

Thermodynamics of Born-Infeld Black Holes

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Chapter 1

Introduction

1.1 Black Holes in Theoretical Physics

Black holes have been objects of astrophysical interest since it was shown in 1931 by Chandrasekhar that they are the inevitable outcome of complete gravitational collapse of a massive star or a collision of stars [1]. Though in theoretical physics black holes were not immediately regarded to be very interesting.

The black hole paradigm was that of a perfect one-way membrane with simple point charge potentials. No information can ever pass such a membrane from the inside and in the presence of an electromagnetic field, the black hole is characterized completely by three scalar quantities: its mass, its charge and its angular momentum. This can be seen in the following way. During a black hole formation process matter and spacetime behave in a heavily dynamical way. The metric and the matter fields evolve very rapidly in time. In the course of time these movements will damp and spacetime will come to rest. In this way eventually each black hole will evolve into a stationary state [2]. All such stationary electrically charged black holes can be described by the class of Kerr-Newman metrics [3] which are characterized by the above three quantities. There is no room for any (further) dependence on the presumed internals of the black hole and we conclude that black holes act as (spinning) point charge source consisting of a massive one-way membrane. This result is called the 'no hair theorem'.

From the theoretical point of view this black hole concept does not raise a lot of burning questions. Although the black hole concept allows for some new fundamental processes in physics, its implications are quite isolated: There is the peculiar prediction that a part of spacetime can be cut off from the rest of the universe in the process of black hole formation. Further there is the new phenomenon that particles can disappear from our causal world by falling into a black hole. The last striking prediction is that an infinite curvature should occur in the center of the black hole. The first two phenomena are perfectly well described (and understood) by general relativity but the infinity bothers since infinities never have been measured. However general relativity did not predict the infinity of a black hole to be measurable. The construction of black hole concept guarantees the infinity is hidden behind the event horizon and there is no experiment that tells us anything about what is going on behind that

horizon. The conjecture that infinities of physical solutions should be shielded by a horizon is known under the name 'cosmic censorship conjecture'.

In theoretical high energy physics interest in black hole physics increased after the discovery that, due to quantum processes, black holes emit radiation [4]. This Hawking radiation has the Planck spectrum which is characteristic for a black body radiating at a certain temperature. In this way black holes turned to be subject to the laws of thermodynamics. In other, more familiar, settings the theory of statistical mechanics explained thermodynamic behavior very successfully in terms of collective phenomena appearing from the dynamics of some microscopic entities. It seemed reasonable to believe that within a black hole the situation is similar. The black hole is conjectured to have an internal structure, which enables us to think about what is going on inside a black hole.

The common believe is that in the black hole interior known physics is not completely accurate. There is a singularity predicted which cannot be ignored anymore once the black hole interior becomes physically relevant. In black hole physics small distances and high energy densities meet. Both quantum- and gravitational phenomena are involved but it is impossible to use general relativity and quantum mechanics together. They are both as what Albert Einstein called "Theories of principle". With this he referred to a theory that sets up the framework that makes a formulation of the laws of nature possible. One way out of this problem would be the invention of a unifying theory. That is a single theory substituting for quantum mechanics and general relativity at the same time. Various candidates for such a theory have been proposed, and many are still being investigated.

What makes black hole theory so interesting is that it may provide insight in the processes of establishing a unified theory of quantum gravity. On the purely theoretical side, black holes can act as a benchmark for candidate theories. One criterion is that they should not produce measurable infinities. Experiments with black holes seem to be out of the question. We have no sources for black holes available and the situation for direct black hole observations does not give much hope either. Hawking radiation is far too weak to observe. When it comes to indirect observations, there is more hope. Many astronomical observations suggest the existence of black holes. However, an unambiguous proof for this statement is still lacking. Neither event horizons nor intrinsic curvature singularities have been observed by means of astronomical techniques [5].

1.2 Black Hole Entropy

Now that the context of black hole research in theoretical physics has been sketched, it is time to zoom in at the subject of this text, black hole entropy. Entropy is a measure for disorder of a system at microscopic level. With a single state of classical fields, or quantum fields, no entropy is associated. When black holes are simply solutions of Einstein's equations they should not process entropy. But as argued above, black holes do show thermodynamic behavior and they are likely to have internal structure which is simply not completely governed by general relativity.

Thermodynamics is an important guide in the search for a model to describe black hole physics. We can not compare two black hole models just on their internal micro dynamics. The micro dynamics is the basic assumption of the

model. What can be compared is thermodynamic behavior. If we have a classical theory of gravity that correctly describes spacetime outside a black hole then the quantum gravity model of the black hole should predict the same thermodynamics and vice versa. For this reason thermodynamics is still a pillar of black hole research. We also see there are two fronts in the field of black hole thermodynamics. The microscopic approach aims to model the internal (quantum) dynamics of black holes while the other branch investigates black holes in the setting of classical field theories of space and time; theories like general relativity which seem, on basis of observations and experiment, to give a quite accurate description of space and time at more moderate energy scales. In this text the classical perspective is adopted.

We investigate the entropy of black holes in connection with a specific classical field theory. Born and Infeld invented in 1931 a nonlinear theory of electrodynamics which gained new interest a few years ago, after it was shown that string theory predicts Born-Infeld correction terms to the Maxwell Lagrangian of electromagnetism. Their theory is introduced in chapter 4. Here we try to find an expression for the entropy of a rotating Born-Infeld charged black hole. A static black hole solution with a Born-Infeld charge is known, but the presumed rotating equivalent has not been found. Still it might be possible to determine the entropy of such a black hole on forehand. We introduce three tools that we use in later attempts to find this entropy expression. These are Walds entropy formula and Sen's entropy formalism. We discuss also the algorithm of Janis and Newman, that can be used to generate rotating black hole solutions from static ones. We attempt to apply these techniques directly, but each time this proves to be very hard. As an alternative we work on a simpler problem and we try to find an analogy of the Newman-Janis algorithm in the near-horizon setting. This is covered in chapters 5 to 7. Chapter 2 offers a short overview of classical black hole mechanics while chapter 3 introduces the specific techniques needed in the later chapters. The first two chapters are supplemented by Appendix A, which contains a brief summary on general relativity and related theories.

Chapter 2

Black Hole Mechanics

Black hole mechanics covers everything related to the dynamics of black hole horizons. Four laws play a central role. They are quite similar to the laws of thermodynamics when the entropy, temperature and pressure are replaced by quantities of the black hole horizon: area, surface gravity and angular momentum. Striking is that general relativity produces these laws without making any reference to micro states. A priori black hole thermodynamics has nothing to do with thermodynamics at all. But the parallel is probably no coincidence. A variety of studies provide support for a micro interpretation of the laws of black hole mechanics. This is why the terms "black hole mechanics" and "black hole thermodynamics" are commonly interchanged.

The first micro dynamical interpretation came from Stephen Hawking. In 1974 Hawking discovered that black holes radiate [4]. As a result of quantum particle creation effects, a black hole radiates all species of particles with a perfect black hole body spectrum, at temperature

$$T = \frac{\kappa}{2\pi}. \quad (2.1)$$

The surface gravity κ is not just a quantity analogous to temperature, (2.1) identifies it as the physical temperature of the black hole. A curious fact is that Hawking's derivation does not make use of any gravitational field equation. It merely regards the behavior of quantized fields outside a black hole configuration. The result is thus quite general. Since Hawking's discovery many more connections between black hole mechanics and thermodynamics have been established. For example, in string theory, for some classes of black holes solutions, the entropy can be derived from the micro statistical dynamics. However, this text deals only with classical black hole mechanics, for which this chapter provides an introduction.

2.1 Classical Theories of Gravity

Historically, the field of classical black hole mechanics has its origins in the theory of general relativity. In the course of time lots of alternatives and generalisations for this theory have been proposed. The context of classical black hole mechanics has broadened to a wide class of diffeomorphism invariant theories. General relativity, as proposed by Einstein, being just one of them.

Einstein searched for theories obeying both Mach's principle and the equivalence principle¹. He found a large class of theories that satisfied these, and, motivated by his strong personal belief in aesthetics, he argued that the right theory of gravitation should be the simplest of them. This theory we call general relativity. Up till now experiment did not disagree on the choice of Einstein, but with precision of experiment ever increasing, there may come a moment at which the experiment outruns the accuracy of general relativity.

These days, Einstein's condition that a theory of gravity should satisfy both Mach's principle and the equivalence principle has been translated into the language of geometry. In a natural way this results in a well defined class of classical theories of physics: the class of diffeomorphism covariant theories². These theories represent spacetime as a set (M, g_{ab}) consisting of a manifold M and metric g_{ab} defined on M . The mathematical construction of a manifold is such that on any point $p \in M$ there exist a subset $U \subset M$, $p \in U$ and a function $\psi : U \rightarrow \mathbb{R}^d$ which is a diffeomorphism. The natural number d is independent of choice of the point p on the manifold, and is referred to as the dimension of the manifold. Note that the definition of manifold can be rephrased to the statement that, locally, M is diffeomorphic with \mathbb{R}^d . The metric is a dynamical field on the manifold. In general relativity its behavior is governed by the Einstein equation which also incorporates the stress-energy tensors of charge- and matter fields.

2.2 Black Hole spacetime

In classical black hole mechanics, a black hole is a region of space from which nothing can escape. Using the language of sets, the black hole region \mathcal{B} of an asymptotically flat spacetime (M, g_{ab}) is defined as

$$\mathcal{B} = M - I^-(\mathcal{I}^+), \quad (2.2)$$

where \mathcal{I}^+ denotes null future infinity and I^- denotes the chronological past³. The event horizon \mathcal{H} is defined as the boundary of \mathcal{B} and is a null hyper surface, and the area of the horizon \mathcal{H} will never increase [6].

As with most physics, black holes become easier to analyze when they have more symmetry. An important symmetry is stationarity since a lot of theorems on black holes require this property. If an asymptotically flat spacetime (M, g_{ab}) contains a black hole region \mathcal{B} , then \mathcal{B} is said to be stationary if there exists a one-parameter group of isometries on (M, g_{ab}) generated by a Killing vector field ξ^a which is timelike. The black hole is called static, if in addition ξ^a is hyper surface orthogonal.

In an attempt to understand these definitions, let us introduce a coordinate system in which the first coordinate t is the evolution parameter of the vector field ξ^a . The latter is timelike, so the coordinate t is a time coordinate. The killing condition on the vector field ξ , $\mathcal{L}_\xi g_{ab} = 0$ can be written as $\frac{\partial}{\partial t} g_{\mu\nu} = 0$ in this coordinate system.⁴ The interpretation is clear: the metric is time independent. To investigate the significance of the condition of hypersurface

¹These principles are reviewed in Appendix A.1.1

²More on the relation between the two principles and diffeomorphism invariance can be found in Appendix A.1.1

³Appendix A.1.3 deals with this notation

⁴The Lie-derivative \mathcal{L} is explained in appendix A.2.4.

orthogonality, we call the hyper surface that is orthogonal to $t^a \Sigma$. By Frobenius theorem the orthogonality is equivalent to the statement

$$\xi_{[a} \nabla_b \xi_{c]} \Big|_{\Sigma} = 0. \quad (2.3)$$

In a neighborhood of Σ every point lies on a single geodesic orbit starting from Σ . This can be used to define so called "Gaussian normal coordinates" in this neighborhood. Consider an arbitrary coordinate system on Σ . This comprises a set of smooth function $x^\mu : \Sigma \rightarrow \mathbb{R}$. Since each point p in the neighborhood lies on exactly one geodesic curve passing through Σ , p can be specified by the unique combination of the final time parameter and the starting point on Σ of the geodesic arriving at p . This gives rise to the family of hyper surfaces Σ_t defined as the collection of points whose time coordinates have the value t . This set is identical to the image of Σ under the flow the isometry ϕ_t generated by the vectorfield ξ^a . In these coordinates the line element takes the form

$$ds^2 = -V^2(x^1, x^2, x^3) dt^2 + \sum_{\mu, \nu=1}^3 h_{\mu\nu}(x^1, x^2, x^3) dx^\mu dx^\nu, \quad (2.4)$$

where $V^2 = -\xi_a \xi^a$ is known as the redshift factor (squared). The metric is independent of time, just as expected. Notice moreover that there appear no cross terms $dt^\mu dx^\nu$. Under an isometry all inner products are conserved. We know that Σ is orthogonal to ξ^a and thus so should be Σ_t . In summary, a static spacetime does not allow for any time evolution at all, whereas a stationary spacetime does allow for rotations.

2.3 Killing Horizons and Surface Gravity

The event horizon \mathcal{H} as defined in the previous section and in A.1 involves a global notion of spacetime. This contrasts with the killing horizon \mathcal{K} , which is defined as

a null hypersurface whose generators coincide with the orbits of a one-parameter group of isometries.

The null surface and the killing field that should be present in a neighborhood of that null surface are local entities. A hypersurface is called a null manifold if and only if its generators⁵ are null vectors. The definition of Killing horizon can be rephrased as:

Consider a Killing field ξ on M and let K a set of points where ξ is null (thus $\xi^a \xi_a = 0$). A connected subset $\mathcal{K} \subset K$ which is a null surface is called a Killing horizon.

Carter proved in 1973 that \mathcal{K} and \mathcal{H} coincide for static and axisymmetric stationary black holes without demanding any restrictions on the field equations. For a static black hole the Killing horizon is a null surface with respect to the

⁵ With "the generators of a null surface" (the set of) all vectors orthogonal to the surface is meant.

Killing field t^a . For a stationary axisymmetric black holes Carter showed the existence of a Killing field ξ which has the form

$$\xi^a = t^a + \Omega\phi^a, \quad (2.5)$$

and which is normal to both the event horizon and the Killing horizon [7]. The constant Ω is called the angular velocity of the horizon. Statements relating \mathcal{H} with \mathcal{K} are called rigidity theorems. The plural used here suggests Carters rigidity theorem is not unique. Indeed there is a second rigidity theorem known, that is due to Hawking [8]. He proved that in vacuum or electro vacuum of general relativity the event horizon of any (not necessary axisymmetric) stationary black hole must be a Killing horizon. Consequently if t^a fails to be normal to the horizon, then there must exist an additional Killing field, ξ^a , which is normal to the horizon. This implies the existence of another killing vector $\Omega\phi^a = \xi^a - t^a$, comprising an axial symmetry. When one assumes general relativity, this statement is more general then that of Carter. According to Hawking every stationary black hole is either axisymmetric or static [9], [10], [11].

On any Killing horizon \mathcal{K} with a corresponding Killing vector field ξ^a , we have $\xi^a\xi_a = 0$, and so $\xi_a\nabla^a(\xi^b\xi_b) = 0$. The vector field $\nabla^a(\xi^b\xi_b)$ should be of the form

$$\nabla^a(\xi^b\xi_b) = -2\kappa\xi^a, \quad (2.6)$$

since the null subspace of the tangent space is one-dimensional on a manifold with Lorentz signature. The function $\kappa : \mathcal{K} \rightarrow \mathbb{R}$ is called surface gravity. The interpretation justifying this name can be given after we explored both the function κ and the construction of surface gravity. Notice that κ must be constant along each null geodesic generator of \mathcal{K} , while it may vary from generator to generator. Using Killing's equation once, and using equation 2.6 twice, we obtain

$$\begin{aligned} -4\xi_a\nabla^a\xi_b\xi_c\nabla^c\xi^b &= 4\xi_a\nabla^a(\xi_b\xi_c\nabla^b\xi^c) = 2\xi_a\nabla^a(\xi_b\nabla^b(\xi_c\xi^c)) \quad (2.7) \\ &= -2\kappa\xi_a\nabla^a(\xi_b\xi^b) = 4\kappa^2\xi^a\xi_a, \end{aligned}$$

and thus κ has to satisfy the relation

$$\kappa^2 = -\left. \frac{\nabla_\xi\xi^a\nabla_\xi\xi_a}{\xi^a\xi_a} \right|_{\mathcal{K}}. \quad (2.8)$$

Before making the identification with surface gravity, we have to define what surface gravity actually is. Surface gravity is the force that is needed to keep some object in place at the black hole horizon. In a static spacetime this notion is well defined. There exists a timelike Killing vector field ξ^a , representing time translation and this vector field is normal to the horizon of the black hole. When no force is exerted, the object will follow an orbit of the Killing field ξ^a . The acceleration of such an orbit is

$$a^b = \frac{\xi^a\nabla_a\xi^b}{-\xi^c\xi_c} = \frac{\xi^a\nabla_a\xi^b}{V^2}, \quad (2.9)$$

where the redshift factor $V = \sqrt{-\xi^c\xi_c}$ appears when normalizing the Killing vector field such that the time parameter of the normalized Killing vector flow is identical to the proper time of an observer following the flow. Now we expressed

gravitation in terms of force, we like to have a connection between force and acceleration, but in curved spacetime associating a force with acceleration is not as trivial as it is in flat space. The lack of a fixed background that makes it impossible to design a useful notion of local strength of the gravitational field. This is explained in Appendix A.1.1. In a space that is asymptotically flat this problem can be circumvented. The global structure allows constructions that replace the preferred local reference which is lacking in background independent theories. For asymptotically flat spaces, the structure of spacetime at asymptotic infinity is identical to the structure of flat spacetime in special relativity. In principle an observer residing in this part of spacetime can exert a force at an object near the black hole. Heuristically he or she uses a long rope or stick and starts pulling. Now the surface gravity can be defined as the force that the observer will need to exert in order to keep a test particle in its position. Of course we take the limit that the object is arbitrary close to the black hole's horizon. In curved spacetime for the energy and time related vector components, a redshift factor $V = \sqrt{-\xi^c \xi_c}$ appears. The force which should be exerted at asymptotic infinity picks up a redshift factor and finally the surface gravity is defined in an unambiguous way through $\kappa = \lim_{p \rightarrow \infty} V \sqrt{a^b a_b}|_p$. Plugging in the acceleration (2.9), we get

$$\kappa^2 = \left. \frac{\xi^a \nabla_a \xi^b \xi^c \nabla_c \xi_b}{V} \right|_p. \quad (2.10)$$

This is formula 2.8, and that justifies the identification of κ as surface gravity.

2.4 Bifurcate Killing Horizons

Closely related to Killing horizons are bifurcate Killing horizons:

A 2-dimensional space like submanifold \mathcal{C} that is the intersection of a pair of hypersurfaces $\mathcal{K}_A \neq \mathcal{K}_B$ is called a bifurcate Killing horizon if \mathcal{K}_A and \mathcal{K}_B are Killing surfaces with respect to the same Killing field ξ^a .

The bifurcation surface \mathcal{C} forms the border of two Killing horizons associated with the same vector field. An important result is that on \mathcal{C} , the Killing field ξ^a vanishes. Being a null surface, the Killing field should be generated by its tangent vector field as pointed out in Appendix A.2.6. The tangent vector field is the Killing field ξ^a . Following the reasoning of both the Appendix and the previous section we see that ξ^a should satisfy the geodesic equation

$$\xi^b \nabla_b \xi^a = \kappa \xi^a. \quad (2.11)$$

At the bifurcation surface the vector at the right-hand side of eq 2.11 should be tangent to the generating curves of both \mathcal{K}_A and \mathcal{K}_B . However the intersection \mathcal{C} , being a space like surface, does not contain any null vectors in its tangent space. Nonetheless equation 2.11 still holds for both Killing horizons. The only way out is

$$\kappa|_{\mathcal{C}} = 0, \quad (2.12)$$

the surface gravity on the bifurcate horizon is zero.

2.5 The Laws of Black Hole Mechanics

In 1973 Bardeen, Carter and Hawking published four laws concerning the dynamics of black holes in general relativity [12]. They pointed out a striking similarity with the laws of thermodynamics. In the same year an article about the similarities between black hole physics and thermodynamics based on information theory appeared from the hand of Jacob Bekenstein. He was the first to conjecture that black hole entropy is proportional to the area of its event horizon divided by the Planck area [13]. We can also arrive at this conjecture from the laws of black hole mechanics.

Law 0 (The zeroth law). *The surface gravity of a stationary black hole is constant over the event horizon.*

This law parallels the zeroth law of thermodynamics which states the temperature is constant throughout a system in thermodynamic equilibrium.

The first law relates changes in the energy E of the black hole with changes in its area A , its angular momentum J and its charge Q .

Law 1 (The first law).

$$dE = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ. \quad (2.13)$$

The constants have the following interpretations: κ equals surface gravity as defined in the previous section, Ω represents angular velocity and Φ is the electric potential. This law is an analog of the first law of thermodynamics which is usually stated as $dE = TdS - PdV$.

The second law concerns the surface area A of the event horizon.

Law 2 (The second law).

$$dA \geq 0. \quad (2.14)$$

The area never decreases with time. Neither does the entropy in thermodynamics according to its second law. The third law of black hole mechanics states that

Law 3 (Third Law of black hole mechanics). *The surface gravity κ cannot be zero.*

This resembles weak form of the third law of thermodynamics which states that it is impossible to reach absolute zero temperature in any system. It is not in accordance with the strong version of the third law which claims that the entropy approaches zero when the temperature is brought to zero.

We see that the four laws of black hole mechanics are (almost) identical to the four laws of thermodynamics when we make the identification

$$E = E, T = \frac{\kappa}{2\pi}, S = \frac{A}{4}. \quad (2.15)$$

Quite surprisingly this black hole mechanical temperature is precisely the Hawking temperature we started this chapter with, (2.1). After the articles of Bekenstein and Bardeen *et al* it took only a year before Hawking published the crucial discovery that black holes radiate.

Chapter 3

Techniques

3.1 Wald's Entropy Formula

In the nineties Robert Wald published a number of articles in which he constructed a firm geometrical foundation for the first law of black hole mechanics [14], [15], [16]. The basics of his approach are described in an article together with Lee in 1989. They introduce a Lagrangian formalism that is compatible with diffeomorphism invariant theories, including general relativity [17]. Four years later a new derivation of the first law appeared [14]. In this paper Wald shows that black hole entropy can always be expressed as a local geometric quantity integrated over a space like cross-section of the horizon. His expression for (black hole mechanical) entropy

$$S = -2\pi \oint_{\Sigma} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd} \sqrt{|h|}, \quad (3.1)$$

, which is known as "Wald's formula". The integration domain Σ can be any spacelike two-dimensional cross section of the event horizon.

3.1.1 Outline of the derivation

Lagrangian Formulation and Noether Current

Essential to Wald's approach is the Noether current that can be associated with each diffeomorphism on the n -dimensional spacetime manifold M . As explained in Appendix A.2.1, a diffeomorphism induces a transformation on a manifold. Such transformations are like coordinate transformations: the description of the physical system changes while the physics is unaffected. At least this is how we like theories of nature to behave. Models that have this property are called diffeomorphism invariant. One of the consequences is that the Lagrangian of such a theory should be diffeomorphism invariant. The Lagrangian is a scalar function on the spacetime M , but it also depends on a collection of tensor fields on M . It is a tensor density depending on some static fields γ and some dynamical fields ϕ . A tensor density of weight one is equivalent to a tensor field

¹. Hence we may write

$$\mathbf{L} = L\epsilon; \quad (3.2)$$

where ϵ is the natural volume element. The volume element and the tensor field \mathbf{L} , which is an n -form, together constitute the tensor density (L, ϵ) . The product of the two is a tensor field \mathbf{L} on M ; this is an n -form. In [17] and [15] the dynamical (tensor)fields are shown to transform according to

$$\delta\mathbf{L} = \mathbf{E}\delta\phi + d\Theta, \quad (3.3)$$

under a first order transformation of the dynamical fields ϕ . When assuming compact support for these fields, the equations of motions can be written as $\mathbf{E} = 0$ ². The dynamical fields ψ are said to be on shell whenever this equality holds. If we assume the above transformation law and make sure that δ vanishes at the boundary (compact support) then the current Θ is conserved, i.e.

$$d\Theta(\phi, \delta\phi, \gamma) = 0, \quad (3.4)$$

as long as the dynamical fields ψ are on shell. We use this result and we assume that the variation $\delta\phi$ of the dynamical fields ϕ is induced by a diffeomorphism. Under diffeomorphisms the set of on-shell field configurations is mapped onto itself since by assumption the theory under consideration is diffeomorphism invariant. Further we know there is a one-to-one relation between diffeomorphisms and smooth vectorflows on the manifold M . Combining this we see for any smooth vector field ξ the quantity $\Theta(\phi, \mathcal{L}_\xi\phi)|_{E(\phi)=0}$ is conserved,

$$d\Theta(\phi, \mathcal{L}_\xi\phi)_{E(\phi)} = 0. \quad (3.5)$$

The associated Lie-derivative of the Lagrangian can be written as a total derivative [17]

$$\mathcal{L}_\xi\mathbf{L} = \mathbf{E}\xi\phi + d\Theta(\phi, \mathcal{L}_\xi\phi), \quad (3.6)$$

and when we define the Noether current $(n-1)$ -form as

$$\mathbf{J}^a = \Theta^a(\mathcal{L}_\xi\psi) - \xi^a L, \quad (3.7)$$

we see that this it is a conserved current as we substitute (3.6),

$$\nabla_a \mathbf{J}^a = 0, \quad (3.8)$$

when the dynamical fields are on shell. It is possible to find the related conserved charge $Q[\xi]$, which is a $(n-2)$ -form called Noether charge

$$\mathbf{J} = d\mathbf{Q}[\xi]. \quad (3.9)$$

Unlike in Euclidean spacetime, in curved spacetime it is far from trivial to prove the existence of such a charge. The interested reader can find a rather technical

¹The transformation rules for tensors and first order tensor densities are the same. For higher order tensor fields this is not true, and it is impossible to replace the tensor density by a tensor.

