

## 5.2 Variations concerning the Riemann-tensor

We also need variations of the Riemann-tensor. It will turn out that it is not difficult to find such an expression, but eventually we want to express it in terms of $\delta g_{\mu\nu}$ and that will be a little harder. A nice way of deriving tensor identities is to jump to a coordinate system in which $\Gamma^{\mu}_{\nu\lambda} \stackrel{\circ}{=} 0$, called "Riemann normal coordinates". In this coordinate system we have $\nabla \rightarrow \partial$. The Riemann-tensor becomes in this particular point

$$R^{\alpha}{}_{\beta\mu\nu} \stackrel{\circ}{=} \partial_{\mu}\Gamma^{\alpha}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}_{\beta\mu} \tag{5.8}$$

Now the connection is varied. The connection itself is not a tensor, because otherwise it couldn't be used to define a covariant derivative in the first place. But the variation $\delta\Gamma^{\alpha}_{\mu\nu}$ *is* a tensor. This can be checked directly by writing down its transformation; the inhomogenous terms in the transformation will cancel. The variation of the connection is simply

$$\delta\Gamma^{\alpha}_{\beta\mu} = \Gamma'^{\alpha}_{\beta\mu} - \Gamma^{\alpha}_{\beta\mu} \tag{5.9}$$

and this induces a variation in the Riemann tensor

$$\boxed{\delta R^{\alpha}{}_{\beta\mu\nu} \stackrel{\circ}{=} \nabla_{\mu}\delta(\Gamma^{\alpha}_{\beta\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\beta\mu})} \tag{5.10}$$

Because tensorial equations hold for every point, this equation holds in every coordinate system. It is called the Palatini equation. It gives us also straight away the variation in the Ricci tensor:

$$\boxed{\delta R_{\mu\nu} = \nabla_{\alpha}(\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\mu\alpha})} \tag{5.11}$$

Ofcourse this expression can also be derived by a straight variation of $R_{\mu\nu}$. If we had done this, we would need an expression for the variation of the connection. We can guess its form by noticing that $\delta\Gamma^{\alpha}_{\mu\nu}$ is a tensor and by the form of $\Gamma^{\alpha}_{\mu\nu}$ itself. What we also can do is to write out explicitly the variation of $\delta\Gamma^{\alpha}_{\mu\nu}$,

[1] After all, that is how a cofactor is defined in the first place.

permute some indices, use the symmetry of the connection and then subtracting the terms. The result is just what can be expected:

$$\boxed{\delta\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\nabla_\mu \delta g_{\nu\rho} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu})} \tag{5.12}$$

which is a tensor, because $\delta g_{\mu\nu}$ is a tensor.

Now we will derive an expression for $\delta R_{\mu\nu\rho\sigma}$ in terms of $\delta g_{\alpha\beta}$, which we will do by some clever guesswork concerning the symmetries. The reason for this, is that a direct variation is very messy and takes several wallpapers of calculations. The variation is defined as $\delta R_{\mu\nu\rho\sigma} = R^{'}_{\mu\nu\rho\sigma} - R_{\mu\nu\rho\sigma}$, so $\delta R_{\mu\nu\rho\sigma}$ has the same symmetries as $R_{\mu\nu\rho\sigma}$. Thse symmetries for the components are:

- $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$
- $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$
- $R_{\mu[\nu\rho\sigma]} = 0$
- $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

The last equation is the Bianchi identity. First we write

$$R_{\mu\nu\rho\sigma} = g_{\sigma\lambda} R_{\mu\nu\rho}{}^{\lambda} \rightarrow \delta R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho}{}^{\lambda} \delta g_{\sigma\lambda} + g_{\sigma\lambda} \delta R_{\mu\nu\rho}{}^{\lambda} \tag{5.13}$$

If the variation has the same symmetry-properties, we expect that $\delta R_{\mu\nu\rho\sigma}$ contains a term like $R_{\mu\nu[\rho}{}^{\lambda}\delta g_{\sigma]\lambda}$ and a term like $g_{\sigma\lambda}\delta R_{\mu\nu\rho}{}^{\lambda}$, where in the last term we have to do an antisymmetrization to conserve the antisymmetry-properties. Looking at the Palatini-identity we want an expression for $\nabla(\delta\Gamma)$. Using metric compatibility,

$$\nabla_\mu \delta\Gamma^{\lambda}_{\nu\rho} = \frac{1}{2} g^{\lambda\alpha}[\nabla_\mu\nabla_\nu \delta g_{\rho\alpha} + \nabla_\mu \nabla_\rho \delta g_{\nu\alpha} - \nabla_\mu \nabla_\alpha \delta g_{\nu\rho}] \tag{5.14}$$

and with this the Palatini-equation becomes

$$\begin{aligned} \delta R_{\mu\nu\rho}{}^{\lambda} &= \frac{1}{2} g^{\lambda\alpha}[\nabla_\mu\nabla_\nu \delta g_{\rho\alpha} + \nabla_{\mu\rho} \delta g_{\nu\alpha} - \nabla_\mu \nabla_\alpha \delta g_{\nu\rho}] \\ &- \frac{1}{2} g^{\lambda\alpha}[\nabla_\nu\nabla_\mu \delta g_{\rho\alpha} + \nabla_\nu \nabla_\rho \delta g_{\mu\alpha} - \nabla_\nu \nabla_\alpha \delta g_{\mu\rho}] \end{aligned} \tag{5.15}$$

Multiplying by $g_{\sigma\lambda}$ we get rid of the metric in front of the expression. This looks messy, but still an antisymmetrization of $\rho$ and $\sigma$ is required. If this is worked out, we finally obtain the result

$$\boxed{\delta R_{\mu\nu\rho\sigma} = R_{\mu\nu[\rho}{}^{\lambda}\delta g_{\sigma]\lambda} + 2\nabla_{[\mu}\nabla_{[\rho}\delta g_{\sigma]\nu]}} \tag{5.16}$$

This notation can be confusing; we only antisymmetrize here the indices $\mu$ and $\nu$, and $\rho$ and $\sigma$, to obtain four terms in total. The second term contains second order derivatives of the metric, but when the action is defined, we will see that these terms can be converted by partial integrations and Stokes' theorem.

## 5.3 An explicit Lie derivative

We will need the Lie derivative of the Riemann tensor. Here such a calculation will be briefly explained and the generalization should be straight forward. The only thing which is needed is the transformation law for tensors and Taylor's theorem. The Lie derivative of the Riemann tensor is defined as

$$\begin{aligned} \pounds_X R_{\mu\nu\rho\sigma} &= \lim_{\delta s \to 0} \frac{R_{\mu\nu\rho\sigma}(x^{'}) - R^{'}_{\mu\nu\rho\sigma}(x^{'})}{\delta s} \\ &\equiv \delta_X R_{\mu\nu\rho\sigma} \end{aligned} \tag{5.17}$$

where the infinitesimal coordinate transformation

$$x^{'} = x + \delta s X \tag{5.18}$$

defines the vectorfield $X$. Using Taylor's theorem and the transformation law, we can write for the variation

$$\delta_X R_{\mu\nu\rho\sigma}(x) = \frac{\partial x^{'\alpha}}{\partial x^\mu}\frac{\partial x^{'\beta}}{\partial x^\nu}\frac{\partial x^{'\gamma}}{\partial x^\rho}\frac{\partial x^{'\epsilon}}{\partial x^\sigma}[R_{\alpha\beta\gamma\epsilon}(x) + \delta s X^\lambda \partial_\lambda R_{\alpha\beta\gamma\epsilon}(x)] - R_{\mu\nu\rho\sigma}(x) \tag{5.19}$$

Now calculate all the transformations; it gives us a bunch of delta functions and terms with $\delta s$. Keeping only first order terms it becomes clear after some calculating that

$$\delta_X R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\epsilon}\nabla_\sigma X^\epsilon + R_{\mu\nu\epsilon\sigma}\nabla_\rho X^\epsilon + R_{\mu\epsilon\rho\sigma}\nabla_\nu X^\epsilon + R_{\epsilon\nu\rho\sigma}\nabla_\mu X^\epsilon + X^\epsilon \nabla_\epsilon R_{\mu\nu\rho\sigma} \tag{5.20}$$

Because of the symmetries mentioned earlier of the Riemann tensor, this can be simplified to give

$$\delta_X R_{\mu\nu\rho\sigma} = X^\epsilon \nabla_\epsilon R_{\mu\nu\rho\sigma} + 2\nabla_\mu X^\epsilon R_{\epsilon\nu\rho\sigma} + 2\nabla_\rho X^\epsilon R_{\mu\nu\epsilon\sigma} \tag{5.21}$$

## 5.4 Variations and covariant derivatives

In varying the Lagrangian, we encounter variations of the covariant derivative. Unlike partial derivatives, covariant derivatives and variations don't commute due to the connection. Let's write the covariant derivative of a general tensor field $\phi$ in compact notation as

$$\nabla_i \phi = \partial_i \phi + \sum_{[i]} \Gamma_i \phi \tag{5.22}$$

where we understand the contractions between the connections and the fields and a minus sign for every covariant index. Here the $[i]$ is *not* a tensorial or summation index. It just tells us which connections, summations and partial derivatives belong to which covariant derivative if we consider higher order derivatives. If we vary this derivative, we obtain

$$\begin{aligned} \delta(\nabla_i \phi) &= \partial_i \delta\phi + \sum_{[i]} \Gamma_i \delta\phi + \sum_{[i]} \delta\Gamma_i \phi \\ &= \nabla_i(\delta\phi) + \sum_{[i]} \delta\Gamma_i \phi \end{aligned} \tag{5.23}$$

The second line is obtained due to the linear character of the covariant derivative and the product rule. So the commutator of $\delta$ and $\nabla_i$ is

$$[\delta, \nabla_i]\phi = \sum_{[i]} \delta\Gamma_i \phi \sim \nabla \delta g \tag{5.24}$$

where $g$ denotes the metric field. Note that this equation makes sense because $\delta\Gamma_i$ is a tensor. In this way variations of derivatives can be rewritten. For example,

$$\begin{aligned} \delta\nabla_1\nabla_2\phi &= (\nabla_1\delta + \sum_{[1]} \delta\Gamma_1)\nabla_2\phi \\ &= \nabla_1(\nabla_2\delta + \sum_{[2]} \delta\Gamma_2)\phi + \sum_{[1]} \delta\Gamma_1)\nabla_2\phi \\ &= \nabla_1\nabla_2\delta\phi + \nabla_1(\sum_{[2]} \delta\Gamma_2\phi) + \sum_{[1]} \delta\Gamma_1)\nabla_2\phi \end{aligned} \tag{5.25}$$

In a similar manner it can be derived that

$$\begin{aligned} \delta\nabla_1\nabla_2\nabla_3\phi &= \nabla_1\nabla_2\nabla_3\delta\phi + \nabla_1\nabla_2(\sum_{[3]} \delta\Gamma_3\phi) + \nabla_1\nabla_2\phi(\sum_{[3]} \delta\Gamma_3) \\ &+ \nabla_1(\nabla_3\phi \sum_{[2]} \delta\Gamma_2) + \nabla_2\nabla_3\phi \sum_{[1]} \delta\Gamma_1 \end{aligned} \tag{5.26}$$

Calculations as these can be useful when we want to rewrite for instance the variation of Lagrangians.

# Chapter 6

# Black hole mechanics and -thermodynamics

To understand the equations of motion of black holes it is necessary to understand some basic ideas about black holes. Those are reviewed in this chapter. In comparing the equations of motion with the familiar thermodynamic relations the similarity between the entropy and the black hole surface area will become clear. It will be made plausible that these aren't just nice coincidences. It turns out that the area can give a measure for the black hole entropy, and this makes the prediction by Hawking that black holes quantummechanically viewed radiate consistent. Proving the second law for this black hole entropy for more complex Lagrangians will turn out to be not that easy.

## 6.1 Basis properties of black holes

Black holes are mathematically viewed certain solutions of the field equations of Einstein which possess a horizon from where null geodesics cannot extend to spatial infinity. There are a number of different solutions of the field equations which give black holes, for example:

- The Schwarzschild metric, which is spherically symmetric and static
- The Kerr metric, which describes a rotating black hole
- The Reissner-Nordström metric, which describes a charged black hole
- The Kerr-Newman metric, which describes a rotating and charged black hole

These solutions imply that a black hole can be totally described by its mass $M$, its angular momentum $J$, its electric charge $Q$ and the area of the event horizon $A$. There are some technical subtleties about how to actually define such quantities because of the equivalence principle. But if we want to generalize the first law, we need to pay some attention to this. The fact that no more quantities are needed to specify a black hole is called the "no hair" theorem. It is quite remarkable, because the star from which the black hole is formed contains much more information. We also assume that every singularity contains an event horizon, so that there will be no naked singularities. This is called "cosmic censorship".

The Kerr-Newman metric is the most general solution, and the other solutions can be obtained by setting $Q = 0$, $J = 0$, or both. To see if there are singularities involved, we need to do a maximal analytic extension. In this process, all coordinate singularities are removed. The Schwarzschild solution for the line element in spherical coordinates is given by

$$\boxed{ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)} \tag{6.1}$$

The Schwarzschild solution contains certain symmetries; it is static[1] and it describes a spherically symmetric black hole. So we have a symmetry in the time direction and in the $\phi$ direction. This gives us the Killing vectors $k^\mu = (1, 0, 0, 0)$ and $k^\mu = (0, 0, 0, 1)$ respectively, and this gives us in turn two conserved charges:

$$\begin{aligned} E &= m\frac{dt}{d\tau}g_{00} = -m(1 - \frac{2M}{r})\frac{dt}{d\tau} \\ L &= m\frac{d\phi}{d\tau}g_{33} = mr^2\frac{d\phi}{d\tau} \end{aligned} \tag{6.2}$$

which have opposite signs due to the signature of the metric. This kind of analysis can also be used for the other solutions; it is a matter of identifying the symmetries and the resulting Killing vectors.

## 6.2 Coordinate choices

Physics doesn't depend on the choice of coordinates, but a particular choice can make life considerably more easy. Remember that the Schwarzschild solution has a singularity at $r = 2M$. This makes one believe there is something very special going on at $r = 2M$, but don't be fooled: it's just a bad choice of coordinate system! It's a matter of cleverly rewriting the metric to avoid these singularities. To see if there are singularities involved, we need to do a maximal analytic extension. Physical singularities can be recognized by the divergence of scalar terms constructed out of the Riemann-tensor like $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. For instance, the Schwarzschild solution contains a coordinate singularity at $r = 2M$. The maximal analytic extension of this solution is called the Kruskal extension. With these maximal analytic extensions we can describe the evolution of geodesics all through space-time in a continuous way ... if it weren't for physical singularities. Physical singularities can be recognized by the divergence of scalar terms constructed out of the Riemann-tensor like $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The procedure is found in allmost all books on general relativity, so we will be brief here, and take the Schwarzschild solution as example. We want to investigate radial null

[1] A solution is called stationary if it is time-independent, and static if it is not evolutionary. Static metrics can't contain cross-terms due to the invariance under $x^0 \to -x^0$. The subtle difference between these two properties can be understood by an analogy: a pipe with water flowing at a constant velocity would be the stationary case, and a pipe with no waterflow at all would be the static case. A space-time is defined to be static if it has a timelike Killing vector field which is hypersurface-orthogonal. A static space-time is always stationary, but the contrary isn't always true, as can be seen by looking at the Kerr-metric.

geodesics[2] going past the Schwarzschild radius.

The procedure is to note that with a new coordinate $r^* = r + 2M \ln|\frac{r-2M}{2M}|$ we have that $d(t \pm r^*) = 0$ on our radial null geodesics. Ingoing coordinates are defined by $v = t + r^*$ and outgoing coordinates by $u = t - r^*$. Now perform the coordinate transformation $(t, r, \theta, \phi) \to (v, r, \theta, \phi)$ and rewrite the metric. To do this, we have to rewrite $dt^2$:

$$\begin{aligned} dt &= dv - dr^* \\ &= dv - \Big(1 - \frac{2M}{r}\Big)^{-1} dr \\ \to dt^2 &= dv^2 - 2\Big(1 - \frac{2M}{r}\Big)^{-1} drdv + \Big(1 - \frac{2M}{r}\Big)^{-2} dr^2 \end{aligned} \tag{6.3}$$

Remember that our coordinate transformations are diffeomorphisms, and that in rewriting the line element we can use the fact that $\frac{\partial x}{\partial y} = \left(\frac{\partial y}{\partial x}\right)^{-1}$ for two dependent coordinates $x$ and $y$. Plugging this into the Schwarzschild line element, we get

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dv^2 + 2drdv + r^2 d\Omega^2 \tag{6.4}$$

and see that the metric is not singular anymore for $r = 2M$ due to the cross-term $drdv$. So we can analytically continuate the metric for $r > 0$. If we rewrite this to $drdv = \ldots$, it is clear that for $ds^2 < 0$ we have $drdv < 0$. Because $dv > 0$ for future-directed worldlines, we must have $dr \leq 0$. So wordlines with $r < 2M$ are forced to go to $r = 0$. The same can be done for the outgoing coordinate $u = t - r^*$. The line element becomes

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)du^2 - 2drdu + r^2 d\Omega^2 \tag{6.5}$$

Here we have that $drdu \geq 0$ for $ds^2 < 0$, so $dr \geq 0$ for future-directed wordlines because $du > 0$. This seems quite strange; an object with a surface $r < 2M$ expands until $r = 2M$ is reached. This is a white hole. Both are solutions of Einstein's field equations, because these equations are time-symmetric. We will only consider black holes to be physical, but in maximal analytic extensions these solutions will both appear.

We can also take the transformation $(t, r, \theta, \phi) \to (u, v, \theta, \phi)$ to obtain the elegant-looking line element

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dudv + r^2 d\Omega^2 \tag{6.6}$$

So with these coordinates we can extend a radial null geodesic from $r = 2M$ to $r = 0$. But the Schwarzschild solution is not geodesically complete; a light ray can come from spatial infinity and terminate at $r = 0$ due to the physical singularity. A solution is called maximal if, at every space-time point, one can extend a geodesic $x^\mu(\lambda)$ to infinite values of the affine parameter $\lambda$ along both directions, or that it terminates on a physical singularity. One example of this is Minkowski space-time. Our rewritten Schwarzschild solution isn't maximal. To

[2] So $dx^\mu = (dt, dr, 0, 0)$ and $ds^2 = 0$.

obtain a maximal solution Kruskal used the following coordinate transformation to rewrite eq.(6.6):

$$\begin{aligned}
U &= -e^{-u/4M} = -e^{(-t+r)/4M}\sqrt{\frac{r-2M}{2M}} \\
V &= e^{v/4M} = e^{(t+r)/4M}\sqrt{\frac{r-2M}{2M}} \\
UV &= -\Big(\frac{r-2M}{2M}\Big)e^{r/2M} \\
ds^2 &= \frac{-32M^3}{r}e^{-r/2M}dUdV + r^2 d\Omega^2 \qquad (6.7)
\end{aligned}$$

So we maximalized the Schwarzschild solution by the transformation $(t, r, \theta, \phi) \rightarrow (U, V, \theta, \phi)$. In these coordinates, we can describe both black and white holes, and we can extend every geodesic in both space-direction and time-direction, except if we encounter the $r = 0$ singularity. The region $r = 2M$ is described by $UV = 0$ and the region $r = 0$ is described by $UV = 1$. A constant $U$ corresponds to outgoing radial null geodesics, and a constant $V$ corresponds to ingoing radial null geodesics, as is clear from the form of those two. We see two

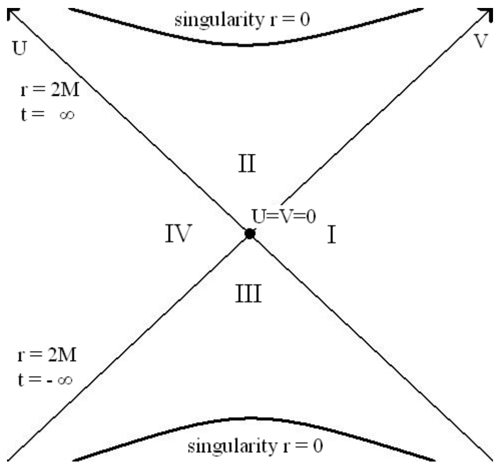


Figure 6.1: *Space-time diagram of the Kruskal solution*

singularities in the figure: a singularity in the past, $t = -\infty$ and a singularity in the future, $t = +\infty$. The regions $I$ and $II$ are covered by $v$, and describe the black hole. The regions $I$ and $III$ are relevant for describing the white hole. Region $I$ alone corresponds to the Schwarzschild solution with $r > 2M$. Perhaps the most curious region is $IV$; it's geometrically equivalent to $I$, and the connecting topology between those two is the very hypothetical Einstein-Rosen bridge. We won't consider those here, and in the forthgoing we will only assume the existence of black holes.

In spherical coordinates we had the isometry $t \rightarrow t + c$ with Killing vector $\xi = (\partial/\partial t, 0, 0, 0)$; this expresses the stationary character of the Schwarzschild solution. In Kruskal coordinates this Killing vector field becomes

$$\xi = \frac{1}{4M}(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}) \tag{6.8}$$

We also see that the point $(U, V) = (0, 0)$ is a fixed point [3] of $\xi$ and this point corresponds to a 2-sphere. One could think that the geodesically incomplete character of solutions could be due to the spherically symmetric collapse, but Hawking and Penrose showed with their singularity theorems that this incompleteness is a general feature of gravitational collapse. For these singularity theorems, one can consult [4], chapter eight.

## 6.3 Black hole surfaces

In the equations of motion, a couple of black hole properties are very important for us. Those properties are the surface area and the surface gravity. A hypersurface can be described as the set of points $\{x \mid S(x) = 0\}$ for $S(x) : M \to \mathbb{R}$, where $M$ is the manifold and $x \in M$. The components of the vector field normal to this hypersurface are then given by

$$\boxed{l^\mu = f(x)g^{\mu\nu}\partial_\nu S(x)} \tag{6.9}$$

Here $f(x) : M \to \mathbb{R}$ is an arbitrary function. If $l_\mu l^\mu = 0$ for a particular member of the set, then that element is called a null-hypersurface. Take for instance the Schwarzschild solution in spherical coordinates and a specific $S$. We know that it contains a coordinate singularity, but continuity implies that we don't have to switch to other coordinate systems like Eddington-Finkelstein coordinates. We choose $S(r) = r - 2M$, so different values of $S$ denote different 2-spheres with arbitrary $t$ and $r = 2M + S(r)$:

$$l_\mu l^\mu = g^{\mu\nu}\partial_\mu S\ \partial_\nu S = g^{11}\partial_1 S\partial_1 S = (1 - \frac{2M}{r}) \tag{6.10}$$

The condition that $l$ is null gives that the surface $r = 2M$ is a null hypersurface:

$$\begin{aligned} l_\mu l^\mu &= 0 \to r = 2M \\ l &= f(x)\Big(1 - \frac{2M}{r}\Big)\frac{\partial}{\partial r} \end{aligned} \tag{6.11}$$

Now let's take a closer look at those null hypersurfaces. The normal of this null hypersurface $\Sigma$ is $l$. Because $l_\mu l^\mu = 0$, our $l$ is also a tangent vector[4] of $\Sigma$. If $x^\mu(\lambda)$ is a curve in $\Sigma$, we can write

$$l^\mu = \frac{dx^\mu}{d\lambda} = f(x)g^{\mu\nu}\partial_\nu S(x) \tag{6.12}$$

[3] For a fixed point $\lambda$ of a function $f(\lambda)$ the equation $f(\lambda) = \lambda$ holds.

[4] Remember that due to the signature of the metric one and the same vector can both be normal and tangent to a hypersurface.

