On problems in de Sitter spacetime physics

scalar fields, black holes and stability

J. Hartong

July 5, 2004

Thesis supervisor:
Prof. Dr. M. de Roo
Center for Theoretical Physics
University of Groningen

Abstract

Various coordinate systems and the conformal structure of de Sitter and Schwarzschild-de Sitter spacetime are discussed. A de Sitter space is shown to have a horizon and no spatial infinity. The spacetime is essentially non-stationary.

Scalar particles are introduced on a fixed de Sitter background and quantized using the method of covariant quantization. First it is shown that there exists a two-parameter family of inequivalent vacua. Then it is argued that the only physically acceptable vacuum is the euclidean vacuum. The Green's functions defined with respect to this vacuum are periodic in imaginary time and are thus thermal Green's functions. The temperature being the same for all observers is an intrinsic property of the de Sitter spacetime. The appearance of this temperature is an example of the generality of the Hawking radiation experienced by a horizon. A geometric interpretation of the temperature of the black hole and cosmological event horizons appearing in the Schwarzschild-de Sitter space will be given in terms of the surface gravity of the horizon, euclidean sections and the removal of conical singularities that appear in static coordinate systems. An important result in this is that the black hole horizon temperature is always higher than the temperature of the cosmological event horizon. Then it is shown that there exist scalar particles which have no Minkowskian analogue in the sense that their mass parameter cannot be related to the mass parameter of the Poincaré group.

Finally, the question of the stability of de Sitter space is discussed. A de Sitter space is stable with respect to any classical perturbation. However, an instability occurs in the semi-classical regime. In a semi-classical approximation of the partition function a sum over the finite temperature gravitational instantons is obtained. One such instanton is $S^2 \times S^2$. This has an unstable fluctuation which appears as an imaginary contribution to the partition function of de Sitter space. De Sitter space undergoes thermally induced topological changes in which the entire spacetime decays into a Schwarzschild-de Sitter spacetime. However, the black hole will evaporate and de Sitter spacetime is left behind.

Contents

Introduction

Recently, in 1998, a group of astronomers has claimed to have observed that our universe is currently undergoing accelerate expansion [11] which is attributed to the existence of a positive cosmological constant. The idea that nature contains a cosmological constant stems from Newton. Newton being rather religious believed that the universe must be infinite in extend, must have existed at all times and must be static. However, gravity attracts causing such a space to be unstable. He therefore postulated that there must be some repulsive mechanism leading to a static universe. At that time and in centuries to follow nothing much was known about the universe and physicists ignored these ideas. Einstein being quite religious too also believed that the universe must the static. However his theory of gravity led a dynamic universe, and he therefore in 1916 reintroduced the cosmological constant. At that time Einstein lived in Germany and could not because of World War I send his letters of correspondence to England and the USA. Still, he was able to send them to the Netherlands which was neutral at that time. The person to receive these letters was W. de Sitter who would then send them to whomever they were addressed to. In this way de Sitter became part of the cosmological debate that was held in those days and was therefore one of the first the hear of Einstein's idea to reintroduce the cosmological constant. In 1917 de Sitter showed that for an empty space this new constant leads to a universe which undergoes accelerate expansion.

Often the cosmological constant is considered to be related to the vacuum energy density of some scalar field. Whatever, the origin might be it will in this thesis be assumed to exist and its effects be discussed.

A de Sitter spacetime describes an empty universe which has a positive cosmological constant. In the following chapters the theory of scalar particles and the properties of black holes defined on such a spacetime will be treated. An outline will be given of some of the main problems that arise in describing them. Then in the final chapter the question of the stability of de Sitter spacetime will be discussed.

This thesis was written with the aim to obtain a master degree in theoretical physics. I would like to express my gratitude to Mees de Roo who was not only responsible for selecting this interesting topic, but with whom

I also had many interesting discussions, and I hope that in the near future this will continue and that it may lead to a valuable contribution to modern theoretical high energy physics.

Tot slot wil ik graag mijn ouders meer dan hartelijk bedanken voor hun vertrouwen en sterke geloof in mij, aan wie ik veel meer dan deze studie te danken heb.

Jelle Hartong

Groningen, July 5, 2004

Chapter 1

Geometry of de Sitter and Schwarzschild-de Sitter spacetime

We shall be working in d dimensions and with the signature $(- + \cdots +)$ except for the discussion of the Schwarzschild-de Sitter space which will be in 4 dimensions. The following index notation will be used. Lower case Latin letters will be used as spacetime indices. Upper case Latin letters will be used as spacetime indices for embedding spaces. Then, for indices which refer to spatial dimensions lower case Greek letters will be employed.

1.1 Introducing de Sitter spacetime

Consider the Einstein equation in relativistic units with cosmological constant, Λ,

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}. \tag{1.1}$$

In the absence of matter one finds the vacuum Einstein equation

$$G_{ab} + \Lambda g_{ab} = 0. \tag{1.2}$$

It is readily seen that for an empty space with positive cosmological constant the solution is a homogeneous Einstein space of constant positive curvature, for taking the trace of equation 1.2 we find

$$R = \frac{2d}{d-2}\Lambda, \tag{1.3}$$

so that the Ricci tensor is proportional to the metric. Since the Einstein equation is fully governed by the metric and the Ricci scalar the same is true for the Riemann tensor. Therefore the Weyl tensor must vanish, that is, our spacetime is conformally flat. This together with the Ricci tensor being proportional to the metric implies that the spacetime is maximally symmetric.