²The equations of motions are obtained by extremizing the action. The integral over a total derivative does not contribute to the variation. With this assumption the condition $\delta S = \int \mathbf{E}\delta\phi = 0$ for all $\delta\phi^a$, result in these equations of motion.

proof in [18]. The last current we need to define is called symplectic. The Noether current \mathbf{J} was related to a single variation δ , but now we consider two subsequent variations, induced by two different diffeomorphisms associated with the vector fields ξ_1 and ξ_2 . The corresponding variations of the dynamical fields are notated as $\delta_1\phi$ and $\delta_2\phi$. With this notation the symplectic current is defined as

$$\omega = \delta_1[\theta(\phi, \delta_2\phi)] - \delta_2[\theta(\phi, \delta_1\phi)]. \quad (3.10)$$

Hamiltonian Formulation

In fact Wald's formula is nothing else than an application of the conservation law theorem applied to some specific Noether current. To show this, we need to define the concepts energy, mass and angular momentum. After all, these are the quantities first law deals with. As explained in the Appendix, it is in no way trivial to define energy in general relativity. The same can be said about the related quantities (mass and angular momentum), but nonetheless there are several ways to define such quantities. All these constructions have shortcomings. What sort of definition is used depends on the situation. For asymptotically flat spacetime the most used convention is referred to as the ADM formalism. It is named after Arnowitt, Deser and Misner who invented a way to construct a Hamiltonian on such spacetimes [19]. It is no coincidence that their method is bound to this class of asymptotically flat manifolds. A Hamiltonian is a scalar function $H(\mathbf{q}, \mathbf{p})$ (on phase space) that governs the dynamics of the physical system through the Hamilton equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \quad (3.11)$$

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}. \quad (3.12)$$

In these equations the dot assigns a special role to time: it represents differentiation with respect to the time coordinate. In general there is no preferred time vector field on a given spacetime manifold but asymptotically flat spacetime is an exception. It can be foliated into spacelike hypersurface Σ_t , labeled by a time coordinate t . This time coordinate gives rise a timelike vector field ξ^a on M , indicating the time flow through the manifold. The coordinate t is the parameter of the diffeomorphism of the time vector field.

From Current to Hamiltonian

Assume the dynamical fields ϕ are on shell and consider the variations $\delta\phi$ which are an arbitrary deviation from the solution to the field equations. The variation induced by ξ on the Noether current $\mathbf{Q}(\phi, \delta\phi)$ is

$$\delta_\xi(J) = \delta_\xi[\Theta(\phi, \delta\phi) - \epsilon \cdot \delta_\xi \mathbf{L}] \quad (3.13)$$

$$(3.14)$$

Using (3.6), it takes only a few lines to show that

$$\delta\mathbf{J} = \omega(\phi, \delta\phi, \nabla_\xi\phi) + d(\epsilon \cdot \Theta). \quad (3.15)$$

The intermediate steps do not matter, it is more important to notice that 3.15 are equivalent to Hamilton's equations 3.11 and 3.12, which can be written alternatively as

$$\delta H = \frac{\partial H}{\partial \mathbf{p}} \delta \mathbf{q} - \frac{\partial H}{\partial \mathbf{q}} \delta \mathbf{p}. \quad (3.16)$$

The variation of H should satisfy

$$\delta H = \delta \int_C \mathbf{J} - \int_C d(\xi \cdot \Theta); \quad (3.17)$$

where C is a compact Cauchy surface for the unperturbed solution. This shows that up to a surface term the Noether current \mathbf{J} acts as a Hamiltonian density for the (time) evolution vector field ξ . It is possible to replace $\delta \mathbf{J}$ by $\delta d\mathbf{Q}$ which expresses the Hamiltonian as a surface integral. In the ADM formalism conserved quantities are related to the Hamiltonian in the familiar way, $E = H$. Here the Hamiltonian is the surface integral at infinity of the Hamiltonian density that corresponds to the time symmetry vector ξ . This situation applies equally to other symmetry vector fields. Thirteen years after the paper of Arnowitt, Deser and Misner it was shown by Regge and Teitelboim that in an asymptotically flat spacetime the value of the surface contribution of the Hamiltonian density at infinity is a conserved charge. Thus if we are given a symmetry vector field together with the Hamiltonian density that generates this vector field, we may write the corresponding conserved quantity as a surface integral. Here we are not interested in the conserved quantities themselves, but in their variations. Of course variations of the time conserved quantities can be expressed as surface integrals as well. We may write the variation of the canonical energy \mathcal{E} , associated with an asymptotic time translation t^a and the variation of the canonical angular momentum \mathcal{J} , associated with an asymptotic rotation ϕ^a as

$$\delta \mathcal{E} = \int_{\infty} (\delta \mathbf{Q}[t] - \mathbf{t} \cdot \Theta), \quad (3.18)$$

$$\delta \mathcal{J} = \int_{\infty} \delta \mathbf{Q}[\phi]; \quad (3.19)$$

where integration over a $n-2$ dimensional surface at infinity is understood. The term $\phi \cdot \Theta$ does not contribute to the surface integral since the Killing vector field ϕ is orthogonal to the sphere. Recall that for a rotating black hole the Killing vector can be written as

$$\xi^a = \mathbf{t}^a + \omega \phi. \quad (3.20)$$

The variation of $Q[\delta_1 \phi, \delta_2 \phi]$ is linear in both $\delta_1 \phi$ and $\delta_2 \phi$. We may decompose $Q[t]$ into a sum and rewrite (3.18) and (3.19) as

$$\delta \int_{\Sigma} \mathbf{Q} = \delta \mathcal{E} - \omega \delta \mathcal{J}; \quad (3.21)$$

where \mathbf{Q} is the Noether charge with respect to the Killing vector field ξ . This expression has exactly the same form as the first law of black hole mechanics. Starting with a Lagrangian theory, the Noether current can be expressed in terms of variations on the Lagrangian. When the Lagrangian is of the form $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \psi, \nabla_{\mu}\psi)$, Wald showed that the (Killing) Noether charge can

be written as $Q^{\mu\nu} = \mathcal{L}^{\mu\nu\rho\sigma} \epsilon_{\rho\sigma}$ [15]. This can be used to obtain Wald's formula that was presented at the beginning of this chapter

$$S = -2\pi \oint_{\Sigma} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd} \sqrt{|h|}. \quad (3.22)$$

3.2 Entropy Function Formalism

The entropy formula of Wald has some formidable qualities: it covers a wide class of black hole solutions, it has a strong mathematical foundation and it allows for a clear interpretation. The shape of 3.1 is convenient, but when we require explicit expressions for the entropy we might encounter some practical difficulties. It is not always an easy task to determine the binormal of a black hole solution. A more substantial limitation is that Wald's formula requires a black hole solution. It is impossible to know the binormal or the integration surface when the black hole solution is not known, and consequently Wald's formula cannot be used to arrive at an entropy expression. Unfortunately in general it is hard to find black hole solutions to a given theory. There are other methods to calculate black hole entropy, but these too require the black hole solution to be known. It does not seem possible to predict the entropy of a presumed black hole solution in a nontrivial class of Lagrangian theories of gravity without having to solve any system of equations. At least no such constructions are known. There is however a procedure that allows to calculate entropy without having to solve the full black hole metric; This is the 'entropy function formalism' by Sen. He showed that the entropy of extremal black holes can often be calculated without knowing the full black hole solution. It is sufficient to know the near-horizon configuration of the black hole, which is usually easier to obtain.

Before we take a closer look at the entropy formalism we discuss some of the concepts we shall need. The first section of this chapter deals with the class of extremal black holes. The use of the entropy formalism is restricted to this class. Then section 3.2.3 introduces the concept of near-horizon limits which proves to be crucial in the discussion of the entropy function formalism in section 3.2.4. The section that follows contains some remarks about attractor behavior of extremal black holes.

3.2.1 Extremal black holes

An extremal black hole is a black hole that possesses two event horizons that coincide. Black holes of this class have some special properties: unlike other black holes their surface gravity vanishes on the event horizon, and they possess the minimum amount of mass that can be compatible with the given charges and angular momentum. An extremal black hole is on the verge of becoming a naked singularity. If the mass decreases any further the particular solution will not have a horizon. In supersymmetry extremal black holes are special since they are often invariant under the action of supercharge operators. Such black holes emit no Hawking radiation. They are stable and the event horizon will not disappear. It has been known for a while that in supersymmetry extremal

black holes often show attractor behavior [20] [21] [22]. Sen discovered that non-supersymmetric extremal black holes can also show this behavior. This is discussed in section 3.2.4.

Extremal Spherical Black Holes

Consider the spherically symmetric line element

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2, \quad (3.23)$$

Each zero of f corresponds with an event horizon. If $f(r)$ has two simple zeros $r_1 \neq r_2$ the metric will have two event horizons. The function $f(r)$ can be factorized as $f(r) = (r - r_1)(r - r_2)g(r)$ with the function $g(r)$ satisfying $g(r_1) \neq 0$, $g'(r_1) \neq 0$, $g(r_2) \neq 0$ and $g'(r_2) \neq 0$. For an extremal black hole $r_1 = r_2$ and the single zero becomes a double zero. f can then be written as $f(r) = (r - r_1)^2g(r)$ and we have $f(r_1) = f'(r_1) = 0$.

The Extremal Reissner-Nordström Black Hole

The metric of the Reissner-Nordström solution is given by³

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}} + r^2d\Omega^2; \quad (3.24)$$

where m is the black hole mass and q is the electric charge. $d\Omega^2$ equals $r^2(d\theta^2 + \sin^2\theta d\phi^2)$ is understood. The factor $\left(1 - \frac{m}{r} + \frac{Q^2}{r^2}\right)$ has the same role as the function f in equation 3.23; we rewrite it as

$$f(r) = \left(\frac{r^2 - 2mr + Q^2}{r^2}\right) \quad (3.25)$$

to make clear that for $Q^2 = m^2$ it factors into $f(r) = \frac{(r-m)^2}{r^2}$. Then the two horizons coincide, the black hole will be extremal and the metric becomes

$$ds^2 = -\frac{(r-m)^2}{r^2}dt^2 + \frac{r^2}{(r-m)^2}dr^2 + r^2d\Omega^2. \quad (3.26)$$

This metric has a double horizon at $r = m$. Notice the function $f(r)$ is a parabola with a U-shape. When the charge is taken even bigger, $Q^2 > m^2$, the parabola will not hit the r-axis anymore and the event horizon disappears leaving a naked singularity.

3.2.2 Extremal Rotating Black Holes

In one of the previous sections we used a model for spherically symmetric black holes, equation 3.23, to explain the properties of extremal black holes. At first glance this model seems too simple to be able to cover also rotating black holes. The rotating black hole metrics can be written in Boyer-Lindquist form,

$$ds^2 = f(r)dt^2 - e^{2\psi}(d\phi - \omega dt)^2 - f^{-1}(r)(dr^2)^2 - e^{2\mu}(d\theta^2)^2, \quad (3.27)$$

³The convention for the charge unit is taken such as to absorb Coulomb's force constant $\frac{1}{4\pi\epsilon_0}$ into the charge Q .

and this does not fit into equation 3.23. The $(d\phi - \omega dt)^2$ term that appears in formula 3.27 gives rise to an extra dt^2 contribution which destroys the symmetry that was crucial in the analysis of section 3.2.1. We see this when we absorb that term into the regular 'time part' of the line element and compare the result with (3.23), but fortunately we are not bound to write the metric like this: we may ignore the $(d\phi - \omega dt)^2$ term in the above metric provided that we can argue the term will not be of influence when it concerns the radial structure of the horizon. Since the analysis of section 3.2.1 only involves the radial structure of the black hole then we may argue that we can take over the analysis of section 3.2.1 unaltered. The argument is simple: in some sense (a shift of) ω in the cross term can be interpreted as a translation of the angular velocity of the observers in the vicinity of the horizon. For non-rotating black holes the angular velocity of a nearby geodesic is $\frac{d\phi}{dt} = 0$, but we can imagine an observer outside the black hole that has angular velocity $\omega \neq 0$. The observer may adopt any angular velocity; the black hole horizon(s) will not be affected⁴. The only difference is that $d\phi \neq 0$.

For geodesics near rotating black holes the 'natural' angular velocity is $\frac{d\phi}{dt} = \omega$. For an observer with this angular velocity the black hole appears not to rotate. The observer cannot distinguish this situation from the previous setting. Again the observer is free to deviate from this 'natural' angular velocity and what is more important: the horizon radii are not effected.

3.2.3 Near-Horizon Geometry of Extremal Black Holes

The analysis of Wald showed that black hole entropy can be expressed as a local quantity integrated over the event horizon. If we want to calculate the entropy of a black hole all information we need is available at the horizon. In this section we describe the near-horizon limit which is an instrument that can be used to extract this information from the horizon. The information gets encoded in a new metric called the 'near-horizon metric'. It represents a black hole with the same entropy as the original black hole, but it has some additional symmetries. When these symmetries are assumed in advance the problem of solving the near-horizon geometry may simplify substantially: it may be easier to tackle this problem than to find a regular black hole solution.

The near-horizon limit

What should be the shape of the near-horizon limit if we require it to have the properties described above? It will become clear that the near-horizon limit has to be constructed from a continuous one-parameter class of coordinate transformations and a limit. The new coordinate parametrization has to allow for a limit in which the original coordinates are fixed to the event horizon of the black hole. After taking the limit a new metric should result. With this in mind we consider a static spherically symmetric extremal black hole whose line element can be put in the form

$$ds^2 = -f(\rho)d\tau^2 + f^{-1}(\rho)d\rho^2 + g(\rho)(d\theta^2 + \sin^2(\theta)d\phi^2). \quad (3.28)$$

⁴Spacetime will only be affected noticeably when the observer involves a substantial amount of energy in comparison with the black hole mass. In this sense ω will be limited.

Let r_0 be the coordinate of the event horizon. Then $f(\rho)|_{\rho=r_0} = 0$. In order to fix the coordinates of the line element to the event horizon we have to take a limit in which $\rho \rightarrow r_0$. The limit that does this job most obviously is $\lim_{\rho \rightarrow r_0}$, but the result does not satisfy. Remember that the goal was to find a new black hole metric with some extra symmetries. If we remove the ρ -freedom by taking this limit we cannot arrive at a black hole metric since there can be no horizon when there is no radial dependence. Taking the limit $\rho \rightarrow r_0$ was a bit naive. To end up with a suitable line element it is necessary to perform a coordinate transformation first. Consider the transformation

$$\begin{aligned} t &= \beta^{-1}\rho, \\ r &= \alpha^{-1}\lambda^{-1}\rho - r_0, \end{aligned} \iff \begin{aligned} \tau &= \beta t, \\ \rho &= \alpha\lambda r + r_0. \end{aligned} \quad (3.29)$$

In the limit $\lambda \rightarrow 0$, ρ goes to r_0 regardless of the value of r . ρ is fixed to the horizon, but in return there is an extra freedom: r is a new free coordinate. Our task is to determine the constants α and β such that in the new coordinates the resulting line element is as simple as possible. For an extremal black hole the function $f(r)$ can be written as $f(\rho) = (\rho - a)^2 \tilde{f}(\rho)$; where \tilde{f} is some function satisfying $\tilde{f}(\rho) > 0$ for all $\rho > 0$. We use this when expressing the line element in the new coordinates

$$\begin{aligned} ds^2 &= -(\alpha\lambda r)^2 \tilde{f}(\alpha\lambda r + r_0) \beta^2 dt^2 \\ &\quad + (\alpha\lambda r)^{-2} \tilde{f}^{-1}(\alpha\lambda r + r_0) (\alpha\lambda)^2 dr^2 \\ &\quad + g(\alpha\lambda r + r_0) (d\theta^2 + \sin^2(\theta) d\phi^2). \end{aligned} \quad (3.30)$$

The differentials are not yet expressed in the coordinates. The new coordinates depend on λ ; so strictly speaking we may not take the limit in λ at this stage; We have to anticipate by leaving $\beta^* = \lambda\beta$ untouched when we take the limit $\lambda \rightarrow 0$. The line element reduces to

$$\lim_{\lambda \rightarrow 0} ds^2 = -(\alpha\beta^* r)^2 \tilde{f}(r_0) dt^2 + \tilde{f}^{-1}(r_0) dr^2 + g(r_0) (d\theta^2 + \sin^2(\theta) d\phi^2). \quad (3.31)$$

The most simple geometry is obtained for $\alpha = 1$, $\beta = 2(f''(r_0))^{-1}$. With this choice the coordinate transformation becomes

$$\begin{aligned} t &= 2f''(r_0)\rho, \\ r &= \lambda^{-1}\rho - r_0, \end{aligned} \iff \begin{aligned} \tau &= \frac{1}{2f''(r_0)}t, \\ \rho &= \lambda r + r_0, \end{aligned} \quad (3.32)$$

and the near-horizon geometry that results is

$$ds^2 = v_1(-r^2 dt^2 + \frac{1}{r^2} dr^2) + v_2(\sin^2(\theta) d\phi^2 + d\theta^2); \quad (3.33)$$

where $v_1 = \frac{2}{f''(r_0)}$ and $v_2 = g(r_0)$ are constants. Note that here we used $f''(r_0) = 2\tilde{f}(r_0)$. Although the function $f(\rho)$ in (3.28) might have been very complicated, the geometry that is left after taking the near-horizon limit is certainly not. This is a significant property of the near-horizon limit, but it is not crucial for the entropy formalism to function. The main point of the entropy formalism is that near-horizon geometry of a unknown black hole solution is almost completely known a priori. If we would have no information about f the only unknown functions in the near-horizon geometry of 3.28, would be v_1 and v_2 : two scalars. We have to find just two numbers to know the near-horizon geometry completely.

Near-Horizon Geometry

The resulting space admits an $SO(2,1)$ isometry and an S^2 isometry. The spherical metric described above is not the only type of metric which has a $SO(2,1)$ isometry in the near-horizon limit. More of them have been found [23] and for extremal rotating black holes similar symmetries appear in the near-horizon limit [24]. Ashoke Sen developed the entropy function formalism after he did this observation. The first article on this subject was published in 2005. Then (of course) there was no theorem guaranteeing the existence of such symmetries in the near-horizon limit of an arbitrary metric, but in 2007 Kunduri, Lucietti and Reall presented such a theorem [25]. They showed that these symmetries occur for any extremal black hole that has the same number of rotational symmetries as known four- and five-dimensional black hole solutions. Their result is valid for any general two-derivative theories of gravity coupled to Abelian vectors and uncharged scalars. It completes the derivation of the entropy formalism that was originally given by Sen.

The Reissner-Nordström Near-Horizon Geometry

In later chapters we will need an expression for the near-horizon metric of the Reissner-Nordström black hole. In section 3.2.1 we arrived at an expression for the metric of the extremal Reissner-Nordström metric

$$ds^2 = -\frac{(r-m)^2}{r^2}dt^2 + \frac{r^2}{(r-m)^2}dr^2 + r^2d\Omega, \quad (3.34)$$

and here we will compute its near-horizon limit. For this metric $f(r) = \frac{(r-m)^2}{r^2}$, $r_0^2 = m^2$ and $f''(r_0) = \frac{2}{m^2}$. Transformation 3.32 becomes

$$\begin{aligned} t &= m^2\rho, \\ r &= \lambda^{-1}\rho - m, \end{aligned} \iff \begin{aligned} \tau &= m^{-2}t, \\ \rho &= \lambda r + r_0. \end{aligned} \quad (3.35)$$

The near-horizon limit of equation is

$$ds^2 = m^2(-r^2dt^2 + \frac{1}{r^2}dr^2) + m^2d\Omega. \quad (3.36)$$

We see that for the Reissner-Nordström black hole the constants v_1 and v_2 in the general ansatz for near-horizon geometries, equation 3.33, are both equal to equal m^2 .

3.2.4 Entropy Function Formalism

Sen conjectured that the metric of each extremal black hole solution reduces to a $AdS_2 \times SO(2)$ space in the near-horizon limit. We saw that in this limit the radial coordinate tends to the horizon of the black hole. In combination with a suitable coordinate transformation a near-horizon metric results. The symmetry that is revealed bears a drastic reduction on the variety of near-horizon geometries that can possibly occur. An important fact is that the entropy associated with the near-horizon black hole metric is identical to the entropy of the associated ordinary black hole. Instead of finding the full black hole solution Sen proposes to solve the near-horizon metric which is easier. He

then applies Wald's formula on this configuration to obtain the entropy [23]. The theory under consideration is a four dimensional one coupled to a set of Abelian gauge fields $A_\mu^{(i)}$, a set of neutral scalar fields ϕ_s and of course there is the metric g_{ab} . The Lagrangian density is denoted as $\sqrt{-\det g}\mathcal{L}$. The most general field configuration compatible with the $SO(2, 1) \times SO(3)$ symmetry is

$$ds^2 = v_1(-r^2 dt^2 + \frac{dr^2}{r^2} + v_2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad (3.37)$$

$$\phi_s = u_s, \quad (3.38)$$

$$F_{rt}^{(i)} = e_i, \quad (3.39)$$

$$F_{\theta\phi}^{(i)} = \frac{4\pi}{\sin(\theta)}. \quad (3.40)$$

Sen defines a new function f as

$$f(\mathbf{u}, \mathbf{v}, \mathbf{e}, \mathbf{p}) = \int d\theta d\phi \sqrt{-\det g}\mathcal{L}, \quad (3.41)$$

and he shows that the near-horizon configuration can be obtained by extremizing f with respect to \mathbf{u} and \mathbf{v} . The gauge fields should satisfy the gauge field equations and the Bianchi equations. These conditions appear when extremizing the so called entropy function,

$$\mathcal{E}(\mathbf{u}, \mathbf{v}, \mathbf{e}, \mathbf{p}, \mathbf{q}) \equiv 2\pi(e_i q_i - f(\mathbf{u}, \mathbf{v}, \mathbf{e}, \mathbf{p})) \quad (3.42)$$

with respect to \mathbf{u} , \mathbf{v} , and \mathbf{e} . Here q represents the electric charge of the black hole. Once the near-horizon configuration is obtained the entropy can be calculated using Walds formula

$$S_{BH} = -8\pi \int_H d\theta d\phi \frac{\delta \mathcal{L}}{\delta R_{rt}} \sqrt{g_{rr}g_{tt}}. \quad (3.43)$$

Using the scaling properties of f it is possible to show that

$$2\pi \left(e_i \frac{\partial f}{\partial_i} - f \right) = \mathcal{E}(\mathbf{u}, \mathbf{v}, \mathbf{e}, \mathbf{q}, \mathbf{p}). \quad (3.44)$$

To calculate the entropy of a black hole in a certain Lagrangian theory it suffices to concentrate on the entropy function. First it has be constructed out of the ansatz and the Lagrangian; then it should be extremized, which is the most difficult part, and to obtain the entropy it has to be evaluated.