These curves $x(\lambda)$ are actually geodesics! This can be seen by a straightforward calculation of $l^\sigma \nabla_\sigma l^\mu$:

$$\begin{aligned} l^\sigma \nabla_\sigma l^\mu &= (l^\sigma \partial_\sigma \ln f) l^\mu + f g^{\mu\nu} l^\sigma \nabla_\nu \partial_\sigma S \\ &= \frac{d \ln f}{d\lambda} l^\mu + l^\sigma f \nabla^\mu (\frac{1}{f} l_\sigma) \\ &= \frac{d \ln f}{d\lambda} l^\mu + \frac{1}{2} \partial^\mu (l_\sigma l^\sigma) - (\partial^\mu \ln f) l_\sigma l^\sigma \end{aligned} \tag{6.13}$$

In the first line we used that our space-time is torsion free ($\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$) and in the second line we used the fact that $l$ is a tangent vector. Now, $l_\sigma l^\sigma$ is a constant on $\Sigma$, so $t^\mu \partial_\mu (l_\sigma l^\sigma) = 0$ for any tangent vector $t$. Because $l$ was also a tangent vector, we see that $\partial_\mu (l_\sigma l^\sigma) \propto l_\mu$. With this we have proven that

$$\boxed{l^\sigma \nabla_\sigma l^\mu \propto l^\mu} \tag{6.14}$$

and that the curve $x(\lambda)$ is a geodesic with tangent vector $l$ on $\Sigma$. Such geodesics are called the generators of $\Sigma$. These curves are important to us, because they can be used to describe the evolution of the hypersurfaces which they generate. Later on we will see that this evolution is described by the Raychaudhuri equation, and with this we can prove the famous area theorem of Hawking.

## 6.4 Killing Horizons and the zeroth law

A null hypersurface $\Sigma$ is called a Killing horizon of a Killing vector field $\xi$ if this Killing vector field is normal to the hypersurface. If we use the field equations of Einstein all event horizons are Killing horizons under the assumptions that the dominant energy condition holds and that the matter fields have a well defined Cauchy description; we will come to that later. Now pick again the normal vector $l$ to $\Sigma$ and choose an affine parameter. We then have $l^\sigma \nabla_\sigma l^\mu = 0$. Because the killing vector field $\xi$ is proportional to $l$, we can write $\xi^\mu = h(x) l^\mu$ for an arbitrary function $h(x)$. If this is plugged in our geodesic equation, we obtain

$$\xi^\sigma \nabla_\sigma \xi^\mu = (-\frac{1}{h} \xi^\sigma \partial_\sigma \frac{1}{h}) \xi^\mu = \kappa \xi^\mu \tag{6.15}$$

where we have defined a very important quantity:

$$\boxed{\kappa = \xi^\sigma \partial_\sigma (\ln h)} \tag{6.16}$$

It is called the surface gravity of $\Sigma$. It turns out [5] that this $\kappa$ is the acceleration of a particle near $\Sigma$ as measured at spatial infinity. So that explains the name, but what are the properties of this surface gravity? From the expression it is clear that it is a function in general. However, it can be proven that the surface gravity is actually constant on $\Sigma$! To prove this, we use Frobenius' theorem [6] and eq.(6.15). Frobenius' theorem tells us that for a normal vector $\xi$ of $\Sigma$ the constraints are

$$\xi_{[\mu} \nabla_\nu \xi_{\rho]} |_\Sigma = 0 \tag{6.17}$$

It is emphasized here that the constraints only apply on the hypersurface $\Sigma$. But the field $\xi$ is a Killing vector field and has thus constraints on its own, namely $\nabla_\mu \xi_\nu = \nabla_{(\mu}\xi_{\nu)}$. Frobenius' theorem then yields

$$\begin{aligned} \xi_{[\mu}\nabla_\nu \xi_{\rho]}|_\Sigma &= 2(\xi_\rho \nabla_\mu \xi_\nu + \xi_\mu \nabla_\nu \xi_\rho + \xi_\nu \nabla_\rho \xi_\mu)|_\Sigma \\ &= 2(\xi_\rho \nabla_\mu \xi_\nu + \xi_\mu \nabla_\nu \xi_\rho - \xi_\nu \nabla_\mu \xi_\rho)|_\Sigma \\ &= 0|_\Sigma \end{aligned} \tag{6.18}$$

If we contract this with $\nabla^\mu \xi^\nu$ and use eq.(6.15) twice, we see that

$$\kappa^2 = -\frac{1}{2}\nabla^\mu \xi^\nu \nabla_\mu \xi_\nu|_\Sigma \tag{6.19}$$

Now we will prove that $\kappa$ is constant on orbits of $\xi$. The vector field $\xi$ is both normal and tangent to $\Sigma$. So we have

$$\begin{aligned} \xi^\rho \nabla_\rho \kappa^2 &= -\xi^\rho \nabla^\mu \xi^\nu \nabla_\rho \nabla_\mu \xi_\nu|_\Sigma \\ &= -\nabla^\mu \xi^\nu \xi^\rho R_{\nu\mu\rho\sigma}\xi^\sigma|_\Sigma \\ &= 0|_\Sigma \end{aligned} \tag{6.20}$$

We used eq.(3.32) and $R_{\nu\mu\rho\sigma} = -R_{\nu\mu\sigma\rho}$. This proves that $\kappa$ is constant on orbits which are induced by the Killing vector $\xi$, and this implies that it is constant on the whole hypersurface $\Sigma$. This is called the zeroth law of black hole mechanics: for a time-independent black hole the surface gravity is a constant. We can compare this with the zeroth law of thermodynamics: for a system in thermal equilibrium the temperature is a constant.

But we still haven't defined $\kappa$ uniquely. From eq.(6.19) we see that a Killing vector $c\xi$ gives us a surface gravity $c^2\kappa$ for $c \in \mathbb{R}$. Because $\xi_\mu \xi^\mu = 0$ on $\Sigma$ we impose a normalization at spatial infinity. We will always deal with asymptotically flat space-times[5] so we want the normalization of our Killing vector connected to the time symmetry to become[6] 1 and the normalization connected to spatial symmetries to become +1. So

$$\lim_{r\to\infty} \xi_\mu \xi^\mu = \pm 1 \tag{6.21}$$

The sign of $\kappa$ then is fixed by the requirement that $\xi$ is future-directed.

To conclude this section, we will calculate explicitly the surface gravity for a Schwarzschild black hole and see that the zeroth law holds. We can write the Killing vector normal to the horizon as

$$\xi^\mu = \delta^\mu_0, \qquad \xi_\mu = g_{\mu 0} \tag{6.22}$$

The surface gravity can be calculated with different expressions, but here eq.(6.19) is used in a coordinate base with spherical coordinates $(t, r, \theta, \phi)$. It is evident that

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\lambda_{\mu\nu}\xi_\lambda = -\nabla_\nu \xi_\mu \tag{6.23}$$

[5]For our purpose it is enough to look at asymptotically flat space-times as space-times which can't be distinguished from Minkowski space-time at spatial infinity. If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then $\lim_{r\to\infty} h_{\mu\nu} = O(\frac{1}{r})$, $\lim_{r\to\infty} \partial_\lambda h_{\mu\nu} = O(\frac{1}{r^2})$, etc.

[6]With our signature of the metric as $(-, +, +, +)$.

Looking at our Killing vector we are interested in $\nabla_1 \xi_0$:

$$\nabla_1 \xi_0 = \partial_1 \xi_0 - \Gamma^{\alpha}_{10} \xi_\alpha = \partial_1 \xi_0 - \Gamma^0_{10} \xi_0 \tag{6.24}$$

This becomes with the metric connection,

$$\nabla_1 \xi_0 = \partial_1 \xi_0 - \frac{1}{2} g^{0\rho} (\partial_1 g_{0\rho} + \partial_0 g_{\rho 1} - \partial_\rho g_{10}) = \frac{1}{2} \partial_1 g_{00} \tag{6.25}$$

Plugging this in, we find that

$$\begin{aligned} \kappa^2 &= -g^{00} g^{11} \nabla_0 \xi_1 \nabla_0 \xi_1 \\ \rightarrow \kappa^2 &= \frac{1}{4} (\partial_1 g_{00})^2 \end{aligned} \tag{6.26}$$

At the Schwarzschild radius $r = 2M$ this becomes

$$\boxed{\kappa = \frac{M}{r^2} = \frac{1}{4M}} \tag{6.27}$$

This is indeed the same for every angle $(\theta, \phi)$. Later on we will encounter the surface gravity for the Kerr-Newman case.

## 6.5 Bifurcation horizons and binormals

The concept of a bifurcation horizon will become important in our treatment of a generalized first law. We saw earlier that in Kruskal coordinates the point $U = V = 0$ was a fixed point of the Killing vector $\xi$. This point corresponds to a 2-sphere which bifurcates the space-time into 4 regions. This is in fact the bifurcation horizon of the Kruskal solution. It can be shown that in general the Killing horizon of a black hole can contain a 2-dimensional space-like cross-section $B$, on which the Killing vector field $\xi$ vanishes: $\xi|_B = 0$. We saw that in the Kruskal solution the bifurcation horizon was a fixed point of the Killing flow, and this is a general feature of a bifurcation horizon. The full Killing horizon of the Kruskal solution exists of two null hypersurfaces, which correspond to $U = 0$ and $V = 0$, and the bifurcation horizon lies at the intersection of these two hypersurfaces. The condition for a bifurcation horizon to exist, is that the generators of the Killing horizon are geodesically complete to the past, and $\kappa \neq 0$.

Technically $B$ can be regarded as follows. Take a Killing vector field $\xi = \partial_\mu$ and look at the orbits of $\xi$. At these orbits we can write $x = x(\lambda)$ for an affine parameter $\lambda$. The Killing vector becomes

$$\begin{aligned} \xi &= \partial_\mu = \frac{d\lambda}{dx} \frac{d}{d\lambda} = fl \\ f &= \frac{d\lambda}{dx} \\ l &= \frac{d}{d\lambda} = \frac{dx^\mu(\lambda)}{d\lambda} \partial_\mu \end{aligned} \tag{6.28}$$

We already know that we can express the surface gravity $\kappa$ as $\kappa = \partial_\mu \ln f$, and we just have seen that this $\kappa$ is actually constant on orbits at the null hypersurface. As a result, $f = f_0 e^{\kappa x}$ at orbits of $\xi$, and $\lambda$ is given by

$$\lambda = \pm e^{\kappa x} \tag{6.29}$$

We used here our freedom to shift $x$ by a constant without altering the equations. For $x \in (-\infty, +\infty)$ there are 2 portions of the generators of $\Sigma$ covered, namely $\lambda > 0$ and $\lambda < 0$. But the point $\lambda = 0$ is a fixed point of $\xi$, and so $\xi$ vanishes at this point. This is an $(n-2)$-dimensional sphere in $n$ dimensions, and it's our sought-after bifurcation horizon $B$. With the foregoing it can be easily seen that $\kappa^2$ is a constant on $B$.

Now to the concept of binormals. For this we take a space-like cross section of some 4-dimensional world volume $\Delta$ to obtain a 3-dimensional space-like surface $\Sigma_{hor}$. The orthogonal complement $\Sigma_{\perp hor}$ of this space has signature $-+$ and it can be spanned by two null vectors. Because $\Sigma_{hor} \subset \Delta$, one of these null vectors can be chosen to be the normal $n$ of $\Delta$, and with this it will be proportional to the Killing vector $\xi$. The other null vector $N$ can be chosen to satisfy $N^\mu n_\mu = -1$. It's a little hard to make a picture in your mind of this situation, because we can't associate an angle with two vectors of which one of them is null. But with the vectors $n$ and $N$ we can construct an antisymmetric two-indexed tensor $\epsilon$ which we call the binormal. It has the following properties:

$$\begin{aligned}
\epsilon_{\mu\nu} &= 2N_{[\mu} \otimes n_{\nu]} = N_\mu n_\nu - N_\nu n_\mu \\
\epsilon_{\mu\nu}\epsilon^{\mu\nu} &= -2 \\
N_\mu N^\mu &= n_\mu n^\mu = 0 \\
N_\mu n^\mu &= -1 \\
n &\propto \xi
\end{aligned} \tag{6.30}$$

It's called $\epsilon_{\mu\nu}$ to remind of the fact that it can be seen as the volume form of the two-surface orthogonal to $\Delta$. Take for example a static and spherically symmetric black hole solution. In spherical coordinates $\{t, r, \phi, \theta\}$ the line element can be put into the form

$$\boxed{ds^2 = -e^{2g(r)}dt^2 + e^{2f(r)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)} \tag{6.31}$$

The directions of $t$ and $r$ are both normal to $\Sigma_{hor}(\theta, \phi)$, and we obtain

$$\boxed{\epsilon_{01} = -\epsilon_{10} = e^{g(r)+f(r)}} \tag{6.32}$$

Because the Killing field is hypersurface orthogonal at the Killing horizon, the antisymmetric tensor $\nabla_{[\mu}\xi_{\nu]}$ can be composed as

$$\nabla_{[\mu}\xi_{\nu]} = \kappa\epsilon_{\mu\nu} + t_{[\mu}\xi_{\nu]} \tag{6.33}$$

in which $t$ is tangential to $\Sigma$. At the bifurcation horizon we observe now that we have the equality

$$\nabla_\mu \xi_\nu = \kappa\epsilon_{\mu\nu} \tag{6.34}$$

Note that we would expect from from eq.(6.19) that $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2\kappa^2$ on the bifurcation horizon, but we can simply rescale the Killing vector because $\kappa$ is constant on the event horizon.

## 6.6 The event horizon and the first law

The first law of black hole mechanics relates the change of $M$, $J$ and $A$. To get this relation, we need the Kerr-Newman metric and calculate all the relevant physical quantities in terms of these three parameters. The Kerr-Newman metric is a solution of the Einstein-Maxwell equations with no electric current $j$:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 2\Big(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} \mathbf{F}^2\Big) \\ \nabla_\mu F^{\mu\nu} &= 0 \end{aligned} \tag{6.35}$$

These equations can be derived from the Einstein-Maxwell action

$$S = \frac{1}{16\pi G} \int \sqrt{|g|}[R - 4\pi G\, \mathbf{F}^2] d^4x \tag{6.36}$$

The solution is the celebrated Kerr-Newman line element (see for example [2]):

$$\begin{aligned} ds^2 &= -\frac{\Delta^2}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - adt]^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\phi^2 \\ \Delta^2 &= r^2 - 2Mr + a^2 + Q^2 \\ \rho^2 &= r^2 + a^2\cos^2\theta \\ a &= \frac{J}{M} \end{aligned} \tag{6.37}$$

This solution has two Killing vectors, namely $\xi = \frac{\partial}{\partial t}$ and $\xi = \frac{\partial}{\partial \phi}$. These are connected with time translations and rotational translations, and they give the conserved quantities $p_0$ and $p_3$. Since the Lie-derivative is a linear operator we can consider the linear combination

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi} \tag{6.38}$$

where $\Omega_H$ defines the angular velocity of the black hole.

Ofcourse, dealing with electromagnetic fields, we are curious about the vector potential. If there are no source terms the Maxwell equations are invariant under duality transformations[7] $\mathbf{F} \rightarrow^* \mathbf{F}$. It turns out that electric and magnetic charges $q$ and $p$ can be defined via

$$q = \frac{1}{4\pi} \oint {}^*\mathbf{F}, \quad p = \frac{1}{4\pi} \oint \mathbf{F} \tag{6.39}$$

where the integrations surface completely surrounds the sources of $\mathbf{F}$. The charge $Q$ in the metric can then be written as $\sqrt{p^2+q^2}$, and the solution for the vector potential becomes

$$\mathbf{A} = \frac{(qr - ap\cos\theta)}{\rho} dt - \frac{(aqr\sin^2\theta - (r^2+a^2)p\cos\theta)}{\rho} d\phi \tag{6.40}$$

[7] Here ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

In our case we assume that the magnetic charge is 0, so that $Q = q$. To make the calculations in this section more transparant we explicitly note the metric components:

$$\begin{aligned}
g_{00} &= -\frac{\Delta^2 - a^2\sin^2\theta}{\rho^2}, \qquad g_{03} = g_{30} = -a^2\sin^2\theta\frac{r^2+a^2-\Delta^2}{\rho^2} \\
g_{11} &= \frac{\rho^2}{\Delta^2} \\
g_{22} &= \sin^2\theta\frac{(r^2+a^2)^2 - \Delta^2 a^2\sin^2\theta}{\rho^2} \\
g_{33} &= \rho^2
\end{aligned} \tag{6.41}$$

We see from this expression that the metric contains two event horizons when the coordinate singularity $\Delta = 0$ is considered, namely

$$r^2 - 2Mr + a^2 + Q^2 = 0 \quad \rightarrow \quad r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{6.42}$$

These are the surfaces of infinite gravitational redshift. Another coordinate singularity is given by $\rho = 0$, so

$$r^2 + a^2\cos^2\theta = 0 \quad \rightarrow \quad (r, \theta) = (0, \frac{\pi}{2}) \tag{6.43}$$

This gives a ring singularity, which will not be considered here. A similar calculation as in the last section gives us again the surface gravity $\kappa$:

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)} = \frac{\sqrt{M^2 - Q^2 - a^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - a^2}} \tag{6.44}$$

With some identities of the metric obtained via[8] $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$ an expression for $\Omega_H$ can be deduced:

$$\begin{aligned}
\Omega_H &= \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{p^3}{p^0} = \frac{g^{3\nu}p_\nu}{g^{0\nu}p_\nu} = \frac{-g_{30}}{g_{33}} \\
&= \frac{a}{M^2[2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}]}
\end{aligned} \tag{6.45}$$

Now we'll look at the outer horizon $r_+ = M + \sqrt{M^2 - a^2 - Q^2}$. To calculate the area of the event horizon, we take $dt = dr = 0$ in the line element. The line element can be written as

$$dl^2 = g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 = h_{ij}dx^i dx^j \tag{6.46}$$

where $h_{ij}$ are the components of the induced metric on the surface. The area of the event horizon with radius $r_+$ is then

$$\begin{aligned}
A(r) &= \int_{S^2}\sqrt{|h|}d\Sigma \\
&= \int_0^{2\pi}d\phi\int_0^\pi\sqrt{g_{22}g_{33}}d\theta \\
&= 4\pi(r_+^2 + a^2) \\
&= 8\pi(M^2 - \frac{1}{2}Q^2 + M\sqrt{M^2 - a^2 - Q^2})
\end{aligned} \tag{6.47}$$

[8]Be aware of the cross-terms in the metric!

Now we change the black hole state from $(M, Q, J)$ to $(M+\delta M, Q+\delta Q, J+\delta J)$. This induces a change in $A = A(M, J, Q)$:

$$\delta A = \frac{\partial A}{\partial M}\delta M + \frac{\partial A}{\partial Q}\delta Q + \frac{\partial A}{\partial J}\delta J \tag{6.48}$$

The derivatives can be calculated to give

$$\frac{\partial A}{\partial M} = \frac{8\pi}{\kappa}, \qquad \frac{\partial A}{\partial J} = \frac{-8\pi\Omega}{\kappa}, \qquad \frac{\partial A}{\partial Q} = \frac{-8\pi\Phi}{\kappa} \tag{6.49}$$

where $\Phi = A^0$ is the scalar potential of the electromagnetic field. If this is plugged in eq.(6.48) and rewritten, it looks quite similar to the first law of thermodynamics:

$$\boxed{\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J + \Phi\delta Q \quad \leftrightarrow \quad \delta E = T\delta\mathcal{S} - P\delta V} \tag{6.50}$$

Ofcourse, at first sight this could just as well be a coincidence. But in the next three sections we will make plausible that in the theory of general relativity the area behaves just like an entropy; the area never decreases.

## 6.7 Causal structure of space-time

Because every black hole entropy candidate should obey the second law of black hole thermodynamics, we will give a sketchy derivation of this theorem in the case of Einstein's field equations. It's sketchy, because we'll need a fair amount of mathematical machinery to make the theorem plausible. We first specify a little more the causal structure of space-time, where we skip the details (see for instance [2] or [4]). Space-time here is considered to be the set $(M, g_{\mu\nu})$.

We already are familiar with the notion of lightcones in space-time; the lightcone consists of the points which are causally connected with one point, in the future and in the past. Timelike or null vectors in the upper halve of the cone are future directed, and timelike or null vectors in the lower halve are past directed. The causal character of curves $x(\lambda)$ is specified by their tangent vectors $t$. The curve is called future directed timelike (fdtc) if at every $p \in \lambda$, $t$ is future directed and timelike ($t^2 < 0$). If, in the same case, $t$ is timelike or null, the curve is called future directed causal ($t^2 \leq 0$), abbreviated fdcc. Now we can define what is called the chronological future $I^+(p)$ of $p \in M$ : it is the set of events which can be reached by an fdtc starting from $p$:

$$I^+(p) = \Big(\forall q \in M \mid \exists\, x_{fdtc}(\lambda),\ x_{fdtc}(0) = p,\ x_{fdtc}(1) = q\Big) \tag{6.51}$$

For a subset $\Omega \subset M$ we define then naturally $I^+(\Omega) = \bigcup_{p\in\Omega} I^+(p)$. A same definition goes for the causal future $J^+(p)$ of $p$:

$$J^+(p) = \Big(\forall q \in M \mid \exists\, x_{fdcc}(\lambda),\ x_{fdcc}(0) = p,\ x_{fdcc}(1) = q\Big) \tag{6.52}$$

Here we also included null-like $t$'s. And ofcourse, $J^+(\Omega) = \bigcup_{p\in\Omega} J^+(p)$. Chronological and causal pasts $I^-$ and $J^-$ are defined similarly. The boundaries of $I$

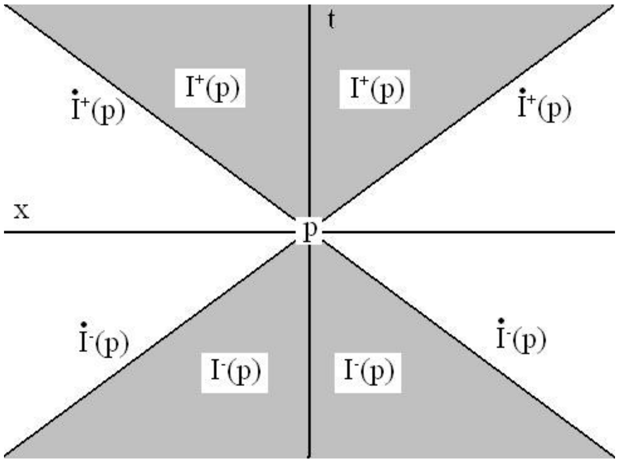


Figure 6.2: *The causal structure of Minkowski space-time*

and $J$ are indicated by a dot. A picture of this is given in figure (6.2). For Minkowski space-time these definitions are quite simple. Here, $\dot{I}^+$ is generated by all future directed null geodesics starting from $p$. But we could build quite artificial space-times in which these statements become more complicated; for example, one could remove a point $r \in \dot{I}^+(p)$ between $p$ and $q$, and no causal curve can be constructed which connect $p$ and $q \in \dot{I}^+(p)$, with as result that $q \notin J^+(p)$.

Next, we assume that $(M, g_{\mu\nu})$ is globally hyperbolic. Global hyperbolicity is a very general assumption about a manifold with a causal structure, which enables one to prove that $\dot{I}^+(\Omega)$ has an endpoint in $\Omega$. This implies the existence of Cauchy-hypersurfaces, which are space-like or light-like hypersurfaces which intersect each time-like curve in a given congruence just once. In this way the initial value problem of a theory can be well-defined: with information from one Cauchy-hypersurface $\Sigma_1(t_1)$ we can uniquely predict what happens at a Cauchy Hypersurface $\Sigma_2(t_2 > t_1)$.