Now, we appeal to the following theorem (see [26] for the proof).

theorem 1.1 *Given two maximally symmetric metrics with the same Ricci tensor R and the same number of eigenvalues of each sign, it is always possible to find a coordinate transformation that carries one metric into the other.*

To find the eigenvalues of a metric at a particular spacetime point it must be diagonalized at that point. It can be shown that the number of positive and negative eigenvalues, that is, the signature, is an invariant of a particular spacetime. This implies that our maximally symmetric solution of the vacuum Einstein equation with positive cosmological constant is unique, and it is called de Sitter spacetime. We shall denote d-dimensional de Sitter spacetime by dS_d, and d-dimensional Minkowski spacetime by $\mathcal{M}^d$.

It is known that a de Sitter spacetime is a special case of an empty Robertson-Walker (RW) spacetime. Hence, coordinate systems exist in which the line element is of the Robertson-Walker form

$$ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2 d\Omega_{d-2}^2), \tag{1.4}$$

where $k = 0, \pm 1$ and where $d\Omega_{d-2}^2$ is the line element of a $(d-2)$-dimensional unit sphere. In this context a $k = 0$ empty RW spacetime is called an open de Sitter space and an empty $k = 1$ RW space a closed de Sitter space (in both cases the line element describes the whole of the spacetime). The Friedmann equations for an empty universe with cosmological constant Λ are

$$(\frac{\dot{a}}{a})^2 = \frac{2}{(d-2)(d-1)}\Lambda - \frac{k}{a^2} \tag{1.5}$$

$$\frac{\ddot{a}}{a} = \frac{2}{(d-2)(d-1)}\Lambda. \tag{1.6}$$

The Hubble parameter, H, is defined by

$$H(t) = \frac{\dot{a}}{a},$$

so that using equation 1.5 we have when $k = 0$,

$$\Lambda = \frac{(d-2)(d-1)}{2}H^2.$$

It is noted that only for a $k = 0$ RW spacetime Hubble's law is exact.

The right-hand side of equation 1.5 is always positive for the cases $k = 0, -1$. For the case where $k = 1$ it must be that at all times $a > a_{cr}$ with

$$a_{cr}^2 = \frac{(d-2)(d-1)}{2\Lambda}. \tag{1.7}$$

This value is often termed the de Sitter radius. Note that $a_{cr} = 1/H$. De deceleration parameter of a RW space is defined as

$$q = -a\ddot{a}/\dot{a}^2 = -\frac{\frac{\ddot{a}}{a}}{(\frac{\dot{a}}{a})^2}$$

Clearly, the deceleration parameter is negative for all values of k meaning that a de Sitter space is undergoing continuous accelerative expansion and/or contraction.

1.2 Embedding of dS_d in $\mathcal{M}^{d+1}$

From the Einstein equation 1.2 it follows that the Ricci scalar is given by

$$R = \frac{2d}{d-2}\Lambda, \tag{1.8}$$

and with equation 1.7 this becomes

$$R = \frac{d(d-1)}{l^2}, \tag{1.9}$$

where we have set $a_{cr} = l$. Next, consider the following theorem.

theorem 1.2 *Let (M, g_{ab}) be a conformally flat C^k-manifold*[1] *of dimension $\leq d$. Then M is an embedded C^k-submanifold of $\mathbb{R}^{d+1}$.*

To prove this it has to be shown that there exists a mapping ϕ which maps M homeomorphically into $\mathbb{R}^{d+1}$. (All mappings and curves are assumed to be of the class C^k). As sets, $M \subset \mathbb{R}^{d+1}$, where M has the topology induced from $\mathbb{R}^{d+1}$. Let $p \in M$ and consider Riemannian normal coordinates at p. Then it follows immediately that there exists an open neighborhood O of p and a homeomorphic mapping $\phi : O \to \mathbb{R}^{d+1}$. So all that has to be proven is that the domain of ϕ can be extended to M. Consider a point $q \in M$ and a differentiable curve γ joining p and q. Now, the idea is that, because the metric can be written as $g_{ab} = \Omega^2\eta_{ab}$ with the conformal factor Ω always positive, we can transport the system of Riemannian normal coordinates at p into a system of Riemannian normal coordinates at q. The origin of these coordinate systems is transported along the curve γ. This is to say, for each

[1]By a C^k manifold we mean that the coordinate patches are mapped into each other, in their overlap region, by k times continuously differentiable functions.

point on any one of the coordinate axes of the Riemannian normal coordinate system at q we can find one and only one point on the Riemannian normal coordinate system at p joined by the curve $\tilde{\gamma}$ which can be continuously deformed into γ. In this way can construct a map ψ from O to $\psi[O]$ around q which maps the Riemannian normal coordinates at p into the Riemannian normal coordinates at q. Then the mapping ϕ is through ψ extendible to M.■

As a corollary we have that dS_d can be embedded in $\mathbb{R}^{d+1}$. In 1956 Schrödinger showed that de Sitter space may be visualized as the embedded hyperboloid

$$\eta_{AB}X^A X^B = l^2 \tag{1.10}$$

in $\mathcal{M}^{d+1}$ with the metric $\eta_{AB} = \mathrm{diag}(-1, +1, \cdots, +1)$ where $A, B = 0, 1, \cdots, d$. This will be shown by calculation of the induced metric on the hyperboloid and the corresponding Ricci scalar. Applying the differential d to equation 1.10 one finds

$$\eta_{ab}(dX^a)X^b + \eta_{ab}X^a(dX^b) - 2(dX^d)X^d = 0,$$

so that

$$dX^d = \frac{\eta_{ab}X^a dX^b}{X^d} = \pm\frac{\eta_{ab}X^a dX^b}{\sqrt{l^2 - \eta_{cd}X^c X^d}}.$$