Attractor Behavior

The entropy is given by the extremum of entropy function \mathcal{E} . Before it is evaluated \mathcal{E} contains only information from the Lagrangian density; it has no dependence on the scalar fields. The parameters of the near-horizon configuration are determined by minimising \mathcal{E} . If \mathcal{E} has a unique minimum at \mathbf{u} , \mathbf{v} , \mathbf{e} , \mathbf{q} and \mathbf{p} then both the near-horizon geometry and the entropy \mathcal{E} are fixed by these parameters. It is clear that the entropy is independent of the moduli fields. However when \mathcal{E} has flat directions not all parameters can be determined by extremizing \mathcal{E} . There is no unique minimum for the entropy function and consequently the near-horizon geometry cannot be determined. If we are interested only in entropy this is not a problem. The near-horizon geometries that minimize \mathcal{E} all have the same entropy since this is what \mathcal{E} represents.

3.2.5 Entropy Function for Rotating Black Holes

The above analysis does not apply when the black hole is not static. For rotating black holes the symmetry that is found is $SO(2, 1) \times U(1)$. This is proven in a more general setting by Kunduri, Lucietti and Reall [25]. An explicit example can be found in section 7.2.1 where the near-horizon limit of the Kerr-Newman metric is calculated.

The most general field configuration consistent with the $SO(2, 1) \times U(1)$ symmetries is

$$ds^2 = v_1(\theta) \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + \beta^2 d\theta^2 + \beta^2 v_2(\theta) (d\phi + \alpha r dt)^2, \quad (3.45)$$

$$\phi_s = u_s(\theta), \quad (3.46)$$

$$\frac{1}{2} F_{\mu\nu}^i dx^\mu dx^\nu = (e_i + \alpha b_i(\theta)) dr \wedge dt + \partial_\theta b_i(\theta) d\theta \wedge (d\phi + \alpha r dt); \quad (3.47)$$

where α , β and e_i are constants and v_1 , v_2 , u_s and b_i are functions of θ . Again a function f depending on the Lagrangian density is defined:

$$f(\alpha, \beta, \mathbf{u}(\theta), v(\theta), e) = \int d\theta d\phi \sqrt{-\det g \mathcal{L}}. \quad (3.48)$$

The equations of motion now correspond to

$$\frac{\partial f}{\partial \alpha} = J, \quad \frac{\partial f}{\partial \beta} = 0, \quad \frac{\partial f}{\partial e_i} = q_i, \quad \frac{\partial f}{\partial v_1} = 0, \quad \frac{\partial f}{\partial v_2} = 0, \quad \frac{\partial f}{\partial u_i} = 0, \quad \frac{\partial f}{\partial b_i} = 0, \quad (3.49)$$

which is equivalent to minimising the entropy function \mathcal{E} , defined as

$$\mathcal{E}[J, q, \alpha, \beta, e, v_1(\theta), v_2(\theta), (u)(\theta), b(\theta)] = 2\pi(J\alpha + qe - f(\alpha, \beta, u(\theta), v(\theta), e)), \quad (3.50)$$

with respect to its arguments. With Wald's entropy formula the entropy can be shown to be equal to \mathcal{E} when it is extremized. The derivation can be found in the review of Sen [23], which pays special attention to the periodic boundary conditions in θ . Extremizing \mathcal{E} determines the near-horizon configuration, and the value of the entropy function equals the entropy of the black hole. Again we have attractor behavior as explained in section 3.2.4.

3.3 The Newman-Janis Algorithm

The Newman-Janis algorithm can be used to generate rotating black hole solutions from static ones. Below we introduce the algorithm and give an example, but first we take a closer look at rotating black holes.

3.3.1 Rotating Black Holes

In classical mechanics it is no problem to make a body rotate. The procedure is to set up a transformation that maps the coordinates onto a rotating coordinate system. Then the equations of motion of the body have to be re-expressed in terms of the new coordinates. From the perspective of the new reference frame,

the body is rotating. This procedure applies to Newtonian mechanics as well to general relativity, but not to black holes. A black hole is not a body in the sense meant above. The term "black hole" refers to properties of space-time itself. Then what is a rotating black hole? To answer this question we have to impose some conditions on space-time: we assume it asymptotically flat and contains a black hole region. The asymptotical flatness allows us to define a 'non-rotating reference frame' at asymptotic infinity in an unambiguous way. This frame we call "inertial frame". When an object is not rotating with respect to this frame, we call it non-rotating. A curious phenomenon in general relativity is that spacetime can have regions in which objects are forced to rotate. A black hole solution may have such regions, and then it is called a 'rotating black hole'. We discuss the properties of rotating black holes on basis of the Kerr-Newman black hole and then we generalize to arbitrary rotating axisymmetrical black holes.

The Kerr-Newman Metric

The Kerr-Newman metric is given by [3]

$$\begin{aligned} ds^2 &= -\frac{\Delta\rho^2}{\Sigma^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\Sigma^2}{\rho^2}\sin^2(\theta)\left(d\phi - \alpha\frac{(r^2 + \alpha^2 - \Delta)dt}{\Sigma^2}\right)^2 + \rho^2d\theta^2, \\ \rho^2 &= r^2 + \alpha^2\cos^2(\theta), \\ \Delta &= r^2 - 2Mr + \alpha^2 + Q^2, \\ \Sigma^2 &= (r^2 + \alpha^2)^2 - \Delta a^2\sin^2(\theta). \end{aligned} \tag{3.51}$$

The metric is singular in $\rho = 0$ and for $\Delta = 0$. Calculation of the Riemann invariant $R^{abcd}R_{abcd}$ reveals that the only intrinsic singularity occurs at $\rho = 0$. This equation describes a 'singular' ring around $r = \rho$. The event horizon is defined by $\Delta = 0$. The function Δ can be factorized as $\Delta = (r - r_+)(r - r_-)$, indicating that there are two event horizons. The $dt d\phi$ cross-term that appears in this metric is the fingerprint of a rotating black hole. To show this we first generalise the metric to get rid of all complicated terms.

Rotating Metric

Stationary rotating axisymmetric metrics that are invariant under simultaneous inversion of space and time fit into the form [3]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_2}(dx_2)^2 + e^{2\mu_3}(dx_3)^2, \tag{3.52}$$

where ν, ψ, μ_2, μ_3 are smooth functions of r and θ . The requirement of invariance under simultaneous inversion of space and time is to assure that the energy stress tensor of matter in this spacetime resembles the energy stress tensor of a rotating body. When both the direction of time and the direction of the rotation angle are changed, nothing happens to the motion of matter.

Imagine all physical trajectories outside the event horizon in a black hole spacetime. To realize some of them we would need unprecedentedly strong rockets, while others (geodesics) would not require any propulsion. Whether propulsion is needed or not, all physical curves have in common that $ds^2 \leq 0$. When moving closer to the event horizon this has a surprising consequence. The

term $e^{2\nu}$ in front of $-dt^2$ goes to zero, causing a relative growth of the term $e^{2\nu}$ in front of the rotation term. Since ds^2 should be equal or less than zero the limitation on the ranges of $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ will get more strict. If $e^{2\nu}$ keeps decreasing it will reach the point that

$$-e^{2\nu}dt^2 + e^{2\psi}(d\phi^2 - \omega dt)^2 + e^{2\mu_2}(dx_2)^2 + e^{2\mu_3}(dx_3^2)^2 = 0. \quad (3.53)$$

This equation defines a hypersurface called "ergo surface". In principle only massless object can stay here in a non-rotating orbit. When moving any closer to the black hole the condition $ds^2 \leq 0$ will be violated for every trajectory unless it satisfies $\frac{d\phi}{dt} \geq \omega$. The region between the static limit surface and the event horizon is called ergo⁵ region. In this region all objects rotate.

3.3.2 Advanced Null Coordinate Systems

We are ready to describe the algorithm but first we take a look at the coordinate basis that appears in the algorithm and through the rest of this text. Advanced null coordinates (also called Eddington-Finkelstein coordinate systems) are adapted to radial null geodesics near the black hole. One of its advantages is that it can make coordinate singularities disappear. Many textbooks on general relativity use this coordinate system to show that the singularity at the event horizon of a Schwarzschild metric is just a coordinate singularity. The appearance of the singularity is a consequence of the choice of the coordinate system and is not physical. To explore the advanced null coordinate systems we consider a black hole metric in Boyer-Lindquist coordinates

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + h(r, \theta, \phi, d\theta, d\phi), \quad (3.54)$$

where f, g, h are smooth functions. A radial lightlike geodesic satisfies $ds^2 = 0$ and $d\theta = d\phi = 0$; so the equation of motion is

$$dt = \frac{g(r)}{f(r)}dr. \quad (3.55)$$

The null coordinate system is adapted to this motion

$$r^* = \int \frac{g(r)}{f(r)}dr, \quad dr^* = \frac{g(r)}{f(r)}dr, \quad (3.56)$$

$$u = t + r^*, \quad du = dt + \frac{g(r)}{f(r)}dr^*, \quad (3.57)$$

$$v = t - r^*, \quad dv = dt - \frac{g(r)}{f(r)}dr^*; \quad (3.58)$$

such that the ingoing null-geodesic equation becomes $du = 0$ and the prescription of the outgoing null-geodesic is $dv = 0$. In the new coordinates the metric is written as

$$ds^2 = -f^2(r)du^2 + 2f(r)g(r)dudr + h(r, \theta, \phi, d\theta, d\phi), \quad (3.59)$$

$$ds^2 = -f^2(r)dv^2 - 2f(r)g(r)dvdv + h(r, \theta, \phi, d\theta, d\phi). \quad (3.60)$$

A typical feature of null coordinates is the lack of a dr^2 term and the appearance of a cross-term $dudr$ or $dvdv$.

⁵Ergo is the latin word for "work". Roger Penrose pointed out the theoretically possible to extract energy from a black hole. The ergo region plays an important role in the process he described.

Reissner-Nordström Metric in Advanced Null Coordinates.

As an illustration we rewrite the Reissner-Nordström metric into advanced null coordinates⁶ The metric is given by

$$ds^2 = - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2, \quad (3.61)$$

so we have $f^2(r) = \frac{1}{g^2(r)} = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)$. subsequently the advanced null coordinates are

$$r^* = \int \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)^{-1} dr, \quad dr^* = \frac{g(r)}{f(r)} dr, \quad (3.62)$$

$$u = t + r^* \quad du = dt + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)^{-1} dr, \quad (3.63)$$

$$v = t - r^* \quad dv = dt - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)^{-1} dr, \quad (3.64)$$

and the metric is written as

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) du^2 + 2dudr + r^2 d\Omega. \quad (3.65)$$

3.3.3 The Algorithm

From section 3.3.1 it is clear that it is impossible to arrive from a static spherically symmetric metric at a stationary axisymmetric metric by means of a coordinate transformation. The metrics are just inequivalent. Surprisingly there is a sort of complex coordinate transformation that can do something like this [26]. Ten years after the discovery of the Kerr metric in 1963 Newman and Janis published an algorithm that is able to generate the Kerr metric from the Schwarzschild solution. The algorithm does not only work with the Schwarzschild metric as seed, but for many non-rotating seed metrics it is able to produce the corresponding rotating version. However in many other cases the algorithm does not work. An overview of the algorithm is presented in the article [27], where 5 steps are distinguished:

1. Start with the spherically symmetric seed element in advanced null coordinates. (see section 3.3.2).
2. Express the covariant form of the metric in terms of a null tetrad. This null tetrad should be a set of four vectors $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ such that

$$\begin{aligned} l_\mu l^\mu &= m_\mu m^\mu = 0, \\ l_\mu m^\mu &= n_\mu m^\mu = 0, \end{aligned} \quad (3.66)$$

$$\begin{aligned} l_\mu n^\mu &= -m_\mu \bar{m}^\mu = 1, \\ g^{\mu\nu} &= l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu. \end{aligned} \quad (3.67)$$

⁶Some text make a distinction between ingoing and outgoing null coordinates in there terminology: the ingoing coordinates are called retarded null coordinates and the outgoing coordinates are called advanced null coordinates. Here we don't make the distinction.

3. Extend the real coordinates x^μ to a set of complex coordinates $\tilde{x}^\mu = x^\mu + iy^\mu(x)$.
4. Perform a coordinate transformation

$$\tilde{x}^\mu = x^\mu + i\alpha \cos(\theta)(\delta_0^\mu - \delta_1^\mu). \quad (3.68)$$

The basis vectors of the tetrad transform according to the tensor transformation rules and the transformed tetrad generates a new metric through relation 3.67.

5. Transform the new metric to Boyer-Lindquist form. This is the coordinate system in which the metric has only one off-diagonal component: $g_{rt} \neq 0$. See section 3.3.5 for a description of the procedure.

3.3.4 The Algorithm Applied to the Reissner-Nordström Metric

It is instructive to discuss the details of this procedure on basis of an example. In the original paper of Janis and Newman the Kerr metric was obtained from the Schwarzschild metric [26]. Here we 'derive' the Kerr-Newman metric starting from the Reissner-Nordström metric. We will use this calculation again in chapter 7, where some of the intermediate are needed in an attempt to take over the Newman-Janis algorithm to the near-horizon setting.

1. The Reissner-Nordström solution will function as seed metric. In section 3.3.2 we found this metric can be written as

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{Q^2}{r^2}\right)du^2 + 2dudr - r^2(d\theta^2 + \sin(\theta)^2 d\phi), \quad (3.69)$$

in advanced null coordinates. For the algorithm we need the inverse metric:

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -\frac{Q^2+r^2-2Mr}{r^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{\csc^2(\theta)}{r^2} \end{pmatrix}. \quad (3.70)$$

2. The corresponding null tetrad is

$$l^\mu = \delta_1^\mu, \quad (3.71)$$

$$n^\mu = \delta_0^\mu - \frac{1}{2} \left(1 - \frac{2m}{r} - \frac{Q^2}{r^2}\right) \delta_1^\mu, \quad (3.72)$$

$$m^\mu = \frac{1}{\sqrt{2}r} \left(\delta_2^\mu + \frac{i}{\sin(\theta)} \delta_3^\mu\right). \quad (3.73)$$

3. We extend the real coordinates \tilde{x}^μ to a new set of complex coordinates $x^\mu = x^\mu + iy^\mu(x)$.
4. Perform a coordinate transformation in the complex plain:

$$\tilde{x}^\mu = x^\mu + i\alpha \cos(\theta)(\delta_0^\mu - \delta_1^\mu); \quad (3.74)$$

again we labeled the old vectors and coordinates with a tilde. The tetrad basis vectors transform according to $X_i^\mu = \tilde{X}_i^\nu \frac{\partial X_i^\mu}{\partial x^\nu}$ and new tetrad is

$$\begin{aligned} l^\mu &= \delta_1^\mu, \\ n^\mu &= \delta_0^\mu - \frac{(Q^2 + \alpha^2 \cos^2(\theta) + r(r - 2M))}{2(r^2 + \alpha^2 \cos^2(\theta))} \delta_1^\nu, \\ m^\mu &= \frac{i\alpha \sin(\theta)}{\sqrt{2}(r - i\alpha \cos(\theta))} \delta_0^\mu - \frac{i\alpha \sin(\theta)}{\sqrt{2}(r - i\alpha \cos(\theta))} \delta_1^\mu \\ &\quad + \frac{1}{\sqrt{2}(r - i\alpha \cos(\theta))} \delta_2^\mu + \frac{i \csc(\theta)}{\sqrt{2}(r - i\alpha \cos(\theta))} \delta_3^\mu. \end{aligned} \quad (3.75)$$

With equation 3.67 we obtain a new metric from the transformed tetrad

$$g^{\mu\nu} = \begin{pmatrix} -\frac{\alpha^2 \sin^2(\theta)}{r^2 + \alpha^2 \cos^2(\theta)} & \frac{r^2 + \alpha^2}{r^2 + \alpha^2 \cos^2(\theta)} & 0 & -\frac{\alpha}{r^2 + \alpha^2 \cos^2(\theta)} \\ \frac{r^2 + \alpha^2}{r^2 + \alpha^2 \cos^2(\theta)} & -\frac{Q^2 + r^2 + \alpha^2 - 2Mr}{r^2 + \alpha^2 \cos^2(\theta)} & 0 & \frac{\alpha}{r^2 + \alpha^2 \cos^2(\theta)} \\ 0 & 0 & -\frac{1}{r^2 + \alpha^2 \cos^2(\theta)} & 0 \\ -\frac{\alpha}{r^2 + \alpha^2 \cos^2(\theta)} & \frac{\alpha}{r^2 + \alpha^2 \cos^2(\theta)} & 0 & -\frac{\csc^2(\theta)}{r^2 + \alpha^2 \cos^2(\theta)} \end{pmatrix}. \quad (3.76)$$

Its inverse is

$$g_{\mu\nu} = \begin{pmatrix} \frac{Q^2 + r^2 + \alpha^2 \cos^2(\theta) - 2Mr}{r^2 + \alpha^2 \cos^2(\theta)} & 1 & 0 & -\frac{(Q^2 - 2Mr)\alpha \sin^2(\theta)}{r^2 + \alpha^2 \cos^2(\theta)} \\ 1 & 0 & 0 & -\alpha \sin^2(\theta) \\ 0 & 0 & -r^2 - \alpha^2 \cos^2(\theta) & 0 \\ -\frac{(Q^2 - 2Mr)\alpha \sin^2(\theta)}{r^2 + \alpha^2 \cos^2(\theta)} & -\alpha \sin^2(\theta) & 0 & g_{\phi\phi} \end{pmatrix}, \quad (3.77)$$

with $g_{\phi\phi} = -\sin^2(\theta) \left(r^2 + \alpha^2 + \frac{2Mr - Q^2}{r^2 + \alpha^2 \cos^2(\theta) + \alpha^2 \sin^2(\theta)} \right)$. This metric is written in a somewhat unusual form whereas it has a $d\phi dt$ cross term as well as a $d\phi dr$ component.

5. We transform the new metric into the more familiar Boyer-Lindquist coordinates which appears more familiar to us. A general procedure is described in the next section, and this particular case is treated as an example. We find the well-known expression 3.51 for the Kerr-Newman metric which describes a rotating charged black hole.

$$ds^2 = -\frac{\Delta \rho^2}{\Sigma^2} dt^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\Sigma^2}{\rho^2} \sin^2(\theta) \left(d\phi - \alpha \frac{(r^2 + \alpha^2 - \Delta) dt}{\Sigma^2} \right)^2 + \rho^2 d\theta^2.$$

The shape 3.77 of the Kerr-Newman black hole is less well known, but nevertheless it happens to be the original description of the Kerr-Newman black hole presented in 1965 by Newman, Couch, Chinnapared, Exton, Prakash and Torrence [28]. They used their new algorithm, very similar to what we did here. Moreover the authors could show that their result satisfies the Einstein-Maxwell equation. More about this can read in the master thesis of Emiel dreth that will appear in September.

3.3.5 The Boyer-Lindquist Form

The Newman-Janis algorithm delivers rotating metrics in a some what disguised form. In the case of the Reissner-Nordström black hole we encountered a $drdt$ cross term and a $drd\phi$ cross term. This allways happens. In section 3.3.1 we saw that in general a rotating metric can be written into the well recognizable Boyer-Lindquist⁷ form

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi} (d\phi - \omega dt)^2 + e^{2\mu_2} (dx^2)^2 + e^{2\mu_3} (dx^3)^2. \quad (3.78)$$

Here we construct a procedure to rewrite metrics like 3.76 into Boyer-Lindquist coordinates. We start from a metric that can be written as

$$g_{\mu\nu} = \begin{pmatrix} g_{uu} & g_{ur} & 0 & g_{u\phi} \\ g_{ur} & 0 & 0 & g_{r\phi} \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{u\phi} & g_{r\phi} & 0 & g_{\phi\phi} \end{pmatrix} \quad (3.79)$$

. We search for a coordinate transformation that removes the $g_{r\phi}$ terms from this metric. In matrix notation a coordinate transformation on a vector has the form of a multiplication

$$v \rightarrow v' = A \cdot v. \quad (3.80)$$

The vectors are made up of the elements of the differential coordinate basis. We transform to a new differential coordinate basis by taking linear combinations of these elements. The $r - \phi$ component should be eliminated, together with the $u - r$ component. We use a coordinate transformation of the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ h & 1 & 0 & f \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad (3.81)$$

where the elements f and h still have to be determined. The metric $g_{\mu\nu}$ is written in matrix form, which suggests to use the transformation rules for matrices from linear algebra. This suggestion is misleading. In linear algebra a square matrix M represents a linear map from a vector space V to the same vector space V . Our metric in contrary is a linear map form $V \times V \rightarrow \mathbb{R}$ where V is the vector space spanned by the elements of differential coordinates basis. If we transform this basis, the outcomes in \mathbb{R} should not change. We can use matrix notation, but the transformation rules need some attention. If u and v are vectors in V than their inner product will be

$$\langle u, v \rangle = u^T \cdot g \cdot v \quad (3.82)$$

in matrix notation. When u and v and g transform according to 3.80 the inner product becomes

$$\langle u', v' \rangle = u \cdot A^T \cdot g' \cdot A \cdot v \quad (3.83)$$

⁷In 1967, Boyer and Lindquist published a fascinating article in which they obtained a maximal analytic extension of the Kerr metric [29]. In this article they introduced a generalisation of Kruskal's transformation, now refered to as Boyer-Lindquist transformation.

We require $\langle u', v' \rangle = \langle u, v \rangle$ and this is guaranteed for $g' = (A^T)^{-1} \cdot g \cdot A^{-1}$. This matrix A is very simple. The inverse matrix equals minus A , so we may write

$$g' = A^T \cdot g \cdot A. \quad (3.84)$$

The metric 3.79 under this coordinate transformation A is

$$g'_{\mu\nu} = \begin{pmatrix} g_{uu} & g_{ur} + hg_{uu} + fg_{u\phi} & 0 & g_{u\phi} \\ g_{ur} + hg_{uu} + fg_{u\phi} & g'_{rr} & 0 & g'_{r\phi} \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{u\phi} & g_{r\phi} + hg_{u\phi} + fg_{\phi\phi} & 0 & g_{\phi\phi} \end{pmatrix}, \quad (3.85)$$

with $g'_{rr} = g_{\phi\phi}f^2 + 2g_{r\phi}f + 2hg_{u\phi}f + 2hg_{ur} + h^2g_{uu}$ and $g'_{r\phi} = g_{r\phi} + hg_{u\phi} + fg_{\phi\phi}$. The new $r - \phi$ component is $g_{r\phi} + hg_{u\phi} + fg_{\phi\phi}$ and the new $t - r$ component is $g_{ur} + hg_{uu} + fg_{u\phi}$. Both of them need to be zero, so we solve

$$\begin{pmatrix} g_{uu} & g_{u\phi} \\ g_{u\phi} & g_{\phi\phi} \end{pmatrix} \cdot \begin{pmatrix} h \\ f \end{pmatrix} = 0, \quad (3.86)$$

and we find $h = \frac{g_{r\phi}g_{u\phi}}{g_{uu}g_{\phi\phi} - g_{u\phi}^2} - \frac{g_{ur}g_{\phi\phi}}{g_{uu}g_{\phi\phi} - g_{u\phi}^2}$ and $f = \frac{g_{ur}g_{u\phi}}{g_{uu}g_{\phi\phi} - g_{u\phi}^2} - \frac{g_{r\phi}g_{uu}}{g_{uu}g_{\phi\phi} - g_{u\phi}^2}$. Substituting this into equation 3.85 gives

$$g'_{\mu\nu} = \begin{pmatrix} g_{uu} & 0 & 0 & g_{u\phi} \\ 0 & \frac{g_{uu}g_{r\phi}^2 - 2g_{ur}g_{u\phi}g_{r\phi} + g_{ur}^2g_{\phi\phi}}{g_{u\phi}^2 - g_{uu}g_{\phi\phi}} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{u\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix}, \quad (3.87)$$

which is quite close to the well known Boyer-Lundquist form for rotating metrics. All unwanted cross terms are gone, but we said nothing about the new coordinates them self. We only talked about the basis of differential coordinates. In most text about general relativity explicit mappings are derived whenever a coordinate transformation is performed. These mappings send the new coordinates to the old ones and this makes it possible to express the components of the new metric as functions of the new coordinates. In practice metric components usually are written as function of the coordinates that corresponds to the specific vector basis associated with that metric. But this is not strictly necessary. In fact we already saw an example of an exception. In section 3.3.2 a transformation from Boyer-Lindquist coordinates to advanced null coordinate is introduced where the transformed line element, equation 3.59, is written in terms of the old coordinates. We used this expression to transform the metric of the Reissner-Nordström black hole into the advanced null shape. Subsequently we used this expression as a seed metric for the Newman-Janis algorithm. During all steps we expressed the functions in the metric in terms of the old coordinates. In the last step of the Newman-Janis algorithm we transformed back to Boyer-Lindquist coordinates. Since the metric components are already (or still) written as functions of Boyer-Lindquist coordinates this is fine.