We also need the notion of conjugate points. Remember that a geodesic is an extremal of the length functional $l[\gamma] = \int \sqrt{\pm g(\dot{\gamma}, \dot{\gamma})}dt$ on a manifold $M$ with metric $g$. This can be a curve of minimal length or maximal length, which depends on the signature. For example, we can go from the north pole to the equator right away, but also via the south pole. Both curves are geodesics on the 2-sphere. How can we see that the longer geodesic isn't the shortest? The idea is to consider such a geodesic $\gamma$ going through the points $p$, $q$, and $r$, with $r$ lying between $p$ and $q$. If there is another geodesic $\gamma'$ lying infinitesimal close to $\gamma$ which goes through $p$ and $r$, then $\gamma$ can vary to a longer geodesic. On our 2-sphere, $p$ lies on the north pole, $q$ on the south pole, and $r$ on the equator. It's clear from the picture that the geodesic going through $r$ is the longest geodesic with respect to the Euclidean metric. Figure (6.3) makes this clear. All the geodesics going from $p$ to $r$ have the same conjugate point $q$. This indicates that

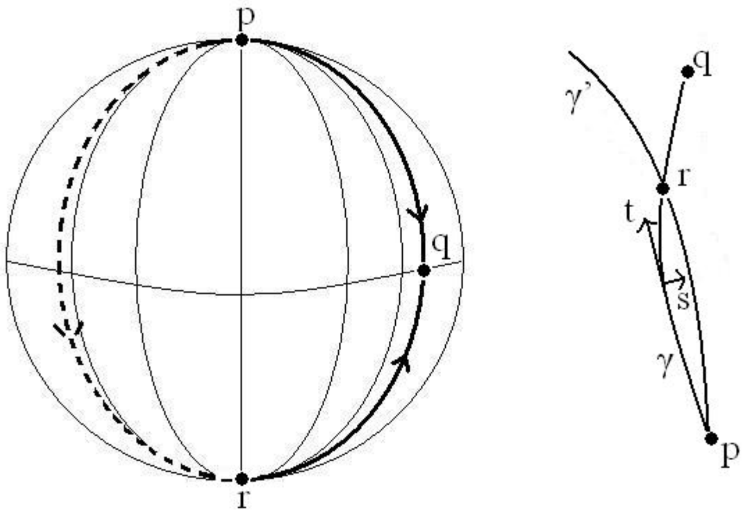


Figure 6.3: *Left: the 2-sphere. Right: 2 conjugated points r of q of a geodesic* $\gamma$.

this particular geodesic doesn't minimize the length, but maximizes it.[9] One could frown his (or her!) eyebrows by the concept of 'infinitesimal close curves', but this concept is really well defined by the equation of geodesic deviation. A congruence can be parametrized by $\gamma_\alpha(\lambda)$. For each $\alpha \in \mathbb{R}$, $\gamma_\alpha(\lambda)$ is a geodesic parametrized by the affine parameter $\lambda$. Coordinates on this congruence are then $x^\mu(\alpha, \lambda)$, and two natural vector fields on this congruence can be defined:

$$t^\mu = \frac{\partial x^\mu}{\partial \lambda} \qquad s^\mu = \frac{\partial x^\mu}{\partial \alpha} \tag{6.53}$$

In the picture $s$ points from $\gamma$ to $\gamma'$, and $t$ is just the flow of the geodesic curve $\gamma$. With this a relative velocity and acceleration between the geodesics $\gamma$ and $\gamma'$ can be defined via the vector field $s$:

$$v^\mu = (\nabla_t s)^\mu = t^\rho \nabla_\rho s^\mu \qquad a^\mu = (\nabla_t v)^\mu = t^\rho \nabla_\rho v^\mu \tag{6.54}$$

Now with some algebra an expression for $a$ can be obtained ([3],[2]):

$$a^\mu = R^\mu{}_{\nu\rho\sigma} t^\nu t^\rho s^\sigma \tag{6.55}$$

This gives a physical interpretation to the Riemann tensor; it tells us that the relative acceleration between $\gamma$ and $\gamma'$ depends on the curvature via the Riemann tensor. Note that at $p$ and $q$ the vector field $s$ disappears. It's also called a Jacobi field. Now we can rephrase the idea of conjugate points, also called focal points. Two points $p$ and $r$ on $\gamma$ are conjugate if there is a Jacobi field $s^\mu \neq 0$ between $p$ and $r$, but which vanishes at $p$ and $r$ themselves. So this

[9] Note that in a Cartesian space a geodesic *minimizes* the length $l = \sqrt{(\delta_{ij} x^i x^j)}$ but that in a space-time with a Lorentzian signature a timelike geodesic *maximizes* the proper time $\tau$! This is also the origin of the twin-paradox; accelerating brings you away from a geodesic, and therefore you experience less proper time.

means that the congruence of which $\gamma$ takes part of is converging. This converging is a result of the curvature of $M$, or gravity.

As two last definitions before jumping to the next section we will define future domains and achronalicity. Whereas $I^+(M)$ or $J^+(M)$ concerned events which could be influenced by $M$, we now specify space-time regions which are determined by such a set. The future domain of dependence $D^+(\Sigma)$ of a hypersurface $\Sigma \subset M$ consists of all points $p$ which are intersected by past inextendible causal curves $\gamma$. A definition of the past domain of dependence $D^-$ replaces

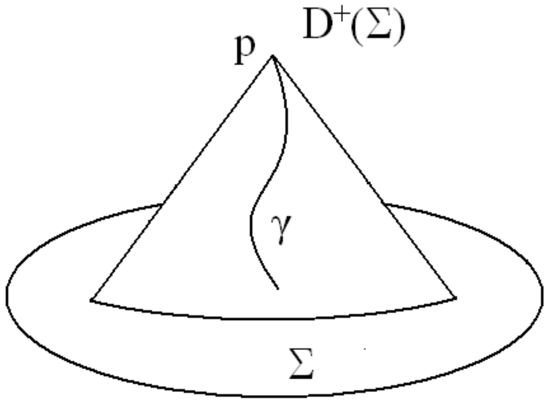


Figure 6.4: *An artistic impression of* $D^+(\Sigma)$

causal curves by future curves. So we always have $S \subset D^+(\Sigma) \subset J^+(\Sigma)$. The idea behind these domains are that data on $\Sigma$ only determines the solutions of differential equations on $D^+(\Sigma)$. If not a single causal curve has an end-point in the past, then all the solutions inside $D^+(\Sigma)$ are uniquely determined by $\Sigma$. This brings us back to the notion of Cauchy hypersurfaces; they have as domain of dependence $D^+(\Sigma) \cup D^-(\Sigma)$ the manifold $M$, and in this case $M$ is globally hyperbolic.

Finally, a set $S \subset M$ is called achronal if $I^+(S) \cap S = \emptyset$. This means that there are no points $(p, q)$ for which $q \in I^+(p)$; $p$ and $q$ can only be connected by a null curve. So without formal proof it is clear that the sets $\dot{I}^\pm$ and $\dot{J}^\pm$ are achronal.

## 6.8 Causal structure of black holes

We already stated the idea of black holes in the beginning of this chapter, but here we will be a little more precise. We characterize black holes by the impossibility of null geodesics to escape to future null infinity $\mathcal{J}^+$. So we should look at the causal past $J^-(\mathcal{J}^+)$. With this definition we involve infinities, which are not part of our original space-time $(M, g)$. But with a conformal transformation we can add these points to form the "unphysical space-time"

$(\tilde{M}, \tilde{g})$[10], and we have to say something about the causal structure of that artificial manifold. We assume that there is a $\tilde{N} \subset \tilde{M}$, such that $(\tilde{N}, \tilde{g})$ is globally hyperbolic. It must have the property that the closure of $M \cap J^-(\mathcal{J}^+)$ is a subset of $\tilde{N}$; $\overline{M \cap J^-(\mathcal{J}^+)} \subset \tilde{N}$. The space-time $(M, g)$ contains a black hole if $M$ isn't contained in $J^-(\mathcal{J}^+)$. The black hole region $B$ and the event horizon $\mathcal{H}$ are defined by

$$\begin{aligned} B &= M - J^-(\mathcal{J}^+) \\ \mathcal{H} &= \dot{J}^-(\mathcal{J}^+) \cap M \end{aligned} \tag{6.56}$$

We turn our attention to the event horizon $\mathcal{H}$[4]. First of all, it is an achronal subset of $M$. This means that two points on $\mathcal{H}$ can only be connected by a null curve. Second, the generators of $\mathcal{H}$ have no future end points, so they never leave $\mathcal{H}$. This is the mathematical statement of the idea that the event horizon consists of photons which cannot escape from the black hole, but also don't continuate into the singularity. But the generators may have past end points; these photons could come from a region outside $B$.

## 6.9 The second law in general relativity

We are now ready to prove the second law for Einstein's equations. The assumptions are

- The space-time $(M, g)$ is strongly asymptotically predictable
- For null vectors $k$ we have the null energy condition: $T_{\mu\nu}k^\mu k^\nu \geq 0$

We further introduce the following conditions and definitions:

- $\mathcal{H}$ is generated by null geodesic generators $x(\tau)$
- $\Sigma(\tau) = \Sigma_1,\ \Sigma(\tau') = \Sigma_2$ are spacelike Cauchy surfaces with $\Sigma_1 =\subset I^+(\Sigma_2)$ for the globally hyperbolic region $\tilde{N}$
- $H_i = \mathcal{H} \cap \Sigma_i$ for $i = 1, 2$

We look at the evolution of $H_1 \to H_2$. The statement of the second law is that the area of $H_1$ is smaller or equal to that of $H_2$. In order to proof this we show that the generators of $H$ diverge everywhere, so $\theta \geq 0$. This is because any (n-2)-dimensional area element $A(\tau)$ evolves via[11]

$$\frac{dA}{d\tau} = \theta A \tag{6.57}$$

and the statement becomes basically that two infinitesimal neighbouring geodesics can't be conjugated on $\mathcal{H}$.

[10] One can write the original metric via a conformal transformation as $\tilde{g} = \Omega^2(x)g$, where $\Omega(x)$ is a function, in which infinities of the original space-time now lie on finite coordinates in the new space-time. With these kind of transformations the original causal structure of space-time is preserved. Often one uses functions like $\tan^{-1}(x) : (\infty, +\infty) \to (-\frac{\pi}{2}, +\frac{\pi}{2})$, which are one-to-one.

[11] See the derivation of the Raychaudhuri equation in the appendix for an explanation of this statement.

Well, let's assume the opposite and see what that implies: we put $\theta < 0$ at a point $y \in \mathcal{H}$. According to the null energy condition and the Raychaudhuri equation, we have that

$$\frac{d\theta}{d\tau} \leq -\frac{1}{2}\theta^2 \tag{6.58}$$

So the null generators going through $y$ reach a point at finite distance at which $\theta \to -\infty$, and beyond this point the generators will have an intersection point. This is also called a caustic. Proposition 4.5.12 of [4] states that that if there is a point $r \in (p, q)$ conjugate to $q$ along a curve $x(\tau)$ then we can vary $x(\tau)$ in such a way that we obtain a timelike curve between $q$ and $p$. The proof of this statement consists of showing that there is a variation $\delta x(\tau)$ for which the tangent vector is time-like everywhere in $(q, p)$. However, due to the achronic nature of $\mathcal{H}$ this is impossible, so this point $p$ cannot exist and hence $\theta \geq 0$. According to eq.(6.57) this implies that $\mathcal{H}(\tau)$ is a non-decreasing function of $\tau$.

So the three laws of black hole thermodynamics are given by

- 0th law: $\delta\kappa = 0$
- 1st law: $\delta M = \frac{\kappa}{8\pi}\delta A + \Phi\delta Q + \Omega\delta J$
- 2nd law: $\delta A \geq 0$

As was noted earlier, the two identifications $\kappa \sim T$ and $A \sim \mathcal{S}$ can be made. Historically, the second identification was first made by Bekenstein, and the first identification was first proposed by Hawking, which results in the so called Hawking effect. Hartle and Gibbson discovered the deeper reason for this connection between the temperature and the surface gravity, which will be treated in the next section. Now raises the question: how general are these laws? If the field equations of Einstein are adopted, one can show two things [4]: First, the event horizon is always a Killing horizon. Second, the metric of the black hole is either static, or it is stationary, axisymmetric and $g_{30} = g_{03} = 0$ in spherical coordinates. These results have to be assumed if a more general theory of gravitation is considered. The derivation of the Hawking effect fortunately doesn't depend on the exact form of the field equations.

## 6.10 Black hole temperature

For black hole thermodynamics we need the notion of a black hole temperature, and Hawking found this effect back in 1974. The so-called Euclidean method will briefly be discussed for the Schwarzschild solution, and with this the reader hopefully has a more comfortable feeling by the notion of black hole temperature in the forthgoing. This is by no means a rigorous justification; the full calculation (see for instance [2], chapter fourteen) contains the treatment of quantum field theory in curved space-time. But the Euclidean method shows how this full calculation can be explained in a different manner [7].

The method consists of an analytic continuation of the time parameter $t$ into the complex plane. This is a technique also used in quantum field theory called Wick-rotation, and it ensures that the path integral will converge. The new time parameter will be written as $\tau = it$. The new line-element $ds_E^2$ obtains a Euclidean signature, but ofcourse it still contains the singularity at $r = 2M$. We rewrite it in a new coordinate $x$ by the usual rewriting of differentials:

$$\begin{aligned} \tau &= it, \quad x = 4M\Big(1 - \frac{2M}{r}\Big)^{1/2} \\ ds_E^2 &= x^2\Big(\frac{d\tau}{4M}\Big)^2 + \Big(\frac{r}{2M}\Big)^4 dx^2 + r^2 d\Omega^2 \end{aligned} \tag{6.59}$$

The spherical coordinate singularity $r = 2M$ corresponds now to $x = 0$. It reminds us of the coordinate singularity we have in the two-dimensional plane with polar coordinates $(r, \theta)$ which we then compare with $(x, \tau/4M)$.[12] The 'natural' thing to do is to impose a periodicity on $\tau/4M$ just as in the polar case, and see how this turns out to be. It's natural, because we expect the singularity to be just a coordinate singularity. So we put

$$\frac{\tau}{4M} \in (0, 2\pi] \quad \rightarrow \quad \tau \in (0, 8\pi M] \tag{6.60}$$

With this we compactified $\tau$. The topology of this Euclidean space-time is $\mathbb{R}^2(x,\tau) \times S^2(\theta,\phi)$. Note that for the Schwarzschild case we had $\kappa = 1/4M$, so $8\pi M = 2\pi/\kappa$. If we now have physical fields $\phi(t, \mathbf{x})$ on a Schwarzschild background and we look at them in imaginary time, $\phi(\tau, \mathbf{x})$, the fields will also have the periodicity in $\tau$: $\phi(\tau, \mathbf{x}) = \phi(\tau + \mathbf{8\pi M}, \mathbf{x})$. At first sight this seems not very ground-shaking, but this imaginary time periodicity actually characterizes the thermal state of $\phi$!

The idea is to consider the amplitude for a scalar field $\phi(x)$ and perform $t \rightarrow \tau$ in this amplitude. Let's take a look at the amplitude of $\phi(x)$ going from the Cauchy-hypersurfaces $\Sigma_1(t_1)$ to $\Sigma_2(t_2)$. The amplitude is given by

$$\begin{aligned} < \phi(\Sigma_2(t_2)) | \phi(\Sigma_1(t_1)) > &= < \phi(\Sigma_2) | e^{-iH(t_2 - t_1)} | \phi(\Sigma_1) > \\ &= \int \mathcal{D}\phi e^{iS[\phi]} \end{aligned} \tag{6.61}$$

Now we do an analytic continuation, $t \rightarrow i\tau$. We put $(\tau_2 - \tau_1) = 8\pi M$ such that $\phi(\Sigma_2) = \phi(\Sigma_1)$. The left-hand of the equation becomes then simply, with the complete set of eigenstates $\{\phi_n\}$,

$$Z[\phi_n] = \sum_n < \phi_n | e^{-H 8\pi M} | \phi_n > \tag{6.62}$$

This corresponds to the partition function of a thermal state with $\beta = 8\pi M$, so $T = 1/8\pi M$, or

$$\boxed{T = \frac{\kappa}{2\pi}} \tag{6.63}$$

which is indeed the imaginary time period. We see that the fields in the neighbourhood of the black hole are behaving as if they are in thermal equilibrium with temperature $T$, where this temperature depends on the surface gravity of the black hole. This gives an extra reason to take the full calculation very seriously.

[12]We are comparing the $(r, \theta)$-plane with the $(x, \tau)$-plane here.

## 6.11 Intrinsic entropy of the gravitational field

Now the question which arises naturally is: Why does a gravitational field can have an intrinsic entropy? A detailed discussion of this is out of the scope of this thesis, but it's important to realize what we are actually calculating. The entropy of a system is normally associated with the number of ways we can permute the microstates, $\Omega$, of that system to obtain the given macrostate. We can't measure those microstates, but only the macrostate. The entropy is given by

$$\mathcal{S} = \ln \Omega \tag{6.64}$$

with $k_B = 1$. It's an extensive state function, and allows us to make predictions about the reversibility of thermodynamics processes. But here we associate an entropy with the event horizon of a black hole, which is just a vacuum: $T_{\mu\nu} = 0$. This entropy is as such associated with the metric tensor, or the gravitational field $g_{\mu\nu}$. It means that there are several configurations associated with one and the same gravitational field. Yang-Mills fields[13] don't have this property, so what distinguishes the gravitational field from the other force fields?

The answer is topology. It turns out that a theory of gravity is not scale invariant, while Yang-Mills theory is. Scale invariance means basically that the same equations of motion are implied by the extremum of the action if we transform the metric by a constant factor $\Omega$, $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. If we approach a theory of gravity by the Euclidean method, it turns out that one and the same gravitational field allows different topologies. So here we could say that we can't measure the different topologies, but only the resulting gravitational field. The difference between these topologies can be calculated, and this equals $\frac{1}{2}\beta E = \frac{1}{4}A$. A discussion of this approach can be seen in [9].

[13] See for a calculation of the Noether charge of Yang-Mills theory [8]; the situation becomes more subtle if Chern-Simons terms are included in the Lagrangian.

# Chapter 7

# Lagrangian and Hamiltonian field theory

When people first encounter the theory of general relativity, a Lagrangian or Hamiltonian treatment is often skipped. But to generalize the laws of black hole thermodynamics to general theories of gravitation, this treatment is necessary. We will see that these points of view will make the theoretical structure more clear, but we will also encounter some subtleties. This subtleties especially arise in the Hamiltonian treatment; in this case the separation of space-time into time and space is necessary, whereas in the Lagrangian case we can continue to formulate in a covariant way. Curiously enough, we won't need the specific form of the Hamiltonian as is given in, for example, [2]. We will give a general formula for the Hamiltonian, in terms of symplectic currents.

## 7.1 Lagrangian field theory

Here we will review the most important aspects of the original treatment by Wald and Lee, which was used to generalize the first law of black hole mechanics. The review here will be quite technical, but at the end we will see some physical examples from it which should clarify business. Details can be found in [10] and [11]. One assumes that space-time has a topology of $\mathbb{R} \times \Sigma$, where $\Sigma$ is an orientable (n-1)-dimensional manifold which describes the spatial part of the space-time considered. This space-time is assumed to be globally hyperbolic.

The collection of physical fields are maps $\phi : M \to M^{'}$, where $M^{'}$ is finite dimensional. For instance, a complex scalar field $\phi$ is seen as a map $\phi : \mathbb{R} \times \Sigma \to \mathbb{C}$. These fields and their variations have compact support in $M$: there is a compact submanifold $N \subset M$ for which $\phi[M \backslash N] \subset \emptyset$ and $\delta\phi[M \backslash N] \subset \emptyset$. This is just a nice way of saying that these fields are localized, or have boundary conditions. In our case the collection of dynamical fields is $\phi^a = (g_{\mu\nu}, \psi)$. The derivative operator $\nabla$ has the property that $\nabla \epsilon = 0$, where $\epsilon$ is the volume-form.

We will restrict ourselves to more specific cases. We assume that the Lagrangian is a scalar density of the form

$$\mathscr{L}[\phi] = \mathscr{L}(\phi^a, \nabla_\mu \phi^a, \ldots, \nabla_{(\mu_1} \cdots \nabla_{\mu_k)} \phi^a) \tag{7.1}$$

There could also be nondynamical background fields in $\mathscr{L}$; an example would be the Minkowski metric $\eta$ in special relativistic field theories. Note that we use derivatives which are totally symmetric in the indices. This is possible, since every antisymmetric part of those derivatives can be expressed via the Riemann-tensor[2], see eq.(3.31) for the second order case. We define the derivatives of $\mathscr{L}$ as

$$\begin{aligned} \mathscr{L}_a &= \frac{\partial \mathscr{L}}{\partial \phi^a} \\ \mathscr{L}_a^\mu &= \frac{\partial \mathscr{L}}{\partial(\nabla_\mu \phi^a)} \\ \mathscr{L}^{\mu_1 \cdots \mu_k} &= \frac{\partial \mathscr{L}}{\partial(\nabla_{(\mu_1} \cdots \nabla_{\mu_k)} \phi^a)} \end{aligned} \tag{7.2}$$

which are all symmetric in the indices. The first variation of the Lagrangian takes the form

$$\delta \mathscr{L} = \sum_{j=0}^{k} \mathscr{L}_a^{\mu_1 \cdots \mu_j} \delta(\nabla_{(\mu_1} \cdots \nabla_{\mu_j)} \phi^a) \tag{7.3}$$

There is no derivative acting on $\delta\phi^a$ in the $j = 0$ case. If we now perform this variation, we end up with terms like $\delta\phi^a$ and higher order derivatives of them. An important step now is to see that $\delta\mathscr{L}$ can always be rewritten as

$$\boxed{\delta \mathscr{L} = E_a \delta \phi^a + \nabla_\rho \Theta^\rho(\phi, \delta\phi)} \tag{7.4}$$

We saw this earlier in chapter four, eq.(4.24), and later in this chapter we will look at a derivation of this. The $E_a$ are locally constructed out of the dynamical fields $\phi$ and their derivatives, and $\Theta$ is locally constructed out of $\phi$, $\delta\phi$ and their derivatives. It's also linear in $\delta\phi$. When we perform the variation under an integral sign, then $\Theta$ is just the expression obtained by removing derivatives from $\delta\phi$ by partial integration. The term $E_a$ sure looks like the Euler-Lagrange equations for the dynamical field. That's no coincidence; we stated that $\delta\phi$ had compact support in $M$, and so has $\Theta$. As a result, the integral over $\nabla_\mu \Theta^\mu$ is equal to zero and we get

$$\int_M E_a \delta \phi^a = \int_M \delta \mathscr{L} \tag{7.5}$$

We see that the equations of motion are obtained in a fancy way. So if $\phi_a$ lies in the space of on-shell solutions, we impose the boundary conditions $\delta\phi_a|_{\partial M} = 0$, and we see that $\nabla_\rho \Theta^\rho = 0$. So under the symmetrygroup of diffeomorphisms the current $\Theta$ is conserved.