Substituting this into

$$ds^2 = \eta_{AB}dX^A dX^B$$

the induced metric on dS_d is obtained as

$$g_{ab} = \eta_{ab} + \frac{X_a X_b}{l^2 - \eta_{cd}X^c X^d}. \tag{1.11}$$

This enables us to define intrinsic coordinates x^a for dS_d by saying that $x^a \in dS_d$ if X^A, pointing from the origin of $\mathcal{M}^{d+1}$ to x^a, satisfies the embedding equation. Then

$$ds^2 = \eta_{AB}dX^A dX^B = g_{ab}dx^a dx^b.$$

The inverse metric is

$$g^{ab} = \eta^{ab} - \frac{1}{l^2}X^a X^b.$$

One then finds for the connection coefficients

$$\Gamma^c_{ab} = \frac{1}{l^2}(\eta_{ab}X^c + \frac{X_a X_b X^c}{l^2 - \eta_{de}X^d X^e}),$$

and finally for the Ricci scalar

$$R = \frac{d(d-1)}{l^2},$$

in agreement with equation 1.9.

Writing equation 1.10 as

$$(X^1)^2 + \cdots (X^d)^2 = l^2 + (X^0)^2, \tag{1.12}$$

then $\sqrt{l^2 + (X^0)^2}$ may be seen as the radius of a $(d-1)$-dimensional sphere. In fact the hyperboloid is the surface of revolution of $\sqrt{l^2 + (X^0)^2}$ around the X^0 axis in the ambient space $\mathcal{M}^{d+1}$, see figure 1.1.

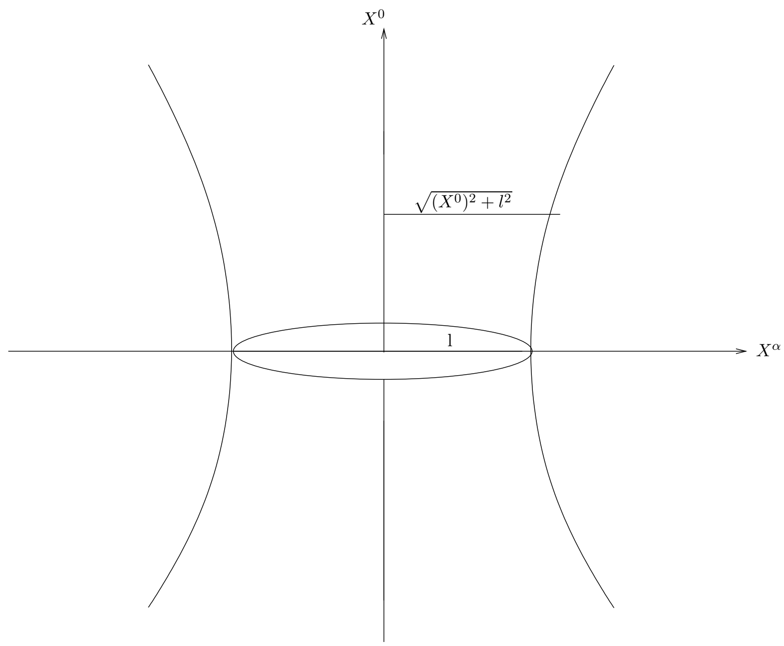

Figure 1.1: Embedding of the hyperboloid in $\mathcal{M}^{d+1}$. All spatial dimensions are represented by one line, the X^α-axis, $\alpha = 1 \cdots d$.

We end this section by noting that the inhomogeneous Lorentz group of $\mathcal{M}^{d+1}$, $O(1,d)$, leaves invariant equation 1.10. From the d-dimensional point of view, seen from within the hyperboloid, $O(1,d)$ is called the full disconnected de Sitter group. We shall return to it at the end of this chapter. de Sitter spacetime has the same number of Killing vectors as d-dimensional Minkowski spacetime.

1.3 Various coordinate systems

In this section we will frequently use the embedding equation to find various coordinate systems for de Sitter space.

1.3.1 The global coordinate system (τ, θ_i)

Consider equation 1.12. It may be parameterized by setting

$$\begin{aligned} X^\alpha &= l\omega^\alpha \cosh(\tau/l) \quad \text{for } \alpha = 1, \cdots, d \quad \text{and} \\ X^0 &= l \sinh(\tau/l) \end{aligned}$$

with $-\infty < \tau < \infty$. The ω^α parameterize an S^{d-1} sphere of unit radius. In spherical coordinates this parametrization becomes

$$\begin{aligned} \omega^1 &= \cos\theta_1 \\ \omega^2 &= \sin\theta_1 \cos\theta_2 \\ &\vdots \\ \omega^{d-1} &= \sin\theta_1 \cdots \sin_{d-2} \cos\theta_{d-1} \\ \omega^d &= \sin\theta_1 \cdots \sin_{d-2} \sin_{d-1}, \end{aligned}$$

where $0 \leq \theta_i < \pi$ for $i = 1, \cdots, d-2$ and $0 \leq \theta_{d-1} < 2\pi$. Then the line element becomes

$$ds^2 = -(dX^0)^2 + \sum_{\alpha=1}^{d} (dX^\alpha)^2 = -d\tau^2 + l^2 \cosh^2(\tau/l)\, d\Omega_{d-1}^2, \qquad (1.13)$$

where $d\Omega_{d-1}^2 = \sum_{j=1}^{d-1} (\prod_{i=1}^{j-1} \sin^2\theta_i) d\theta_j^2$ is the line element of S^{d-1}. This line element corresponds to the Robertson-Walker line element 1.4 with $k = 1$ and $r = \sin\theta$. One Killing vector is manifest, $(\frac{\partial}{\partial\theta_{d-1}})^a$. The coordinate system therefore has axial symmetry.