Kerr-Newman from Advanced Null Coordinates into Boyer-Lindquist Coordinates

Equation 3.85 tells us how to transform the metric that was left in the previous section after after we applied the Newman-Janis algorithm on the Reissner-

Nordström black hole. Equation 3.77 says

$$\begin{aligned} g_{uu} &= \frac{Q^2 + r^2 + \alpha^2 \cos^2(\theta) - 2Mr}{r^2 + \alpha^2 \cos^2(\theta)}, & g_{ur} &= 1, \\ g_{u\phi} &= -\frac{(Q^2 - 2Mr)\alpha \sin^2(\theta)}{r^2 + \alpha^2 \cos^2(\theta)}, & g_{r\phi} &= -\alpha \sin^2(\theta), \\ g_{\phi\phi} &= -\sin^2(\theta) \left(r^2 + \alpha^2 + \frac{2Mr - Q^2}{r^2 + \alpha^2 \cos^2(\theta) + \alpha^2 \sin^2(\theta)} \right). \end{aligned}$$

For the transformation of the differential basis

$$\begin{pmatrix} u \\ r \\ \theta \\ \phi \end{pmatrix} \rightarrow A \cdot \begin{pmatrix} u \\ r \\ \theta \\ \phi \end{pmatrix}, \quad (3.88)$$

we find

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{r^2 + \alpha^2}{Q^2 + r^2 + \alpha^2 - 2Mr} & 1 & 0 & -\frac{\alpha}{Q^2 + r^2 + \alpha^2 - 2Mr} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.89)$$

If we mark the old coordinates with a tilde this transformation can be written alternatively as

$$d\tilde{r} = d\tilde{r} - \frac{r^2 + \alpha^2}{Q^2 + r^2 + \alpha^2 - 2Mr} du - \frac{\alpha}{Q^2 + r^2 + \alpha^2 - 2Mr} d\phi. \quad (3.90)$$

The elements of the transformed metric 3.87 become

$$g_{tt} = \frac{Q^2 + r^2 + \alpha^2 \cos^2(\theta) - 2Mr}{r^2 + \alpha^2 \cos^2(\theta)}, \quad (3.91)$$

$$g_{t\phi} = -\frac{(Q^2 - 2Mr)\alpha \sin^2(\theta)}{r^2 + \alpha^2 \cos^2(\theta)}, \quad (3.92)$$

$$g_{rr} = -\frac{r^2 + \alpha^2 \cos^2(\theta)}{Q^2 + r^2 + \alpha^2 - 2Mr}, \quad (3.93)$$

$$g_{\theta\theta} = -r^2 - \alpha^2 \cos^2(\theta), \quad (3.94)$$

$$g_{\phi\phi} = \left(-r^2 - \alpha^2 - \frac{(2Mr - Q^2)\alpha}{r^2 + \alpha^2 \cos^2(\theta)} \right) \sin^2(\theta). \quad (3.95)$$

This can be written as

$$g_{\mu\nu} = \begin{pmatrix} \frac{\Delta \rho^2}{\Sigma^2} + 2\frac{(r^2 + \alpha^2 - \Delta)^2}{\rho \Sigma^2} & 0 & 0 & \frac{\Sigma^2}{\rho} \sin^2(\theta) \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{\Sigma^2}{\rho} \sin^2(\theta) & 0 & 0 & \frac{\Sigma^2}{\rho^2} \sin^2(\theta) \end{pmatrix}, \quad (3.96)$$

$$\rho^2 = r^2 + \alpha^2 \cos^2(\theta), \quad (3.97)$$

$$\Delta = r^2 - 2Mr + \alpha^2 + Q^2,$$

$$\Sigma^2 = (r^2 + \alpha^2)^2 - \Delta a^2 \sin^2(\theta),$$

in which we recognise the usual Boyer-Lindquist form of the Kerr-Newmann metric.

Chapter 4

Born-Infeld Black Holes

4.1 Born-Infeld theory

Born-Infeld theory is an alternative for Maxwell's theory of electrodynamics. Maxwell's theory predicts the electrical field strength to blow up like $1/r^2$ as one moves toward a point charge. Born searched for a theory that does not allow for an unbounded field strength. Together with his student Infeld he arrived in 1935 at a nonlinear theory [30] with the Lagrangian

$$\mathcal{L}_{\text{EBI}} = \sqrt{-\det(g)}R + \frac{4}{b^2} \left(\sqrt{-\det(g)} - \sqrt{-\det(g + bF)} \right). \quad (4.1)$$

Unlike Maxwell's equations, the equations of motion of Born-Infeld are non-linear, and the electric field for a point charge is bounded. Moreover, the limit $b \rightarrow 0$ yields the Maxwell Lagrangian as we will demonstrate below. In Maxwell's theory there is an $SO(2)$ symmetry of rotations between the electric and the magnetic field. In non-linear theories for electromagnetism this symmetry is not present in general, but Born-Infeld theory is one of the few exceptions [31]. In general the field equations and the Bianchi identities for a theory of electromagnetism can be written as

$$\partial_{[\alpha} \star G_{\mu\nu]} = 0, \quad (4.2)$$

$$\partial_{[\alpha} F_{\mu\nu]} = 0; \quad (4.3)$$

where $F_{\mu\nu}$ is the field strength tensor, \star is the hodge dual and

$$G^{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}. \quad (4.4)$$

The Bianchi identities and the field equations are invariant under the electromagnetic duality rotation

$$F_{\mu\nu} \rightarrow \cos(\alpha)F_{\mu\nu} + \sin(\alpha)G_{\mu\nu}, \quad (4.5)$$

$$G_{\mu\nu} \rightarrow \cos(\alpha)G_{\mu\nu} - \sin(\alpha)F_{\mu\nu}.$$

For Maxwell's theory, which is linear, $G_{\mu\nu} = \star F_{\mu\nu}$ and the electromagnetic duality rotation reduces to the hodge duality rotation

$$F_{\mu\nu} \rightarrow \cos(\alpha)F_{\mu\nu} + \sin(\alpha) \star F_{\mu\nu}, \quad (4.6)$$

$$\star F_{\mu\nu} \rightarrow \cos(\alpha) \star F_{\mu\nu} - \sin(\alpha)F_{\mu\nu}, \quad (4.7)$$

In a local orthonormal coordinate frame $E_i = F_{i0}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. \mathbf{E} is the electric field strength vector and \mathbf{B} is the magnetic field strength vector. Under the Hodge operator these transform into each other

$$\mathbf{E} \rightarrow \star\mathbf{E} = -\mathbf{B}, \quad (4.8)$$

$$\mathbf{B} \rightarrow \star\mathbf{B} = +\mathbf{E}. \quad (4.9)$$

From the electromagnetic duality rotation invariance it follows that Maxwell's theory is invariant under the action of the Hodge duality rotation

$$\mathbf{E} \rightarrow \cos(\alpha)\mathbf{E} - \sin(\alpha)\mathbf{B}, \quad (4.10)$$

$$\mathbf{B} \rightarrow \sin(\alpha)\mathbf{B} + \cos(\alpha)\mathbf{E}. \quad (4.11)$$

For an arbitrary theory of electromagnetism it is not generally true that $G_{\mu\nu} = \star F_{\mu\nu}$. To clarify the symmetry of the electromagnetic duality rotation, equation 4.6, we define the electric induction \mathbf{D} and the magnetic intensity \mathbf{H} in Minkowski spacetime by $D_i = G_{i0}$ and $H_i = \frac{1}{2}\epsilon_{ijk}G_{jk}$. The theory is invariant under the electromagnetic duality rotation, equation 4.6; in Minkowski spacetime it equivalent to

$$\mathbf{E} \rightarrow \cos(\alpha)\mathbf{E} - \sin(\alpha)\mathbf{H}, \quad (4.12)$$

$$\mathbf{H} \rightarrow \sin(\alpha)\mathbf{H} + \cos(\alpha)\mathbf{E}, \quad (4.13)$$

$$\mathbf{D} \rightarrow \cos(\alpha)\mathbf{D} - \sin(\alpha)\mathbf{B}, \quad (4.14)$$

$$\mathbf{B} \rightarrow \sin(\alpha)\mathbf{B} + \cos(\alpha)\mathbf{D}. \quad (4.15)$$

To show the identification with Maxwell for zero Born-Infeld parameter we perform a Taylor expansion of the root in \mathcal{L}_{EBI} . Up to order b^3 we obtain

$$\mathcal{L}_{EBI} = -F_{ab}F^{ab} + \frac{b^2}{4}((F_{ab}F^{ab})^2 + (F_{ab}\star F^{ab})^2), \quad (4.16)$$

which reduces to $-F_{ab}F^{ab}$, the usual Maxwell Lagrangian, when $b = 0$.

4.2 Black Hole Solution

The static black hole solution for Einstein-Born-Infeld theory is presented in [32]. In canonical coordinates (t, r, ϕ, θ) this solution can be written in terms of Legendre elliptic functions [33]; the metric and the field tensor are

$$ds^2 = -\psi dt^2 + \psi^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)), \quad (4.17)$$

$$\psi = 1 - \frac{2M}{r} - \frac{\lambda r^2}{3} + \frac{2}{3}b^2 r^2 \left(1 - \sqrt{1 + \frac{Q^2}{b^2 r^4}} \right) \quad (4.18)$$

$$+ \frac{2Q^2}{3r} \sqrt{\frac{b}{Q}} F\left(\arccos\left(\frac{br^2/Q - 1}{br^2/Q + 1}\right), \frac{1}{\sqrt{2}}\right), \quad (4.19)$$

$$F_{rt} = Q \left(r^4 + \frac{Q^2}{b^2} \right)^{-\frac{1}{2}}. \quad (4.20)$$

F is an elliptic function, $F(\beta, k) := \int_{\beta}^{\infty} (1 - k^2 \sin^2(s))^{\frac{1}{2}} ds$. M is the mass parameter of the system, and Q is the electric charge. The solution can be

given a magnetic charge by applying the electromagnetic duality transformation defined above. As r approaches zero, the electric field strength F_{rt} does not blow up, and as the Born-Infeld parameter b goes to zero, the Reissner-Nordström metric is recovered: just as we expected. In the limit that $r \rightarrow \infty$ the energy-momentum tensor converges to the energy-momentum tensor of the Reissner-Nordström black hole [33]. The horizon structure of this solution is determined by the behavior of the function $\psi(r)$, and the parameter space can be divided in three regions according to the behavior of $\psi(r)$:

1. The region where $M > \frac{2Q^2}{3r} \sqrt{\frac{b}{Q}} F(\arccos(\frac{br^2/Q-1}{br^2/Q+1}), \frac{1}{\sqrt{2}})$, ψ diverges to $-\infty$ as $r \rightarrow 0$, like the Schwarzschild metric.
2. In the region $M < \frac{2Q^2}{3r} \sqrt{\frac{b}{Q}} F(\arccos(\frac{br^2/Q-1}{br^2/Q+1}), \frac{1}{\sqrt{2}})$, ψ diverges to $+\infty$ when $r \rightarrow 0$ and ψ has two zero's like the Reissner-Nordström metric.
3. When $M = \frac{2Q^2}{3r} \sqrt{\frac{b}{Q}} F(\arccos(\frac{br^2/Q-1}{br^2/Q+1}), \frac{1}{\sqrt{2}})$, ψ has a double zero, and the black hole is extreme.

The third class of solutions comprises extremal EBI black holes, so we may apply the entropy formalism of Sen; this is done in the next section.

4.3 Entropy of the Born-Infeld Black Hole

To calculate the entropy of an extremal Einstein-Born-Infeld black hole we can use the black hole solution 4.17 and calculate the area of the horizon. Instead we choose to apply the entropy formalism which gives the same result. - Lagrangian of the theory is

$$\mathcal{L}_{EBI} = \sqrt{-\det(g)}R + \frac{4}{b^2} \left(\sqrt{-\det(g)} - \sqrt{-\det(g+bF)} \right), \quad (4.21)$$

and the ansatz for the near-horizon configuration given by Sen is

$$ds^2 = v_1(-r^2 dt^2 + \frac{dr^2}{r^2}) + v_2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad (4.22)$$

$$\phi_s = u_s, \quad (4.23)$$

$$F_{rt} = e_i, \quad (4.24)$$

$$F_{\theta\phi} = \frac{4\pi}{\sin(\theta)}. \quad (4.25)$$

First we have to write down the entropy function which is defined by

$$f(u, v, e, p) = \int d\theta d\phi \sqrt{-\det(g)} \mathcal{L}. \quad (4.26)$$

The determinant of the metric equals $-v_1^2 v_2^2 \sin^2(\theta)$, and with Riemann tensor $R = \frac{2}{v_2} - \frac{2}{v_1}$, the entropy function $\mathcal{E} = 2\pi(eq - f)$ becomes

$$\mathcal{E} = 2\pi \left(16\pi qe - 8\pi(v_1 - v_2) + \frac{4}{b^2} \left(v_1 v_2 - \sqrt{(v_1^2 - b^2 e^2)(v_2^2 + b^2 p^2)} \right) \right). \quad (4.27)$$

By minimizing this function with respect to e , v_1 and v_2 we obtain a system of ordinary differential equations

$$\begin{aligned} 0 &= \frac{\partial \mathcal{E}}{\partial e} = 32\pi^2 \left(q - \frac{q(b^2 p^2 + x^2)}{w} \right), \\ 0 &= \frac{\partial \mathcal{E}}{\partial v_1} = -\frac{16\pi^2}{b^2} \left(b^2 + 2x - \frac{2z(b^2 p^2 + x^2)}{w} \right), \\ 0 &= \frac{\partial \mathcal{E}}{\partial v_2} = 0 - \frac{16\pi^2}{b^2} \left(-b^2 + 2z + \frac{2x(b^2 q^2 - z^2)}{w} \right), \end{aligned}$$

where

$$\begin{aligned} w &= \sqrt{(b^2 p^2 + x^2)(x^2 - b^2 q^2)}, \\ x &= p^2 + q^2 - \frac{b^2}{4}, \\ y &= p^2 + q^2 + \frac{b^2}{4}, \\ z &= p^2 + q^2. \end{aligned}$$

The solution to this system is

$$\begin{aligned} q &= q, \\ v_1 &= q^2 + p^2 + \frac{1}{4}b^2, \\ v_2 &= q^2 + p^2 - \frac{1}{4}b^2. \end{aligned}$$

This solution is presented by de Roo, Chemissany and Panda [34]. Moreover, they were able to arrive at an exact expression for the entropy of a (non-rotation) EBI black hole when axion and dilaton fields are present. The above solution can be substituted in the entropy function which then gives the entropy of the black hole

$$\mathcal{E} = 16\pi^2 \left(q^2 + p^2 - \frac{1}{4}b^2 \right). \quad (4.28)$$

The b^2 term is new compared to the Kerr-Newman case: the Born-Infeld structure changes the entropy relation, but only with constant term $-\frac{b^2}{4}$. As a consequence, in absence of charge the black hole will have negative entropy. Notice that the entropy is manifestly invariant under electric-magnetic charge duality; just as it should be.

Chapter 5

Entropy function for the rotating Born-Infeld black Hole

In the previous chapter we saw a static black hole solution for the Einstein-Born-Infeld Lagrangian. Unlike the Einstein-Maxwell case rotating solutions are not known. There is an approximate solution due to Lomardo [35], but it is valid only in the limit of slow rotation. In 2004 Lombardo undertook another attempt. He applied the Newman-Janis Algorithm with the static EBI black hole solution as seed, and obtained a new metric [36]. Unfortunately the stress-energy tensor of this solution does not correspond to a rotating Born-Infeld charge, and Lombardo was forced to conclude that the Newman-Janis algorithm does not work for (the nonlinear) Born-Infeld theory. Although a rotating EBI black hole solution is not known, with the tools of chapter 3 we still have a good chance to arrive at an expression for its entropy. In this chapter we employ the entropy formalism again, now for the rotating case. It should not be hard to derive the entropy function; the main challenge will be to solve the system of differential equations that appears when extremizing this entropy function.

5.1 From Lagrangian to System of Equations

In order to obtain an expression for the entropy of a rotating EBI black hole we apply the entropy formalism of Sen, see section 3.2.5. We begin with the calculation of the entropy function. The EBI Lagrangian is given by

$$\mathcal{L}_{EBI} = \sqrt{-\det(g)}R + \frac{4}{d^2} \left(\sqrt{-\det(g)} - \sqrt{-\det(g + dF)} \right). \quad (5.1)$$

The attentive reader will notice that here the Born-Infeld parameter is called d in stead of b , like in section 4.3. We stick to the common notation as much as possible, but here we have to make a choice. We prefer the notation of Sen, who reserved the symbol b for a function in the ansatz for the near-horizon

configuration of rotating black holes

$$ds^2 = v_1(\theta) \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + \beta^2 d\theta^2 + \beta^2 v_2(\theta) (d\phi^2 + \alpha r dt^2)^2, \quad (5.2)$$

$$\phi_s = u_s(\theta), \quad (5.3)$$

$$\frac{1}{2} F_{\mu\nu}^i dx^\mu dx^\nu = (e_i + \alpha b_i(\theta)) dr \wedge dt + \partial_\theta b_i(\theta) d\theta \wedge (d\phi + \alpha r dt). \quad (5.4)$$

The symbols α , β and e_i represent constants, and v_1 , v_2 , u_s and b_i are functions of θ . Compared with the static case, the Riemann curvature becomes a bit more complicated in this ansatz:

$$\begin{aligned} R = & \frac{e^{-4\psi(\theta)} \beta \alpha^2}{2\Omega(\theta)} - \frac{2\Omega(\theta) \psi'(\theta)^2}{\beta} + \frac{2\Omega'(\theta)^2}{\beta \Omega(\theta)} \\ & - 2\beta \Omega(\theta) - \frac{2\psi'(\theta) \Omega'(\theta)}{\beta} - \frac{2\Omega(\theta) \psi''(\theta)}{\beta} - \frac{4\Omega''(\theta)}{\beta}. \end{aligned} \quad (5.5)$$

Now we turn to the entropy function. For rotating geometries it is defined as $\mathcal{E} = 2\pi (eq + J\alpha - 2\pi \int_0^\pi \mathcal{L} d\theta)$. After substituting the ansatz, the term $\sqrt{-\det(g)} R$ under the integral can be simplified: it contains some total derivatives which leave only boundary terms after integration. Boundary terms should vanish since the configuration is supposed to resemble empty space time at infinity. To prevent us ending up in a notational mess we will not write everything out in full, in stead we confine to following notation

$$\begin{aligned} \mathcal{L} = & \frac{e^{2\psi(\theta)} \beta \Omega(\theta)^3}{d^2} - \frac{2\psi'(\theta)^2 \Omega(\theta)}{\beta} - 2\beta \Omega(\theta) + \frac{2\Omega'(\theta)^2}{\beta \Omega(\theta)} + \frac{e^{-4\psi(\theta)} \alpha^2 \beta}{2\Omega(\theta)} \\ & - \frac{\sqrt{-(-e^{4\psi(\theta)} \Omega(\theta)^4 + d^2 e^2 + d^2 \alpha^2 b(\theta)^2 - 2d^2 e \alpha b(\theta)) (\beta^2 \Omega(\theta)^2 + d^2 b'(\theta)^2)}}{d^2}, \\ \mathcal{E} = & 2\pi (eq + J\alpha - 2\pi \int_0^\pi \mathcal{L} d\theta). \end{aligned} \quad (5.6)$$

Deleting total derivatives from the Lagrangian, as we did here, is actually a bit of a cheat, but for our purpose of minimizing the entropy function it does not matter. Indeed the next step is to extremize \mathcal{E} with respect to $\Omega(\theta)$, $\Psi(\theta)$, $b(\theta)$, α , β , and e . The equations of optimization are the usual Euler-Lagrange equations. For example, to minimize with respect to $\Omega(\theta)$, we have

$$\frac{\partial \mathcal{E}}{\partial \Omega} - \frac{\partial}{\partial \theta} \frac{\partial \mathcal{E}}{\partial \Omega'} = 0, \quad (5.7)$$

where a functional derivative is understood. In this way we obtain a system of six coupled differential and integral equations. Since this system is quite bulky, it is accommodated in Appendix B. Together equations B.1 till B.6 determine the exact form of the near-horizon configuration. In the following section we try to solve this configuration.

5.2 Dealing with the System of Differential Equations

The system B.1 - B.6 is rather complicated. It consists of six equations, and each of them involves three unknown functions of θ and three unknown constants.

Moreover, fractions and complicated square roots appear, and the first three equations contain integrals as well. This makes it quite a challenge to find a solution. Without doubt it will be necessary to separate this problem into a number of easier sub-problems. We will take the following road. From the complicated system we try to extract a single differential equation. This one should be as simple as possible. In particular we hope to arrive at an uncoupled differential equation which we can solve. What will left is a system of five equations involving two unknown θ -dependencies and three unknown constants. For the θ -dependencies, we try the same procedure again. If we succeed in obtaining all three θ -dependences, we are done. Only the constants still have to be dealt with. A black hole is characterised by the magnitudes of its mass, its electric charge, and its angular momentum, and so should be our solution. This means that the constants α , β , and e have to be expressed in terms of m , q and J . For this purpose three equations are available, but if it turns not possible to solve α , β , and e it would be no big deal: through equations B.4 and B.5 we can parametrise m , q and J in terms of α , β , and e . In this way it is still possible to determine the characteristics of each solution. Moreover, the theorem of local inversion from calculus tells us the parametrization is locally invertible in regions where it is smooth.

Now we have sketched the road we like to follow, it is time to set off the journey. The first step is to eliminate as much root terms as possible, and also we want to get rid of the integrals. In the following subsections we try several strategies that could lead to such simplifications.

5.2.1 Conserved currents eliminate the integrals

Equations B.4-B.6 are all of the form

$$q = \int_0^\pi I(\theta) d\theta. \quad (5.8)$$

For our purpose this shape is quite useless. Integration followed by an evaluation always yields a constant. This form is only useful for telling how to find the amount of electric charge of the near-horizon configuration once we know the θ -dependences. In order to extract some useful information we have to get rid of the integral. For q and I we like to use the terms 'charge' and 'current'. This perfectly reflects the structure of the equation. The charges associated with equations B.4-B.6 are respectively the electric charge q , the angular momentum J and the zero charge 0. The first two are indeed charges in the usual meaning. The last one is not. Notice that it is a bit unusual to call I a current, since it is a function that depends only on the angular variable θ . This does not dissuade us from introducing another term in addition to these two. When $\frac{\partial}{\partial \theta} I(\theta) = 0$ we will talk about a 'conserved current'. In that case, I will be a constant, and equation 5.8 tells us that $I = \frac{q}{2\pi}$. In this way we get rid of the integral.