We need one more quantity, the so-called symplectic current density $\Omega$. This can be derived by considering two different variations of $\mathscr{L}$, namely $\delta_1$ and $\delta_2$. If we consider the second variation of $\mathscr{L}$, we get

$$\delta_1 \delta_2 \mathscr{L} = \delta_1 E_a \delta_2 \phi^a + E_a \delta_1 \delta_2 \phi^a + \nabla_\mu \delta_1 \Theta_2^\mu \tag{7.6}$$

A similar expression holds for $\delta_2\delta_1\mathscr{L}$, and these two second variations are identical. This defines our symplectic current density as

$$\boxed{\Omega^\mu(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\Theta_2^\mu - \delta_2\Theta_1^\mu} \tag{7.7}$$

and the divergence of it becomes

$$\nabla_\mu\Omega^\mu = \delta_2 E_a \delta_1\phi^a - \delta_1 E_a \delta_2\phi^a \tag{7.8}$$

Because the various variations $\delta_i$ commute, the symplectic current will depend linearly on $\delta\phi^a$ and its derivatives. It is antisymmetric in the variations.

We now write all of this in terms of forms and consider $\mathbf{L} \in \Lambda^n(M)$ and $\mathbf{\Theta} \in \Lambda^{n-1}(M)$. In that case we state in compact notation that[1]

$$\delta\mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta} \tag{7.9}$$

Both notations will be mixed up here; it's straightforward to convert eq.(7.4) into eq.(7.9). Here we leave out the explicit sum over $a$ and in the right hand side we understand the contraction between the first indices of the tensors involved. We see that $\mathbf{\Theta}$ is only determined up to a closed (n-2)-form $\mathbf{Y}(\phi, \delta\phi)$; the substitution $\mathbf{\Theta} \to \mathbf{\Theta} + d\mathbf{Y}$ doesn't change the equations of motion. The symplectic current $\mathbf{\Omega}$ becomes

$$\mathbf{\Omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1[\mathbf{\Theta}(\phi, \delta_2\phi)] - \delta_2[\mathbf{\Theta}(\phi, \delta_1\phi)] \tag{7.10}$$

The ambiguity in $\mathbf{\Theta}$ introduces the ambiguity $\mathbf{\Omega} \to \mathbf{\Omega} + d[\delta_1\mathbf{Y}(\phi, \delta_2\phi) - \delta_2\mathbf{Y}(\phi, \delta_1\phi)]$. Now we consider a vector field $\xi$ and the variation it induces. In the end we promote it to a Killing vector field, but first we consider it as being arbitrary. The Lie-derivative of $\mathbf{L}$ with respect to this vector is

$$\begin{aligned} \pounds_\xi\mathbf{L} &= \mathbf{E}\pounds_\xi\phi + d\mathbf{\Theta}(\phi, \pounds_\xi\phi) \\ &= \xi \cdot d\mathbf{L} + d(\xi \cdot \mathbf{L}) \\ &= d(\xi \cdot \mathbf{L}) \end{aligned} \tag{7.11}$$

because $d\mathbf{L} = 0$[2]. We can associate a Noether current $\mathcal{J} \in \Lambda^{n-1}(M)$ to each $\xi$ by

$$\boxed{\mathcal{J} = \mathbf{\Theta}(\phi, \pounds_\xi\phi) - \xi \cdot \mathbf{L}} \tag{7.12}$$

This definition is chosen this way due to the properties it has if the equations of motion are satisfied. Namely, the exterior derivative of this current is

$$\begin{aligned} d\mathcal{J} &= d\mathbf{\Theta} - d(\xi \cdot \mathbf{L}) \\ &= \delta\mathbf{L} - \mathbf{E}\delta\phi - d(\xi \cdot \mathbf{L}) \\ &= -\mathbf{E}\pounds_\xi\phi \end{aligned} \tag{7.13}$$

So if $\phi$ is on-shell then $\mathcal{J}$ is closed for all $\xi$. In that case there exists according to the Poincaré theorem at least locally a $\mathbf{Q} \in \Lambda^{n-2}(M)$ constructed from the

[1] Note that from this it is immediately clear that if we define the action on a compact manifold via field-localization, the variation of this gives the equations of motion due to $\int_\Omega d\mathbf{\Theta} = \int_{\partial\Omega}\mathbf{\Theta} = \int_\emptyset \mathbf{\Theta} = 0$.

[2] Diffeomorphism invariance can then be stated as $\delta\phi = \pounds_\xi\phi \to \delta_\xi\mathbf{L} = d(\xi \cdot \mathbf{L})$.

fields which appear in $\mathbf{L}$ and $\xi$, such that if we consider the on-shell solutions we have

$$\boxed{d\mathcal{J} = 0 \rightarrow \mathcal{J} = d\mathbf{Q}} \tag{7.14}$$

$\mathbf{Q}$ is the Noether charge associated with the vector field $\xi$. Being an (n-2)-form, we can integrate it over an (n-2)-dimensional surface $\Sigma$ to obtain the Noether charge of $\Sigma$, relative to $\xi$. But for that it has to exist globally. In [12] it is proven that in the general case of $\mathbf{E} \neq 0$ the Noether current takes the form

$$\mathcal{J}[\xi] = d\mathbf{Q}[\xi] + \xi \cdot \mathbf{C} \tag{7.15}$$

in which $\mathbf{C} \in \Lambda^{n-1}(M)$. The condition $\mathbf{C} = 0$ are then the constraint equations for the equations of motion to hold.

## 7.2 Three examples

Now it is time to put all of this in a familiar context. As noticed before, it is often much easier to do the variation directly, and that's what we are going to do. The previous formalism will be applied in the case of scalar fields, metric fields and electromagnetic fields. We will split up these fields and look at them individually. Combinations of them, like the Einstein-Maxwell action, are then easily analyzed. An explicit algorithm to calculate $Q$ from a given $\mathcal{J}$ is not given here, but can be found in [13]. In rewriting the Lagrangian in forms, we use the following identity for two p-forms $\alpha$ and $\beta$ with corresponding antisymmetric p-vectors $A$ and $B$:

$$\begin{aligned} *\alpha \wedge \beta &= \alpha \wedge *\beta \\ &= \frac{1}{2}\alpha^{\mu_1\cdots\mu_p}\beta_{\mu_1\cdots\mu_p}\epsilon_{1\cdots n}dx^1 \wedge \ldots \wedge dx^n \\ &= \frac{1}{2}(A \cdot B)\epsilon \end{aligned} \tag{7.16}$$

### 7.2.1 Scalar fields

First we look at the free massless scalar field and consider its action:

$$S[\phi] = \frac{1}{2}\int \sqrt{|g|}g_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi \; d^4x \tag{7.17}$$

where we will abbreviate $\nabla^2 = g^{\mu\nu}\nabla_\nu\nabla_\mu$. If we vary with respect to the field $\phi$ we obtain the equations of motion plus the boundary term, and if we vary with respect to the metric we obtain the energy momentum tensor. The variation can be written as $\delta\mathscr{L} = \delta_m\mathscr{L} + \delta_g\mathscr{L}$ with

$$\begin{aligned} \delta_m\mathscr{L} &= \sqrt{|g|}\Big(\nabla_\mu(\delta\phi\nabla^\mu\phi) - (\nabla^2\phi)\delta\phi\Big) \qquad (7.18) \\ \delta_g\mathscr{L} &= \sqrt{|g|}\Big(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla^\mu\phi - \nabla^\mu\nabla^\nu\phi\Big)\delta g_{\mu\nu} \end{aligned}$$

So the equation of motion and the boundary term are

$$\begin{aligned} E_{\mu\nu\rho\sigma} &= \nabla^2\phi\epsilon_{\mu\nu\rho\sigma} \\ \Theta_{\mu\nu\rho}(\phi,\delta\phi) &= (\delta\phi\nabla^\sigma\phi)\epsilon_{\sigma\mu\nu\rho} \end{aligned} \tag{7.19}$$

The components of the Noether current are given by

$$\begin{aligned}\mathcal{J}_{\mu\nu\rho} &= \Theta_{\mu\nu\rho} - \xi^\sigma L_{\sigma\mu\nu\rho} \\ &= \delta\phi \nabla^\sigma \phi \epsilon_{\sigma\mu\nu\rho} - \nabla^2\phi\xi^\sigma\epsilon_{\sigma\mu\nu\rho} \\ &= [-\xi^\lambda\nabla_\lambda\phi\nabla^\sigma\phi - \nabla^2\phi\xi^\sigma]\epsilon_{\sigma\mu\nu\rho} \end{aligned} \tag{7.20}$$

if we remember that $\delta\phi = -\xi^\mu\nabla_\mu\phi$. Imposing the equations of motion we get

$$\mathcal{J}_{\mu\nu\rho} = -\Big(\xi^\lambda\nabla_\lambda\phi\nabla^\sigma\phi\Big)\epsilon_{\sigma\mu\nu\rho} \tag{7.21}$$

Note that in looking at $\mathbf{\Theta}$ as a vector density and using eq.(4.24), $\mathbf{\Theta}$ would be given by

$$\begin{aligned}\Theta^\mu &= \mathscr{L}^\mu\delta\phi \\ &= \frac{\partial(\sqrt{|g|}\mathscr{L})}{\partial(\nabla_\mu\phi)}\delta\phi \\ &= -(\nabla^\mu\phi)\xi^\nu\nabla_\nu\phi \end{aligned} \tag{7.22}$$

How about the Noether charge? If we regard $\phi \in \Lambda^0(M)$ we can write[3], using eq.(7.16),

$$\mathbf{L} = d\phi \wedge *d\phi \tag{7.23}$$

The variation becomes

$$\delta\mathbf{L} = d(\delta\phi \wedge *d\phi) + \delta\phi \wedge d * d\phi + d\phi \wedge \delta * d\phi \tag{7.24}$$

We can't commute the $*$-operator and $\delta$; their commutator is not zero, but

$$[\delta, *] = \frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta} * \tag{7.25}$$

which can be easily derived in components. These terms are going to give us the terms concerning $\delta g_{\alpha\beta}$ which will result in the energy-momentum tensor for $\phi$. The first term in eq.(7.24) gives us $\Theta$, and the second term gives us the equations of motion for $\phi$. The Noether current becomes

$$\begin{aligned}\mathcal{J} &= \delta\phi \wedge *d\phi - \xi \cdot \mathbf{L} \\ &= -\xi \cdot d\phi \wedge *d\phi - d\phi \wedge \xi \cdot *d\phi \end{aligned} \tag{7.26}$$

if we use $\delta_\xi\phi = -\xi \cdot d\phi$, and if we look at the exterior derivative of this we indeed obtain

$$\begin{aligned} d\mathcal{J} &= d(\delta_\xi\phi \wedge *d\phi) - \delta_\xi\mathbf{L} \\ &= -\delta_\xi\phi \wedge d * d\phi \end{aligned} \tag{7.27}$$

where $d * d\phi = 0$ are the equations of motion. However, it can be seen that this term is not exact[4] and so the Noether charge can't be defined globally.

[3]For a massive scalar field we would have obtained $\mathbf{L} = d\phi \wedge *d\phi - m^2\phi \wedge *\phi$.

[4]This is by looking at the appearance of $\xi$ in $\mathcal{J}$.

### 7.2.2 The Hilbert action

Second we consider the Hilbert action,

$$S[g_{\mu\nu}] = \int \sqrt{|g|} g^{\mu\nu} R_{\mu\nu} d^4x \tag{7.28}$$

where we have put the coupling to 1 for convenience. The variation of the Lagriangian density $\mathscr{L}$ becomes

$$\delta\mathscr{L} = \delta\sqrt{|g|} g^{\mu\nu} R_{\mu\nu} + \sqrt{|g|}\delta g^{\mu\nu} R_{\mu\nu} + \sqrt{|g|} g^{\mu\nu}\delta R_{\mu\nu} \tag{7.29}$$

With eq.(5.7) and eq.(5.11) this variation becomes

$$\delta\mathscr{L} = \sqrt{|g|}(\frac{1}{2}Rg^{\alpha\beta} - R^{\alpha\beta})\delta g_{\alpha\beta} + \sqrt{|g|} g^{\mu\nu}(\nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\mu\alpha}) \tag{7.30}$$

According to our formalism we also have $\delta\mathscr{L} = E^{\alpha\beta}\delta g_{\alpha\beta} + \nabla_\alpha \Theta^\alpha$. If we put the metric under the covariant differentiation and some indices are relabeled, the following can be recognized:

$$\begin{aligned} E^{\alpha\beta} &= \sqrt{|g|}(\frac{1}{2}Rg^{\alpha\beta} - R^{\alpha\beta}) = \sqrt{|g|}G^{\alpha\beta} \\ \Theta^\alpha &= \sqrt{|g|}(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu_{\mu\nu}) \end{aligned} \tag{7.31}$$

This $\Theta$ can be rewritten using eq.(5.12) as

$$\begin{aligned} \Theta^\alpha &= \sqrt{|g|}\frac{1}{2} g^{\mu\nu} g^{\alpha\rho}(\nabla_\mu \delta g_{\nu\rho} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu}) \\ &- \sqrt{|g|}\frac{1}{2} g^{\mu\alpha} g^{\nu\rho}(\nabla_\mu \delta g_{\nu\rho} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu}) \\ &= \sqrt{|g|} g^{\mu\nu} g^{\alpha\rho}(\nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu}) \end{aligned} \tag{7.32}$$

If we write this symplectic potential as a form and put back the coupling $\frac{1}{16\pi}$ it becomes

$$\Theta_{\mu\nu\rho}(g, \delta g) = \frac{1}{16\pi}\epsilon_{\alpha\mu\nu\rho} g^{\alpha\lambda} g^{\sigma\theta}(\nabla_\sigma \delta g_{\lambda\theta} - \nabla_\lambda \delta g_{\sigma\theta}) \tag{7.33}$$

If we again rewrite this quantities in forms, we can calculate the Noether current:

$$\begin{aligned} \mathcal{J}_{\mu\nu\rho} &= \Theta_{\mu\nu\rho} - \xi^\alpha L_{\alpha\mu\nu\rho} \\ &= [g^{\alpha\lambda} g^{\sigma\theta}(\nabla_\sigma \delta g_{\lambda\theta} - \nabla_\lambda \delta g_{\sigma\theta}) - R\xi^\alpha]\epsilon_{\alpha\mu\nu\rho} \end{aligned} \tag{7.34}$$

This can be written in a more elegant form. By the definition of the Einstein tensor we are able to rewrite $R\xi^\alpha$, namely

$$G_{\alpha\lambda} = R_{\alpha\lambda} - \frac{1}{2}Rg_{\alpha\lambda} \to R\xi^\alpha = 2(R^\alpha{}_\lambda \xi^\lambda - G^\alpha{}_\lambda \xi^\lambda) \tag{7.35}$$

If we then also plug in the explicit variation of the metric, $\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$, the Noether current becomes

$$\mathcal{J}_{\mu\nu\rho} = \frac{1}{8\pi}\Big(\nabla_\lambda(\nabla^{[\lambda}\xi^{\alpha]}) + G^\alpha{}_\lambda \xi^\lambda\Big)\epsilon_{\alpha\mu\nu\rho} \tag{7.36}$$

The Noether charge which can be derived from this current if $G_{\mu\nu} = 0$ is

$$\boxed{Q_{\mu\nu} = -\frac{1}{16\pi}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho \xi^\sigma} \tag{7.37}$$

### 7.2.3 Electromagnetic fields

Finally, we look at the electromagnetic field. The Lagrangian $\mathscr{L}[\mathbf{A}, g]$ is

$$\begin{aligned} \mathscr{L} &= -\frac{1}{4}\sqrt{|g|}F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{4}\sqrt{|g|}\mathbf{F}^2 \end{aligned} \tag{7.38}$$

The metric part is going to give the electromagnetic energy-momentum tensor, while the vector potential part is going to give us the equations of motion and the sought-after boundary term for $\mathbf{A}$. The total variation can be written as $\delta\mathscr{L} = \delta_g\mathscr{L} + \delta_m\mathscr{L}$, with

$$\begin{aligned} \delta_g\mathscr{L} &= -\frac{1}{4}\sqrt{g}\Big(\frac{1}{2}g^{\rho\lambda}\mathbf{F}^2 + F^{\rho}_{\ \nu}F^{\lambda\nu} + F_{\mu}^{\ \rho}F^{\mu\lambda}\Big)\delta g_{\rho\lambda} \\ \delta_m\mathscr{L} &= -\sqrt{|g|}\Big(\nabla_\mu(F^{\mu\nu}\delta A_\nu) - \nabla_\mu F^{\mu\nu}\delta A_\nu\Big) \end{aligned} \tag{7.39}$$

where partial derivatives were replaced by covariant ones. In this case we can read of that

$$\Theta_{\mu\nu\rho}(\mathbf{A}_,\delta\mathbf{A}) = -F^{\lambda\sigma}\delta A_\sigma\epsilon_{\lambda\mu\nu\rho} \tag{7.40}$$

The variation in the vector potential is rewritten as

$$\begin{aligned} \delta A_\sigma &= -\pounds_\xi A_\sigma \\ &= -(\xi^\alpha\nabla_\alpha A_\sigma + A_\alpha\nabla_\sigma\xi^\sigma) \\ &= -2\xi^\nu\nabla_{[\nu}A_{\sigma]} - \nabla_\sigma(\xi^\nu A_\nu) \end{aligned} \tag{7.41}$$

and with this the Noether current can be written down again:

$$\mathcal{J}_{\mu\nu\rho} = \Big(F^{\lambda\sigma}\Big(\xi^\nu\nabla_{[\nu}A_{\sigma]} + \nabla_\sigma(\xi^\nu A_\nu)\Big) + \frac{1}{4}\xi^\lambda\mathbf{F}^2\Big)\epsilon_{\lambda\mu\nu\rho} \tag{7.42}$$

Now it's the question if there is a Noether charge corresponding to this Noether current. To explore this question, we rewrite the calculation in forms. The Lagrangian can be written as

$$\mathbf{L} = -\frac{1}{2}\mathbf{F}\wedge *\mathbf{F} \tag{7.43}$$

The variation $\delta * \mathbf{F}$ is going to give us

$$\delta * \mathbf{F} = *\delta\mathbf{F} + \frac{1}{2}(g^{\alpha\beta}\delta g_{\alpha\beta}) * \mathbf{F} \tag{7.44}$$

We can forget about the metric-part of this equation because we are only interested in $\mathbf{A}$. Varying eq.(7.43) gives then

$$\begin{aligned} \delta\mathbf{L} &= -\frac{1}{2}\Big(\delta\mathbf{F}\wedge *\mathbf{F} + \mathbf{F}\wedge\delta * \mathbf{F}\Big) \\ &= -\frac{1}{2}\Big(d(\delta\mathbf{A}\wedge *\mathbf{F}) + \delta\mathbf{A}\wedge d * \mathbf{F} + \mathbf{F}\wedge\delta * \mathbf{F}\Big) \end{aligned} \tag{7.45}$$

and we see that $\mathbf{\Theta} = -\frac{1}{2}\delta_\xi\mathbf{A}\wedge *\mathbf{F}$. So our Noether current becomes

$$\mathcal{J} = -\frac{1}{2}\Big(\delta_\xi\mathbf{A}\wedge *\mathbf{F} + \xi\cdot(\mathbf{F}\wedge *\mathbf{F})\Big) \tag{7.46}$$

This can be rewritten with $\delta_\xi \mathbf{A} = -(\xi \cdot \mathbf{F} + d(\xi \cdot \mathbf{A}))$ if we use eq.(3.23):

$$\begin{aligned}
\mathcal{J} &= -\frac{1}{2}\Big(-\xi \cdot \mathbf{F} \wedge *\mathbf{F} - d(\xi \cdot \mathbf{A}) \wedge *\mathbf{F} + \xi \cdot \mathbf{F} \wedge *\mathbf{F}\Big) \\
&= \frac{1}{2} d(\xi \cdot \mathbf{A}) \wedge *\mathbf{F} \\
&= \frac{1}{2} d(\xi \cdot \mathbf{A} \wedge *\mathbf{F}) \qquad (7.47)
\end{aligned}$$

which is justified by the equations of motion $d * \mathbf{F} = 0$. So the Noether charge is given by

$$\boxed{\mathbf{Q} = \frac{1}{2}\xi \cdot \mathbf{A} \wedge *\mathbf{F}} \qquad (7.48)$$

It's important to realize that this Noether charge is linear in $\xi$, and doesn't contain any derivatives of $\xi$.

## 7.3 Hamiltonian systems

We saw the notions of mass, charge and angular momentum appearing in the derivation of the first law. What we did basically was to consider the event horizon given by the metric, induce variations, and relate these variations. The final form of this relation looked just like the equations known from thermodynamics. It was evident that the mass, charge and angular momentum from these equations were the ones of the black hole. However, in a more general treatment of the first law we need some more general notions of these physical quantities. These can be defined via the Hamiltonian, if it is possible to construct it. That this is not so trivial was already noted; the equivalence principle prevents us from giving a clear local notion of the energy of a gravitational field. So the best thing we can do is to let go of the hope to define a local expression for the gravitational energy, and consider global regions of space-time. In this way it *is* possible to define something that describes the evolution of a system along a certain vector flow. However, it is not always clear at once that this Hamiltonian can be coupled to a sensible notion of energy.

In classical mechanics, the Hamiltonian is obtained via the Lagrangian with the Legendre transformation $H(p, q) = p\dot{q} - L(q, \dot{q})$. The coordinates $(t, q^i)$ are exchanged for $(p^i, q^i)$, and this results in the well-known Poisson-brackets which obey the Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ and are antisymmetric. To describe this geometrically, we can use the symplectic structure which will now be considered. First we take the simple one-dimensional case and then we generalize it.

The divergence of a vector is formally defined in $\mathbb{R}^2$ in the following way. Take an open n-dimensional manifold $M \subseteq \mathbb{R}^2$ with its volume form $\epsilon = dx \wedge dy$. For a vector field $\xi \in T_x M$ we have that $\xi \cdot \epsilon \in \Lambda^1(M)$ and so $d(\xi \cdot \epsilon) \in \Lambda^2(M)$. Being a two-form, we must have that $d(\xi \cdot \epsilon) = f\ \epsilon$, where $f : M \to \mathbb{R}$. This function $f$ is defined as the divergence of $\xi$, denoted as $\nabla \cdot \xi$:

$$d(\xi \cdot \epsilon) \equiv (\nabla \cdot \xi)\epsilon \qquad (7.49)$$

This should be compared with eq.(3.64). Taking $\xi = \xi_1 e_{(1)} + \xi_2 e_{(2)}$ we get the familiar result $\nabla \cdot \xi = \frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial y}$. If now $\nabla \cdot \xi = 0$, it is clear that $\xi \cdot \epsilon$ is closed. If, on top of that, it is also exact, then there is a form $\mathbf{H}(x, y) : M \to \mathbb{R}$, for which

$$\xi \cdot \epsilon = d\mathbf{H} \tag{7.50}$$

This is defined as the Hamiltonian. It folllows that

$$\begin{aligned} \xi \cdot \epsilon &= -\xi_2 dx + \xi_1 dy \\ &= \frac{\partial \mathbf{H}}{\partial x} dx + \frac{\partial \mathbf{H}}{\partial y} dy \to \\ \xi &= \frac{\partial \mathbf{H}}{\partial y} e_{(1)} - \frac{\partial \mathbf{H}}{\partial x} e_{(2)} \end{aligned} \tag{7.51}$$

This is equivalent with the system of differential equations[5]

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x} \end{aligned} \tag{7.52}$$

This should look familiar if $(x, y)$ is identified with the coordinate and momentum of a system. The vector field $\xi$ is called the Hamiltonian vector field. $\mathbf{H}$ is conserved along the flow of $\xi$:

$$\dot{\mathbf{H}} = \xi(\mathbf{H}) = d\mathbf{H}(\xi) = \xi \cdot \epsilon(\xi) = \epsilon(\xi, \xi) = 0 \tag{7.53}$$

We can easily extend this idea to $\mathbb{R}^n$: Take $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and define $\epsilon = dx_i \wedge dy^i$. In this case the Hamiltonian $\mathbf{H} : \mathbb{R}^{2n} \to \mathbb{R}$ and the accompanying Hamiltonian vector field are defined via

$$d\mathbf{H} = \xi \cdot \omega = \omega(\xi, -) \tag{7.54}$$

Again we end up with a system of differential equations

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial y_i} \\ \dot{y}_i &= -\frac{\partial H}{\partial x_i} \end{aligned} \tag{7.55}$$

and again with an expression of energy conservation, $\dot{\mathbf{H}} = 0$. So we used symplectic forms and symmetries to define the Hamiltonian. This suggests to look for symplectic forms and the Noether potential in theories of gravity. Also, in other theories one can construct the Hamiltonian from the Lagrangian, so the description in the last section looks appropriate. If we associate with the variation $\delta$ a vector field $X$, we saw that we can write the variation of the Lagrangian as $\delta_X \mathbf{L} = \mathbf{E}\delta_X \phi + d\mathbf{\Theta}$. The Noether current is defined as $\mathcal{J} = (\mathbf{\Theta}(\mathbf{L}, \pounds_\xi \phi) - \xi \cdot \mathbf{L}) \in \Lambda^{n-1}(M)$, and $\delta \mathcal{J}$ becomes

$$\begin{aligned} \delta_X \mathcal{J} &= \delta_X \mathbf{\Theta}(\mathbf{L}, \pounds_\xi \phi) - \xi \cdot \delta_X \mathbf{L} \\ &= \delta_X \mathbf{\Theta}(\mathbf{L}, \pounds_\xi \phi) - \xi \cdot (\mathbf{E}\delta_X \phi) - \xi \cdot d\mathbf{\Theta}(\mathbf{L}, X) \\ &= \delta_X \mathbf{\Theta}(\mathbf{L}, \pounds_\xi \phi) - \pounds_\xi \mathbf{\Theta}(\mathbf{L}, X) - \xi \cdot (\mathbf{E}\delta_X \phi) + d(\xi \cdot \mathbf{\Theta}(\mathbf{L}, X)) \\ &= \mathbf{\Omega}(\mathbf{L}, X, \pounds_\xi \phi) - \xi \cdot (\mathbf{E}\delta_X \phi) + d(\xi \cdot \mathbf{\Theta}(\mathbf{L}, X)) \end{aligned} \tag{7.56}$$

[5]Remembering that a vector field $\xi$ can be seen as a smooth map $\xi : \mathbb{R}^n \to \mathbb{R}^n$ with the flow $\dot{x} = \xi(x)$.