The main features of this coordinate system are listed. First of all, apart from the coordinate singularities at $\theta_i = 0, \pi$ for $i = 1, 2, \cdots, d-2$, the coordinates $(\tau, \theta_1, \cdots, \theta_{d-1})$ cover the whole of de Sitter spacetime and are therefore referred to as global coordinates. At fixed τ the line element 1.13 describes the spacelike hypersurfaces of $(d-1)$-dimensional spheres of radius $l\cosh(\tau/l)$ which is infinitely large at $\tau = \pm\infty$, and of minimum length at $\tau = 0$. The parameter τ is often called the world or cosmic time because observers moving along world lines parameterized by τ are comoving with the expansion/contraction.

The points lying on the left hyperboloid in fig. comprise the time evolution of what will be called the north pole of de Sitter space and likewise the points lying on the right hyperboloid describe the time evolution of the south pole. If θ_1 denotes the angle between ω^1 and the line joining the north and south poles of S^{d-1} then the north pole is defined to be at $\theta_1 = 0$ and the south pole at $\theta_1 = \pi$.

From the line element in global coordinates it can concluded that the topology of de Sitter space is $\mathbb{R} \times S^{d-1}$. It is noted that using the global coordinate system one can prove [28] that de Sitter spacetime is geodesically complete, that is, the affine parameter of any geodesic passing through any point can be extended to reach arbitrary values.

If one tries to find the Killing vector of time-translational symmetry in this coordinate system, it will turn out that no change in the spatial coordinates can compensate for the change in the time parameter meaning that this coordinate system does not possess a Killing vector of time translational symmetry. Since the global coordinates cover the whole of the space it must be concluded that a de Sitter space is a non-stationary spacetime.

1.3.2 The static coordinate system (t, r, θ_i)

Writing the embedding equation 1.10 in terms of two constraints

$$\begin{aligned} -(\frac{X^0}{l})^2 + (\frac{X^d}{l})^2 &= 1 - (\frac{r}{l})^2 \\ (\frac{X^1}{l})^2 + \cdots + (\frac{X^{d-1}}{l})^2 &= (\frac{r}{l})^2, \end{aligned}$$

where the first describes a hyperbola of radius $\sqrt{1-(\frac{r}{l})^2}$ and the second a $(d-2)$-dimensional sphere of radius r, it may be parameterized by

$$\begin{aligned} \frac{X^0}{l} &= -\sqrt{1-(\frac{r}{l})^2}\sinh(\frac{t}{l}) \\ \frac{X^d}{l} &= -\sqrt{1-(\frac{r}{l})^2}\cosh(\frac{t}{l}) \\ \frac{X^\alpha}{l} &= \frac{r}{l}\omega^\alpha \quad \text{for } \alpha = 1, \cdots, d-1, \end{aligned}$$

where $0 \le r < l$ and $-\infty < t < \infty$. Noting that $-X^0 + X^d \le 0$ and $X^0 + X^d \le 0$, it is seen that the coordinates cover only one quarter of the de Sitter space as shown in figure 1.2.

In this parametrization of the embedded hyperboloid the line element becomes

$$ds^2 = \eta_{AB} dX^A dX^B = -(1-(\frac{r}{l})^2)dt^2 + \frac{dr^2}{1-(\frac{r}{l})^2} + r^2 d\Omega^2_{d-2}. \quad (1.14)$$

It has two manifest Killing vectors $(\frac{\partial}{\partial t})^a$ and $(\frac{\partial}{\partial \theta_{d-2}})^a$, and therefore has axial and time-translational symmetries. From the g_{tt} part of the metric it can be seen that the timelike Killing vector, $(\frac{\partial}{\partial t})^a$, becomes null when

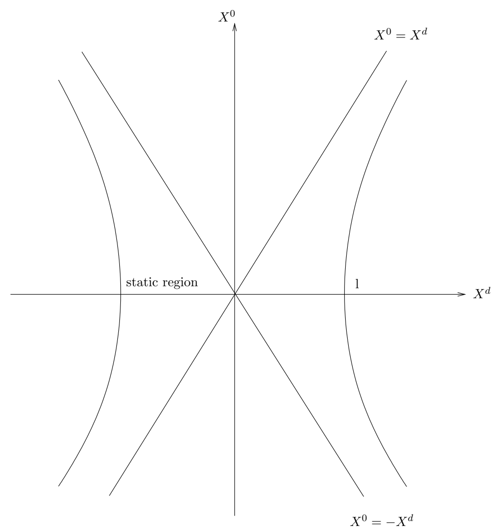

Figure 1.2: Part of the hyperboloid covered by the static coordinates. The lines $X^0 = X^d$ and $X^0 = -X^d$ represent the horizons bounding as seen by an observer on the north, that is, the left hyperboloid.

$r = l$ and spacelike[2] when $r > l$. The region of spacetime where the radial coordinate r becomes equal to l is called the horizon. The topology of the horizon is S^{d-2}. The origin and implications of this will be explained in section 1.4.

This coordinate system is suited for an observer who is fixed at a particular point in space. Such an observer will be termed static, and it follows that his view is limit to $0 \leq r < l$, where r is the static observer's radial coordinate. Still, the limit $r \to \infty$ exists. This is in contrast to the spatial sections of the global coordinates which are compact.