Our system contains three equations like 5.8: each of them involves an integral we hope to remove. For convenience we name the three currents after the charges they are associated with: I_q , I_J and I_0 . In accordance with the above paragraph we start working out the derivative, $I'(\theta)$, of each of the currents. We arrive at three pretty complicated expressions, see equations B.7 and B.8 in the supplement. To determine whether these expressions yield zero, we try

to rewrite them in terms equations B.1-B.3. For the current I_0 this works out well: we can show that it's derivative is equal to

$$I'_0 = \frac{\Omega'(\theta)(B.1) + \psi'(\theta)(B.2) + b'(\theta)s.B.3}{\beta} = 0. \quad (5.9)$$

The first equality came about with some puzzle work, and the second equality is a consequence of the right-hand sides of equations B.1-B.3 being zero. According to the above paragraph, now we have got rid of one of the integrals. We may replace equation B.6 by

$$I_0 = 0. \quad (5.10)$$

Unfortunately I_0 is the only current that has a vanishing derivative. Before we can make this statement hard, we have to broaden the discussion a bit. Till now we have concentrated on three separate equations that fit in the form of equation 5.8. We passed over the fact that any linear combination of such equations will also take the form of 5.8. This means that all linear combinations of the three currents have to be treated as candidates for conserved currents. We will consider each of them, and check whether its derivative equals zero or not. Since we already know that $I'_0 = 0$, we only have to check (linear) combinations of I'_q and I'_j , but non the less the expression for I'_j may be used for simplification purposes. Before trying any (linear) combination, the expressions for I'_q and f'_j should be simplified as much as possible. For this task we deploy equations B.1 - B.3. We make use of the fact that each of them contains a second derivative of precisely one function of θ . There are two other equations that we can use for simplification purposes: these are I_0 and I'_0 , given by equations 5.9 and 5.10. The expression for I'_0 contains second derivatives of all three functions of θ while the expression for I_0 does not contain any double derivatives. We start trying to simplify I'_q . The expression for I'_q , (B.7), contains one double derivative, $b''(\theta)$. Equation B.3 can be used to eliminate this term. The coefficient of $b''(\theta)$ in I'_e is ¹

$$\frac{d^2(e - \alpha b(\theta))b'(\theta)}{\sqrt{(e^{4\psi(\theta)}\Omega(\theta)^4 - d^2e^2 - d^2\alpha^2b(\theta)^2 + 2d^2e\alpha b(\theta))(\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}}, \quad (5.11)$$

while the coefficient of $b''(\theta)$ in equation B.3 is

$$\frac{\beta^2\Omega(\theta)^2(e^{4\psi(\theta)}\Omega(\theta)^4 - d^2e^2 - d^2\alpha^2b(\theta)^2 + 2d^2e\alpha b(\theta))}{\sqrt{(e^{4\psi(\theta)}\Omega(\theta)^4 - d^2e^2 - d^2\alpha^2b(\theta)^2 + 2d^2e\alpha b(\theta))(\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}}. \quad (5.12)$$

This means that the combination we aim at cannot be of linear nature. We are forced to use the following combination

$$\begin{aligned} & \beta^2\Omega(\theta)^2 \left(e^{4\psi(\theta)}\Omega(\theta)^4 - d^2e^2 - d^2\alpha^2b(\theta)^2 + 2d^2e\alpha b(\theta) \right) \text{ B.7} \\ & - d^2v(e - \alpha b(\theta))b'(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) \text{ B.3.} \end{aligned} \quad (5.13)$$

This combination yields a fraction with the usual root term in the denominator. The numerator happens to be very complicated, see ???. The story for simplifying I'_j happens to be completely analogue. Again we find a slightly simpler

¹0 = (coef) $b''(\theta)$ + ...

expression by taking a nonlinear combination with B.3. The combination we take is

$$\beta^2 \Omega(\theta)^2 \left(e^{4\psi(\theta)} \Omega(\theta)^4 - d^2 e^2 - d^2 \alpha^2 b(\theta)^2 + 2d^2 e \alpha + b(\theta) \right) \cdot I'_J \quad (5.14)$$

$$-d^2 b(\theta) (\alpha b(\theta) - e) b'(\theta) (-\beta^2 \Omega(\theta)^2 - d^2 b'(\theta)^2) \cdot B.3. \quad (5.15)$$

At this stage the expressions for I'_q and I'_J are simplified as much as possible. First we will take a look at linear combinations that might vanish. Because the simplifications we just made involved nonlinear combinations we should consider the original expressions for I'_q and I'_J . The coefficients of $b''(\theta)$ of the two differ enormously: in a linear combination these coefficients can never yield zero. We have to focus on nonlinear combinations of the currents. Such combinations are of the form

$$I_{combi} = f_q(\theta) \cdot I_q(\theta) + f_J(\theta) \cdot I_J(\theta), \quad (5.16)$$

where f_q and f_J are some smooth functions periodic in θ . This kind of products introduce a complication: the integral operator and the differential operator do not commute with the operation of multiplication with a non-constant function of θ . To arrive at an expression for the associated charge q_{combi} we need to use the rule for partial integration, and to arrive at I'_{combi} , the product rule of differentiation is needed. The product rule tells to search for functions $f_q(\theta)$ and $f_J(\theta)$ satisfying

$$I'_{combi} = f_q(\theta) I'_q(\theta) + f'_q(\theta) I_q(\theta) + f_J(\theta) I'_J(\theta) + f'_J(\theta) I_J(\theta) = 0. \quad (5.17)$$

A glimpse at the equations in the supplement learns that expressions like these easily grow bigger than one page. Therefore, from this point on, we will merely summarize the results of our search for a solution to equation 5.16, and not include intermediate steps in the Appendix. What we do is writing out expression 5.17 and rephrase it into a single fraction. Then we isolate the numerator and dived away the common factors. What left is a complicated expression made up of functions of θ and their derivatives. We don't know any of the θ -dependencies and the expression is way to heavy to tackle as a whole. We tried to split the problem up into smaller peaces, but this did not yield a solution.

5.2.2 The roots natural enemy

The first thing to try to get rid of a square root is to take its square. To naturalize the denominators we multiply equations B.1 - B.3 with a root-term, and then we bring the resulting root terms to the left-hand side. We observe the following:

- From equation B.1 we obtain a long expression.
- Form equation B.2 we obtain an other long expression.
- The part left at the right side of equation B.3 is way to long to yield a usefull expression after squaring.

And again we start trying combinations of different expressions; this time we combine the two long expressions we obtained after the squaring. We arrive at a simpler equation when we take the combination

$$16\Omega^2(\theta)\text{square1} - \text{square2}. \quad (5.18)$$

The result is

$$\begin{aligned}
& -\Omega(\theta)^2 \left(+d^2 e^2 \beta^2 + d^2 \alpha^2 \beta^2 b(\theta)^2 - 2d^2 e \alpha \beta^2 b(\theta) \right)^2 \\
& - \Omega(\theta)^2 \left(-3e^{4\psi(\theta)} \beta^2 \Omega(\theta)^4 - 2d^2 e^{4\psi(\theta)} b'(\theta)^2 \Omega(\theta)^2 \right) \\
& - \frac{e^{-8\psi(\theta)} \left(-e^{4\psi(\theta)} \Omega(\theta)^4 + d^2 e^2 + d^2 \alpha^2 b(\theta)^2 - 2d^2 e \alpha b(\theta) \right) \left(\beta^2 \Omega(\theta)^2 + d^2 b'(\theta)^2 \right)}{4d^4 \beta^2 \Omega(\theta)^4} \\
& \cdot \left(-6e^{6\psi(\theta)} \beta^2 \Omega(\theta)^4 + 4d^2 e^{4\psi(\theta)} \left(\beta^2 + \psi'(\theta)^2 \right) \Omega(\theta)^2 + 8d^2 e^{4\psi(\theta)} \Omega''(\theta) \Omega(\theta) \right. \\
& \left. + d^2 \left(\alpha^2 \beta^2 - 4e^{4\psi(\theta)} \Omega'(\theta)^2 \right) \right)^2.
\end{aligned}$$

It is quite long, but we lost the root terms.

5.2.3 Other ways to lose the roots

Equation 5.10 contains only one root term; it can simply be multiplied away. What is left is

$$\begin{aligned}
0 = & \Omega(\theta)^3 \left(-e \alpha \beta^4 + \alpha^2 b(\theta) \beta^4 + 2d^2 e^{4\psi(\theta)} b'(\theta)^3 \psi'(\theta) \right) \quad (5.19) \\
& + e^{4\psi(\theta)} \beta^2 b'(\theta) \Omega'(\theta) \Omega(\theta)^4 + d^2 \beta^2 (e - \alpha b(\theta))^2 b'(\theta) \Omega'(\theta) \\
& + e^{4\psi(\theta)} \beta^2 (2b'(\theta) \psi'(\theta) + b''(\theta)) \Omega(\theta)^5 + 2d^2 e^{4\psi(\theta)} b'(\theta)^3 \Omega'(\theta) \Omega(\theta)^2 \\
& - d^2 \beta^2 (e - \alpha b(\theta)) \Omega(\theta) (\alpha b'(\theta)^2 + (e - \alpha b(\theta)) b''(\theta)).
\end{aligned}$$

The rest of the equations contain both terms with and without roots. For these equations it is no use to multiply them.

5.2.4 Combining all we have

On the way we found three equations that are free from root terms and integrals. From these equations we hope to extract a single differential equation that we can solve. Again we try to make linear combinations of equations to arrive at simpler ones. Notice that in the last two equations b appears in a complicated way. We found that it is impossible to remove this dependence with the use of relation one. This means the combination we are looking for can only contain b and its derivatives, and so we should be able to delete ω and ϕ in some combination. We investigated this possibility, but we found no way to accomplish this.

We are forced to conclude that we were not able to extract, from the complicated system, a simple differential equation in one function of θ . This blocks our attempts to solve the system that appears from minimizing the entropy function of the rotating EBI black hole.

Chapter 6

A Simpler Problem

Before we can work out a relation for the entropy of a rotating Born-Infeld black hole, we need a mathematical description of its near-horizon geometry. In the previous chapter we calculated the entropy function for a Rotating Born-Infeld Black hole and, by minimizing this function, we tried to find the corresponding near-horizon geometry. Unfortunately we were not able to solve the system of differential and integral equations that emerged. In particular the huge root terms made this hard. The minimizing equations for the entropy function involve first- and second order derivatives. Applied to the root terms this results in equations with a lot of terms.

In this chapter we want to develop a taste for how to solve complex systems equations like the one of the previous chapter. For this purpose it might be fruitful to study a simplified problem. For example, we can expand the Lagrangian in the Born-Infeld parameter (d in the previous chapter); then the roots drops out completely, and the analysis simplifies a lot. The cost of a simplification is the disappearance of similarity with the original problem. Another way to reduce the complexity of our problem, is to constrain the ansatz for the near-horizon geometry further. In this way we keep the root terms but decrease the complexity. In this chapter we explore the consequences of constraining the ansatz.

6.1 A Simplified Ansatz

Before considering what constraint we might impose on the ansatz for the near-horizon geometry, let us remind how the ansatz of Sen came about. Sen's ansatz encompasses the most general near-horizon configuration for an extremal rotating black hole, and in section 3.2.3 we saw it can be obtained by taking near-horizon limit of the most general metric for a rotating black hole. When we confine Sen's ansatz, it is eligible to impose constraints on the original black hole metric, and not on its near-horizon geometry directly. We are more comfortable with black hole solutions than with near-horizon geometries; if we want to formulate constraints that make sense, we better stick to black hole geometries.

In a Born-Infeld black hole spacetime, at asymptotic infinity the electric charge should not influence the local geometry of spacetime: far away from the black hole, space time appears just like a Kerr-Newman spacetime. In no case we

may restrict the black hole ansatz in such a way that we exclude Kerr-Newman geometries. Our constraints should respect this requirement. The most general metric for a rotating black hole is

$$ds^2 = \frac{dt^2 \Delta \rho^2}{\Sigma^2} - d\theta^2 \rho^2 - \frac{\rho^2 dr^2}{\Delta} - \frac{\Sigma^2 \sin^2(\theta) (d\phi (\alpha (r^2 + \alpha^2 - \Delta)) dt)}{\rho^2 \Sigma^2}. \quad (6.1)$$

This metric contains three (a priori) unknown functions of θ : ρ , Δ and Σ . Every axisymmetric rotating black hole metric can be written in this shape. If the metric under consideration represents the Kerr-Newman black hole then we know all θ -dependencies, and it is possible to rewrite the metric into Kerr-Schild form [3]

$$ds^2 = \frac{\Delta \sin^2(\theta) (dt - \alpha \sin^2(\theta))^2}{\rho^2} - \frac{\rho^2 dr^2}{\Delta} - d\theta^2 \rho^2 - \frac{\sin^2(\theta) ((r^2 + \alpha^2) d\phi - \alpha dt)^2}{\rho^2}. \quad (6.2)$$

We decide to use this form for our ansatz, and in addition we require that it can be rewritten into the Boyer-Lindquist form for rotating metrics. This makes it necessary to fix ρ and Σ on forehand: we have to take ρ and Σ as in the Kerr-Newman metric. Such an assumption is unrealistic, but it should be stressed that this undertaking should be seen just as an exercise that might lead us to the right track. We expect to arrive at a simpler system containing root terms. At the same time we are sure not to spoil the desired asymptotic behavior: the Kerr-Newman solution fits perfectly in this ansatz. We summarize the proposed constraints on the black hole ansatz, equation 6.2,

$$\rho^2 = r^2 + \alpha^2 \cos^2(\theta), \quad (6.3)$$

$$\Delta = (r - r_h)^2 G(r_h, \theta), \quad (6.4)$$

$$\Sigma^2 = (r^2 + \alpha^2)^2 - \delta \alpha^2 \sin^2(\theta), \quad (6.5)$$

and for convenience we rewrite the metric into Kerr-Schild form

$$ds^2 = \frac{\Delta \sin^2(\theta) (dt - \alpha \sin^2(\theta))^2}{\rho^2} - d\theta^2 \rho^2 - \frac{\rho^2 dr^2}{\Delta} - \frac{\sin^2(\theta) ((r^2 + \alpha^2) d\phi - \alpha dt)^2}{\rho^2}; \quad (6.6)$$

Note (again) that this rephrasing is in general impossible, but with the constraints of (6.3) it is allowed.

6.2 The Near-Horizon Limit

Here we determine what the near-horizon limit for the ansatz (6.6) looks like. We start with a coordinate transformation

$$dt \rightarrow \lambda^{-1} dt, \quad (6.7)$$

$$dr \rightarrow \lambda dr, \quad (6.8)$$

$$d\theta \rightarrow d\theta, \quad (6.9)$$

$$d\phi \rightarrow h \lambda^{-1} dt. \quad (6.10)$$

$$(6.11)$$

The constant h will be determined below; it has the same function as in section 3.2.2: it is used to make sure that all negative powers of λ disappear before we take the limit $\lambda \rightarrow 0$. Under this transformation

$$\rho^2 \rightarrow \alpha^2 \cos^2(\theta) + r_h^2, \quad (6.12)$$

$$\Delta \rightarrow \lambda^2 r^2 G(\theta), \quad (6.13)$$

$$\Sigma^2 \rightarrow (\alpha^2 + r_h^2)^2, \quad (6.14)$$

and

$$\begin{aligned} ds^2 \rightarrow & \frac{G(\theta)r^2 (\alpha^2 \cos^2(\theta) + r_h^2)}{(\alpha^2 + r_h^2)^2} dt^2 - \frac{(\alpha^2 \cos^2(\theta) + r_h^2)}{r^2 G(\theta)} dr^2 - \rho^2 d\theta^2 \\ & - \frac{(\alpha^2 + r_h^2)^2 \sin^2(\theta) \left(d\phi - \frac{\alpha((\lambda r - r_h)^2 - \lambda^2 r^2 G(\theta)) dt}{(\alpha^2 + r_h^2)^2 \lambda} + \frac{dt h}{\lambda} \right)}{(\alpha^2 \cos^2(\theta) + r_h^2)^2} \end{aligned} \quad (6.15)$$

$$\begin{aligned} = & \frac{G(\theta)r^2 (\alpha^2 \cos^2(\theta) + r_h^2)}{(\alpha^2 + r_h^2) dt^2} - \frac{(\alpha^2 \cos^2(\theta) + r_h^2) dr^2}{r^2 G(\theta)} dt^2 \\ & - \frac{(\alpha^2 + r_h^2)^2 \sin^2(\theta) \left(d\phi + \frac{dt(2\alpha r_h r)}{(\alpha^2 + r_h^2)^2} \right)}{(\alpha^2 \cos^2(\theta) + r_h^2)^2} - (\alpha^2 \cos^2(\theta) + r_h^2) d\theta^2 \end{aligned} \quad (6.16)$$

$$\begin{aligned} = & \frac{1}{\alpha^2 + r_h^2} (\alpha^2 \cos^2(\theta) + r_h^2) \left(-\frac{(\alpha^2 + r_h^2) dr^2}{r^2 G(\theta)} - (\alpha^2 + r_h^2) d\theta^2 + \frac{dt^2 r^2 G(\theta)}{\alpha^2 + r_h^2} \right) \\ & - \frac{(\alpha^2 + r_h^2)^2 \sin^2(\theta) \left(d\phi + \frac{dt(2\alpha r_h r)}{(\alpha^2 + r_h^2)^2} \right)}{(\alpha^2 \cos^2(\theta) + r_h^2)^2}, \end{aligned} \quad (6.17)$$

where we took $h = \frac{\alpha}{r_h^2 + \alpha^2}$ to make all terms of order λ^{-1} cancel.

After a second coordinate transformation,

$$dt \rightarrow \frac{(\alpha^2 + r_h^2) dt}{G(\theta)}, \quad (6.18)$$

the metric is written as

$$\begin{aligned} ds^2 = & (\alpha^2 \cos^2(\theta) + r_h^2) \frac{(-dr^2 + dt^2 r^2 - d\theta^2 G(\theta))}{G(\theta)} \\ & - (\alpha^2 \cos^2(\theta) + r_h^2)^2 \sin^2(\theta) (\alpha^2 + r_h^2)^2 \left(d\phi + \frac{2dtr\alpha r_h}{(\alpha^2 + r_h^2)^2} \right). \end{aligned} \quad (6.19)$$

This is the (constrained) ansatz for the near-horizon geometry over which we will minimize the entropy function.

6.3 The Entropy Function

In section 3.2.4 the entropy function was defined as

$$f(\hat{u}, \hat{v}, \hat{e}, \hat{p}) = \int d\theta d\phi \sqrt{-\det g\mathcal{L}}. \quad (6.20)$$

We take for \mathcal{L} the Einstein-Born-Infeld Lagrangian that was written down in section 5.1,

$$\mathcal{L}_{EBI} = \sqrt{-\det(g)}R + \frac{4}{d^2} \left(\sqrt{-\det(g)} - \sqrt{-\det(g + dF)} \right). \quad (6.21)$$

It is convenient to write the metric in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{2} \left(\frac{2r^2 x}{G(\theta)} - \frac{8r^2 \alpha^2 \sin^2(\theta) r_h^2}{y^2 x} \right) & 0 & 0 & -\frac{2r\alpha \sin^2(\theta) r_h}{x} \\ 0 & -\frac{x}{G(\theta)} & 0 & 0 \\ 0 & 0 & -x & 0 \\ -\frac{2r\alpha \sin^2(\theta) r_h}{x} & 0 & 0 & -\frac{\sin^2(\theta) (\alpha^2 + r_h^2)^2}{x} \end{pmatrix}, \quad (6.22)$$

where

$$x = (\alpha^2 \cos^2(\theta) + r_h^2), \quad (6.23)$$

$$y = (\alpha^2 + r_h^2). \quad (6.24)$$

The determinant is

$$\det(g_{\mu\nu}) = -\frac{r^2 \sin^2(\theta) (\alpha^2 + r_h^2)^2 (\alpha^2 \cos^2(\theta) + r_h^2)^2}{G(\theta)^2}. \quad (6.25)$$

For the calculation of the Riemann-curvature we use a computer algebra package (Mathematica). The computer takes about an hour to come up with a very complicated answer. We investigated the system of equations that arises when substituting this Riemann curvature in our ansatz, but the complexity of the Riemann scalar frustrated all our attempts. Sen used a slightly different appearance for his line-element ansatz and it is clear why: with his parametrization of the metric the the Riemann tensor is substantially simpler. The way out seems clear: our ansatz has to be rewritten in such a way that it resembles Sen's ansatz. This is possible since the near-horizon geometry of any rotating extremal black hole can be rewritten into that shape (see section 3.2.3, page 19). It is not hard to see that a suitable transformation on the θ coordinate is sufficient to make (6.19) fit into 3.45,

$$d\theta \rightarrow dG(\theta)d\theta. \quad (6.26)$$

We are free to do this, but we end up with a practical problem. Some of the θ -dependencies are already fixed. therefore we have to keep track of what happens the θ coordinate closely. The transformation of $d\theta$ does not bother us, even though G not known on forehand. For θ this becomes a problem because we cannot evaluate the integral

$$\theta \rightarrow \theta' = \int G(\theta)d\theta, \quad (6.27)$$

explicitly. We do not want to have the abstract inverse function of this integral to appear in the metric This will make the minimization of the entropy function even more problematic. This forces us to conclude this ansatz is a dead end.

6.4 Discussion

The failure of the attempts in this chapter lead us to think. It is clear that the problem is complicated and moreover that success depends heavily on the choice of the simplifications and on the way in which the ansatz is written. At this stage it is wise to stick to the paved roads as close as possible. There is a lot to learn by investigating the consequences of small modifications. An example of such an approach the Taylor expansion mentioned in the beginning of this chapter. Instead of using the Born-Infeld Lagrangian which includes root terms, we can make an expansion in the Born-Infeld parameter d^2 . In section 4 we already did this up to order d^2 . The result was

$$\mathcal{L}_{EBI} = -F_{ab}F^{ab} + \frac{d^2}{4} \left((F_{ab}F^{ab})^2 + (F_{ab} \star F^{ab})^2 \right); \quad (6.28)$$

which clearly yields a nonlinear complication with respect to the Maxwell Lagrangian. It would be very interesting to try whether we can minimize the associated entropy function. We have to admit that this simplification is far more promising than the simplification we just worked out. First of all the restrictions on the ansatz we chose were rather artificial. It is certainly true that in the far away limit and in the limit of a small Born-Infeld parameter the metric (locally) should look like the metric of Kerr-Newman, but the metric we look for will also have a certain d -dependence, and by fixing parts of the ansatz we exclude a large class of possible d -dependencies on forehand. Moreover from the very beginning it was clear that the kind of equations we deal with get complicated very rapidly. The analysis for the simplest rotating black hole near-horizon geometries is already quite involved: in the article of Sen we see that adding a non-constant scalar field is enough to make the system of equations practically unsolvable [24].

Nonetheless the essential of our attempt may be useful provided that we take in account the insights that came a bit to late.

1. The complexity of the system of equations that appears can be reduced by imposing certain extra conditions on the ansatz.
2. In the limit of slow rotation, we should obtain the usual Kerr-Newman results. We have to take this into account.
3. We should depart from the preconceived roads as less as possible. This means changing a minimum number of parameters at the same time.

For example we could fix the near-horizon geometry as Kerr-Newman, and start changing the electromagnetic part of the Lagrangian bit by bit while we keep track of the changes in the equations for the field strength tensor and its solutions. On the other hand we could fix the field strength tensor and assume simple modifications to the metric; now the near-horizon geometry should be adapted accordingly.

Chapter 7

Algorithm in the Near-Horizon Limit

In 2000, Grilo Lombardo, employing the Newman-Janis algorithm, tried to find a rotating Born-Infeld black-hole metric. As mentioned in chapter 5, the metric he found cannot represent a rotating-Born-Infeld-charge. In this chapter, we turn to the Newman-Janis algorithm again, but we moderate our goal a bit: Instead of targeting on a EBI black hole, we aim to find an expression for the near-horizon geometry. We investigate whether the Newman-Janis Algorithm generate the near- horizon configuration of a rotating EBI black hole. The idea is to start with a static EBI black hole solution, determine it's near-horizon geometry and use it as seed metric in the Newman-Janis algorithm. In this way we hope to arrive at the near-horizon metric of a rotating EBI black hole. Then, to obtain the entropy, we can evaluate the entropy function (see chapter 5) for this geometry. As far as we know this combined approach has never been tried before.