We considered the field $\xi$ to be a fixed background field, and so $\delta\xi = 0$. The symplectic current is defined as

$$\boxed{\mathbf{\Omega}(\mathbf{L}, X, Y) = \delta_X \mathbf{\Theta}(\mathbf{L}, Y) - \delta_Y \mathbf{\Theta}(\mathbf{L}, X)} \tag{7.57}$$

with respect to the vector fields $X$ and $Y$ which induce the variations. The variation of the Noether charge is defined via

$$\begin{aligned} \delta_X \mathbf{Q}_\Sigma(\mathbf{L}, \xi) &= \int_\Sigma (\delta_X \mathcal{J}(\mathbf{L}, \xi) - d(\xi \cdot \mathbf{\Theta}(\mathbf{L}, X))) \\ &= \int_\Sigma (\mathbf{\Omega}(\mathbf{L}, X, \pounds_\xi \phi) - \xi \cdot \mathbf{E}(\mathbf{L}, X)) \end{aligned} \tag{7.58}$$

Here $\Sigma$ is an (n-1) dimensional hypersurface. This is the basis of the so-called Arnowitt-Deser-Misner formalism, or ADM-formalism in short. If the vector field $\xi$ is transverse to $\Sigma$ the variation of the Hamiltonian $\mathbf{H}(L, \xi, \Sigma)$ is defined by

$$\delta_X \mathbf{H}(\mathbf{L}, \xi, \Sigma) = \delta_X \mathbf{Q}_\Sigma(\mathbf{L}, \xi) \tag{7.59}$$

What we basically did here, is to split up space-time in space and time; the Cauchy-hypersurfaces define the spatial part and the vector field $\xi$ defines the time direction. Now the role of the symplectic current becomes clear; as physicists most of the time we are interested in the case $\mathbf{E} = 0$:

$$\boxed{\delta_X \mathbf{H}(\mathbf{L}, \xi, \Sigma) = \int_\Sigma \mathbf{\Omega}(\mathbf{L}, X, \pounds_\xi \phi) \quad \text{for} \ \ \mathbf{E} = 0} \tag{7.60}$$

This will be a crucial identity in the next section!

## 7.4 The first law revisited

Having made ourselves familiar with the mechanism, we are now in a position to rederive the first law of black hole mechanics in the case of neutral charged black holes, and identify the black hole entropy in terms of the Lagrangian of the theory. If we want to succeed in this, we have to find expressions for the energy and angular momentum of the black hole, expressed via $\mathbf{Q}$ and $\mathbf{\Theta}$. It will turn out that this is indeed possible. For our derivation we take an arbitrary variation $\delta\phi$ for a $\phi$ which is on-shell. We saw that the variation $\delta\mathcal{J}$ can be written as ( dropping the vector associated with the variation )

$$\begin{aligned} \delta\mathcal{J} &= \delta\mathbf{\Theta}(\phi, \pounds_\xi \phi) - \xi \cdot \delta\mathbf{L} - \delta\xi \cdot \mathbf{L} \\ &= \delta\mathbf{\Theta}(\phi, \pounds_\xi \phi) - \xi \cdot (\mathbf{E}\delta\phi + d\mathbf{\Theta}) - \delta\xi \cdot \mathbf{L} \\ &= \delta\mathbf{\Theta}(\phi, \pounds_\xi \phi) - \pounds_\xi \mathbf{\Theta} + d(\xi \cdot \mathbf{\Theta}) \end{aligned} \tag{7.61}$$

Here we assumed again that $\delta\xi = 0$ and we used $\mathbf{E} = \mathbf{0}$ and $\xi \cdot d\mathbf{\Theta} = \pounds_\xi \mathbf{\Theta} - d(\xi \cdot \mathbf{\Theta})$. Now the first two terms on the right hand side can be recognized as the symplectic current $\mathbf{\Omega}(\phi, \delta\phi, \pounds_\xi \phi)$ with one variation specified as the Lie-derivative with respect to $\xi$, and the variation becomes

$$\delta\mathcal{J} = \mathbf{\Omega}(\phi, \delta\phi, \pounds_\xi \phi) + d(\xi \cdot \mathbf{\Theta}) \tag{7.62}$$

The crucial step now is the following: if the evolution of a system is described by the vector $\xi$ and we can find a Hamiltonian which describes this evolution, then Hamilton's equations of motion are given by

$$\boxed{\delta \mathbf{H} = \int_C \mathbf{\Omega}(\phi, \delta\phi, \mathcal{L}_\xi \phi)} \tag{7.63}$$

This identification should be clear from the last section. So if the dynamics are generated by $\xi$ and there is a Hamiltonian describing this, we obtain

$$\begin{aligned} \delta \mathbf{H} &= \delta \int_C \mathcal{J} - \int_C d(\xi \cdot \mathbf{\Theta}) \\ &= \delta \int_C \mathcal{J} - \int_{\partial C} \xi \cdot \mathbf{\Theta} \end{aligned} \tag{7.64}$$

Remember that $\phi$ satisfies the equations of motion, so we can write $\mathcal{J} = d\mathbf{Q}$. If we plug this in, we observe that the Hamiltonian is a surface term if it is rewritten by Stokes' theorem:

$$\boxed{\delta \mathbf{H} = \int_{\partial C} (\delta \mathbf{Q} - \xi \cdot \mathbf{\Theta})} \tag{7.65}$$

The variation can safely be put under the integral sign. This Hamiltonian is going to give us the relevant physical quantities which appear in the first law. We rename them for convenience: $\Xi$ is the energy, and $\Upsilon$ is the angular momentum. Being derived from a Hamiltonian, they are called canonical. The canonical energy $\Xi$ is associated with a time translation vector $t$ and the canonical angular momentum $\Upsilon$ is associated with a rotation vector $\varphi$. To get these quantities, we integrate $\delta\mathbf{H}$ over a hypersurface lying at infinity, which is Cauchy and (n-2)-dimensional. The rotation vector $\varphi$ is tangent to this sphere, and by Stokes' theorem the integral $\int_\infty \varphi \cdot \Theta = 0$. The variations of these canonical quantities are then given by

$$\begin{aligned} \delta \Xi &= \int_\infty (\delta \mathbf{Q}(t) - t \cdot \mathbf{\Theta}) \\ \delta \Upsilon &= -\int_\infty (\delta \mathbf{Q}(\varphi) + \varphi \cdot \mathbf{\Theta}) \\ &= -\int_\infty \delta \mathbf{Q}(\varphi) \end{aligned} \tag{7.66}$$

Notice the opposite sign convention of $\delta\Xi$ and $\delta\Upsilon$; they are a consequence of the signature of the metric, just like in eq.(6.2). It is also useful to note that we split up $\delta\mathbf{Q}$ in a $t$-piece and a $\varphi$-piece. Since $\mathbf{Q}$ depends linearly on $\xi$ and its derivatives we can take the linear combination $at + b\varphi$ with $a, b \in \mathbb{R}$, and then $\delta\mathbf{Q}(\xi) = a\delta\mathbf{Q}(t) + b\delta\mathbf{Q}(\varphi)$. For the charge itself we can write

$$\boxed{Q_{\mu\nu} = A_{[\mu}\xi_{\nu]} + B\nabla_{[\mu}\xi_{\nu]}} \tag{7.67}$$

by eliminating all higher derivatives with the Killing identity (3.32). The functions $A_\mu$ and $B$ are local functions which depend on the fields appearing in the

Lagrangian.

Now the integration surface and $\xi$ are going to be more specified. We are going to apply the foregoing discussion to stationary black holes with a bifurcate Killing horizon $\Sigma$. Specifying to four dimensions, the Killing vector field $\xi$ can be written as

$$\xi^\mu = t^\mu + \Omega_H \varphi^\mu \tag{7.68}$$

which is just eq.(6.38) slightly rewritten. Since $\xi$ is a Killing vector field, $\pounds_\xi \phi = 0$. If $\delta\phi$ is also a solution for the equations of motion, we have that

$$\delta\mathbf{\Theta}(\phi, \delta\phi) = \pounds_\xi \mathbf{\Theta}(\phi, \delta\phi) = 0 \tag{7.69}$$

and as a result, eq.(7.61) yields

$$\begin{aligned} \delta\mathcal{J} &= \delta(d\mathbf{Q}) \\ &= d(\delta\mathbf{Q}) \\ &= d(\xi \cdot \mathbf{\Theta}) \end{aligned} \tag{7.70}$$

Note that this implies that $\delta\mathbf{H} = 0$ if $\xi$ is a Killing vector, just what we expect. Ofcourse, with such an expression we are eager to integrate it. The hypersurface $C$ will be chosen such that it extends from asymptotic infinity up to the bifurcation horizon $\Sigma$ of the black hole, see figure (7.1). So we obtain

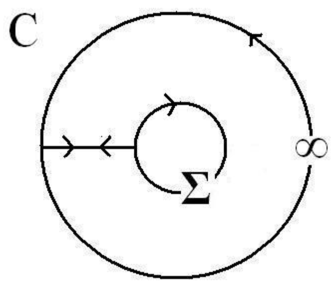


Figure 7.1: *The integration domain*

$$\begin{aligned} \int_C d(\xi \cdot \mathbf{\Theta}) &= \int_{\partial C} \xi \cdot \mathbf{\Theta} \\ &= \int_\infty \xi \cdot \mathbf{\Theta} - \int_\Sigma \xi \cdot \mathbf{\Theta} \\ &= \int_\infty \xi \cdot \mathbf{\Theta} \end{aligned} \tag{7.71}$$

Now we see something truly nice! For $\delta\mathbf{Q}(\xi)$ we can write $\delta\mathbf{Q}(t) + \Omega_H \delta\mathbf{Q}(\varphi)$, and finally we obtain something that looks very much like the first law of black hole mechanics:

$$\boxed{\delta \int_\Sigma \mathbf{Q} = \int_\infty \delta\mathbf{Q} - \int_\infty \xi \cdot \mathbf{\Theta} = \delta\Xi - \Omega_H \delta\Upsilon} \tag{7.72}$$

Note that we have connected quantities defined at asymptotic infinity with quantities defined at $\Sigma$. It looks like this is a result from the bifurcation horizon, but later we'll see that the integral actually doesn't depend on the horizon cross section. Comparing this result with the first law, it is very tempting to regard the left hand side as the entropy. This entropy should be a geometrical quantity defined at $\Sigma$, and the potential $\mathbf{Q}$ describing this entropy should be invariant under diffeomorphisms $\phi : \Sigma \to \Sigma$. We already noted that an arbitrary derivative of a Killing vector $\xi$ can be described as a linear combination of $\xi$ and $\nabla\xi$ and that we can write $\mathbf{Q} = \mathbf{Q}(\xi, \nabla\xi)$. Furthermore, on the bifurcation horizon $\xi = 0$ and according to eq.(6.34) the derivative of $\xi$ can also be eliminated via $\nabla_\mu \xi_\nu = \kappa \epsilon_{\mu\nu}$. So then $\mathbf{Q}$ only depends on the fields which appear in the Lagrangian. In this way we can make the identification

$$\boxed{\mathcal{S} = 2\pi \oint_\Sigma \mathbf{Q}} \tag{7.73}$$

where we have normalized the horizon Killing field to have unit surface gravity. This expresses the entropy of a black hole in terms of the Lagrangian of the theory, and it should also be noted that the Euclideanization of a path integral didn't appear in the derivation. Now we return to our Lagrangian $\mathscr{L}$ which was a function of the metric, the Riemann-tensor, a matter field and its derivative, $\mathscr{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \psi, \nabla_\mu \psi)$. In that case we have $Q^{\mu\nu} = \mathscr{L}^{\mu\nu\rho\sigma} \epsilon_{\rho\sigma}$ and so

$$\mathcal{S} = 2\pi \int_\Sigma \frac{\partial \mathscr{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \sqrt{|h|} d\Omega \tag{7.74}$$

Here $|h|$ is the determinant of the induced metric $h$ on the (n-2) dimensional surface over which we integrate. Now let's check this expression in the case of adopting Einstein's field equations. There we have that $\mathscr{L} = \frac{1}{16\pi} R$, and from eq.(8.37) we obtain $\mathscr{L}^{\mu\nu\rho\sigma} = \frac{1}{32\pi} g^{\mu[\rho} g^{\sigma]\nu}$. We already computed the binormal for a static black hole in spherical coordinates, and so the contraction becomes

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} &= \mathscr{L}^{0101} \epsilon_{01} \epsilon_{01} + \mathscr{L}^{1010} \epsilon_{10} \epsilon_{10} + \mathscr{L}^{1001} \epsilon_{10} \epsilon_{01} + \mathscr{L}^{0110} \epsilon_{01} \epsilon_{10} \\ &= \frac{1}{8\pi} (\epsilon_{01})^2 g^{00} g^{11} \end{aligned} \tag{7.75}$$

Remembering that $\epsilon_{01} = e^{f(r)+g(r)}$,$g^{00} = -e^{-2g(r)}$ and $g^{11} = e^{-2f(r)}$ we arrive at

$$\begin{aligned} \mathcal{S} &= \frac{2\pi}{8\pi} \int_\Sigma \sqrt{|h|} d\Omega \\ &= \frac{A}{4} \end{aligned} \tag{7.76}$$

We see that the more general expression for the black hole entropy gives us exactly what we want.

Maybe it's a good idea to summarize what we did so far. We saw that in the case of Einstein's field equations we can perform some variations on black hole solutions of these field equations. If we varied the area associated with the

Schwarzschild radius, we saw some equations that remind us of the second law of thermodynamics. In this section we explored diffeomorphism invariance of general relativity; this invariance enabled us to rewrite $\delta\mathbf{L}$ as $\delta\mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta}$ and this led to a Noether charge. We saw that with some mathematics we could turn the variation of these charges into a form which, again, reminds us of the second law of thermodynamics if we could only identify the symmetry charge with the black hole entropy. And indeed, this identification is possible. All this gives us the remarkable result that the black hole entropy has a deep connection with the geometry of the black hole surface. It has to be emphasized that we made heavy use of the Killing vector field $\xi$ which describes the symmetries of our black hole. So the entropy formula given here can only be used for stationary black holes. These can be characterized by the fact that on the event horizon $\mathcal{H}$ the expansion of the generators of $\mathcal{H}$ is zero: $\theta = 0$. There have been proposals for an extention to dynamical black holes, but those will not be considered here, see for example [11] or [14].

## 7.5 Looking more closely at the entropy

First we answer the question: what if we had chosen another surface $\Sigma'$ instead of the bifurcation surface $\Sigma$? To answer this question, we look at the difference of the surface integrals over $\Sigma'$ and $\Sigma$ of the Noether charge $\mathbf{Q}$, where the two surfaces enclose a 3-volume $C$, $\partial C = \Sigma' - \Sigma$:

$$\begin{aligned}\int_{\Sigma'} \mathbf{Q} - \int_{\Sigma} \mathbf{Q} &= \int_C d\mathbf{Q} \\ &= \int_C \mathcal{J} \\ &= \int_C (\mathbf{\Theta}(\phi, \pounds_\xi \phi) - \xi \cdot \mathbf{L}) \\ &= 0 \end{aligned} \tag{7.77}$$

We remember that $\xi$ was chosen in such a way that it was a symmetry field for $\phi$ and that it was tangent to the surface $C$. So, $\pounds_\xi \phi = 0$ and with that it's clear that the integral is zero. It can be concluded that the definition of the black hole entropy in this way doesn't depend on the particular surface chosen. This is important for several reasons; the bifurcate horizon may not be part of the space-time, and if we would like to extend our entropy formula to dynamic black holes, we would like the answer to be independent of the surface used. As a practical matter, sometimes we don't have a coordinate system which covers also the bifurcate horizon.

Now we will look more closely to how to obtain the entropy and make some remarks with which we can extract more easily the relevant terms from $\delta\mathbf{L}$ which are important to us. We saw that for $\mathbf{E} = 0$ we are able to write $\mathcal{J} = d\mathbf{Q}$. This implies that if $\mathbf{Q}$ contains up to order $k$ derivatives, then $\mathcal{J}$ contains at most up to order $k+1$ derivatives of $\xi$. Also, the term in $\mathbf{Q}$ which contains the order $k$ derivatives is fixed by the term in $\mathcal{J}$ which contains order $k+1$ derivatives. In using eq.(7.73) we don't have to worry about the terms in $\mathbf{Q}$ which are linear in $\xi$ because $\xi = 0|_\Sigma$ where $\Sigma$ is the bifurcation horizon. If we consider $\mathcal{J}$ we only have to consider second order derivatives of $\xi$. We already saw this; in the Einstein-Hilbert case these terms arise from that part of $\delta\mathbf{L}$ involving second order derivatives of the field variations $\delta g_{\mu\nu}$. If $\mathbf{L} = \mathbf{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \psi, \nabla\psi \ldots)$ these terms only can be given by $\delta R_{\mu\nu\rho\sigma}$, and that was demonstrated in eq.(7.74). To summarize this:[6]

$$\begin{aligned} \mathbf{Q} &= \mathbf{Q}\{\nabla\xi\} \ \rightarrow \ d\mathbf{Q} = d\mathbf{Q}\{\nabla^2\xi\} \ \rightarrow \\ \mathcal{J} &= \mathcal{J}\{\nabla^2\xi\} \ \rightarrow \ \mathbf{\Theta} = \mathbf{\Theta}\{\nabla^2\xi\} \rightarrow \\ d\mathbf{\Theta} &= d\mathbf{\Theta}\{\nabla^3\xi\}, \quad \delta\mathbf{L} = \delta\mathbf{L}\{\nabla^3\xi\} \rightarrow \\ \delta_\xi\phi &= \delta_\xi\phi(\nabla\xi) \rightarrow \boxed{\nabla^2\delta_\xi\phi} \end{aligned} \tag{7.78}$$

The term we are talking about in this particular case is $2\nabla_{[\mu}\nabla_{[\rho}\delta g_{\sigma]\nu]}$. Taking this into account, we can write

$$\begin{aligned} \delta\mathbf{L} &= 2\frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\nabla_\mu\nabla_\rho\delta g_{\nu\sigma}\epsilon + \ldots \\ &= \nabla_\mu\Big(2\frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\nabla_\rho\delta g_{\nu\sigma}\epsilon\Big) + \ldots \end{aligned} \tag{7.79}$$

where the dots are the terms which we won't need to calculate $S$. This implies

$$\Theta = 2\frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\nabla_\rho\delta g_{\nu\sigma}\epsilon + \ldots \tag{7.80}$$

and

$$\mathcal{J} = 2\frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\nabla_\rho\nabla_{(\nu}\xi_{\sigma)}\epsilon + \ldots \tag{7.81}$$

This yields the result given earlier if we work out the implications of this calculation, which can be checked as a nice exercise.