1.3.3 The planar coordinate system (t, x^α)

The planar coordinate system is obtained by foliating de Sitter space with infinite spatial planes. It corresponds to a Robertson-Walker metric with $k = 0$. Thus, it describes the open de Sitter space of section 1.1 The line element is

$$ds^2 = -dt^2 + a^2(t)((dx^1)^2 + \cdots (dx^{d-1})^2), \tag{1.15}$$

where $a(t) = e^{\lambda t}$ solves equations 1.5 and 1.6 with $\lambda^2 = \frac{2\Lambda}{(d-2)(d-1)}$, so that $\lambda = \pm\frac{1}{l}$. The part of the de Sitter space undergoing expansion has $\lambda = \frac{1}{l}$

[2]We have in fact only constructed static coordinates with r restricted to $0 \leq r < l$. However they may be analytically extended to cover the whole range of r values, see equations 1.27 and 1.28.

and the part undergoing contraction has $\lambda = -\frac{1}{l}$.

In the literature on cosmology, especially where it involves the inflationary model, one often considers this coordinate system. It is sometimes referred to as the inflationary coordinate system. In this picture the de Sitter space comes about through the working of the inflaton field whose potential effectively plays the role of the cosmological constant. The full de Sitter space as described by the global coordinate system is not well-suited because it describes a spatially closed space, whereas the outcome of the inflationary period is a flat universe. In this context one distinguishes between the closed, $k = 1$, and open, $k = 0$, de Sitter spaces of section 1.1.

In order to find the Killing vector field of time translations we let $t \to t + \delta t$ in the above metric. This leads to

$$ds^2 \to -dt^2 + a^2(t)((dx^1)^2 + \cdots (dx^{d-1})^2)(1 + 2\lambda\delta t).$$

To make the metric invariant under this time translation the spatial coordinates, x^α, must be dilated

$$x^\alpha \to x^\alpha - \lambda x^\alpha \delta t.$$

In general a Killing vector ξ^a is an infinitesimal isometry,

$$x^a \to x^a + \epsilon\xi^a,$$

where ϵ is infinitesimal. Setting $\epsilon = \delta t$ we read off that the Killing vector which generates time translations in the planar coordinate system is

$$\xi^a = (1, -\lambda x^\alpha).$$

The norm is given by

$$\xi^2 = g_{ab}x^a x^b = -1 + a^2\lambda^2|\vec{x}|^2.$$

Hence the horizon is given by $a^2\lambda^2|\vec{x}|^2 = 1$, and ξ^a is timelike in the inner region. Therefore in the expanding part of the de Sitter space the horizon is located at a distance $|\vec{x}| = le^{-t/l}$ away from an observer.

It is noted that this metric was used in the steady-state model of the universe. This model has been discarded for various reasons one being that it requires Hubble's law to be exact which is not confirmed by experiment. Still, it follows from the analogy that this coordinate patch is geodesically incomplete in de past. All timelike geodesics emanate from one point in the infinite past. This point would then correspond to the big bang of the steady state model.

We now have three relevant coordinate systems at our disposal: the global, static and planar coordinate systems. The global coordinate system is suitable for describing processes comoving with the expansion/contraction of

dS_d, but lacks a Killing vector field of time translational symmetry, and so does not display the presence of an event horizon clearly. The static coordinate system is relevant for processes as observed by one particular observer. Last, the planar coordinate system describes the expanding part of dS_d using infinite spatial planes and is non-singular at the horizon whereas as the static coordinate system is.

Next, we will maximally extend the static coordinates, that is, we shall be looking for Kruskal type coordinates. Further we shall be interested in the properties of de Sitter space at infinity. To that end we now first consider the conformal coordinates.

1.3.4 Conformal coordinates (T, θ_i)

de Sitter space is conformally flat. To find a coordinate system which manifestly exhibits this structure the line element is written as

$$ds^2 = F^2(T/l)(-dT^2 + l^2 d\Omega_{d-1}^2),$$

where $F > 0 \ \forall \ T$ with T a timelike coordinate which has a finite range. Comparison with the line element of the global coordinate system gives

$$\begin{aligned} F^2(\frac{T}{l}) &= \cosh^2(\frac{\tau}{l}) \geq 1 \quad \forall \, \tau \\ 1 &= F^2(\frac{T}{l})(\frac{dT}{d\tau})^2. \end{aligned}$$

Differentiating the first expression with respect to T and then eliminating the $\frac{dT}{d\tau}$ using the second equation one obtains

$$\frac{d \ln F}{dT} = \pm\sqrt{F^2 - 1}.$$

The initial value of F may be set to $F(0) = 1$. This first order differential equation is solved by

$$F(\frac{T}{l}) = \sec(\frac{T}{l}) \tag{1.16}$$

with $-\pi/2 < T < \pi/2$. The line element then takes the form

$$ds^2 = \sec^2(\frac{T}{l})(-dT^2 + l^2 d\Omega_{d-1}^2), \tag{1.17}$$

so that de Sitter spacetime is conformal to that portion of the Einstein static universe which is covered by $-\pi/2 \leq T \leq \pi/2$, see figure 1.3. Without the conformal factor it has the topology of a cylinder of finite height embedded in $\mathcal{M}^{d+1}$.

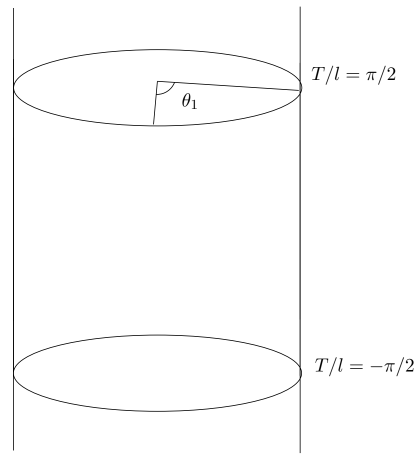

Figure 1.3: The infinite cylinder embedded in $\mathcal{M}^{d+1}$ represents the Einstein static universe. dS_d is conformal to the finite portion shown in the figure where $d-2$ dimensions are suppressed.