In fact, we do not know anything about behavior of the Newman-Janis algorithm in the near-horizon limit. We saw that for Einstein-Maxwell theory black hole solutions can be grouped into four classes. These are the static black holes, the rotating black holes and their near-horizon configurations. For any usual Lagrangian model of gravity that admits rotating axis-symmetric extremal black hole solutions these four types of black hole solutions appear. The static solution is obtained from the rotating solution by setting the angular velocity to zero. The near-horizon geometries are obtained by taking the near-horizon limit. This is a limit over a continuous one-parameter class of solutions to some system of differential equations. Such a limit is always a solution itself, and so is the near-horizon geometry. The following scheme lists the connections that may exist between these black holes solutions when they are extremal. To obtain a static metric from a rotating one it suffices to set the angular velocity to zero. It is sometimes possible to go the other way around: a rotating metric can sometimes be generated from its static counterpart, but the Newman-Janis algorithm works not for all black holes. When the black holes under consideration are extremal we can move down in the scheme, and find the near-horizon geometry by taking the near-horizon limit. Obviously at the left-hand side this results in a static near-horizon metric and at the right-hand side it yields a ro-

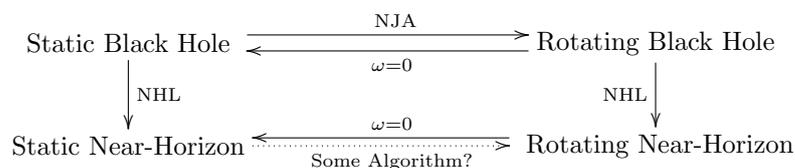


Figure 7.1: Commutative diagram of extremal black hole geometries. NJA stands for 'Newman-Janis algorithm' and NHL means 'near-horizon limit'.

tating near-horizon metric. Also at the bottom of the diagram we can go from the right-hand side to the left-hand side by assuming zero angular momentum, but at the bottom of the diagram we do not know how to go from the left to the right. We speculate that the Newman-Janis algorithm is able to. If it does, we would like to say that the Newman-Janis algorithm and the near-horizon limit 'commute':

$$[\text{Newman-Janis Algorithm, Near-Horizon Limit}] = 0, \quad (7.1)$$

in the sense that the order in which the two are applied does not matter. In this chapter, we will investigate this issue and other issues related to the lower part of the diagram. We start with an exploration of the effects of the Newman-Janis algorithm on the near-horizon geometry of the extremal Reissner-Nordström black hole.

7.1 The Near-Horizon Analogy

The Schwarzschild solution is the simplest static black hole solution. In 1965, Newman and Janis presented, what they called, 'a curious derivation of the Kerr metric by performing a complex coordinate transformation on the Schwarzschild metric' [26]. This 'derivation' proved to work for other black hole solutions as well. In this section, we repeat the calculations of Newman and Janis to check whether the algorithm works for the near-horizon limit of the simplest black hole solution: the Reissner-Nordström black hole.

The near-horizon limit of the Reissner-Nordström metric was calculated in section 3.2.3. We found

$$ds^2 = r_0^2 \left(-r^2 dt^2 + \frac{dr^2}{r^2} + r^2 (d\theta^2 + d\phi^2 \cos^2(\theta)) \right), \quad (7.2)$$

where $r_0 = m$. This metric has to be rewritten into advanced null-coordinates. The parametrization of the outgoing null-coordinate u is

$$u = t - \frac{1}{r}. \quad (7.3)$$

We re-express the time parameter t in terms of the outgoing null-coordinate u . In this coordinate system the line element appears as

$$ds^2 = du^2 r^2 + 2drdu - (d\theta^2 + d\phi^2 \cos^2(\theta)) r^2. \quad (7.4)$$

it can be reproduced¹ with the following tetrad:

$$l^\mu = (0, 1, 0, 0), \quad (7.5)$$

$$n^\mu = \left(1, -\frac{r^2}{2}, 0, 0\right), \quad (7.6)$$

$$m^\mu = \frac{\left(0, 0, 1, \frac{i}{\sin(\theta)}\right)}{\sqrt{2}r}. \quad (7.7)$$

Before we perform a complex coordinate transformation, we extend the domain of the parameter r to the field of complex numbers. We use the same coordinate transformation that Newman and Janis used:

$$\tilde{x}^v = x^v + i\alpha \cos(\theta) (\delta_0^v - \delta_1^v). \quad (7.8)$$

We parametrize $\frac{1}{r}$ by $\frac{1}{\bar{r}}$ and r^2 by $r\bar{r}$, where \bar{r} is the conjugate variable. To see how the metric transforms, we substitute the new coordinates into expression 7.5. Since the tetrad is made up of vectors we should respect the vector transformation law. The result is

This new tetrad generates the following metric

$$g^{\mu\nu} = \begin{pmatrix} -\frac{\alpha^2 \sin^2(\theta)}{r^2 + \alpha^2} & \frac{r^2 + \alpha^2}{\rho} & 0 & -\frac{\alpha}{\rho} \\ \frac{r^2 + \alpha^2}{\rho} & -\frac{\rho^2 + \alpha^2 \sin^2(\theta)}{\rho} & 0 & \frac{\alpha}{\rho} \\ 0 & 0 & -\frac{1}{\rho} & 0 \\ -\frac{\alpha}{\rho} & \frac{\alpha}{\rho} & 0 & -\frac{\csc^2(\theta)}{\rho} \end{pmatrix}, \quad (7.9)$$

where ρ is $r^2 + \alpha^2 \cos^2(\theta)$. We examine also the inverse metric

$$g_{\mu\nu} = \begin{pmatrix} \rho & 1 & 0 & -\alpha(\rho - 1)\sin^2(\theta) \\ 1 & 0 & 0 & -\alpha\sin^2(\theta) \\ 0 & 0 & -\rho & 0 \\ -\alpha(\rho - 1)\sin^2(\theta) & -\alpha\sin^2(\theta) & 0 & \sigma \end{pmatrix}, \quad (7.10)$$

where $g_{44} = \sigma = \alpha^2 \sin^4(\theta) (-r^2 + (r^2 - 2)) + \alpha^2 \sin^2(\theta) \cos^2(\theta) (\alpha^2 \sin^2(\theta) - 1)$. The element g^{11} shows a θ -dependence which is very odd. We expected to end up with a metric in advanced null coordinates; in such coordinates g^{11} is always zero. The algorithm failed to produce the near-horizon geometry of the Kerr-Newman metric.

7.2 The Kerr-Newman Near-Horizon Geometry

In section 3.2.3, we calculated the near-horizon limit of the Reissner-Nordström metric and in the previous section we applied the Newman-Janis algorithm to this near-horizon geometry. In this section, we calculate the near-horizon geometry of the Kerr-Newman metric and we transform it to advanced null coordinates. Then we try to modify the algorithm to make it produce the desired Near-Horizon geometry.

¹see formula 3.67 for the decomposition of the metric into a null tetrad.

7.2.1 Near-Horizon Limit of the Kerr-Newman Metric

We write the Kerr-Newman metric slightly different compared to section 3.3.1,

$$\begin{aligned}
ds^2 &= -\frac{\Delta\rho^2}{\Sigma^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\Sigma^2}{\rho^2}\sin^2(\theta)(d\phi - \omega dt)^2 + \rho^2d\theta^2, \\
\rho^2 &= r^2 + \alpha^2\cos^2\theta, \\
\Delta^2 &= r^2 - 2Mr + \alpha^2 + Q^2, \\
\Sigma^2 &= (r^2 + \alpha^2)^2 - \Delta\alpha^2\sin^2(\theta) \\
\omega &= \frac{\alpha(r^2 + \alpha^2 - \Delta)}{\Sigma}.
\end{aligned} \tag{7.11}$$

These Boyer-Lindquist coordinates are useful to emphasize the terms responsible for the rotation of the black hole. Before taking the near-horizon limit, we have to adjust the parameters such to make the black hole solution extremal. In section 3.2.2, we have seen that the horizons should coincide. The extremal limit corresponds to $\alpha^2 + Q^2 = M^2$: for this choice $\Delta^2 = r^2 - 2Mr + M^2 = (r - M)^2$, indicating that only one event horizon, with a double zero, is left. This horizon is 'rotating' with angular velocity ω at $r = M$.

We start with the coordinate transformation familiar from section 3.2.3, see formula 3.32,

$$\begin{aligned}
r &\rightarrow M + \lambda r, \\
t &\rightarrow \frac{t}{\lambda}, \\
\phi &\rightarrow \phi + \beta t,
\end{aligned} \tag{7.12}$$

In addition, we add a transformation of ϕ in the t -direction. This is used to eliminate a divergent contribution λ^{-1} of the rotation term. The parameter β still has to be determined.

The building blocks of the metric can be expressed in terms of the new coordinates:

$$\begin{aligned}
\rho^2 &\rightarrow (M + \lambda r)^2 + \alpha^2\cos^2\theta, \\
\Delta^2 &\rightarrow (\lambda r)^2, \\
\Sigma^2 &\rightarrow ((M + \lambda r)^2 + \alpha^2)^2 - \Delta\alpha^2\sin^2(\theta), \\
\omega &\rightarrow \frac{\alpha((M + \lambda r)^2 + \alpha^2 - \Delta)}{\Sigma},
\end{aligned} \tag{7.13}$$

and also the differential basis can be rewritten:

$$\begin{aligned}
dt &\rightarrow \lambda^{-1}dt, \\
dr &\rightarrow \lambda dr,
\end{aligned} \tag{7.14}$$

$$d\phi \rightarrow d\phi + \beta\lambda^{-1}dt. \tag{7.15}$$

Before rewriting the complete metric in terms of the new coordinates we concentrate on the rotational terms in order to fix the constant β . We anticipate on the fact that in a moment we will take the limit $\lambda \rightarrow 0$: all terms of the

metric containing λ^n with $n \geq 1$ are omitted. Terms with $n \leq -1$ have to be eliminated in order to prevent infinities to arise from the limit. The rotation term of the metric, $\frac{\Sigma^2}{\rho^2} \sin^2(\theta) (d\phi - \omega dt)^2$, is transformed in

$$\frac{(M^2 + \alpha^2)^2}{M^2 + \alpha^2 \cos^2 \theta} \sin^2(\theta) \left(d\phi + \beta dt - \alpha \frac{M^2 + \alpha^2 + 2M\lambda r}{M^2 + \alpha^2} \lambda^{-1} dt \right)^2.$$

The term $\alpha \lambda^{-1} dt$ can be eliminated by taking $\beta = \alpha$. Now we can safely take the limit, and the near-horizon geometry that is left can be written as

$$\begin{aligned} ds^2 = & \left(1 - \frac{\alpha^2}{r_0^2} \sin^2(\theta) \right) \left[-\frac{r^2}{r_0^2} dt^2 + \frac{r_0^2}{r^2} dr^2 + r_0^2 d\theta^2 \right] \\ & + r_0^2 \sin^2(\theta) \left(1 - \frac{\alpha^2}{r_0^2} \sin^2(\theta) \right)^{-1} \left(d\phi + \frac{2\alpha r M}{r_0^4} dt \right)^2, \end{aligned} \quad (7.16)$$

where $r_0 = M$.

7.2.2 Kerr-Newman Near-Horizon Geometry in Advanced Null Coordinates

We want to know the near-horizon geometry of the Kerr-Newman in advanced null coordinates. Following the procedure of section 3.3.2, we obtain a new (differential) coordinate basis

$$\begin{aligned} dt &= r_0^2 d\tilde{t}, \\ du &= dt + \frac{1}{r^2} d\tilde{r}, \end{aligned} \quad (7.17)$$

where the old coordinates are marked with $\tilde{\cdot}$. The transformation of the t -coordinate makes it possible to factor out the r_0^2 -term of the first part of the metric. With respect to the coordinates (u, r, θ, ϕ) the metric is written as

$$\begin{aligned} ds^2 = & r_0^2 \left(1 - \frac{\alpha^2 \sin^2(\theta)}{r_0^2} \right) [-r^2 du^2 + 2drdu + d\theta^2] \\ & + \frac{r_0^2 \sin^2(\theta)^2}{1 - \frac{\alpha^2 \sin^2(\theta)}{r_0^2}} \left(\frac{dr}{r} - \frac{(2\alpha m)rdu}{r_0} + d\phi \right)^2. \end{aligned} \quad (7.18)$$

We see that we also have to translate $d\phi$ with

$$\phi \rightarrow \phi - \frac{1}{r} dr. \quad (7.19)$$

The metric becomes

$$\begin{aligned} ds^2 = & r_0^2 \left(1 - \frac{\alpha^2 \sin^2(\theta)}{r_0^2} \right) [-r^2 du^2 + 2drdu + \theta^2] \\ & + \frac{r_0^2 \sin^2(\theta)}{1 - \frac{\alpha^2 \sin^2(\theta)}{r_0^2}} \left(d\phi - \frac{2\alpha m r}{r_0} du \right)^2; \end{aligned} \quad (7.20)$$

and in matrix notation

$$g_{\mu\nu} = \begin{pmatrix} \left(4\alpha^2 m^2 \frac{r_0^2 \sin^2(\theta)}{\sigma} - \sigma\right) r^2(\sigma) & \sigma & 0 & -2\alpha m r_0^3 \frac{\sin^2(\theta)}{\sigma} r \\ \sigma & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ -2\alpha m r_0^3 \frac{\sin^2(\theta)}{\sigma} r & 0 & 0 & \frac{r_0^4 \sin^2(\theta)}{\sigma} \end{pmatrix}, \quad (7.21)$$

and

$$g^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{\sigma} & 0 & 0 \\ \frac{1}{\sigma} & \frac{r^2}{\sigma} & 0 & \frac{2mr\alpha}{r_0\sigma} \\ 0 & 0 & \frac{1}{\sigma} & 0 \\ 0 & \frac{2mr\alpha}{r_0\sigma} & 0 & \frac{\csc^2(\theta)r_0^2 - \alpha^2}{r_0^4} \end{pmatrix}, \quad (7.22)$$

where $\sigma = r_0^2 - \alpha^2 \sin^2(\theta)$. We will find a tetrad to parametrize the covariant metric.

7.2.3 Tetrad for the Near-Horizon Metric

The Newman-Janis algorithm does not generate the near-horizon geometry of the Kerr-Newman black hole. In this section we try to modify the algorithm to make it work in the near-horizon setting. Before we start thinking about the algorithm itself, we have to define what has to be its input and output. It is easiest to concentrate on the simplest black hole metrics; those for which the Newman-Janis algorithm proved to work. We like the algorithm to transform the near-horizon geometry of the extremal Reissner-Nordström black hole into the near-horizon geometry of the extremal Kerr-Newman black hole. The first step is to calculate the expressions for these configurations.

What Tetrad can generate the near-horizon geometry of the Kerr-Newman metric (7.22)? We discuss the two simplest tetrads.

The root tetrad

The first is based on the tetrad that we used when we applied the Newman-Janis algorithm to the Reissner-Nordström metric in section 3.3.4. The tetrad we found for the near-horizon limit of the Kerr-Newman near-horizon geometry is

$$\begin{aligned} l^\mu &= (0, 1, 0, 0), \\ n^\mu &= \frac{1}{\sigma} \left(1, \frac{2m^2 \alpha^2 r_0^2 \sin^2(\theta)}{\sigma} - \frac{r^2}{2}, 0, 0 \right) \\ m^\mu &= \left(0, \frac{\sqrt{2} m r_0 \alpha r \sin(\theta)}{\sigma^{3/2}}, \frac{-i}{\sqrt{2}\sigma}, \frac{-\sqrt{\sigma} \csc \theta}{\sqrt{2} r_0^2} \right) \end{aligned}$$

An important difference with the tetrad for Reissner-Nordström and the Kerr-Newman metric is that this tetrad contains root terms. To see what happened,

we reprint the Kerr-Newman metric and tetrad. The Kerr-Newman metric is

$$g^{\mu\nu} = \begin{pmatrix} -\frac{\alpha^2 \sin^2(\theta)}{\rho} & \frac{r^2 + \alpha^2}{\rho} & 0 & -\frac{\alpha}{\rho} \\ \frac{r^2 + \alpha^2}{\rho} & -\frac{Q^2 + r^2 + \alpha^2 - 2mr}{\rho} & 0 & \frac{\alpha}{\rho} \\ 0 & 0 & -\frac{1}{\rho} & 0 \\ -\frac{\alpha}{\rho} & \frac{\alpha}{\rho} & 0 & -\frac{\csc^2(\theta)}{\rho} \end{pmatrix}, \quad (7.23)$$

and the tetrad we used is

$$\begin{aligned} l^\mu &= \delta_1^\mu, \\ n^\mu &= \delta_0^\mu - \frac{(Q^2 + \alpha^2 \cos^2(\theta) + r(r - 2M))}{2(r^2 + \alpha^2 \cos^2(\theta))} \delta_1^\mu, \\ m^\mu &= \left(\frac{i\alpha \sin(\theta)}{\sqrt{2}(r - i\alpha \cos(\theta))} \right) \delta_0^\mu - \frac{i\alpha \sin(\theta)}{\sqrt{2}(r - i\alpha \cos(\theta))} \delta_1^\mu \\ &\quad + \frac{1}{\sqrt{2}(r - i\alpha \cos(\theta))} \delta_2^\mu + \frac{i \csc(\theta)}{\sqrt{2}(r - i\alpha \cos(\theta))} \delta_3^\mu. \end{aligned} \quad (7.24)$$

Also recall the generating formula for the metric:

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu. \quad (7.25)$$

In matrix notation this is a sum of outer products. The outer product \otimes is a map that adds a matrix to each pair of two vectors, $\otimes : \mathbb{R}^n \times \mathbb{R}^n \rightarrow M_{(n,n)}(\mathbb{R})$, such that $(u \otimes v)_{ij} = u_i \cdot v_j$. If the vector u contains k nonzero elements, and the vector v contains l nonzero elements then their outer product $u \otimes v$ is a matrix containing $k \cdot l$ nonzero elements. We see that l and n generate the symmetric upper-left (2×2) part of the matrix $g^{\mu\nu}$. The vector m has four nonzero elements in \mathbb{C} , but the combination of m and \bar{m} generates less than sixteen elements: only the real parts of the products $m \otimes \bar{m}$ and $\bar{m} \otimes m$ survive the addition $m \otimes \bar{m} + \bar{m} \otimes m$. The element m_3 has an extra factor i compared to the other elements, and so $g^{ij} = 0$ for $i = 2, j \neq 2$. No other complex elements in $m \otimes \bar{m}$ appears; the vector m gives a contribution to all the nonzero elements of g .

We try to alter this tetrad such to obtain the metric 7.22, the metric of the Kerr-Newman near-horizon geometry. Looking at the Kerr-Newman metric, (7.23), we see that now also the elements $g^{0\nu}$, $\nu \neq 1$ become zero. We need to have $m_0 = 0$. Now we have all zero elements at the right place, we concentrate on nonzero terms. The denominators that can be generated are either products of two real terms or they are the absolute value of one complex term. In both ways it is not possible to obtain a difference of two squares if we are not using square roots in the tetrad. To produce the difference of squares that appears in (7.23), we arrive at the 'root tetrad'.

The Simple Tetrad

The root tetrad was based on the one that generates the Kerr-Newman metric. To avoid square roots in the tetrad, we build on from scratch. We start with

the upper part and the left part of the metric. Because the number of nonzero terms is limited, and in particular $g^{11} = 0$, we propose

$$l^\mu = (0, 1, 0, 0), \quad (7.26)$$

together with

$$n^\mu = \left(\frac{1}{r_0^2 - \alpha^2 \sin^2(\theta)}, -\frac{r^2}{2(\alpha^2 \sin^2(\theta) - r_0^2)}, 0, \frac{2Mr\alpha}{r_0^3 - \alpha^2 \sin^2(\theta)r_0} \right), \quad (7.27)$$

to have the right $(1, i)$ -terms. These two vectors make up the second row and the second column. Now the vector m has to be determined; it has to generate the diagonal 2×2 matrix in the right lower part of $g^{\mu\nu}$. There should be no cross-terms. To stay in the lower-right corner we have take $m_1 = m_2 = 0$, and to suppress cross-terms, there should be a factor i difference in m_3 and m_4 . We find

$$m^\mu = \left(0, 0, \frac{i}{\sqrt{2}(r_0^2 - \alpha^2 \sin^2(\theta))}, \frac{\csc^2(\theta)r_0^2 - \alpha^2}{\sqrt{2}r_0^4} \right). \quad (7.28)$$

7.2.4 Search for a New Algorithm

The heart of the Newman-Janis algorithm the transformation rule. Since the algorithm does not work in the near-horizon geometry, we try whether we can find an alternative transformation rule that can be used to transform the Reissner-Nordström near-horizon geometry to the Kerr-Newman near-horizon geometry. The line element of the Reissner-Nordström near-horizon geometry is

$$ds^2 = du^2 r^2 - (d\theta^2 + d\phi^2 \cos^2(\theta)) r^2 + 2drdu, \quad (7.29)$$

and the inverse metric is

$$g^{\mu\nu} = \frac{1}{r_0^2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -r^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{\csc^2(\theta)}{r^2} \end{pmatrix}; \quad (7.30)$$

It can be expressed using the following tetrad:

$$\begin{aligned} l^\mu &= \frac{1}{r_0^2} (0, 1, 0, 0), \\ n^\mu &= \frac{1}{r_0^2} \left(1, -\frac{r^2}{2}, 0, 0 \right), \\ m^\mu &= \frac{1}{r_0^2} \frac{1}{\sqrt{2}r} \left(0, 0, 1, \frac{i}{\sin(\theta)} \right). \end{aligned}$$

It should be noted that this tetrad is not unique, but it is the only tetrad that approaches the form of the two tetrads we try to obtain through a coordinate transformation. For convenience, we reprint the tetrad of the Kerr-Newman

near-horizon geometry here again

$$\begin{aligned}
 l^\mu &= (0, 1, 0, 0), \\
 n^\mu &= \left(\frac{1}{r_0^2 - \alpha^2 \sin^2(\theta)}, -\frac{r^2}{2(\alpha^2 \sin^2(\theta) - r_0^2)}, 0, \frac{2Mr\alpha}{r_0^3 - \alpha^2 \sin^2(\theta)r_0} \right), \\
 m^\mu &= \left(0, 0, \frac{i}{\sqrt{2}(r_0^2 - \alpha^2 \sin^2(\theta))}, \frac{\csc^2(\theta)r_0^2 - \alpha^2}{\sqrt{2}r_0^4} \right).
 \end{aligned}$$

It is remarkable that the vector m does not contain any r -dependence whereas in the Reissner-Nordström case m has an overall factor $\frac{1}{r}$. We concentrate on the third component; it is clear that θ should be transformed to a constant c times $r^2\theta$ in order to get rid of the r -dependence. Consequently some r -dependence will appear within the sin-functions elsewhere in the tetrad. In this way we will not arrive at the desired expression, but that means there is no coordinate transformation that can convert the first metric into the second one.

There is nothing we can do except moving on the the square root tetrad, and look again for a coordinate transformation that transforms the Kerr-Newman near-horizon tetrad to the root tetrad

$$l^\mu = (0, 1, 0, 0), \quad (7.31)$$

$$n^\mu = \left(\frac{1}{r_0^2 - \alpha^2 \sin^2(\theta)}, \frac{2M^2 r^2 \alpha^2 \sin^2(\theta) r_0^2}{(\alpha^2 \sin^2(\theta) - r_0^2)^3} - \frac{r^2}{2(\alpha^2 \sin^2(\theta) - r_0^2)}, 0, 0 \right), \quad (7.32)$$

$$m^\mu = \left(0, -\frac{i\sqrt{2}Mr\alpha \sin(\theta)r_0}{\sqrt{\alpha^2 \sin^2(\theta) - r_0^2}(r_0^2 - \alpha^2 \sin^2(\theta))}, \frac{1}{\sqrt{2}\sqrt{\alpha^2 \sin^2(\theta) - r_0^2}} \right) \quad (7.33)$$

$$, \frac{i \csc(\theta) \sqrt{\alpha^2 \sin^2(\theta) - r_0^2}}{\sqrt{2}r_0^2} \quad (7.34)$$

However, the same argument is effective for the root tetrad: there is no r -dependence in m^μ , and this means there is no coordinate transformation that produces this tetrad out of tetrad 7.31.

7.3 Discussion

In this chapter, we searched for an algorithm that makes near-horizon geometries rotate. In particular, we searched for a modification of the Newman-Janis algorithm. In that way, we imposed certain restrictions on the algorithms that we considered. This restrictions made the search process simpler, but also less powerful. Since the Newman-Janis algorithm applies to black hole solutions, it is not unreasonable to take that algorithm as a starting point in the search for a near-horizon algorithm, but we do not know if there exists such an algorithm. The Newman-Janis algorithm is not well understood, and in this respect its ingredients are quite arbitrary. If we change one of the steps the algorithm breaks down. Moreover, in many cases the original algorithm does not work.