## 7.6 Variations of the Lagrangian

Here we will take a closer look to the claim that we can always write $\delta\mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta}$, where $\phi = (g, \psi)$. For this we write the Lagrangian $\mathbf{L} = \mathscr{L}\epsilon$ as

$$\mathbf{L} = \mathbf{L}(g_{\mu\nu}, \nabla_{\lambda_1}R_{\nu\rho\sigma\alpha}, \ldots, \nabla_{(\lambda_1}\cdots\nabla_{\lambda_k)}R_{\nu\rho\sigma\alpha},\ \psi, \nabla_{\lambda_1}\psi, \ldots, \nabla_{(\lambda_1}\cdots\nabla_{\lambda_l)}\psi) \tag{7.82}$$

Ofcourse it can be doubted that we are always able to write the Lagrangian in such a covariant form in the first place; this is proven in [11], and the subtle fact that the Riemann tensor and it's derivates cannot be chosen independently is also noted there. We will simply assume the form given above. We stated that $\mathbf{\Theta} = \mathbf{\Theta}(\phi, \delta\phi)$. Let's be more specific: the claim is that we are always able to

[6] Here the brackets $\{\nabla^n\xi\}$ mean "can contain derivatives up to order $n$".

write

$$
\begin{aligned}
\delta \mathbf{L} &= \mathbf{E}\delta\phi + d\mathbf{\Theta} \\
\mathbf{\Theta} &= 2E_R^{\nu\rho\sigma}\nabla_\sigma \delta g_{\nu\rho} + A^{\mu\nu}(\phi)\delta g_{\mu\nu} + \sum_{i=0}^{k-1} B_i(\phi)^{\mu\nu\rho\sigma\lambda_1\cdots\lambda_i}\delta\nabla_{(\lambda_1}\cdots\nabla_{\lambda_i)}R_{\mu\nu\rho\sigma} \\
&+ \sum_{i=0}^{l-1} C_i(\phi)^{\lambda_1\cdots\lambda_i}\delta\nabla_{(\lambda_1}\cdots\nabla_{\lambda_i)}\psi \\
(E_R^{\nu\rho\sigma})_{\alpha\beta\gamma} &\equiv \frac{\delta\mathscr{L}}{\delta R_{\mu\nu\rho\sigma}}\epsilon_{\mu\alpha\beta\gamma} \qquad (7.83)
\end{aligned}
$$

where $A, B, C$ are tensor fields which depend on $\phi$. Proving this kind of identities is often merely a matter of rewriting derivatives and ennobled accounts keeping, so there is nothing magical going on. Our strategy is to denote the explicit form of $\delta\mathbf{L} = \epsilon\delta\mathscr{L} + \mathscr{L}\delta\epsilon$ in the following compact way[7],

$$
\begin{aligned}
\delta\mathbf{L} &= \Big(\frac{\partial\mathscr{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\delta R_{\mu\nu\rho\sigma} + \sum_1^k \frac{\partial\mathscr{L}}{\partial\nabla_{(\ldots\nabla)}R_{\mu\nu\rho\sigma}}\delta\nabla_{(\ldots\nabla)}R_{\mu\nu\rho\sigma} \\
&+ \sum_1^l \frac{\partial\mathscr{L}}{\partial\nabla_{(\ldots\nabla)}\psi}\delta\nabla_{(\ldots\nabla)}\psi\Big)\epsilon + \frac{1}{2}g^{\mu\nu}L\delta g_{\mu\nu} \qquad (7.84)
\end{aligned}
$$

and look at a general term like

$$
\frac{\partial\mathscr{L}}{\partial\nabla_{(\lambda_1}\cdots\nabla_{\lambda_i)}\psi}\delta\nabla_{(\lambda_1}\cdots\nabla_{\lambda_i)}\psi \qquad (7.85)
$$

First we treat a simple case: $\psi$ is a scalar field and $i = 2$. It's then natural to extend the analysis to more complicated terms. The term of interest is

$$
\frac{\partial\mathscr{L}}{\partial\nabla_{(\mu}\nabla_{\nu)}\psi}\delta\nabla_\mu\nabla_\nu\psi \equiv L^{\mu\nu}\delta\nabla_\mu\nabla_\nu\psi \qquad (7.86)
$$

We rewrite this as

$$
\begin{aligned}
L^{\mu\nu}\delta\nabla_\mu\nabla_\nu\psi &= L^{\mu\nu}\nabla_\mu\delta\nabla_\nu\psi - L^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\nabla_\lambda\psi \\
&= \nabla_\mu\Big(L^{\mu\nu}\delta\nabla_\nu\psi\Big) - (\nabla_\mu L^{\mu\nu})\delta\nabla_\nu\psi - L^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\nabla_\lambda\psi \qquad (7.87)
\end{aligned}
$$

The last term can be written out as

$$
L^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}L^{\mu\nu}\Big(\nabla_\mu\delta g_{\nu\rho} + \nabla_\nu\delta g_{\rho\mu} - \nabla_\rho\delta g_{\mu\nu}\Big)\nabla_\lambda\psi \qquad (7.88)
$$

The first term here which contains a $\nabla_\mu$ can be rewritten as

$$
\frac{1}{2}\nabla_\mu\Big(g^{\lambda\rho}L^{\mu\nu}\nabla_\lambda\psi\delta g_{\nu\rho}\Big) - \frac{1}{2}\nabla_\mu\Big(g^{\lambda\rho}L^{\mu\nu}\nabla_\lambda\psi\Big)\delta g_{\nu\rho} \qquad (7.89)
$$

and the same goes for the other two derivatives $\nabla_\nu$ and $\nabla_\rho$ in eq.(7.88). So what did all this rewriting bring us? Jumping to the form-notation, we have

$$
\begin{aligned}
W_{\lambda\rho\sigma} &= L^{\mu\nu}\delta\nabla_\nu\psi\ \epsilon_{\mu\lambda\rho\sigma} \\
(dW)_{\alpha\lambda\rho\sigma} &= \nabla_\mu\Big(L^{\mu\nu}\delta\nabla_\nu\psi\Big)\epsilon_{\alpha\lambda\rho\sigma} \qquad (7.90)
\end{aligned}
$$

[7] The ... denote the usual contractions.

We recognize this $d\mathbf{W}$ term in our derivation, but it isn't the only term we can write exactly; the terms proportional to $\nabla\delta g$ can also be rewritten as exact forms as eq.(7.88) makes clear. This total exact form is written as $\mathbf{V}$:

$$\mathbf{V} = \mathbf{W} + [\sim \delta g]\epsilon \tag{7.91}$$

where $[\sim \delta g]$ denotes terms proportional to $\delta g$. So we end up with

$$\Big(L^{\mu\nu}\delta\nabla_\mu\nabla_\nu\psi\Big)\epsilon = d\mathbf{V} - \Big(\nabla_\mu(L^{\mu\nu})\ \delta\nabla_\nu\psi\Big)\epsilon + [\sim \delta g]\epsilon \tag{7.92}$$

This can be done for arbitrary fields with arbitrary order of derivatives; if this procedure is iterated, we can rewrite every term in our variation $\delta L$. In the end we get a term proportional to $\delta g$, the equations of motion for $\psi$ and $R_{\mu\nu\rho\sigma}$ in the form of $\Big(E_R^{\mu\nu\rho\sigma}\delta R_{\mu\nu\rho\sigma} + E_\psi\delta\psi\Big)$ pop up and an exact form $d\tilde{\mathbf{\Theta}}$ appears in our $\delta\mathbf{L}$. Ofcourse, the Riemann tensor isn't an independent field, and we can use eq.(5.16) to rewrite the terms concerning this tensor and it's derivatives in terms proportional to $\delta g$ and $\nabla\nabla\delta g$. With two partial integrations we can write the variation of the Lagrangian form as

$$\begin{aligned}\delta\mathbf{L} &= E_g^{\mu\nu}\delta g_{\mu\nu} + E_\psi\delta\psi + d\mathbf{\Theta} \\ \mathbf{\Theta} &= \tilde{\mathbf{\Theta}} + 2E_R^{\nu\rho\sigma}\nabla_\sigma\delta g_{\nu\rho} - 2\nabla_\sigma E_R^{\nu\rho\sigma}\delta g_{\nu\rho}\end{aligned} \tag{7.93}$$

which is exactly the form we had in mind. In the next chapter we will see some an explicit example, and this kind of calculations will be used to derive some identities for one explicit form of the Lagrangian.

## 7.7 Ambiguities in the Noether charge

There are some ambiguities in the theory which are important. First of all, we can always shift the Lagrangian by an exact form without altering the equations of motion;

$$\mathbf{L} \ \rightarrow\ \mathbf{L} + d\mathbf{\Delta}, \qquad \mathbf{\Theta} \ \rightarrow\ \mathbf{\Theta} + \delta\mathbf{\Delta} \tag{7.94}$$

Second, we saw that we can shift $\Theta$ by an exact form without altering the equations of motion;

$$\mathbf{\Theta} \ \rightarrow\ \mathbf{\Theta} + d\mathbf{Y} \tag{7.95}$$

This $\mathbf{Y}$ depends linearly on the variation of the fields, $\mathbf{Y} = \mathbf{Y}(\delta_\xi\phi)$. So this gives the ambiguity $\mathbf{\Theta} \rightarrow \mathbf{\Theta} + \delta\mathbf{\Delta} + d\mathbf{Y}$, and with eq.(3.23) this gives the shift

$$\mathcal{J} \ \rightarrow\ \mathcal{J} + d\mathbf{Y} + d(\xi\cdot\mathbf{\Delta}) \tag{7.96}$$

which, on it's turn, shifts the Noether charge by an amount of

$$\mathbf{Q} \ \rightarrow\ \mathbf{Q} + \mathbf{Y} + \xi\cdot\mathbf{\Delta} \tag{7.97}$$

So, as was already noted, the Noether charge is not unique. However, the entropy is calculated on the bifurcation horizon with the assumption that $\xi$ is a Killing vector field of the black hole. So our expression for $\mathcal{S}$ doesn't change, as it shouldn't: $\int_\Sigma(\mathbf{Y} + \xi\cdot\mathbf{\Delta}) = 0$. Later on we will encounter another ambiguity with a topological nature.

# Chapter 8

# An explicit example

In this chapter we will consider a particular example, and see how an expression for the Noether charge can be obtained. These calculations often are quite tedious; the full calculation of the Noether current will be skipped, but this chapter should be a good guiding line.

## 8.1 General covariance without matter field

The form of the Lagrangian density we will consider is the following:

$$\mathscr{L} = \mathscr{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \psi_{\mu\nu}, \nabla_\lambda \psi_{\mu\nu}) \tag{8.1}$$

where $\psi$ is some matter field with no assumed symmetries. Later we will consider the more general case and put the formalism in a firm mathematical framework. We see the appearance of te Riemann tensor, which we explained in the last chapter. Now we vary the metric and the matter field. As said in section (4.4), this variation will be written as a general coordinate transformation which is multiplied by a test function $\epsilon(x)$. We induce this variation by a vectorfield $\xi(x)$:

$$\begin{aligned}
\delta_\xi g_{\mu\nu} &= -\epsilon(x)\pounds_\xi g_{\mu\nu} \\
&= -\epsilon(x)[\nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x)] \\
\delta_\xi \psi_{\mu\nu} &= -\epsilon(x)\pounds_\xi \psi_{\mu\nu} \\
&= -\epsilon(x)[\nabla_\mu \xi^\sigma \psi_{\sigma\nu} + \nabla_\nu \xi^\sigma \psi_{\mu\sigma} + \xi^\sigma \nabla_\sigma \psi_{\mu\nu}]
\end{aligned} \tag{8.2}$$

For constant $\epsilon$ these transformations can be regarded as gauge-transformations of our gravitational theory. Next we impose boundary conditions on $\epsilon(x)$ and $\partial_\mu \epsilon(x)$ in such a way that the variation of $\mathscr{L}$ gives a term like $\nabla_\mu J^\mu$ plus the equations of motion. To make life more easy, we first consider a Lagrangian which depends only on the Riemann tensor and the metric. After that we consider the variations with a matter field added. But note that if we vary the covariant derivative of the matter field we end up with variations in the connection, which on their turn contain variations in the metric. Our starting point is

$$\mathscr{L} = \mathscr{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}) \tag{8.3}$$

For a diffeomorphism $\delta\mathscr{L}$ should be zero. This variation is written as

$$\begin{aligned}\delta_\xi \mathscr{L} &= \xi^\sigma \nabla_\sigma \mathscr{L} \\ &= \mathscr{L}^{\mu\nu}\delta g_{\mu\nu} + \mathscr{L}^{\mu\nu\rho\sigma}\delta R_{\mu\nu\rho\sigma} \\ &= 0\end{aligned} \tag{8.4}$$

where we will define for convenience

$$\mathscr{L}^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial g_{\mu\nu}}, \quad \mathscr{L}^{\mu\nu\rho\sigma} = \frac{\partial \mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}, \quad \mathscr{L}_\psi^{\rho,\mu\nu} = \frac{\partial \mathscr{L}}{\partial \nabla_\rho \psi_{\mu\nu}}, \quad \mathscr{L}_\psi^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial \psi_{\mu\nu}} \tag{8.5}$$

The symmetries are

$$\mathscr{L}^{\mu\nu\rho\sigma} = \mathscr{L}^{[\mu\nu][\rho\sigma]}, \quad \mathscr{L}^{\mu\nu\rho\sigma} = \mathscr{L}^{\rho\sigma\mu\nu}, \quad \mathscr{L}^{\mu\nu} = \mathscr{L}^{\nu\mu} \tag{8.6}$$

Via the chain rule we write

$$\begin{aligned}\xi^\lambda \nabla_\lambda \mathscr{L} &= \xi^\lambda [\frac{\partial \mathscr{L}}{\partial g_{\mu\nu}} \nabla_\lambda g_{\mu\nu} + \frac{\partial \mathscr{L}}{\partial R_{\mu\nu\rho\sigma}} \nabla_\lambda R_{\mu\nu\rho\sigma}] \\ &= \xi^\lambda \mathscr{L}^{\mu\nu\rho\sigma} \nabla_\lambda R_{\mu\nu\rho\sigma}\end{aligned} \tag{8.7}$$

where the variation with respect to the metric doesn't contribute due to the metric compatibility of our space-time. When we do contractions, we have to take notice of the symmetries involved. Covariance thus implies

$$\begin{aligned}\delta_\xi \mathscr{L} &= 2\mathscr{L}^{\mu\nu}\nabla_\mu \xi_\nu + \mathscr{L}^{\mu\nu\rho\sigma}[\xi^\lambda \nabla_\lambda R_{\mu\nu\rho\sigma} + R_{\mu\nu\rho\lambda}\nabla_\sigma \xi^\lambda + R_{\mu\nu\lambda\sigma}\nabla_\rho \xi^\lambda \\ &+ R_{\mu\lambda\rho\sigma}\nabla_\nu \xi^\lambda + R_{\lambda\nu\rho\sigma}\nabla_\mu \xi^\lambda] = 0 \\ &= \xi^\lambda \mathscr{L}^{\mu\nu\rho\sigma}\nabla_\lambda R_{\mu\nu\rho\sigma}\end{aligned} \tag{8.8}$$

Now remember that $\mathscr{L}^{\mu\nu\rho\sigma}$ has the same symmetries as $R^{\mu\nu\rho\sigma}$. For example,

$$\begin{aligned}\mathscr{L}^{\mu\nu\rho\sigma} R_{\mu\nu\rho\lambda}\nabla_\sigma \xi^\lambda &= \mathscr{L}^{\rho\sigma\mu\nu} R_{\mu\nu\rho\lambda}\nabla_\sigma \xi^\lambda = \\ \mathscr{L}^{\rho\sigma\mu\nu} R_{\rho\lambda\mu\nu}\nabla_\sigma \xi^\lambda &= \mathscr{L}^{\mu\nu\rho\sigma} R_{\mu\lambda\rho\sigma}\nabla_\nu \xi^\lambda\end{aligned} \tag{8.9}$$

With this our expression becomes

$$2[\mathscr{L}^{\mu\nu}\nabla_\mu \xi_\nu + \mathscr{L}^{\mu\nu\rho\sigma} R_{\mu\nu\rho\lambda}\nabla_\sigma \xi^\lambda + \mathscr{L}^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\sigma}\nabla_\rho \xi^\lambda] = 0 \tag{8.10}$$

But if we use the antisymmetric properties of the Riemann tensor twice, we finally obtain

$$\mathscr{L}^{\mu\nu}\nabla_\mu \xi_\nu + 2\mathscr{L}^{\mu\nu\rho\sigma} R_{\mu\nu\rho\lambda}\nabla_\sigma \xi^\lambda = 0 \tag{8.11}$$

This first identity is nothing more than a statement of diffeomorphism invariance.

## 8.2 The equations of motion without matter field

A second identity can be obtained from the action principle:

$$\begin{aligned}\delta S &= \int_\Omega [\delta\sqrt{|g|}\mathscr{L} + \sqrt{|g|}\delta\mathscr{L}] d^4x \\ &= \int [\frac{1}{2}\sqrt{|g|} g^{\mu\nu}\delta g_{\mu\nu}\mathscr{L} + \sqrt{|g|}\delta\mathscr{L}] d^4x = 0\end{aligned} \tag{8.12}$$

where we won't write every time the integration domain $\Omega$. We know that the action is an extremum for arbitrary $\delta g_{\mu\nu}$ so we want to pull out the term $\delta g_{\mu\nu}$ in order to set the integrand in $\delta S$ to zero. We already obtained an expression for $\delta R_{\mu\nu\rho\sigma}$ in eq.(5.16), but this contains second order derivatives of $\delta g_{\mu\nu}$. We can get rid of those via partial integrations and Stokes' theorem. If the expression for $\delta\mathscr{L}$ is filled in, the variation of the action reads

$$\delta S = \int \sqrt{|g|}[(\frac{1}{2}g^{\mu\nu}\mathscr{L} + \mathscr{L}^{\mu\nu})\delta g_{\mu\nu} + \mathscr{L}^{\mu\nu\rho\sigma}\delta R_{\mu\nu\rho\sigma}]d^4x \tag{8.13}$$

The term with the variation in the Riemann tensor reads, with eq.(5.16),

$$\int \sqrt{|g|}[\mathscr{L}^{\mu\nu\rho\sigma}R_{\mu\nu\rho}{}^{\lambda}\delta g_{\sigma\lambda} + 2\mathscr{L}^{\mu\nu\rho\sigma}\nabla_\mu\nabla_\rho\delta g_{\sigma\nu}]d^4x \tag{8.14}$$

Observe that, due to $\mathscr{L}^{\mu\nu\rho\sigma} = \mathscr{L}^{[\mu\nu][\rho\sigma]}$, we can forget about the antisymmetrizing brackets in the contraction. The first term is OK, but the second term needs some attention. We have to take a closer look at our boundary conditions. The second part of the integral can be written as

$$\begin{aligned}&\int \sqrt{|g|}[\mathscr{L}^{\mu\nu\rho\sigma}\nabla_\mu\nabla_\rho\delta g_{\sigma\nu}]d^4x = \\ &\int \sqrt{|g|}\nabla_\mu[\mathscr{L}^{\mu\nu\rho\sigma}\nabla_\rho\delta g_{\sigma\nu}]d^4x - \int \sqrt{|g|}\nabla_\mu\mathscr{L}^{\mu\nu\rho\sigma}\nabla_\rho\delta g_{\sigma\nu}d^4x\end{aligned} \tag{8.15}$$

The first term on the right can be converted into an integral over the hypersurface $\partial\Omega$. We know that $\delta g_{\mu\nu}$ vanishes on $\delta\Omega$, and so does the projection of $\nabla_\rho\delta g_{\mu\nu}$ on the hypersurface. So we end up only with the second term on the right. Doing another partial integration, we get

$$\begin{aligned}&\int \sqrt{|g|}\mathscr{L}^{\mu\nu\rho\sigma}\nabla_\rho\delta g_{\sigma\nu}d^4x = \\ &\int \sqrt{|g|}\nabla_\rho[\nabla_\mu\mathscr{L}^{\mu\nu\rho\sigma}\delta g_{\sigma\nu}]d^4x - \int \sqrt{|g|}\nabla_\rho\nabla_\mu\mathscr{L}^{\mu\nu\rho\sigma}\delta g_{\sigma\nu}d^4x\end{aligned} \tag{8.16}$$

The first term on the right can again be converted into a surface integral, and the fact that $\delta g_{\mu\nu}$ vanishes on $\delta\Omega$ makes it zero. Putting this all together, the requirement that $\delta S = 0$ for arbitrary $\delta g_{\mu\nu}$ gives us the equation of motion for the metric:

$$\frac{1}{2}g^{\mu\nu}\mathscr{L} + \mathscr{L}^{\mu\nu} + \mathscr{L}^{\rho\sigma\lambda(\mu}R_{\rho\sigma\lambda}{}^{\nu)} - 2\nabla_{(\rho}\nabla_{\sigma)}\mathscr{L}^{\rho\mu\nu\sigma} = 0 \tag{8.17}$$

So now we have these equations, we can add the matter field to obtain the expressions for $\mathscr{L} = \mathscr{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \psi_{\mu\nu}, \nabla_\lambda\psi_{\mu\nu})$.

## 8.3 Adding of a matter field: equations of motion revisited

So let's add the matter field. The equations of motion for the matter field are well known; they follow straight from the Euler Lagrange equations:

$$\mathscr{L}_\psi^{\mu\nu} - \nabla_\lambda\mathscr{L}_\psi^{\lambda,\mu\nu} = 0 \tag{8.18}$$

We already have the expression for $\delta\psi_{\mu\nu}$. The expression for $\delta(\nabla_\lambda\psi_{\mu\nu})$ is

$$\delta(\nabla_\rho\psi_{\mu\nu}) = \nabla_\rho(\delta\psi_{\mu\nu}) - \delta\Gamma^\lambda_{\rho\mu}\psi_{\lambda\nu} - \delta\Gamma^\lambda_{\rho\nu}\psi_{\mu\lambda} \tag{8.19}$$

So here we see that the variation of the matter field depends on the variation of the metric. The terms with the metric and the Riemann tensor are already known. Now we are curious about

$$\int \sqrt{|g|}[\mathscr{L}^{\mu\nu}_\psi \delta\psi_{\mu\nu} + \mathscr{L}^{\rho,\mu\nu}_\psi \delta(\nabla_\rho\psi_{\mu\nu})]d^4x \tag{8.20}$$

This is equivalent to

$$\int \sqrt{|g|}[\mathscr{L}^{\mu\nu}_\psi \delta\psi_{\mu\nu} + \mathscr{L}^{\rho,\mu\nu}_\psi \nabla_\rho(\delta\psi_{\mu\nu}) - \mathscr{L}^{\rho,\mu\nu}_\psi \psi_{\lambda\nu}\delta\Gamma^\lambda_{\rho\mu} - \mathscr{L}^{\rho,\mu\nu}_\psi \psi_{\mu\lambda}\delta\Gamma^\lambda_{\rho\nu}]d^4x \tag{8.21}$$

If we want to obtain the equations of motion for the metric, we have to pull out a factor $\delta g_{\mu\nu}$ again. This requires again some partial integrations and Stoke's theorem for each term. After all, $\delta\Gamma^\lambda_{\rho\nu}$ contains terms like $\nabla_\rho\delta g_{\mu\nu}$. Let's zoom in at a term in the integral which contains $\delta\Gamma^\lambda_{\rho\mu}$, and write it out:

$$\int \sqrt{|g|}[\mathscr{L}^{\rho,\mu\nu}\psi_{\lambda\nu}\frac{1}{2}g^{\lambda\alpha}(\nabla_\rho\delta g_{\mu\alpha} + \nabla_\mu\delta g_{\alpha\rho} - \nabla_\alpha\delta g_{\mu\nu})]d^4x \tag{8.22}$$

In eq.(7.88) we did a similar operation: focussing on one term is enough, the rest goes the same way. We write

$$\begin{aligned} &\int \sqrt{|g|}g^{\lambda\alpha}\psi_{\lambda\nu}\mathscr{L}^{\rho,\mu\nu}_\psi \nabla_\rho(\delta g_{\mu\alpha})d^4x \\ =&\int \sqrt{|g|}g^{\lambda\alpha}(\nabla_\rho[\mathscr{L}^{\rho,\mu\nu}_\psi \psi_{\lambda\nu}\delta g_{\mu\alpha}] - \nabla_\rho[\mathscr{L}^{\rho,\mu\nu}_\psi \psi_{\lambda\nu}]\delta g_{\mu\alpha})d^4x \end{aligned} \tag{8.23}$$

The first term on the right can be converted into a surface integral, and the condition that $\delta g_{\mu\nu} = 0$ at $\delta\Omega$ makes it vanish. The other term is just what we wanted. Every $\delta\Gamma^\alpha_{\mu\nu}$-term gives us three terms of $\mathscr{L}^{\rho,\mu\nu}_\psi \psi_{\lambda\beta}$ with contractions between the indices. If we work this out, the equations of motion for the metric become

$$\begin{aligned} \tfrac{1}{2} \quad & \mathscr{L} + \mathscr{L}^{\mu\nu} + \mathscr{L}^{\rho\sigma\lambda(\mu}R_{\rho\sigma\lambda}{}^{\nu)} - 2\nabla_{(\rho}\nabla_{\sigma)}\mathscr{L}^{\rho\mu\nu\sigma} \\ + \quad & \frac{1}{4}\nabla_\lambda\Big[\mathscr{L}^{\lambda,\mu\rho}_\psi \psi^\nu{}_\rho + \mathscr{L}^{\lambda,\rho\mu}_\psi \psi_\rho{}^\nu + \mathscr{L}^{\mu,\lambda\rho}_\psi \psi^\nu_\rho \\ + \quad & \mathscr{L}^{\mu,\rho\lambda}_\psi \psi_\rho{}^\nu - \mathscr{L}^{\mu,\nu\rho}\psi^\lambda{}_\rho - \mathscr{L}^{\mu,\rho\nu}_\psi \psi_\rho{}^\lambda + (\mu \leftrightarrow \mu)\Big] = 0 \end{aligned} \tag{8.24}$$