1.3.5 Kruskal coordinates (U, V, θ_i)

In order to maximally extend the static coordinates and study the conformal structure of dS_d in terms of these, an alternative coordinate system formed from the Kruskal coordinates will be constructed. This system is more convenient because it uses null coordinates. In order to introduce them one first defines the Eddington-Finkelstein coordinates x^+ and x^- in the static region of dS_d,

$$dx^{\pm} = dt \pm \frac{dr}{1-(\frac{r}{l})^2}. \tag{1.18}$$

This equation is solved by

$$x^{\pm} = t \pm \frac{l}{2} \ln \frac{1+\frac{r}{l}}{1-\frac{r}{l}}$$

with $-\infty < x^{\pm} < \infty$. Rewriting the line element of the static coordinate system one finds

$$ds^2 = -\frac{1}{\cosh^2(\frac{x^+-x^-}{2l})} dx^+ dx^- + l^2 \tanh^2(\frac{x^+-x^-}{2l}) d\Omega^2_{d-2}. \tag{1.19}$$

Then in the region with $0 \leq r < l$ the Kruskal coordinates are defined by

$$U = -e^{x^-/l} \tag{1.20}$$
$$V = e^{-x^+/l}, \tag{1.21}$$

so that $U \leq 0$ and $V \geq 0$. The line element now takes the form

$$ds^2 = \frac{l^2}{(1-UV)^2}(-4dUdV + (1+UV)^2 d\Omega_{d-2}^2). \qquad (1.22)$$

The timelike Killing vector $(\frac{\partial}{\partial t})^a$ of the static coordinate system becomes in Kruskal coordinates

$$(\frac{\partial}{\partial t})^a = \frac{\partial U}{\partial t}(\frac{\partial}{\partial U})^a + \frac{\partial V}{\partial t}(\frac{\partial}{\partial V})^a = \frac{U}{l}(\frac{\partial}{\partial U})^a - \frac{V}{l}(\frac{\partial}{\partial V})^a. \qquad (1.23)$$

In the next section we will analyze the conformal and therefore causal structure of the global and static coordinate systems. It is therefore convenient to be able to relate the conformal coordinates to the Kruskal coordinates. Comparing the two metrics two conditions follow,

$$\begin{aligned} \frac{(1+UV)^2}{(1-UV)^2} &= \frac{\sin^2\theta_1}{\cos^2(\frac{T}{l})} \\ \frac{4l^2}{(1-UV)^2} dUdV &= \frac{1}{\cos(\frac{T}{l})}(dT^2 - l^2 d\theta_1^2). \end{aligned}$$

These equations are solved by

$$U = \tan\frac{1}{2}(\frac{T}{l} + \theta_1 - \frac{\pi}{2}) \qquad (1.24)$$

$$V = \tan\frac{1}{2}(\frac{T}{l} - \theta_1 + \frac{\pi}{2}). \qquad (1.25)$$

These Kruskal coordinates are the maximal analytic extension of the ones defined by 1.20 and 1.21, and hence describe the whole of de Sitter spacetime.

1.4 Conformal infinity and causality

Consider a "physical" spacetime manifold M with the line element ds. The problem of describing the causal structure of M at infinity is nicely stated by Penrose in [20][3]:

> The idea is to construct another "unphysical" manifold $\tilde{M}$ with a boundary $\mathcal{I}$ and metric $d\tilde{s}$, such that M is conformal to the interior of $\tilde{M}$ with $d\tilde{s} = \Omega^2 ds$, and so that the "infinity" of M is represented by the "finite" hypersurface $\mathcal{I}$. This last property is expressed by the condition that $\Omega = 0$ on $\mathcal{I}$; that is to say, the metric at $\mathcal{I}$ is stretched by an infinite factor in the passage from $\tilde{M}$ to M so $\mathcal{I}$ gets mapped to infinity.

It can be proven that null geodesics are conformally invariant [25]. Therefore the local light cone structure of M can be differed from $\tilde{M}$.

[3] Penrose denotes the physical manifold by $\tilde{M}$ and the unphysical manifold by M.

theorem 1.3 *Every null geodesic in $\tilde{M}$ originates and terminates on $\mathcal{I}$.*

To prove this take an arbitrary null geodesic[4], λ, in M. Then assume that for $\Omega > 0$ this null geodesic corresponds to a null geodesic, $\tilde{\lambda}$, which lies in the interior of $\tilde{M}$. Now, due to the simple connectedness of M, we may take λ in M to be past and future inextendible[5]. Then, since infinity of M is mapped onto $\mathcal{I}$ ($\Omega = 0$), we must have that $\tilde{\lambda}$ originates and terminates on $\mathcal{I}$. ■

The time orientability of M is carried over to $\tilde{M}$, so that the boundary $\mathcal{I}$ may be decomposed into $\mathcal{I}^+$ and $\mathcal{I}^-$, the future and past null infinity, respectively. Then it follows immediately that all null geodesics in $\tilde{M}$ originate on $\mathcal{I}^-$ and terminate on $\mathcal{I}^+$.