In order to find a new algorithm, we looked for different complex coordinate transformations, we investigated the ways to choose the tetrad vectors and we checked the influence of the coordinate transformation. It turned out that in advanced null coordinates the matrix representation of the covariant metric (with induces up) is as simple as possible. Furthermore, the structure of the matrix often allows for simple tetrad vectors. There are always many ways to choose the tetrad vectors, and there are no strict selection criteria apart from the orthogonality conditions. When we searched for an algorithm this was no problem: we knew we wanted to arrive at a tetrad compatible with the Kerr-Newman near-horizon geometry, and we investigated all possible tetrads using this criterion.

The combination of different complex coordinate transformations, different tetrads and different coordinate representation did not yield anything. To arrive at a working algorithm, possibly we need some other modifications as well. Things that might be worth investigating are

- Other decompositions of the metric into a tetrad basis. This means changing the generating formula for the metric, or the so called frame metric.
- Maybe it is necessary to use not only coordinate transformations on the tetrad, but also direct substitutions, additions and multiplications. In this way we can transform any arbitrary tetrad into any other tetrad. We can only speak about an algorithm if we have one rule that works for several cases. Taking this into account we would have to investigate as much as possible near-horizon geometries.

We saw that the Newman-Janis algorithm and the near-horizon limit do not commute. Is there an explanation for this? After all, coordinate transformations and limits commute as long as they are continuous. The coordinate transformation of the near-horizon limit is

$$\begin{aligned} r &\rightarrow M + \lambda r, \\ t &\rightarrow \frac{t}{\lambda}, \end{aligned}$$

As $\lambda \rightarrow 0$, t becomes singular. In the line element all singularities are canceled, but for the tetrad this is not the case. Since u scales with λ^{-1} , and r scales with λ , the vector transformation rule dictates

$$x^\mu = (x^0, x^1, x^2, x^3) \rightarrow \left(\frac{1}{\lambda}x^0, \lambda x^1, x^2, x^3\right). \quad (7.35)$$

This has consequences for the tetrad vectors: x^1 and x^2 become singular in the near-horizon limit if they are nonzero unless they have a factor $\lambda^{n>0}$. We consider the tetrad vectors for the Reissner-Nordström geometry in the extremal case

$$\begin{aligned} l^\mu &= (0, 1, 0, 0), \\ n^\mu &= \left(1, -\frac{1}{2} \frac{(r-m)^2}{r^2}, 0, 0\right), \\ m^\mu &= \frac{\left(0, 0, 1, \frac{i}{\sin(\theta)}\right)}{\sqrt{2}r}, \end{aligned}$$

and we see what went wrong: n^0 will pick up a factor λ^{-1} from the near-horizon coordinate transformation, which will produce a singularity in the limit $\lambda \rightarrow 0$. At the other hand the second component, n^2 will scale as λ^3 , and therefore it will vanish.

When we calculated the near-horizon geometry of the Reissner-Nordström metric no terms were wiped away, and no terms blew up. The reason is that we used the line element, and not the metric tensor, to construct the near horizon geometry. In the line element all overall factors of λ vanished because the line element transforms as a scalar. In this section we used tetrad vectors to parametrize the metric, and these transform as vectors. This explains why we are left with diverging and vanishing overall factors of λ .

Is it possible to find a tetrad for the extremal Reissner-Nordström metric that will not become singular? The upper-left part of the covariant metric, in Advanced null coordinates, is

$$\begin{pmatrix} 0 & 1 \\ 1 & \frac{(r-m)^2}{r^2} \end{pmatrix}. \quad (7.36)$$

The components of the metric have to be constructed from a product of the components of the tetrad vectors. To prevent singularities to appear in the near-horizon limit, the first component of these vectors must be of order λ^1 or equal zero. The second elements must be of order λ^{-1} or equal zero; otherwise the second components vanish in the limit $\lambda \rightarrow 0$. It is not possible to find such a tetrad, even if we allow the generating formula of the metric to be adapted: although we can generate the 1-elements with $l = (r, 0, 0, 0)$, $m = (0, \frac{1}{r}, 0, 0)$, there is no way to construct $(r - m)^2/r^2$ and at the same time having λ^{-1} in the second component of both vectors involved.

In advanced null-coordinates the upper-left part of the metric will always cause this problem. What about the other coordinate representations? Is it possible find a coordinate representation in which we have tetrad vectors that do not become singular? To answer this question we considers the extremal Reissner-Nordström metric in Boyer-Lindquist coordinates. A vector $x^\mu = (x^0, x^1, x^2, x^3)$ transforms to $(\frac{1}{\lambda}x^0, \lambda x^1, x^2, x^3 + \frac{1}{\lambda}\alpha x^1)$. This means that the first components of the tetrads are zero or $x^0 \propto r$ and/or $x_1 \propto \frac{1}{r}$. The upper-left part of the metric (indices up) is

$$\begin{pmatrix} -\frac{r^2}{(r-m)^2} & 0 \\ 0 & \frac{(r-m)^2}{r^2} \end{pmatrix}. \quad (7.37)$$

Here we have the similar problem: After the coordinate transformation the one-one component scales as λ^{-2} . There is no way to write this a product of two terms with common factor r . The problem is not resolved when we consider the contra variant metric, or when we use an other coordinate representation. In the case of a smooth coordinate transformation, we can transform the metric back to Boyer-Lindquist form, and transform the generating tetrad accordingly. Therefore there cannot be a non-singular tetrad in an other coordinate system.

Chapter 8

Conclusions

We discussed four ways to investigate the rotating Born-Infeld black hole. The most obvious way is to solve the (modified) Einstein equations and apply Wald's formula. This road does not seem to be fruitful; several people tried to solve the equations, but no one succeeded. therefore we tried to apply Sen's entropy formalism. Usually the system of equations that arises from minimizing the entropy function is less complicated, but in the rotating case the system becomes complicated very rapidly. In the rotating Born-Infeld case the root terms from the Born-Infeld action made the system very complex. We were able to remove integrals and root terms from the system, but it seems impossible to disentangle the system completely. It might be advantageous to study some simpler cases first: small variations on the known cases might provide clues. The third possibility was to employ the Newman-Janis algorithm to obtain the desired rotating black hole solution. This has been tried by Lombardo [35] in 2005, but the unpredictable algorithm failed; the algorithm is known for its whimsicality, in particular when the seed metric gets complicated. But as a matter of fact a near-horizon metric is always very simple. Maybe the algorithm works in this setting; especially in the Born-Infeld case, which is rather complicated, this might simplify the problem a lot. Moreover, if this method works out for one case, it will be likely to work also in other cases; the near-horizon geometries of black holes are identical up to some constants. therefore we tried to apply the Newman-Janis algorithm on the near-horizon geometry of the Reissner-Nordström black hole. This did not work out, and so we started looking for a modification of the algorithm. A difficulty is that the algorithm is not well understood. Several steps of the algorithm are not understood. Therefore it was not immediately clear what steps should be altered to get the algorithm work in the near-horizon setting. The freedom to adapt the complex coordinate transformation and the way in which tetrad vectors are chosen proved not to be sufficient to arrive at a working algorithm. Working in a different coordinate basis seemed only to complicate matters: when we depart from advanced-null coordinates the metric matrices become a lot more complicated, and therefore finding a tetrad becomes (almost) impossible. Unfortunately, it turned out that we have no other option. In advanced null coordinates the tetrad vectors are bound to become singular in the near-horizon limit. This explains why in the near-horizon limit the Newman-Janis algorithm does not function.

We conclude with some recommendations for future work. To adapt the al-

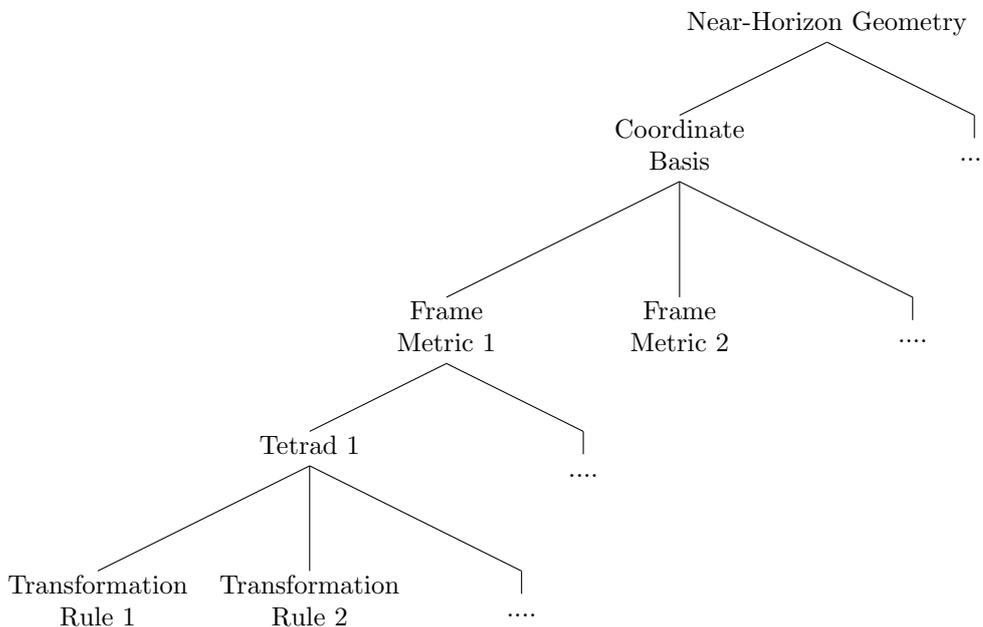


Figure 8.1: Tree of the search for the horizon limit algorithm. Before a transformation rule can be tested, a lot of choices have to be made: the coordinate representation of the metric, the frame metric and the choice of the tetrad.

gorithm successfully, we would like to have a coordinate representation in which we can avoid singularities in the tetrads, but such a representation cannot be obtained from a smooth coordinate transformation. Depending on the coordinate representation, it might be necessary to adapt the frame metric, expressed in the generating formula for the metric, as well. When a suitable coordinate representation and compatible tetrad structure are found, it is time to review the complex coordinate transformation. It has to be chosen such to generate the rotating near-horizon tetrads, in the same coordinate representation, using the same frame metric. Of course, all the possible ways for choosing the tetrad vectors should be taking into account in this last step.

Chapter 9

Acknowledgments

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Appendix A

Geometry of Physics

Modern theories of physics are often stated in terms of geometry. In such a formulation each quantity has a geometrical interpretation; the geometry adds some extra structure to the model. This structure can be useful. We will see that in general relativity it is an ideal basis for a consistent notation. The notation naturally gives rise to an extra consistency check: geometrical consistence.

In this Appendix some of the geometry of physics is reviewed. In the first section the concept of spacetime is defined, and this opens the door for a tensor formulation of the laws of physics, see section A.1.4. In this thesis the notation and definitions of the book by Wald [6] are adapted. This book is recommended as a geometrical introduction to general relativity.

A.1 Spacetime and the Laws of Physics

In special relativity space can be represented by \mathbb{R}^4 with the Lorentzian metric η_{ab} . To allow spacetime to be curved, \mathbb{R}^4 can be substituted by a manifold M . In general relativity, the metric g_{ab} is a dynamical quantity: it is governed by the Einstein equations. therefore a spacetime is represented by a pair (M, g_{ab}) consisting of the manifold M and the metric g_{ab} on M .

The metric is used to define distances on the manifold. In flat spacetime it is often used to give an ad-hoc tensor formulation of the (spacetime) Pythagorean theorem. In this view η_{ab} is nothing more than a symbol that represents the matrix $\pm\text{diag}(1, -1, -1, -1)$. The metric can also be interpreted as a part of a very elegant mathematical construction: it is a tensor field on a manifold. Below we give a short summary of these concepts, and their relation to physics.

A.1.1 The First Principles of General Relativity

Einstein argued that a theory of space and time should respect two principles: the equivalence principle and Mach's principle. The (weak) principle of equivalence states that:

The trajectory of a falling test body depends only on its initial position and velocity, and is independent of its composition.

Mach's principle is the name given by Einstein to a vague hypothesis first supported by the physicist and philosopher Ernst Mach. In special relativity the

notion of inertial motion is not related to the motion of the rest of the matter in the universe. Some found this idea unsatisfactory: when a universe is devoid of matter, inertial motion cannot have any meaning. As a consequence of this idea the matter distribution in the universe should determine the local definition of acceleration and rotation. There are several formulations for this principle. A very general formulation of Mach's principle is

Local physical laws are determined by the large-scale structure of the universe.

An other important principle in general relativity is the equivalence principle. It states that

The trajectory of a falling test particle depends only on its initial position and velocity, and is independent of its composition.

As a consequence of the equivalence principle there is no way to shield a particular observer from the influence of gravity, and therefore there is no direct way to measure gravity. In special relativity there exists a solution to this problem. The fixed flat background allows to define a global inertial frame. Gravitation can be defined as the force needed to keep a test particle in rest with respect to the inertial frame. For a theory obeying Mach's principle there is no such fixed background metric. The procedure to measure gravity breaks down, and there is no global definition for gravitational force. If there would exist such a definition, then it would be possible to define a consistent notion of inertial frames, and this is forbidden by Mach's principle. We can say that

Mach's principle forbids the existence of a consistent global definition of gravitational force.

Then, how can a theory of space and time be described without the concept of concept of gravitational force? A resolution can be found in the following hypothesis

Spacetime does not have to be flat; it can be curved. The world lines of freely falling bodies in a gravitational field are simply the geodesics on the spacetime manifold.

Background observers automatically follow the same geodesics as all other bodies, and therefore the gravitational force cannot be defined. This also means that the interpretation of gravitation as a force field cannot hold. Instead gravitation can be interpreted as an aspect of spacetime geometry itself.

A.1.2 Manifolds

A manifold M is defined as a set such that on any point $p \in M$ there exists a subset $O \subset M$, $p \in O$ and a function $\psi : O \rightarrow U \subset \mathbb{R}^d$ which is a diffeomorphism¹. Such a diffeomorphism can be used to assign a coordinate to each point on the manifold. In general, the subset U above cannot contain the full set M , and multiple pairs of open subsets and diffeomorphisms are used to cover the full manifold. Such a pair of open subsets and maps is called a chart. The collection of them is called an atlas. Obviously there is no limit on the number of atlases that are compatible with a given manifold.

¹The definition of a diffeomorphism is as follows: Given two manifolds N and M , a bijective map $\psi : N \rightarrow M$ is called a diffeomorphism if both ψ and its inverse ψ^{-1} are continuous.

A.1.3 Spacetime Manifolds and Causality

When talking about parts of spacetime, it is convenient to use the language of sets. In special relativity for each point p of spacetime we can define the so called light cone. It indicates in which directions physical trajectories may leave the point p . It is equivalent to a subset of spacetime called causal future of p . This is the union of all points that can be reached by lightlike or timelike trajectories that pass through p . In general relativity spacetime does not have to be flat, so the causal future does not have to be a cone. The causal future of a point, J^+ is defined as

$$J^+(p) = \{q \in M \mid \exists \text{ future directed causal curve } \lambda(t) : \lambda(0) = p, \lambda(1) = q\}.$$

A similar, but smaller, set is the null future infinity

$$I^+(p) = \{q \in M \mid \exists \text{ future directed time like curve } \lambda(t) : \lambda(0) = p, \lambda(1) = q\}.$$

It is clear that $I^+ \subset J^+$ and I^+ is open: any causal curve γ connecting p with a point $q \in I^+ = J^+ - I^+$ is a null geodesic. The sets J^+ and I^+ can easily be modified to cover the past instead of the future. J^- and I^- are defined similar to J^+ and I^+ with a single difference that "future directed" is replaced with "past directed". All four constructs can be easily generalized to apply on sets instead of points. For example, the causal future can be extended to a set S

$$J^+(S) = \cup_{p \in S} J^+(p).$$

Let (M, g_{ab}) be a spacetime. Although M is curved in general, at each point p it is locally isomorphic to \mathbb{R}^n , and so is the tangent space of M at p . Within this tangent space V_p lies the light cone of p . It starts at the origin, and from there it extends into the directions that are causally connected to p . Half of this light cone in V_p is called future, and the other half is called past². The causal future of $p \in M$, denoted $I^+(p)$, is defined as the set of events that can be reached by future directed timelike curves starting from p .

A.1.4 Tensors and the Laws of Physics

Given an n -dimensional vector space V over \mathbb{R} , we can always construct an associated n -dimensional covector space V^* by

$$V^* = \{f : V \rightarrow \mathbb{R} \mid f(\alpha u + v) = \alpha f(u) + f(v), \forall u, v \in V, \alpha \in \mathbb{R}\}. \quad (\text{A.1})$$

A tensor T over a vector space V is defined as a map $T : V^n \times V^{*m} \rightarrow \mathbb{R}$ which satisfies the condition of $(n+m)$ -linearity: the map is linear in each of its slots.

The laws of physics deal with scalar fields and vector fields on spacetime. Spacetime is represented by manifolds, and the vector fields are maps from the manifold to some vector space. In principle it is possible to map the manifold on any vector space one can think of, but this would make no sense. In theoretical physics the vector space is almost exclusively the tangent space of the manifold. Strictly, measurable quantities are always of scalar type. There are no probes that can measure a tensorial quantity directly. Vectors can be used to formalize

²At least for all simply connected spacetimes. Other spacetimes may not be time orientable. In the rest of this text all spacetimes are supposed to be time orientable.

linear behavior of nature. An example is the magnetic field. The value of the field strength that is measured depends on the direction of the probe. A probe can have only one direction at a time; this makes it impossible to measure the complete magnetic field at a point, using one elementary probe. However, we can demonstrate that the field strength depends on the direction of measurement in a linear way. Once the field strength is known in three directions it is known completely by means of extrapolation. Thus at each point the magnetic field can be characterized by three numbers.

Another example is current. A current is associated to a charge, and the charge is a scalar function $q : M \rightarrow \mathbb{R}$ that assigns a number to each point of the manifold M . The current, let's call it I , indicates the rate of the charge flow at each point. To be able to calculate the flow of charge explicitly, we use a chart ψ to introduce a coordinate system $\{x^\mu\}$ on some patch of M . The charge can now be expressed as a smooth function $q : \mathbb{R}^n \rightarrow U \subset \mathbb{R}$, which admits a Taylor expansion around each point x_0 of the open set U . To first order the difference of q in x_0 with q in some point x lying close to x_0 is equal to

$$\delta q = q(x) - q(x_0) = \sum_{\mu=1}^n \partial_\mu q(x) dx^\mu, \quad (\text{A.2})$$

where $dx^\mu = x^\mu - x_0^\mu$. This looks like a product of a vector and a covector. We call $\partial_\mu q$ a vector and dx^μ a covector. The vector space is called the tangent space of M at x_0 . A tangent space differs in general from point to point. Therefore, the vector space associated with a manifold M is defined only locally; it changes when one is moving over the manifold.

Denote the set of smooth functions that map M on \mathbb{R} as \mathcal{F} . A tangent vector v at a point $p \in M$ is defined to be a linear map from \mathcal{F} onto \mathcal{R} that satisfies Leibniz rule, for $v : \mathcal{F} \rightarrow \mathcal{R}$

$$v(af + bg) = av(f) + bv(g) \quad \forall f, g \in \mathcal{F}; a, b \in \mathbb{R} \quad (\text{A.3})$$

$$v(fg) = f(p)v(g) + g(p)v(f) \quad (\text{A.4})$$

These maps make up what is called the tangent space of the manifold at point p , denoted by T_p . It is almost trivial to verify that the tangent space satisfies the conditions mentioned in the definition of a vector space, especially when one recognizes that \mathcal{F} is a vector space since \mathbb{R} is a vector space under addition. We can make this more precise.

Introduce a coordinate basis, and let $\psi : O \rightarrow U$ be a chart of M with $p \in O \subset M$. For $\mu = 1, 2, \dots, n$ define $X_\mu : \mathcal{F} \rightarrow \mathbb{R}$ by

$$X_\mu(f) = \left. \frac{\partial}{\partial x^\mu} (f \circ \psi^{-1}) \right|_{\psi(p)} \quad (\text{A.5})$$

for each $f : M \rightarrow \mathbb{R}$. Each vector v in the tangent space of M can be decomposed in the basis $\{X_\mu\}$,

$$v = \sum_{\mu=1}^n v^\mu X_\mu, \quad (\text{A.6})$$

where $v^\mu = v(x^\mu \circ \psi)$. See for example [6] for a proof. The vectors X_μ and X_ν are easily shown to be linearly independent whenever $\mu \neq \nu$. After fixing the

constants $\alpha_1, \dots, \alpha_n$ it is always possible to choose a function $f \in \mathcal{F}$ such that $f \circ \psi^{-1}|_{\psi(p)} = \sum_{\mu=1}^n \beta_\mu x^\mu$. All one has to do is to take the constants β_μ in a way such that $\left(\sum_{\mu=1}^n \alpha_\mu X^\mu\right)(f) \neq 0$. In other words, it is shown that one cannot possibly construct a null vector in T_p from out of some non-trivial linear combination of X^μ vectors. Since the vectors $X^\mu, \mu = 1, 2, \dots, n$ form a complete and linearly independent basis for the tangent space of M , the dimension of the tangent space is n . In this exploration of the formal aspects of the tangent spaces, the introduction of coordinates appears in a natural way; this make it feel indispensable, but that is not true: in the end, the actual use of coordinate system in formulating the laws of physics can always be avoided. In this view the introduction of coordinates was just a hypothetical possibility that we used to show that the tangent space is n -dimensional. If the laws of physics can be defined in terms of tensors on manifolds, then introducing coordinates on this manifold, it is not necessary for the formulation of laws of physics, but when it comes to predictions there is no way to avoid the introduction of coordinates. The reason for this is exactly that we are only capable of measuring scalar quantities and not tensors.

A.1.5 The Metric

Without mentioning it, in the last section we introduced a notion of distance in the example of charge and currents. The scalar function of charge was evaluated at two different points on M . To measure the (infinitesimal) separation between the two points, the Euclidean distance in coordinatespace was used. Obviously, the distance between two points depends on the coordinate system that is used if we define it in this way. To define distance in a coordinate independent way the manifold has to be equipped with a metric tensor. The metric is a map that adds a 'square infinitesimal distance' to each a "infinitesimal displacement". The notion of "infinitesimal displacement" on the manifold is covered by the concept of tangent vectors. The metric is a map $g : T_p \times T_p \rightarrow \mathbb{R}$. Up to first order, infinitesimal square distances are bi-linear in the displacement, and thus g should be a linear map, a tensor.

$$\delta ds^2 = g_{ab} \delta x^a \delta y^b \quad (\text{A.7})$$

A.1.6 The Abstract Index Notation

In section A.1.4 we saw that it is not necessary to introduce coordinates in order to formulate the laws of physics, but in practice coordinates can be very useful, and when it comes to measurements they are indispensable. The abstract index notation is useful to switch smoothly between tensors and coordinate representations of tensors. An (n, m) tensor, that works on n covectors and on m vectors, is denoted as $T_{b_1, \dots, b_m}^{a_1, \dots, a_n}$. The Roman characters indicates that we deal with a tensor. After introducing a coordinate system, the coordinate representation of the tensor is denoted as $T_{\nu_1, \dots, \nu_m}^{\mu_1, \dots, \mu_n}$; Greek characters are used to refer to a particular coordinate representation.

A.1.7 Asymptotical Flatness

In physics it is usually preferable to study isolated systems. In special relativity this demands the fields to vanish with a certain falloff rate outside a "world tube" of compact spatial support³. In general relativity the notion of isolated system is problematic. time is one of the dynamical fields, and there is no independent background $\eta_{\mu\nu}$ in terms of which the fall-off rates of the curvature of the spacetime can be defined; in particular, there is no natural global inertial coordinate system to define a preferred radial coordinate r that can be used to specify the fall-off rates. In an asymptotically flat spacetime, the curvature vanishes at large distances from any chosen region. This makes it possible to define the fall-off rate of the fields.

A.2 Derivatives, Symmetries and Killing Fields

A.2.1 Vector Fields and one-parameter Groups of Diffeomorphisms

In the previous section, we saw that tangent vectors are associated with displacements on the manifold: they indicate how a scalar function is changing (to first order) under a displacement. Here this idea is made more precise.