## 8.4 Adding of a matter field: general covariance revisited

Now with those matter fields added, the total variation of $\mathscr{L}$ becomes

$$\delta\mathscr{L} = \mathscr{L}^{\mu\nu}\delta g_{\mu\nu} + \mathscr{L}^{\mu\nu\rho\sigma}\delta R_{\mu\nu\rho\sigma} + \mathscr{L}^{\mu\nu}_\psi \delta\psi_{\mu\nu} + \mathscr{L}^{\rho,\mu\nu}_\psi \delta(\nabla_\rho\psi_{\mu\nu}) \tag{8.25}$$

This is equal to $\xi^\sigma\nabla_\sigma\mathscr{L}$, where

$$\nabla_\sigma\mathscr{L} = \mathscr{L}^{\mu\nu}\nabla_\sigma g_{\mu\nu} + \mathscr{L}^{\mu\nu\rho\lambda}\nabla_\sigma R_{\mu\nu\rho\lambda} + \mathscr{L}^{\mu\nu}_\psi \nabla_\sigma\psi_{\mu\nu} + \mathscr{L}^{\lambda,\mu\nu}_\psi \nabla_\sigma\nabla_\lambda\psi_{\mu\nu} \tag{8.26}$$

We already know how to deal with the first 2 terms on each right side, and now we are interested in the contributions of the matter field. The term $\delta(\nabla_\rho\psi_{\mu\nu})$ is just the Lie derivative of $\nabla_\rho\psi_{\mu\nu}$, and we get the equality

$$\begin{aligned}
&\mathscr{L}_\psi^{\mu\nu}[\nabla_\mu\xi^\sigma\psi_{\sigma\nu} + \nabla_\nu\xi^\sigma\psi_{\mu\sigma} + \xi^\sigma\nabla_\sigma\psi_{\mu\nu}] + \\
&\mathscr{L}_\psi^{\rho,\mu\nu}[\nabla_\rho\psi_{\mu\sigma}\nabla_\nu\xi^\sigma + \nabla_\rho\psi_{\sigma\nu}\nabla_\mu\xi^\sigma + \nabla_\sigma\psi_{\mu\nu}\nabla_\rho\xi^\sigma + \xi^\sigma\nabla_\sigma\nabla_\rho\psi_{\mu\nu}] = \\
&\mathscr{L}_\psi^{\mu\nu}\xi^\sigma\nabla_\sigma\psi_{\mu\nu} + \mathscr{L}_\psi^{\lambda,\mu\nu}\xi^\sigma\nabla_\sigma\nabla_\lambda\psi_{\mu\nu} \qquad (8.27)
\end{aligned}$$

We have four terms which cancel with each other here. If now $\mathscr{L} = \mathscr{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \psi_{\mu\nu}, \nabla_\lambda\psi_{\mu\nu})$ is considered again, covariance leads to the identity

$$\begin{aligned}
2 \quad & \mathscr{L}^{\mu\nu}\nabla_\mu\xi_\nu + 4\mathscr{L}^{\mu\nu\rho\sigma}R_{\mu\nu\rho}{}^\lambda\nabla_\sigma\xi_\lambda + \mathscr{L}_\psi^{\mu\nu}[\nabla_\mu\xi^\rho\psi_{\rho\nu} + \nabla_\nu\xi^\rho\psi_{\mu\rho}] \\
+ \quad & \mathscr{L}_\psi^{\rho,\mu\nu}[\nabla_\rho\xi^\rho\nabla_\sigma\psi_{\mu\nu} + \nabla_\mu\xi^\sigma\nabla_\rho\psi_{\sigma\nu} + \nabla_\nu\xi^\sigma\nabla_\rho\psi_{\mu\sigma}] = 0 \qquad (8.28)
\end{aligned}$$

## 8.5 The current and the accompanying Noether potential

We will not explicitly derive the form of the Noether current here, because this simply involves a lot of partial integrations and the freedom to do this makes the calculation quite messy. However, the general idea will be given, and the form itself can partly be intuitively explained. First we consider $\mathscr{L} = \mathscr{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma})$ What we try to do, is to derive the current which is yielded by the variation of the action. We saw in section 4.4 that the variation of the action can be written as

$$\delta S = \int \sqrt{|g|}(\partial_\mu\epsilon(x))J^\mu d^4x \qquad (8.29)$$

The idea now is to plug in the explicit variations of the fields involved under general transformations. The boundary conditions help us to rewrite the terms which involve derivatives of $\epsilon(x)$ via partial integrations. The variation of the action gives

$$\delta S = \int \sqrt{|g|}\Big[(\mathscr{L}^{\mu\nu} + \frac{1}{2}g^{\mu\nu}\mathscr{L})\delta g_{\mu\nu} + \mathscr{L}^{\mu\nu\rho\sigma}\delta R_{\mu\nu\rho\sigma}\Big]d^4x \qquad (8.30)$$

The variations are

$$\delta g_{\mu\nu} = -\epsilon(x)(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) \qquad (8.31)$$

for the metric and

$$\begin{aligned}
\delta R_{\mu\nu\rho\sigma} &= -\frac{1}{2}\epsilon(x)(R_{\mu\nu\rho}{}^\lambda\delta g_{\sigma\lambda} - R_{\mu\nu\sigma}{}^\lambda\delta g_{\rho\lambda}) - \epsilon(x)(\nabla_\mu\nabla_\rho\delta g_{\sigma\nu} - \nabla_\nu\nabla_\rho\delta g_{\sigma\mu}) \\
&- \epsilon(x)(\nabla_\mu\nabla_\rho\delta g_{\sigma\nu} - \nabla_\mu\nabla_\sigma\delta g_{\rho\nu}) \qquad (8.32)
\end{aligned}$$

We see that the expression $\delta R_{\mu\nu\rho\sigma}$ contains second order derivatives of $\delta g_{\mu\nu}$, and therefore up to second order derivatives of $\epsilon$ and third order derivatives of the vector field $\xi$. If we plug everything in, then eventually we want to write the integrand as the derivative of $\epsilon(x)$ times a term which is our current. It was noted before that in this derivation the equations of motion are used. So we expect that the terms $\mathscr{L}^{\mu\nu}$ won't appear in our current.

The extension with matter fields goes in a similar way, where also the equations of motion are used. If this lengthy calculation is performed, the following Noether current arises [15]:

$$\begin{aligned}
J^\mu &= \xi^\mu \mathscr{L} - 2\mathscr{L}^{\mu\nu\rho\sigma}\Big[R_{\lambda\nu\rho\sigma}\xi^\lambda + \nabla_\nu\nabla_\rho\xi_\sigma\Big] + 4\nabla_\rho\mathscr{L}^{\mu\nu\rho\sigma}\nabla_{(\nu}\xi_{\sigma)} \\
&- \mathscr{L}_\psi^{\mu,\rho\sigma}\Big[\nabla_\rho\xi^\lambda\psi_{\lambda\sigma} + \nabla_\sigma\xi^\lambda\psi_{\rho\lambda} + \xi^\lambda\nabla_\lambda\psi_{\rho\sigma}\Big] \\
&+ \frac{1}{2}(\nabla_\lambda\xi_\rho + \nabla_\rho\xi_\lambda)\Big[\mathscr{L}_\psi^{\mu,\rho\sigma}\psi^\lambda{}_\sigma + \mathscr{L}_\psi^{\mu,\sigma\rho}\psi_\sigma{}^\lambda + \mathscr{L}_\psi^{\rho,\mu\sigma}\psi^\lambda{}_\sigma \\
&+ \mathscr{L}_\psi^{\rho,\sigma\mu}\psi_\sigma{}^\lambda - \mathscr{L}_\psi^{\rho,\lambda\sigma}\psi^\mu{}_\sigma - \mathscr{L}_\psi^{\rho,\sigma\lambda}\psi_\sigma{}^\mu\Big]
\end{aligned} \tag{8.33}$$

## 8.6 An explicit example: Reissner-Nördstrom

Now we have an expression for the current, let's take a concrete example: the Reissner-Nördstrom solution. It is derivable from the Lagrangian density

$$\mathscr{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, F_{\mu\nu}) = R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.34}$$

There is no factor $\nabla_\lambda F_{\mu\nu}$ in $\mathscr{L}$, so the current is in this case

$$J^\mu = \xi^\mu \mathscr{L} - 2\mathscr{L}^{\mu\nu\rho\sigma}[R_{\lambda\nu\rho\sigma}\xi^\lambda + \nabla_\nu\nabla_\rho\xi_\sigma] + 4\nabla_\rho\mathscr{L}^{\mu\nu\rho\sigma}\nabla_{(\nu}\xi_{\sigma)} \tag{8.35}$$

It is clear that only an expression for $\mathscr{L}^{\mu\nu\rho\sigma}$ is needed. Without (anti)symmetrization we would say that $\mathscr{L}^{\mu\nu\rho\sigma} = g^{\mu\sigma}g^{\nu\rho}$, but accounting for this (anti)symmetries gives

$$\begin{aligned}
\mathscr{L}^{\mu\nu\rho\sigma} &= \frac{1}{4}\Big(g^{\mu\sigma}g^{\nu\rho} - g^{\nu\sigma}g^{\mu\rho} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}\Big) \\
&= \frac{1}{2}\Big(g^{\mu\sigma}g^{\nu\rho} - g^{\nu\sigma}g^{\mu\rho}\Big)
\end{aligned} \tag{8.36}$$

So,

$$\boxed{\frac{\partial R}{\partial R_{\mu\nu\rho\sigma}} = \frac{1}{2}g^{\mu[\rho}g^{\sigma]\nu}} \tag{8.37}$$

By metric compatibility then, $\nabla_\rho\mathscr{L}^{\mu\nu\rho\sigma} = 0$. The current becomes

$$J^\mu = R\xi^\mu + \frac{1}{4}\xi^\mu F_{\alpha\beta}F^{\alpha\beta} - [g^{\nu\sigma}g^{\mu\rho} - g^{\mu\sigma}g^{\nu\rho}][R_{\lambda\nu\rho\sigma}\xi^\lambda + \nabla_\nu\nabla_\rho\xi_\sigma] \tag{8.38}$$

The Noether potential $Q^{\mu\nu}$ obtains a familiar form:

$$\begin{aligned}
Q^{\mu\nu} &= -2\mathscr{L}^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma \\
&= -[g^{\nu\sigma}g^{\mu\rho} - g^{\mu\sigma}g^{\nu\rho}]\nabla_\rho\xi_\sigma \\
&= 2\nabla^{[\mu}\xi^{\nu]}
\end{aligned} \tag{8.39}$$

which indeed is antisymmetric. Note that this is the same $\mathbf{Q}$ we would obtain if $F_{\mu\nu} = 0$. This coincides with our example (7.2.3), in which we saw that our $\mathbf{Q}$ for the electromagnetic field was linear in $\xi$, and by integrating this term over the bifurcation horizon it doesn't alter the entropy. On this bifurcation horizon we can write $Q^{\mu\nu} = \mathscr{L}^{\mu\nu\rho\sigma}\epsilon_{\rho\sigma}$.

# Chapter 9

# Correction terms to the Hilbert action

We saw a good candidate for the black hole entropy, but one question which rises naturally is: does it obey the second law for more complicated Lagrangians? In the original proof for Einstein's equations, Hawking used a couple of tools. He used the null convergence condition $R_{\mu\nu}X^\mu X^\nu \geq 0$ for null vectors $X$, and also the cosmic censorship. The question is if a similar increase theorem can be formulated for higher derivative gravity. This question is not yet answered in the affirmitive and is complicated by the fact that the explicit equations of motion are needed. Here some examples of higher derivative Lagrangians will be examined. The resulting action consists of the Hilbert action plus additional correction terms. It is important to be aware of the structure of the resultant field equations: for example, the order of derivatives of the metric, and ofcourse the field equations must possess solutions with physical singularities.

## 9.1 Higher derivatives and degrees of R

In this section we first will consider so-called $R^2$-gravity in $n$ dimensions. The action for $R^2$-gravity is

$$\begin{aligned} S_H &= \frac{1}{16\pi}\int d^n x\sqrt{|g|}R \\ S_{R^2} &= \int d^n x\sqrt{|g|}\Big(aR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + bR_{\mu\nu}R^{\mu\nu} + cR^2\Big) \\ S &= S_H + \alpha S_{R^2} \end{aligned} \tag{9.1}$$

with $(a, b, c, \alpha) \in \mathbb{R}$ and $\alpha$ is a coupling. It doesn't contain any derivatives of the Riemann-tensor. In general this will give equations of motion which are of higher degree in derivatives of $g_{\mu\nu}$ than the usual degree of two of $S_H$ alone. The particular choice $(a = c = 1, b = -4)$ is called the Gauss-Bonnet action and appears often in literature about brane worlds because the field equations contain up to second order derivatives of the metric in any dimension. If these field equations are derived and we know the form of static solutions we can use our expression for $\mathcal{S}$ to calculate the entropy.

Now derivative terms will be considered; this treatise basically continues section (7.5). There we saw how we could abstract the right information from the Lagrangian to do explicit calculations. In solving $\mathscr{J} = d\mathbf{Q}$ for $\mathbf{Q}$ we pull back the current $\mathscr{J}$ to the bifurcation horizon where $\xi = 0$, so terms linear in $\xi$ are not important to us. This means that we want the terms linear with second order derivatives of $\xi$ in $\mathscr{J}$ which, via $\mathscr{J} = \mathbf{\Theta} - \xi \cdot \mathbf{L}$, are given by the part of $\mathbf{\Theta}$ with at least second derivatives of $\xi$. Now we are concerned with higher derivative terms of the Riemann-tensor. For this we are going to make use of the following result which we won't prove, but is nevertheless true [11]. We can compose the Noether charge $\mathbf{Q}$ in the following non-unique way:

$$\begin{aligned} \mathbf{Q} &= \mathbf{A}_\mu \xi^\mu + \mathbf{B}^{\mu\nu}\nabla_{[\mu}\xi_{\nu]} + \mathbf{C} + d\mathbf{D} \\ \mathbf{A} &= \mathbf{A}(\phi), \quad \mathbf{B} = \mathbf{B}(\phi), \quad \mathbf{C} = \mathbf{C}(\phi, \pounds_\xi \phi), \quad \mathbf{D} = \mathbf{D}(\phi, \xi) \end{aligned} \tag{9.2}$$

The form $\mathbf{C}$ is linear in $\pounds_\xi\phi$ and the form $\mathbf{D}$ is linear in $\xi$. The important ingredient for us is that we can always choose $\mathbf{B}$,$\mathbf{C}$ and $d\mathbf{D}$ such that

$$(\mathbf{B}^{\mu\nu})_{\rho_3\cdots\rho_n} = -\frac{\delta\mathscr{L}}{\delta R_{\alpha\beta\mu\nu}}\epsilon_{\alpha\beta\rho_3\cdots\rho_n}, \quad \mathbf{C} = 0, \quad d\mathbf{D} = 0 \tag{9.3}$$

Let's make this a little plausible by looking at the Lagrangian

$$\mathscr{L} = \mathscr{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}) \tag{9.4}$$

where we will do the same kind of calculations as we did in section (7.6). From the discussion of section (7.5) it's clear that we are interested in the variations with respect to the Riemann-tensor:

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\delta R_{\mu\nu\rho\sigma} + \frac{\partial\mathscr{L}}{\partial\nabla_\lambda R_{\mu\nu\rho\sigma}}\delta\nabla_\lambda R_{\mu\nu\rho\sigma} \tag{9.5}$$

The first term we already handled. To be explicit, the second term in this variation reads

$$\frac{\partial\mathscr{L}}{\partial\nabla_\lambda R_{\mu\nu\rho\sigma}}\Big(\nabla_\lambda\delta R_{\mu\nu\rho\sigma} - \delta\Gamma^\alpha_{\lambda\mu}R_{\alpha\nu\rho\sigma} - \delta\Gamma^\alpha_{\lambda\nu}R_{\mu\alpha\rho\sigma} - \delta\Gamma^\alpha_{\lambda\rho}R_{\mu\nu\alpha\sigma} - \delta\Gamma^\alpha_{\lambda\sigma}R_{\mu\nu\rho\alpha}\Big) \tag{9.6}$$

but those $\delta\Gamma$-terms aren't going to give us terms like $\nabla^2\delta_\xi g_{\alpha\beta}$. Keeping only the relevant terms we obtain

$$\begin{aligned} \delta\mathscr{L} &= \frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}}\delta R_{\mu\nu\rho\sigma} + \nabla_\lambda\Big(\frac{\partial\mathscr{L}}{\partial\nabla_\lambda R_{\mu\nu\rho\sigma}}\delta R_{\mu\nu\rho\sigma}\Big) - \nabla_\lambda\Big(\frac{\partial\mathscr{L}}{\partial\nabla_\lambda R_{\mu\nu\rho\sigma}}\delta R_{\mu\nu\rho\sigma}\Big) + \ldots \\ &= \Big(\frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\lambda\frac{\partial\mathscr{L}}{\partial\nabla_\lambda R_{\mu\nu\rho\sigma}}\Big)\delta R_{\mu\nu\rho\sigma} + \nabla_\lambda\Big(\frac{\partial\mathscr{L}}{\partial\nabla_\lambda R_{\mu\nu\rho\sigma}}\delta R_{\mu\nu\rho\sigma}\Big) + \ldots \\ &= \nabla_\mu\Big(2(\frac{\partial\mathscr{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\lambda\frac{\partial\mathscr{L}}{\partial\nabla_\lambda R_{\mu\nu\rho\sigma}})\nabla_\rho\delta g_{\nu\sigma}\Big) + \ldots \end{aligned} \tag{9.7}$$

Iterating these steps we will see that the 'equations of motion' for $R_{\mu\nu\rho\sigma}$ pop up in the entropy. The same goes for a Lagrangiang which, in addition, depends on a matter field $\psi$ and it's first derivative $\nabla\psi$. So we can conclude that for $\mathscr{L} = \mathscr{L}(\psi, \nabla_\alpha\psi, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\alpha R_{\mu\nu\rho\sigma}, \nabla_\alpha\nabla_\beta R_{\mu\nu\rho\sigma}, \ldots)$ the entropy becomes

$$\boxed{\mathcal{S} = 2\pi\oint_\Sigma \sqrt{|h|}\frac{\delta\mathscr{L}}{\delta R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}d\Omega} \tag{9.8}$$

where $h$ is the pulled-back metric on $\Sigma$.

## 9.2 The Gauss-Bonnet Lagrangian

We could ask ourselves the question: what if we would add a term to the action, which on varying it, would be a total derivative? With other words,

$$\mathbf{L} \quad \rightarrow \quad \mathbf{L} + \Delta, \qquad \text{with } \ \delta\mathbf{\Delta} = d\mathbf{V} \tag{9.9}$$

for some $\mathbf{V} \in \Lambda^{n-1}(M)$. Notice that this differs from the ambiguity eq.(7.94). This ambiguity means that $\Delta$ wouldn't contribute to the dynamics of the theory, but it surely could contribute to the entropy because $\mathcal{J}$ is shifted by $(\mathbf{V} - \xi \cdot \mathbf{\Delta})$. We can't use the same reasoning from section (7.7) to make these effects irrelevant for $\mathbf{Q}$. These additions to the Lagrangian aren't that artificial; the Gauss-Bonnet theorem gives a way to construct such terms[16]. The theorem states that for a compact manifold $M$ the Gaussian curvature $K$ and the Euler characteristic $\chi(M)$ are linked via

$$\int_M \sqrt{|g|} K d^n x = 2\pi\chi(M) \tag{9.10}$$

It gives a connection between local information of the manifold ($K$) and global information ($\chi$). If $\partial M \neq 0$ then the geodesic curvature of $\partial M$ also contributes to $\chi(M)$. The Euler characteristic is a topological invariant of the manifold. The generalized Gauss-Bonnet Lagrangian density is given by [17]

$$\mathscr{L}_{GB} = \frac{(-1)^m}{2^m}\epsilon_{\mu_1\nu_1\cdots\mu_m\nu_n}\epsilon^{\rho_1\sigma_1\cdots\rho_m\sigma_m} R^{\mu_1\nu_1}{}_{\rho_1\sigma_1}\cdots R^{\mu_m\nu_m}{}_{\rho_m\sigma_m} \tag{9.11}$$

where the number of space-time dimensions is equal to $2m$. For example, $m = 1$ gives $\mathscr{L}_{GB} = -R$, which yields the Hilbert action. This means that for 2 space-time dimensions the Hilbert action doesn't give any dynamics, but merely a boundary term. For space-time with four dimensions the Gauss-Bonnet action is proportional to[18]

$$S = \int d^4x\sqrt{|g|}R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\phi}\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi} \tag{9.12}$$

In four dimensions this action yields the Gauss-Bonnet action. To uncover it, we use the determinant identity which can be seen with eq.(3.44):

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\phi} = \begin{vmatrix} \delta^\alpha_\mu & \delta^\beta_\mu & \delta^\gamma_\mu & \delta^\phi_\mu \\ \delta^\alpha_\nu & \delta^\beta_\nu & \delta^\gamma_\nu & \delta^\phi_\nu \\ \delta^\alpha_\rho & \delta^\beta_\rho & \delta^\gamma_\rho & \delta^\phi_\rho \\ \delta^\alpha_\sigma & \delta^\beta_\sigma & \delta^\gamma_\sigma & \delta^\phi_\sigma \end{vmatrix}$$

Rewriting $R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\phi}\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi} = R^{\mu\nu}{}_{\rho\sigma}R^{\alpha\beta}{}_{\gamma\phi}\epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi}$, plugging in the determinant and writing out all the terms gives us

$$R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\phi}\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi} = 4(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) \tag{9.13}$$

which is four times our wanted Gauss-Bonnet term. This form can be investigated more easily. In the variation we use $\epsilon^{\mu\nu\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}$, and we obtain

$$\begin{aligned} \delta(\mathscr{L}_{GB}\sqrt{|g|}) \quad &= \quad \sqrt{|g|}\Big(\mathscr{L}_{GB}g^{\mu\nu}\delta g_{\mu\nu} + 2R_{\alpha\beta\gamma\phi}\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi}\delta R_{\mu\nu\rho\sigma} \\ &+ \quad 2R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\phi}\epsilon^{\mu\nu\alpha\beta}\delta\epsilon^{\rho\sigma\gamma\phi}\Big) \end{aligned} \tag{9.14}$$

The variation in the Riemann-tensor reads $\delta R_{\mu\nu\rho\sigma} = g_{\mu\xi}(\nabla_\rho \delta\Gamma^\xi_{\nu\sigma} - \nabla_\sigma \delta\Gamma^\xi_{\nu\rho})$, and in the contraction these two $\nabla\delta\Gamma$-terms are minus each-other, so this becomes

$$2\delta R_{\mu\nu\rho\sigma} R_{\alpha\beta\gamma\phi} \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi} = 4 g_{\mu\xi} \nabla_\rho \delta\Gamma^\xi_{\nu\sigma} \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi} \tag{9.15}$$

The variation of the generalized Levi-Civita symbol is $\delta\epsilon^{\alpha\beta\mu\nu} = -\frac{1}{2} g^{\lambda\theta}\delta g_{\lambda\theta}\epsilon^{\alpha\beta\mu\nu}$. Plugging this in and rearranging, the variation becomes finally

$$\delta(\sqrt{|g|}\mathscr{L}_{GB}) = \nabla_\phi \Big( 4\sqrt{|g|} g_{\lambda\mu} \delta\Gamma^\lambda_{\nu\gamma} R_{\rho\sigma\alpha\beta} \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi} \Big) \tag{9.16}$$

Plugging in the explicit variation $\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$ we can identify our vector density $\Theta^\phi$ belonging to the Gauss-Bonnet part of the action as

$$\Theta^\phi = \alpha\sqrt{|g|} g_{\lambda\mu} g^{\lambda\rho} \Big( \nabla_\nu \nabla_{(\gamma}\xi_{\rho)} + \nabla_\gamma \nabla_{(\rho}\xi_{\nu)} - \nabla_\rho \nabla_{(\nu}\xi_{\gamma)} \Big) R_{\rho\sigma\alpha\beta} \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\phi} \tag{9.17}$$

The corresponding Noether charge is [11],[19]:

$$Q_{\alpha_1\cdots\alpha_{n-2}} = -\epsilon_{\mu\nu\alpha_1\cdots\alpha_{n-2}} \Big( \frac{1}{16\pi}\nabla^\mu\xi^\nu + 2\alpha R \nabla^\mu \xi^\nu + 8\alpha \nabla^{[\rho}\xi^{\nu]} R^\mu{}_\rho + 2\alpha R^{\mu\nu\rho\sigma}\nabla_\rho \xi_\sigma \Big) \tag{9.18}$$

To calculate the associated entropy with the added term we could use this result, but instead we will look again at spherically symmetric solutions of the vacuum equations described by eq.(6.31). According to Birkhoff's theorem[2],[6] this solutions are stationary and asymptotically flat if we deal with Einstein's field equations. The derivatives involved are

$$\begin{aligned} \frac{\partial[R^2]}{\partial R_{\alpha\beta\gamma\delta}} &= R g^{\alpha[\gamma} g^{\delta]\beta} \\ \frac{\partial[R^{\mu\nu}R_{\mu\nu}]}{\partial R_{\alpha\beta\gamma\delta}} &= 2 g^{\alpha[\gamma} R^{\delta]\beta} \\ \frac{\partial[R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}]}{\partial R_{\alpha\beta\gamma\delta}} &= 2R^{\alpha\beta\gamma\delta} \end{aligned} \tag{9.19}$$

The second identity comes from $R^{\mu\nu}R_{\mu\nu} = g^{\mu\phi}g^{\nu\xi}g^{\lambda\tau}g^{\kappa\varphi}R_{\lambda\mu\tau\nu}R_{\kappa\phi\varphi\xi}$. With this the derivative $\mathscr{L}^{\alpha\beta\gamma\delta}_{GB}$ becomes

$$\mathscr{L}^{\alpha\beta\gamma\delta}_{GB} = \Big( R g^{\alpha[\gamma} g^{\delta]\beta} - 8 g^{\alpha[\gamma} R^{\delta]\beta} + 2R^{\alpha\beta\gamma\delta} \Big) \tag{9.20}$$

Another calculation gives[1]

$$\begin{aligned} \mathscr{L}^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} &= 4\mathscr{L}^{0101}(\epsilon_{01})^2 \\ \mathscr{L}^{0101} &= \Big( R g^{0[0} g^{1]1} - 8 g^{0[0} R^{1]1} + 2R^{0101} \Big) \\ \rightarrow \mathscr{L}^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} &= \Big( 2R g^{00} g^{11} - 16 g^{00} R^{11} + 8R^{0101} \Big)(\epsilon_{01})^2 \end{aligned} \tag{9.21}$$

With the Hilbert action we had that the different factors in $\mathscr{L}^{\alpha\beta\gamma\delta}$ and $\epsilon_{\rho\lambda}$ cancelled each-other which resulted in an integrand which was simply a numerical

[1]Note that for spherically symmetric solutions we can put $g^{00}g^{11} = 1$ with an appropriate choice of coordinates.