It can be shown that for a spacetime with positive cosmological constant $\mathcal{I}$ is spacelike. Further it is known that to a spacelike $\mathcal{I}^-$ there corresponds a particle horizon and to a spacelike $\mathcal{I}^+$ there corresponds an event horizon. This is the origin of the event horizon that we encountered when discussing the various coordinate systems. That this is so, is readily seen for a de Sitter space in the conformal coordinate system to be discussed next.

1.4.1 Penrose diagram in conformal coordinates

In section 1.3.4 it was shown that de Sitter space is conformal to a finite region of the Einstein static universe depicted in figure 1.3. The condition $\Omega = 0$ determines the hypersurface $\mathcal{I}$. In the conformal coordinate system this gives

$$\Omega = \cos(\frac{T}{l}) = 0,$$

so that the spacelike hypersurfaces $T/l = -\pi/2$ and $T/l = \pi/2$, both having the topology of S^{d-1}, comprise $\mathcal{I}^-$ and $\mathcal{I}^+$, respectively. This confirms the above mentioned fact that when $\Lambda > 0$, $\mathcal{I}$ is spacelike for the case of a de Sitter space. Furthermore, null infinity is described by a compact spacelike manifold. This implies that de Sitter space has no spatial infinity. Any spacelike geodesic starts and ends on $\mathcal{I}$ so no spatial infinity can be reached by going along a spacelike geodesic.

We have taken the left and right hyperboloids of figure 1.1 to describe the time evolution of the north and south poles, respectively. In the Einstein static universe they have become straight vertical lines from $T/l = -\pi/2$ to $T/l = \pi/2$, see figure 1.3. If we cut the cylinder twice along these lines and unwrap it to form a 2-dimensional spacetime diagram we obtain the Penrose diagram of dS_d, figure 1.4. The metric for this diagram is

[4] For our purposes it suffices to consider only spacetimes that are simply connected. In such a spacetime there are no null geodesics which run into a singularity.

[5] This means that λ has no past and/or future accumulation point not on the curve.

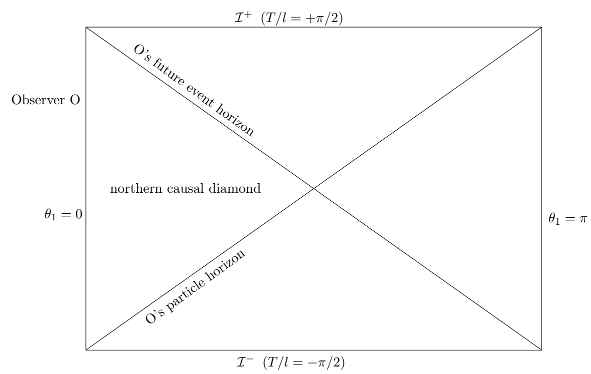

Figure 1.4: Penrose diagram in conformal coordinates

$$ds^2 = -dT^2 + l^2 d\theta_1^2 + l^2 \sin^2\theta_1 d\Omega_{d-2}^2 \tag{1.26}$$

The north and south poles are the timelike lines of $\theta_1 = 0$ and $\theta_1 = \pi$. A horizontal line $T =$ const. is a $(d-1)$-dimensional sphere of radius l. The interior points are $(d-2)$-dimensional spheres of radius $l \sin\theta_1$. The equation for a null geodesic at the north pole and south pole is given by

$$ds^2 = -dT^2 + l^2 d\theta_1^2 = 0.$$

These are the diagonal lines of figure 1.4. They form the event horizon of an observer O at one of the poles. This is because the $\mathcal{I}^-$ and $\mathcal{I}^+$ are spacelike. An observer at the north pole cannot, at any moment in time, see anything from the south pole due to O's particle horizon. Further O can never send any information to the south pole due to O's future event horizon. The set of spacetime points within which O is able to communicate, that is, to which information can be sent and from which information can be received, is called the northern causal diamond.

It is noted that the poles are no privileged points. This is due to the spherical symmetry of the spatial part of the global coordinate system. Further, it is mentioned that stationary observers at the poles are sufficiently general, for all timelike geodesics in dS_d are related to each other by transformations of the isometry group $SO(d,1)$. This follows immediately from the fact that the tangent remains timelike under $SO(d,1)$ and from the fact that the Lagrangian, L, for geodesic motion,

$$L^2 = \frac{ds^2}{d\lambda^2},$$

with λ some affine parameter, is, due to the presence of ds^2, manifestly invariant under the isometry group.

1.4.2 Penrose diagram in Kruskal coordinates

The static coordinates (r, t) and the Kruskal null coordinates (U, V) are related by

$$\frac{r}{l} = \frac{1+UV}{1-UV} \tag{1.27}$$

$$-\frac{U}{V} = e^{2t/l} \tag{1.28}$$

Even though the radial coordinate r of the static coordinate system was restricted to $r < l$, the line element of the static coordinate system has meaning for all values of r. By equations 1.24 and 1.25 the Kruskal coordinates are maximally extended to cover the whole of dS_d. Through equations 1.27 and 1.28 the same is true for the static coordinates.

Relation 1.27 shows that the origin $r = 0$ of the static coordinate system, that is, the north and south poles have $UV = -1$, that the horizon $r = l$ has $UV = 0$, and that $r = \infty$ corresponds to $UV = 1$. From the second relation it follows that the null coordinate line $U = 0$, which is part of the horizon, corresponds to past infinity $t = -\infty$ and that the null coordinate line $V = 0$, which also forms part of the horizon, corresponds to future infinity $t = \infty$. $U = 0$ is called the past event horizon and $V = 0$ the future event horizon. Setting the conformal factor of the line element in Kruskal coordinates, $\Omega^2 = \frac{(1-UV)^2}{l^2}$, equal to zero, the Kruskal designation of $\mathcal{I}^{\pm}$ is obtained as $UV = 1$ $(r = \infty)$. This leads to the following Penrose diagram.