Let M be a manifold. A one-parameter group of diffeomorphisms ϕ_t is a map from $\mathbb{R} \times M \rightarrow M$ such that for fixed $t \in \mathbb{R}$, ϕ_t is a diffeomorphism, and for all t and s we have $\phi_t \circ \phi_s = \phi_{t+s}$. A one-parameter group of diffeomorphisms can be associated with a vector field v and vice versa. To add a vector field to ϕ_t take a point $p \in M$ and generate its flow by varying the time parameter t of $u : \mathbb{R} \rightarrow M$. At each point $p = \phi_t(p_0)$ of this curve a tangent vector $v : \mathcal{F} \rightarrow \mathbb{R}$ is obtained through $v(f) = \frac{\partial}{\partial t} f \circ \phi_t(p_0)$. In this way a smooth vector field is established. It is also possible to go the other way around. We start with a smooth vector field v on M and demand that $v|_p$ is a tangent vector to some curve $u(t)$. After introducing a coordinate chart ϕ , this requirement can be expressed in coordinate components:

$$\frac{\partial}{\partial t} x^\mu(t) = v^\mu(x^1, x^2, \dots, x^n) \quad (\text{A.8})$$

, where $x^\mu(t) = (\pi^\mu \circ \phi \circ u)(t)$, $v(x^1, x^2, \dots, x^n) = v \circ \phi^{-1}(x^1, x^2, \dots, x^n)$ and $\pi^\mu : \mathbb{R}^n \rightarrow \mathbb{R}$ simply projects out the μ component in \mathbb{R}^n . This is a system of ordinary differential equations. Whenever the vector field v is smooth, the differential equations together with the condition $x^\mu(t_0) = \phi(t_0)$ forms a well posed initial value problem. The existence of a unique solution is guaranteed at least locally. In this way a continuous flow $u_p(t)$, such that $u_p(0) = p$, can be generated through every point p on the manifold. The one-parameter group of diffeomorphisms is now defined at each point p on the manifold by $\phi_t(p) = u_p(t)$. Hence, $\phi_t(p)$ is equal to the point that you end up with after following the flow starting at p for a 'time' t .

³In the limit that $r \rightarrow \infty$ along any null geodesic, to be precise

A.2.2 Parallel Transport and Covariant Derivative

In Euclidean space \mathbb{R}^n we frequently make use of the partial derivative $\frac{\partial}{\partial x^\mu}$. After introducing coordinates, this derivative can be used as well for tensor fields on manifolds: we let it act on the coordinate components of the tensor fields. In section A.1.4 this was done in the construction of vectors from zeroth order tensor fields (scalar functions on M). The partial derivative was used as a measure of the change of a scalar function along some direction of the manifold. The partial derivative operator was used for this because it compares the scalar function at two different points that lie infinitely close to each other. Since the laws of physics are supposed to be formulated in terms of tensors on a manifold, it is desirable to extend the derivative functional to the space of tensor fields on the manifold. The generalisation is not as trivial as it may seem. On a non-flat manifold the tangent space differs at each point. Just subtracting coordinate components works, but in this way angles between vectors are not preserved. Another derivative has to be introduced. In principle we can call any functional that maps the tensor fields on M upon some space a derivative. Remember however that the use of the derivative was to measure changes of tensor fields, or to compare tensor fields. The derivatives of interest are functionals that map smooth tensors of type (k, l) on the tangent space of M to smooth tensors type $(k, l + 1)$ on the tangent space of M . Further, a derivative operator, denoted by ∇_a , should satisfy

1. Linearity
2. Leibnitz rule
3. Commutativity with contraction
4. Consistency with the notion of tangent vectors as directional derivatives
5. Torsion free

Operators satisfying those conditions are called covariant derivatives. The partial derivative belongs to this class. In the book of Wald [6] a precise mathematical formulation of these conditions can be found. Using these conditions together with some theorems in calculus, he shows that the difference of two covariant derivatives (in operator space) results in a tensor

$$(\tilde{\nabla}_a - \nabla_a)v_b = C_{ab}^c v_c. \quad (\text{A.9})$$

This tensor is called the the connection tensor. Note that in most books this object is not regarded as a tensor. This is due to a subtle difference in definitions. Here we defined a tensor as a multilinear map, but in most books it is defined as an object whose (coordinate) components transform in a certain way under a coordinate transformation. $C_{ab}^c v_c$ does not transform in the same way, and therefore it is not a tensor according to this definition. For arbitrary tensors relation A.9 is generalized to

$$\nabla_a T_{c_1 \dots c_m}^{b_1 \dots b_n} - \tilde{\nabla}_a T_{c_1 \dots c_m}^{b_1 \dots b_n} = \sum_{i=1}^n C_{ad}^{b_i} T_{c_1 \dots d \dots c_m}^{b_1 \dots b_n} - \sum_{j=1}^m C_{ab_j}^d T_{c_1 \dots d \dots c_m}^{b_1 \dots b_n}. \quad (\text{A.10})$$

A derivation of this relation can also be found in [6].

Now that the class of derivatives has been reduced in a precise way, it becomes easier to find a derivative that respects the inner product. The partial derivatives that statisfie condition 1 through 5, can be written as

$$(\nabla_a - \partial_a)v_b = C_{ab}^c v_c \quad (\text{A.11})$$

, because ∂_a is derivative. To preserve inner products our derivative should satisfy $\nabla_c g_{ab} = 0$. All that is left is to determine a connection C_{ab}^c that respects this condition. This specific connection is called the metric connection or Christoffel symbol, and it is usually denotated as Γ_{ab}^c . Working out the algebra, we obtain a unique expression

$$\Gamma_{bc}^a = g^{cd}(\partial_a g_{db} + \partial_b g_{ac} - \partial_c g_{ab}). \quad (\text{A.12})$$

A.2.3 Maps of Manifolds

Pullback

Let M and N be manifolds and let $\phi : M \rightarrow N$ be a smooth map. In a natural way ϕ pulls back a function on N , $f : N \rightarrow \mathbb{R}$ on N to $f \circ \psi : M \rightarrow \mathbb{R}$ on M . In a similar way tangent vectors can be pulled back from N to M . A tangent vector on M works on functions on M . Given a tangent vector v on M and a function f on N , we can consider $v f \cdot \psi$. This defines a linear map from the functions on N to \mathbb{R} , and thus it defines a vector field on N . We denotate this map as $\phi^* : T(M)_p \rightarrow T(N)_{\phi(p)}$. If ϕ has a smooth inverse, then vectors can also be transformed in the other direction. This is the case when ψ is a diffeomorphism. Hence tensors of all types can be mapped back and forth between the two tensor spaces. For general tensors, these maps are called ϕ^* and $(\phi^{-1})^*$.

Isometry

Suppose $\phi : M \rightarrow M$ is a diffeomorphism and T is a tensor field on M . Then T can be compared with $\phi^* T$. If $\phi^* T = T$ the tensor field is unaffected by the movement induced by ϕ , and ϕ is called a symmetry transformation for the tensor field T . A symmetry transformation for the metric is called an isometry.

A.2.4 Lie Derivatives and Killing Vector Fields

The Lie derivative is a coordinate independent directional derivative which maps the space of tensor fields into itself. Given some manifold M , a tensor (or scalar) field $T_{b_1 \dots b_m}^{a_1 \dots a_n}$ and a one-parameter group of diffeomorphisms generated by some vector field v , the Lie derivative of the tensor field with respect to v is defined as

$$\mathcal{L}_v T_{b_1 \dots b_l}^{a_1 \dots a_k} = \lim_{t \rightarrow 0} \frac{\phi_{-t}^* T_{b_1 \dots b_l}^{a_1 \dots a_k} - T_{b_1 \dots b_l}^{a_1 \dots a_k}}{t}. \quad (\text{A.13})$$

ϕ_t is a symmetry transformation of T if and only if $\mathcal{L}_v T_{b_1 \dots b_l}^{a_1 \dots a_k} = 0$. The Lie derivative can be expressed in terms of covariant derivatives as follows

$$\mathcal{L}_v T_{b_1 \dots b_l}^{a_1 \dots a_k} = v^c \nabla_c T_{b_1 \dots b_l}^{a_1 \dots a_k} - \sum_{i=1}^k T_{b_1 \dots b_l}^{a_1 \dots c \dots a_k} \nabla_c v^{a_i} + \sum_{j=1}^l T_{b_1 \dots c \dots b_l}^{a_1 \dots a_k} \nabla_{a_j} v^c \quad (\text{A.14})$$

A symmetry transformation of the metric is called an isometry, and a vector field generating an isometry is called a Killing vector field. The covariant derivative of the metric equals zero, so each vector field ξ_a satisfying Killing's equation,

$$\nabla_a \xi_b + \nabla_b \xi_a = 0, \quad (\text{A.15})$$

is a Killing vector field. Each symmetry corresponds a conserved quantity. If ξ^a generates an isometry, and u^a is the tangent of a geodesic γ , then the quantity

$$\xi^a u_a \quad (\text{A.16})$$

is conserved along the geodesic γ .

A.2.5 Hypersurfaces and Frobenius's Theorem

To introduce the hypersurface, it is convenient first to define the embedded submanifold. Suppose a manifold M of dimension n , a manifold S of dimension $p < n$ and a smooth map $\phi : S \rightarrow M$ to be given. If ϕ is globally one to one (i.e. $\phi(S)$ does not intersect itself), then $\phi(S)$ is called an embedded submanifold of M . Since the embedded submanifold $\phi(S)$ is a subspace of M , its tangent space W_p is also a subspace of the tangent space V_p of M at each point $p \in \phi(S)$. Moreover the tangent space of the submanifold at a point $q \in \phi(S)$, W_q , is a p -dimensional subspace of the tangent space of M at q . The argument is simple: Since the map ϕ is a diffeomorphism, for each point $q \in S$ the tangent spaces $V_q(S)$ and $V_{\phi(q)}(\phi(S))$ are diffeomorphic to each other. Therefore they must have the same dimension.

A hypersurface N of M now is defined as a $(n - 1)$ -dimensional embedded submanifold. Its tangent space is a $(n - 1)$ -dimensional subspace of the n -dimensional tangent space at each point $p \in N \subset M$. At any point of $M \cap N$, the tangent space of M can be decomposed into a product of the tangent space of N times the one-dimensional space orthogonal to N .

Given some smooth hypersurface in M it is always possible to find a hypersurface orthogonal vector field in the tangentspace of M . The reverse of this statement does not hold. Given some $(m < n)$ -dimensional subspace $W_p \subset T_p(M)$, $p \in M$, for $m > 1$ in general there does not exist an integral submanifold of W for $m > 1$. Frobenius's theorem states

The subspace $W \subset T(M)$ admits an integral submanifold if and only if for all $Y^a, Z^a \in W$ we have $[Y, Z]^a \in W$.

For a tensor field ξ this results in a simple hypersurface orthogonality condition. If

$$\xi_{[a} \nabla_b \xi_{c]} = 0 \quad (\text{A.17})$$

then the vector field admits an integral surface.

A.2.6 Null Hypersurfaces

A hypersurface Σ is called a nullsurface whenever its tangent vectors are null vectors. If ξ^a is the tangent field of Σ , then there exists some scalar function κ such that

$$\xi^b \nabla_b \xi^a = \kappa \xi^a. \quad (\text{A.18})$$

This is the geodesic equation for curves with a non-affine parameter. The tangent vector field of this geodesic is a null vector, and therefore the geodesic is called a null geodesic. Here something counter intuitive happens. The inner product of the tangent vectors of this curve and the normal vector of the hypersurface vanishes, $\xi^a \xi_a = 0$. Although the null vector ξ^a is normal to the hypersurface it generates a curve that lies on the null hypersurface. The hypersurface is generated by its own normal vector field. This goes beyond our intuition based on ordinary 3-space. The metric of this 3-space has signature $(+, +, +)$, excluding the existence of null vectors.

A.2.7 Diffeomorphism Invariance, General Covariance and gauge Freedom

An important principle in physics is general covariance. It can be put as

The metric g_{ab} and quantities deduced from it are the only spacetime quantities appearing in the laws of physics.

This statement is equivalent to the statement that the laws of physics, when expressed in some coordinate representation, should transform in the way that coordinate representations of tensors transform.

In general relativity, Gauge freedom appears from diffeomorphism invariance. Suppose M and N to be manifolds and let ϕ be a diffeomorphism from M to N . Because they are diffeomorphic, M and N have the same structure. Now imagine a law of nature formulated in terms of tensor fields $T^{\{i\}}(M)$ on M . Such a law is formulated as a set of tensor equations. Using the pullback and pullforward, the same equations will hold for $*T^{\{i\}}(M)$ on N . Therefor

(N, ϕ^*T^i) represents the same physics as (M, T^i) does.

This situation might appear more familiar after we introduce an arbitrary coordinate basis. Under a diffeomorphism $\phi : M \rightarrow M$, the coordinate representation of the tensors appearing in any physical law will change according to the tensor transformation laws. Diffeomorphism invariance guarantees that the formulation of the laws do not depend on the coordinate representation.

Appendix B

Formula supplement

In section 5.1 we minimize the entropy function for the rotating Born-Infeld black hole. The system of differential and integral equations that appears is as follows:

$$0 = -\frac{e^{-4\psi(\theta)}\beta\alpha^2}{2\Omega(\theta)^2} + \frac{3e^{2\psi(\theta)}\beta\Omega(\theta)^2}{d^2} - \frac{2\psi'(\theta)^2}{\beta} + \frac{2\Omega'(\theta)^2}{\beta\Omega(\theta)^2} - 2\beta - \frac{4\Omega''(\theta)}{\beta\Omega(\theta)} - \frac{\Omega(\theta)3e^{4\psi(\theta)}\beta^2\Omega(\theta)^4}{d^2w} - \frac{\Omega(\theta)(2e^{4\psi(\theta)}b'(\theta)^2\Omega(\theta)^2 - e^2\beta^2 - \alpha^2\beta^2b(\theta)^2 + 2e\alpha\beta^2b(\theta))}{w}, \quad (\text{B.1})$$

$$0 = \frac{2e^{2\psi(\theta)}\beta\Omega(\theta)^3}{d^2} + \frac{4\psi''(\theta)\Omega(\theta)}{\beta} - \frac{4\psi'(\theta)\Omega'(\theta)}{\beta} - \frac{2e^{-4\psi(\theta)}\alpha^2\beta}{\Omega(\theta)} - \frac{2e^{4\psi(\theta)}\Omega(\theta)^4(\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}{d^2w}, \quad (\text{B.2})$$

$$0 = \Omega(\theta)^3 \left(-e\alpha\beta^4 + \alpha^2b(\theta)\beta^4 + 2d^2e^{4\psi(\theta)}b'(\theta)^3\psi'(\theta) \right) + \Omega(\theta)^3 \left(e^{4\psi(\theta)}\beta^2b'(\theta)\Omega'(\theta)\Omega(\theta)^4 + d^2\beta^2(e - \alpha b(\theta))^2b'(\theta)\Omega'(\theta) \right) + \frac{\Omega(\theta)^3}{d^2w} \left(e^{4\psi(\theta)}\beta^2(2b'(\theta)\psi'(\theta) + b''(\theta))\Omega(\theta)^5 + 2d^2e^{4\psi(\theta)}b'(\theta)^3\Omega'(\theta)\Omega(\theta)^2 \right), \quad (\text{B.3})$$

$$0 = q - \int_0^\pi \frac{2\pi(e - \alpha b(\theta))(\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}{w} d\theta, \quad (\text{B.4})$$

$$0 = J - \int_0^\pi \frac{2e^{-4\psi(\theta)}\pi\alpha\beta}{\Omega(\theta)} d\theta - \int_0^\pi \frac{2\pi b(\theta)(e - \alpha b(\theta))(\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}{w} d\theta, \quad (\text{B.5})$$

$$0 = -2\beta\Omega(\theta)^2 \int_0^\pi \frac{w}{b'(\theta)^2d^4 + \beta^2\Omega(\theta)^2d^2} d\theta + \int_0^\pi \frac{2e^{2\psi(\theta)}\Omega(\theta)^3}{d^2} + \left(\frac{4\psi'(\theta)^2}{\beta^2} - 4 \right) \Omega(\theta) + \frac{e^{-4\psi(\theta)}\alpha^2 - \frac{4\Omega'(\theta)^2}{\beta^2}}{\Omega(\theta)} d\theta, \quad (\text{B.6})$$

where $w = \sqrt{(e^{4\psi(\theta)}\Omega(\theta)^4 - d^2e^2 - d^2\alpha^2b(\theta)^2 + 2d^2e\alpha b(\theta)) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}$ is the root term mentioned in section 5.2. In that section we try to arrive at a single differential equation that we can solve. In one of the procedures we use the derivatives of, what we called, the currents; these are listed here:

$$I'_e(\theta) = \frac{(e - \alpha b(\theta)) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}{2 \left((e^{4\psi(\theta)}\Omega(\theta)^4 - d^2e^2 - d^2\alpha^2b(\theta)^2 + 2d^2e\alpha b(\theta)) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) \right)^{3/2}} \cdot \left[(-\beta^2\Omega(\theta)^2 - d^2b'(\theta)^2) \left(-4e^{4\psi(\theta)}\psi'(\theta)\Omega(\theta)^4 - 4e^{4\psi(\theta)}\Omega'(\theta)\Omega(\theta)^3 - 2d^2e\alpha b'(\theta) + 2d^2\alpha^2b(\theta)b'(\theta) \right) - \left(-e^{4\psi(\theta)}\Omega(\theta)^4 + d^2e^2 + d^2\alpha^2b(\theta)^2 - 2d^2e\alpha b(\theta) \right) (2b'(\theta)b''(\theta)d^2 + 2\beta^2\Omega(\theta)\Omega'(\theta)) \right] - \frac{\alpha b'(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}{w} \quad (\text{B.7})$$

$$I_\beta(\theta) = - \frac{\alpha b'(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}{w} + \frac{(e - \alpha b(\theta)) (2b'(\theta)b''(\theta)d^2 + 2\beta^2\Omega(\theta)\Omega'(\theta))}{w} - \left[(-\beta^2\Omega(\theta)^2 - d^2b'(\theta)^2) \left(-4e^{4\psi(\theta)}\psi'(\theta)\Omega(\theta)^4 - 4e^{4\psi(\theta)}\Omega'(\theta)\Omega(\theta)^3 - 2d^2e\alpha b'(\theta) + 2d^2\alpha^2b(\theta)b'(\theta) \right) - \left(-e^{4\psi(\theta)}\Omega(\theta)^4 + d^2e^2 + d^2\alpha^2b(\theta)^2 - 2d^2e\alpha b(\theta) \right) (2b'(\theta)b''(\theta)d^2 + 2\beta^2\Omega(\theta)\Omega'(\theta)) \right] \cdot \frac{(e - \alpha b(\theta)) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2)}{2 \left((e^{4\psi(\theta)}\Omega(\theta)^4 - d^2e^2 - d^2\alpha^2b(\theta)^2 + 2d^2e\alpha b(\theta)) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) \right)^{3/2}} \quad (\text{B.8})$$

The expression for $I'_\gamma(\theta)$ is far more complicated; we decided not to print it. Form equation B.3 and B.7, it is possible to simplify I'_q , but the result is not that simple:

$$\begin{aligned} & w \cdot \left(\beta^2\Omega(\theta)^2 \left(e^{4\psi(\theta)}\Omega(\theta)^4 - d^2e^2 - d^2\alpha^2b(\theta)^2 + 2d^2e\alpha b(\theta) \right) \right) \cdot (\text{B.7}) \\ & - d^2v(e - \alpha b(\theta))b'(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) \cdot (\text{B.3}) \\ & = \\ & \Omega(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) \left(e^{4\psi(\theta)}\beta^2 (\alpha b'(\theta) + 2(e - \alpha b(\theta))\psi'(\theta))\Omega(\theta)^5 \right) \\ & + \Omega(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) \left(+e^{4\psi(\theta)}\beta^2(e - \alpha b(\theta))\Omega'(\theta)\Omega(\theta)^4 \right) \quad (\text{B.9}) \\ & + \Omega(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) \left(2d^2e^{4\psi(\theta)}(e - \alpha b(\theta))b'(\theta)^2\psi'(\theta)\Omega(\theta)^3 \right) \\ & + \Omega(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) \\ & \left(+2d^2e^{4\psi(\theta)}(e - \alpha b(\theta))b'(\theta)^2\Omega'(\theta)\Omega(\theta)^2 - d^2\alpha\beta^2(e - \alpha b(\theta))^2b'(\theta)\Omega(\theta) \right) \\ & + \Omega(\theta) (\beta^2\Omega(\theta)^2 + d^2b'(\theta)^2) (+d^2\beta^2(e - \alpha b(\theta))^3\Omega'(\theta)). \end{aligned}$$

Bibliography

- [1] S. Chandrasekhar, *Astrophys. J.* **74**, 81 (1931).
- [2] M. Heusler, *Living Reviews in Relativity* **1** (1998).
- [3] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, 1983).
- [4] S. W. Hawking, *Communications in Mathematical Physics* p199, vol.43, 1975.
- [5] A. Mueller, *POS P2GC*, 017 (2007).
- [6] R. M. Wald, *General Relativity*, Chicago and London, The University of Chicago Press (1984).
- [7] B. Carter, Black hole equilibrium states, in *Black Holes*, edited by C. DeWitt and B. DeWitt, pp. 57–214, New York, U.S.A., 1973, Gordon and Breach.
- [8] S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time* Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, U.K., 1973).
- [9] D. Sudarsky and R. Wald, *Phys. Rev. D* **46**, 1453 (1992).
- [10] D. Sudarsky and R. Wald, *Phys. Rev. D* **47**, R5209 (1993).
- [11] P. Chruściel and R. Wald, *Commun. Math. Phys.* **163**, 561 (1994).
- [12] J. Bardeen, B. Carter, and S. Hawking, *Commun. Math. Phys.* **31**, 161 (1973).
- [13] J. D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973).
- [14] R. M. Wald, *Physical Review D* **48**, 3427 (1993).
- [15] V. Iyer and R. M. Wald, *Physical review D*, vol.50, no.2 (1994).
- [16] T. Jacobson and G. Kang, *arXiv:gr-qc/9502009* (1995).
- [17] J. Lee and R. M. Wald, *Journal of Mathematical Physics* **31**, 725 (1990).
- [18] R. M. Wald, *Journal of Mathematical Physics* **31**, 2378 (1990).
- [19] R. Arnowitt, S. Deser, and C. W. Misner, *Phys. Rev.* **122**, 997 (1961).

- [20] S. Ferrara, R. Kallosh, and A. Strominger, *Physical Review D* **52**, 5412 (1995).
- [21] A. Strominger, *Physics Letters B* **383**, 39 (1996).
- [22] S. Ferrara and R. Kallosh, *Physical Review D* **54**, 1514 (1996).
- [23] A. Sen, *Black hole entropy function, attractors and precision counting of microstates*, 2007.
- [24] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, *JHEP* **0610**, 058 (2006).
- [25] H. K. Kunduri, J. Lucietti, and H. S. Reall, *Classical and Quantum Gravity* **24**, 4169 (2007).
- [26] E. T. Newman and A. I. Janis, *J. Math. Phys.* **6**, 915 (1965).
- [27] S. P. D. P. Szekeres, *General Relativity and Gravitation* **32**, 445 (2000).
- [28] E. T. Newman *et al.*, *Journal of Mathematical Physics* **6**, 918 (1965).
- [29] R. H. Boyer and R. W. Lindquist, *Journal of Mathematical Physics* **8**, 265 (1967).
- [30] M. Born and M. Infeld, *Proc. R. Soc. London* **A144**, 425 (1934).
- [31] G. W. Gibbons and D. A. Rasheed, *Nucl. Phys.* **B454**, 185 (1995), hep-th/9506035.
- [32] H. S. Garcia and J. Plebanski, *Nuovo.Cim* **84**, 65 (1984).
- [33] N. Breton, *Born-infeld generalization of the reissner-nordstrom black hole*, 2001.
- [34] W. A. Chemissany, M. de Roo, and S. Panda, *Class. Quant. Grav.* **25**, 225009 (2008), 0806.3348.
- [35] D. J. Cirilo-Lombardo, *General Relativity and Gravitation* **37**, 847 (2005).
- [36] D. Cirilo-Lombardo, *Classical and Quantum Gravity* **21**, 1407 (2004).