4, but here we have to do some more calculations. In calculating the connection symbols, Riemann-tensor and Ricci-tensor we only have to keep track of the $(\partial_1 = \partial_r)$ - terms for the index-values 0 and 1. The only non-zero connection symbols are

$$\Gamma^0_{01} = \frac{1}{2} g^{00}\partial_r g_{00}, \quad \Gamma^1_{00} = -\frac{1}{2} g^{11}\partial_r g_{00} \tag{9.22}$$

With these connections the relevant terms are

$$\begin{aligned} R^{11} &= R_{\alpha\beta} g^{1\alpha} g^{1\beta} = R_{11} g^{11} g^{11} = g^{11} g^{11} \Big( R^0{}_{101} + R^1{}_{111} \Big) \\ R^{0101} &= g^{1\nu} g^{0\rho} g^{1\sigma} R^0{}_{\nu\rho\sigma} = g^{11} g^{00} g^{11} R^0{}_{101} \\ R &= g^{\alpha\beta} R_{\alpha\beta} = g^{00} R^0{}_{000} + g^{00} R^1{}_{010} + g^{11} R^0{}_{101} + g^{11} R^1{}_{111} \end{aligned} \tag{9.23}$$

Finally, the components of the Riemann-tensor are

$$\begin{aligned} R^0{}_{000} &= R^1{}_{111} = 0 \\ R^0{}_{101} &= -\partial_r \Gamma^0_{10} - \Gamma^0_{01}\Gamma^0_{01} \\ &= -\frac{1}{2}[\partial_r g^{00}\partial_r g_{00} + g^{00}\partial_r^2 g_{00}] - \frac{1}{4}[g^{00}\partial_r g_{00}]^2 \\ R^1{}_{010} &= \partial_r \Gamma^1_{00} - \Gamma^1_{00}\Gamma^0_{10} \\ &= -\frac{1}{2}\partial_r[g^{11}\partial_r g_{00}] + \frac{1}{4} g^{00} g^{11}[\partial_r g_{00}]^2 \end{aligned} \tag{9.24}$$

With these results we can explicitly calculate $\mathcal{S}$ if we plug in $g_{00} = -e^{2g(r)}$ and $g_{11} = +e^{2f(r)}$ for a given $f(r)$ and $g(r)$. Let's see what this implies for the Schwarzschild solution in the case of Einstein's field equations with this extra topological term. Being a vacuum solution, it implies that $G_{\mu\nu} = 0$. Taking the trace of this equation implies that $R = 0$, and this in turn implies $R_{\mu\nu} = 0$, which simplifies our entropy quite a bit. First of all, we have for Schwarzschild solutions that

$$\mathscr{L}^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} = 8R^{0101}(\epsilon_{01})^2 \tag{9.25}$$

Plugging in all the explicit terms, performing the derivatives and rearranging gives the following rather elegant result:

$$\mathscr{L}^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} = 8\Big(\frac{M^2 + 2M}{r^3}\Big) \tag{9.26}$$

So the extra entropy term $\mathcal{S}_{GB}$, again evaluated at the event horizon $r = 2M$, is simply

$$\mathcal{S}_{GB} = 2\pi \frac{(M+2)}{M^2} A \tag{9.27}$$

where $A$ is the surface of the event horizon, $A = \int_\Sigma \sqrt{|h|}\, d\Omega$, which is not altered because the field equations are not changed by adding the Gauss-Bonnet term. We see that the term becomes less significant if the mass $M$ of the black hole becomes larger. Due to the Hawking-effect we have eventually that $M \to 0$, but also that $A \to 0$. If $A$ goes to zero fast enough $\mathcal{S}_{GB}$ remains finite, but to say more about that we have to know how $A$ and $M$ change in time.

So how should we interpret this remarkable result? Apparently, the demand that the entropy remains unchanged if the equations of motion remain unchanged is a subtle demand if we look at topological terms. Perhaps we should take a closer look at the boundary conditions we impose on the metric which give the equations of motion, but this is a guess; we won't go into further detail here. Nevertheless it should be clear that according to the calculation above a Gauss-Bonnet term proportional to eq.(9.13) can have a physical effect in four space-time dimensions.

# Chapter 10

# Conclusions and overview

It's time to look back at what is done so far. In this thesis, the physical background and mathematics needed for the method of Wald were investigated, to calculate the black hole entropy $\mathcal{S}$ for stationary black holes with nonvanishing surface gravity, $\kappa \neq 0$. In this formalism, $\mathcal{S}$ is the Noether charge with respect to the horizon Killing field $\xi$. The assumption was that the event horizon of this black hole is a Killing horizon. We saw a close connection between the diffeomorphism invariance of gravitational theories which describe gravity with a dynamical background, and the intrinsic entropy of a gravitational field. Also some black hole thermodynamics was reviewed to put all this in the right context. With this method some examples were calculated. We saw that higher derivative terms in the action add corrections to the classical calculation that for Einstein's field equations yield $\mathcal{S} = A/4$. We further noted the difficulty in extending the formalism to dynamical black holes because we have to let go of the symmetries which were used to derive the entropy formula. In the last chapter we investigated the addition of a topological invariant to the action, which changed the entropy non-trivially in four dimensions.

To end, we note some interesting open questions in this field of research:

- Are the equations of motion for black hole thermodynamic merely nice analogs, or are the entropy and temperature real thermodynamic properties of the black hole?
- How can be proven that a general expression for the entropy obeys the area theorem of Hawking?
- What is the exact connection between the macroscopic notion of entropy discussed here, and the microscopic notion?
- How can the method of Wald be extended to dynamical black holes?
- What is the interpretation of the Gauss-Bonnet term which alters the entropy, but not the field equations?

It can be concluded that Wald's formalism gives a remarkable connection between the geometry of the black hole and it's thermodynamic aspects using diffeomorphism invariance which in it's application region coincides with other formalisms developed. Hopefully this thesis could help by answering the questions above, and other open questions.

# Appendix A

# The transport term

In chapter five we saw two different kinds of variations which will be repeated here for convenience:

$$\begin{aligned}\delta\psi &= \psi^{'}(x) - \psi(x) \\ &= \lim_{t\to 0}\frac{1}{t}[\phi_t^*\psi(\phi_t(x)) - \psi(x)]\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}\tilde{\delta}\psi &= \psi^{'}(x^{'}) - \psi(x) \\ &= \lim_{t\to 0}\frac{1}{t}[\phi_t^*\psi(x) - \psi(x)]\end{aligned} \tag{A.2}$$

together with $\delta x^\mu = x^{'\mu} - x^\mu$. It is clear that $[\delta, \partial_\mu] = 0$ because the coordinates are not changed with $\delta$. However, for $\tilde{\delta}$ this is not true. In deriving the transport term we work up to first order and use Taylor's theorem. First we note that

$$\begin{aligned}\delta(\partial_\mu\psi) &= \frac{\partial\psi^{'}(x)}{\partial x^\mu} - \frac{\partial\psi(x)}{\partial x^\mu} \\ &= \partial_\mu(\delta\psi) \\ \tilde{\delta}(\partial_\mu\psi) &= \frac{\partial\psi^{'}(x^{'})}{\partial x^{'\mu}} - \frac{\partial\psi(x)}{\partial x^\mu}\end{aligned} \tag{A.3}$$

Now consider

$$\begin{aligned}\psi^{'}(x^{'}) &= \psi^{'}(x + \delta x) \\ &\approx \psi^{'}(x) + \delta x^\mu\partial_\mu\psi^{'}(x)\end{aligned} \tag{A.4}$$

Combining this with

$$\begin{aligned}\partial_\mu\psi^{'}(x) &= \partial_\mu\psi(x) + \partial_\mu\delta\psi \\ &\approx \partial_\mu\psi(x)\end{aligned} \tag{A.5}$$

where $\partial_\mu\delta\psi \approx 0$ is used, gives

$$\psi^{'}(x^{'}) \approx \psi^{'}(x) + \delta x^\mu\partial_\mu\psi(x) \tag{A.6}$$

This in turn can be written as

$$\psi^{'}(x^{'}) = \delta\psi + \psi(x) + \delta x^{\mu}\partial_{\mu}\psi(x) \tag{A.7}$$

and so

$$\boxed{\tilde{\delta}\psi = \delta\psi + \delta x^{\mu}\partial_{\mu}\psi(x)} \tag{A.8}$$

where $\delta x^{\mu}\partial_{\mu}\psi(x)$ is the transport term. The variation $\tilde{\delta}\psi$ is separated into a part which accounts for the change in the field and a part which accounts for the change in the coordinates. Notice that a change in coordinates means actually that the physical point is not moved while in keeping the coordinates fixed the physical point *is* moved. If the derivative $\partial^{'}_{\mu}$ is rewritten in terms of $\partial_{\mu}$ up to first order, an explicit expression for the commutator $[\tilde{\delta}, \partial_{\mu}]$ can be found[20]:

$$[\tilde{\delta}, \partial_{\mu}]\psi(x) = -\partial_{\mu}(\delta x^{\nu})\partial_{\nu}\psi(x) \tag{A.9}$$

So as long as the variation $\delta x^{\nu}$ is constant along the space-time region of interest, one is allowed to commute these two operators.

# Appendix B

# The Raychaudhuri equation

The Raychaudhuri equation is a very important equation for proving the singularity theorems and the area theorem, and a derivation is given below. It describes the behaviour of congruences in $\Omega \subset M$ with tangent vector $t$ and connecting vector $s$. So with this, the space-time curvature is investigated via a congruence of geodesics constructed on it. The equation can be derived for timelike ($t^2 = -1$) and null-geodesics ($t^2 = 0$); we will consider only the latter here because we want to apply them to the null generators of hypersurfaces. The relevant situation is sketched in figure (B.9): The vectors $\partial/\partial\alpha$ and $\partial/\partial\lambda$

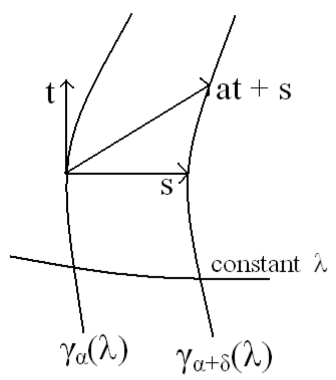


Figure B.1: *Two neighbouring geodesics*

commute, so $[t, s] = \pounds_t s = 0$. If we remember that we can replace partial derivatives by covariant ones in Lie-derivatives, this results in $[t^\mu \nabla_\mu s^\nu - s^\mu \nabla_\mu t^\nu] = 0$. If we define $B_{\mu\nu} = \nabla_\mu t_\nu$, we obtain

$$B_{\mu\nu} \equiv \nabla_\mu t_\nu \quad \rightarrow \quad t^\nu \nabla_\nu s^\mu = B^\mu{}_\nu s^\nu \tag{B.1}$$

So this gives us a nice interpretation of $B_{\mu\nu}$; if $B_{\mu\nu} = 0$ we see that $s$ is parallel-transported along $\gamma_\alpha(\lambda)$, and otherwise it isn't. Second we note that the connecting $s$ vector is not uniquely defined; we see that the vector $at + s$

with $a \in \mathbb{R}$ also points from $\gamma_\alpha(\lambda)$ to $\gamma_{\alpha+\delta}(\lambda)$. If $\gamma_\alpha$ would be timelike, we could impose the condition $t_\mu s^\mu = 0$ and consider the 3-dimensional space spanned by all the vectors orthogonal to $t$. However, tangent vectors of a null geodesic obey $t^2 = 0$ and so $t$ is also included in this orthogonal space. To fix the choice of vectors, we introduce a vector $n$ for which

$$n^2 = 0, \qquad n_\mu t^\mu = -1 \quad \forall\lambda \tag{B.2}$$

This is merely one convenient choice. All the displacement vectors in the congruence have a component in the $n$-direction which is not orthogonal to $t$. Because these conditions are imposed everywhere, we have $\nabla_\mu(n^2) = \nabla_\mu(n_\rho t^\rho) = 0$.

Now the vectors $\eta$ which are both orthogonal to $n$ and $t$ span a two-dimensional spacelike subspace $\Sigma_\perp$ with positive definit metric $h_{\mu\nu}$. If the geodesics are the generator of the null-surface $\Sigma$, then $\Sigma_\perp \subset \Sigma$. Now the metric $g_{\mu\nu}$ can be written as[1]

$$g_{\mu\nu} = h_{\mu\nu} - 2t_{(\mu}n_{\nu)} \tag{B.3}$$

The 'metric' $h_{\mu\nu}$ has to satisfy $h_{\mu\nu} = h_{(\mu\nu)}$ and $h_{\mu\nu}t^\mu = h_{\mu\nu}n^\mu = 0$. That's why the parentheses are used; the tensor is degenerate in $\Sigma_\perp$, a property which an honest metric shouldn't have. The relevance of this 'metric' is that we can define a projection operator with it which projects vectors unto $\Sigma_\perp$:

$$P^\mu_{\ \nu} \equiv g^{\mu\rho}h_{\rho\nu} = \delta^\mu_\nu + n^\mu t_\nu + t^\mu n_\nu \tag{B.4}$$

Ofcourse this operator should obey $P^\mu_{\ \nu}\eta^\nu = \eta^\mu$. With some algebra it can be shown that if $\eta \in \Sigma_\perp$ for some $\lambda$, it stays there: $\{\eta \in \Sigma_\perp \ \forall\lambda\}$.

Having defined all this, we project $B_{\mu\nu}$ unto $\Sigma_\perp$, and indicate this projection with a hat:

$$\hat{B}_{\rho\sigma} \equiv P^\mu_{\ \rho}P^\nu_{\ \sigma}B_{\mu\nu} \tag{B.5}$$

We can decompose it in a trace, an antisymmetric part and a symmetric part, remembering that $\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}$ for the tangent space of four dimensional second rank tensors:

$$\begin{aligned} \theta &\equiv \hat{B}^\mu_{\ \mu}, \quad \text{(the expansion)} \\ \hat{\sigma}_{\mu\nu} &\equiv \hat{B}_{(\mu\nu)} - \frac{1}{2}\theta h_{\mu\nu}, \quad \text{(the shear)} \\ \hat{\omega}_{\mu\nu} &\equiv \hat{B}_{[\mu\nu]}, \quad \text{(the twist)} \end{aligned} \tag{B.6}$$

With this we decompose $\hat{B}_{\mu\nu}$ as

$$\hat{B}_{\mu\nu} = \frac{1}{2}\theta h_{\mu\nu} + \hat{\sigma}_{\mu\nu} + \hat{\omega}_{\mu\nu} \tag{B.7}$$

By using Frobenius' theorem we see that the tangent vectors $t$ are only hypersurface orthogonal iff $\hat{\omega}_{\mu\nu} = 0$. The tensors $\hat{\sigma}_{\mu\nu}$ and $\hat{\omega}_{\mu\nu}$ are both orthogonal to $t$ and are thus purely spatial. We gave these irreducible components names; $\theta$ is a measure for the expansion of the surrounding geodesics, $\hat{\omega}_{\mu\nu}$ measures

[1] It's written such that $g_{\mu\nu}\eta^\nu = h_{\mu\nu}\eta^\nu$ and $g_{\mu\nu}t^\nu = n_\mu + h_{\mu\nu}t^\nu$.

the rotation of these geodesics[2] and $\hat{\sigma}_{\mu\nu}$ measures how geometric forms are deformed being Lie-transported along $t$. That last one is a little harder to imagine.

By now we are close to actually deriving Raychaudhuri's equation. For that we look at how the expansion changes with $\lambda$: we are interested in calculating $\frac{d\theta}{d\lambda} = t^\rho \nabla_\rho \theta$. For that we calculate $t^\rho \nabla_\rho B_{\mu\nu}$:

$$\begin{aligned} t^\rho \nabla_\rho B_{\mu\nu} &= t^\rho \nabla_\nu \nabla_\rho t_\mu + R_{\mu\nu\rho}{}^\sigma t^\rho t_\sigma \\ &= \nabla_\nu (t^\rho \nabla_\rho t_\mu) - (\nabla_\nu t^\rho)(\nabla_\rho t_\mu) + R_{\mu\nu\rho}{}^\sigma t^\rho t_\sigma \\ &= -B^\rho{}_\nu B_{\mu\rho} + R_{\mu\nu\rho}{}^\sigma t^\rho t_\sigma \end{aligned} \tag{B.8}$$

If this equation is projected on $\Sigma_\perp$, the trace of the last equation with the decomposition (B.7) gives us finally

$$\boxed{\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} - R_{\mu\nu}t^\mu t^\nu} \tag{B.9}$$

If this term is negative, then the geodesics converge and vice versa. The last term $R_{\mu\nu}t^\mu t^\nu$ gives us directly the effect of gravity on our geodesics. Now take a look at eq.(4.59). The Einstein equations tell us that the last term in the Raychaudhuri equation is always negative; gravity attracts!

A justification for $\theta$ being some sort of expansion can be given by the following argument. Assume that along the curve $\gamma$ with tangent vector $t$ we drag a basis $X^{(\mu)}$. This implies that $\pounds_t X = 0$ for each basis vector $X$. This results in $\dot{X}^\mu = (\nabla_\nu X^\mu)t^\nu = (\nabla_\nu t^\mu)X^\mu$. So if we regard $[X] = [X^0 X^1 X^2 X^3]$ as a matrix with each column given by a vector $X$, we can write down the equation $[\dot{X}] = B[X]$, where $B_{\mu\nu} = \nabla_\mu t_\nu$. Now consider the trace of $B$, $trace(B) = trace([\dot{X}][X]^{-1}) = \frac{1}{|X|}|\dot{X}|$. This gives an indication how the basis volume changes along the geodesic, and thus a notion of the convergence and divergence of the geodesics.

To conclude, we look at the consequences of eq.(B.9) for hypersurface orthogonal null geodesics. In this case we already argued that $\hat{\omega}_{\mu\nu} = 0$. The term $\hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu}$ is positive, and so we can write down the following inequality:

$$\begin{aligned} \frac{d\theta}{d\lambda} + \frac{1}{2}\theta^2 &\leq 0, \text{ so} \\ \frac{d}{d\lambda}\left[\frac{1}{\theta}\right] &\geq \frac{1}{2} \\ \rightarrow \theta^{-1}(\lambda) &\geq \frac{1}{\theta_0} + \frac{1}{2}\lambda \end{aligned} \tag{B.10}$$

Here $\theta_0$ denotes the initial value of the expansion: $\theta_0 = \theta(\lambda_i)$ if $\lambda \in [\lambda_i, \lambda_f]$. The interesting implication of this inequality is the case for a congruence which converges initially, $\theta_0 \leq 0$. The expansion $\theta$ will then take the value $-\infty$ within proper time $\lambda \leq 2/|\theta_0|$. This implies the existence of a conjugate point before this proper time is reached.

[2]We saw earlier that $B_{\mu\nu}$ gives an indication of the failure of $s$ being parallelly transported along geodesics.

# Appendix C

# Useful geometric identities

Here some useful geometric identities concerning curvature and variations are listed:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\rho\nu}$$

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$$

$$\nabla_\mu X^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} X^\mu)$$

$$\pounds_\xi X^{\mu\cdots}_{\nu\cdots} = \xi^\lambda \nabla_\lambda X^{\mu\cdots}_{\nu\cdots} - X^{\lambda\cdots}_{\nu\cdots} \nabla_\lambda \xi^\mu - \cdots + X^{\mu\cdots}_{\lambda\cdots} \nabla_\nu \xi^\lambda + \cdots$$

$$\delta R_{\mu\nu} = \nabla_\alpha \delta\Gamma^{\alpha}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\alpha}_{\mu\alpha}$$

$$\delta R_{\mu\nu\rho\sigma} = R_{\mu\nu[\rho}{}^{\lambda} \delta g_{\sigma]\lambda} + 2\nabla_{[\mu}\nabla_{[\rho}\delta g_{\sigma]\nu]} = \nabla_\rho \delta\Gamma_{\mu\nu\sigma} - \nabla_\sigma \delta\Gamma_{\mu\nu\rho}$$

$$\delta\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\nabla_\mu \delta g_{\nu\rho} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu})$$

$$\delta R = g^{\mu\nu}\Big(\nabla_\alpha \delta\Gamma^{\alpha}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\alpha}_{\mu\alpha}\Big) - R^{\mu\nu}\delta g_{\mu\nu}$$

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha \Big(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\lambda}_{\mu\lambda}\Big)$$

$$R^{\mu\nu}\delta R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}\Big(\nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\mu\lambda}\Big)$$

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By making the pictures (6.3) and (6.4), the author was highly inspired by [23]