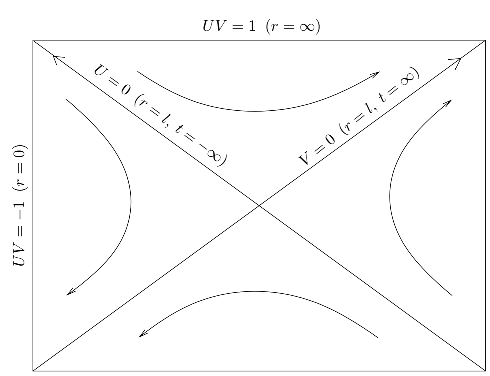

Figure 1.5: Penrose diagram in Kruskal coordinates. The arrows on the horizons indicate that V increases towards the upper left corner and U increases towards the upper right corner. The curved arrows in each of the four (static) coordinate patches indicate the direction in which time increases or equivalently indicate the flow of positive energy.

Let us consider the Killing vector 1.23. We give it here again for convenience

$$(\frac{\partial}{\partial t})^a = \frac{U}{l}(\frac{\partial}{\partial U})^a - \frac{V}{l}(\frac{\partial}{\partial V})^a.$$

Its norm is given by

$$(\frac{\partial}{\partial t})^2 = g_{tt} = -(1-(\frac{r}{l})^2) = \frac{4UV}{(1-UV)^2}.$$

The direction of the vectors $(\frac{\partial}{\partial U})^a$ and $(\frac{\partial}{\partial V})^a$ is indicated by the arrows on the diagonals of figure 1.5, pointing to the upper right and left corner, respectively. Further in each of the four coordinate patches the direction of $(\frac{\partial}{\partial t})^a$ is indicated. In the northern causal diamond it is future-directed, in the southern causal diamond it is past-directed and in the past and future triangles it is spacelike since in these regions we have $UV > 0,\ \ l < r < \infty$.

Finally it is noted that at certain points of the diagram the coordinate designation is ambiguous. This is due to a bad choice of coordinates, in this case of the static coordinate system in the neighborhood of these points. Of course, the Kruskal coordinates form an analytic and unambiguous extension of the static coordinates.

1.4.3 Penrose diagram in planar coordinates

Finally, the Penrose diagram of dS_d in planar or flat coordinates is discussed without deriving its properties. The spacetime diagram is depicted in figure 1.6. The lines of constant r where $r^2 = x^2 + y^2 + z^2$ all emanate from the

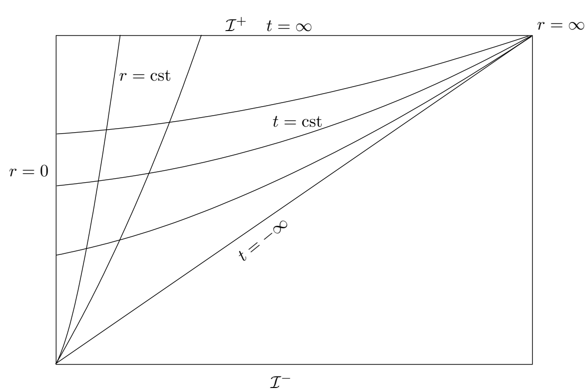

Figure 1.6: Penrose diagram in planar coordinates. Lines of constant t and r are shown.

lower left corner of the diagram and end on future null infinity. Hence the north pole at $t=-\infty$ may be considered as the big bang of the open de Sitter space. If one likes to view this coordinate system from the point of view of the steady-state model then the horizon will be $\mathcal{I}^-$ which is null in this case and the upper right corner would represent spatial infinity. However, considered as a part of dS_d the expanding part, that point belongs to a spacelike future null infinity. Note again the coordinate ambiguity in the time parameter at $r = \infty$. Formally this means that the limit $r \to \infty$ does not exist in this coordinate patch.

1.4.4 Cauchy surfaces of dS_d

It was shown that every null geodesic starts at $\mathcal{I}^+$ and ends at $\mathcal{I}^-$. From the Penrose diagram of dS_d in conformal coordinates it is clear that on their way to future null infinity the null geodesics cross once and only once the horizontal lines of $T = \text{const}$. These are spacelike slices with the topology of S^{d-1}, so that these surfaces are Cauchy surfaces of $\tilde{M}$. Hence they are Cauchy surfaces of $M \cap \tilde{M}$ with $M = dS$. This raises the suspicion that the spacelike hypersurfaces S^{d-1} are in fact the Cauchy surfaces of dS_d. This fact is confirmed by the global coordinate system which foliates dS_d with spacelike S^{d-1} surfaces.

1.5 The geodesic distance

In the next chapter we will more elaborately digress on the de Sitter group. We will need, however, at this stage, some basic facts. As has already been said the de Sitter group is the inhomogeneous Lorentz group, $O(d,1)$, of the ambient space, $\mathcal{M}^{d+1}$, on which is imposed the constraining condition of the embedding equation 1.10. The group $O(d,1)$ contains four disconnected components, that is to say there are no continuous transformations in $O(d,1)$ which can relate elements of these four subsets. This can be understood by noting that the a generic matrix element, g, of $O(d,1)$ can have determinant ± 1 and that it must have either ${g^0}_0 \geq 1$ or ${g^0}_0 \leq -1$. Let $SO(d,1)$ denote the part of $O(d,1)$ which is continuously connected to the identity, that is, it contains those elements g which have determinant $+1$ and ${g^0}_0 \geq 1$.

Following Allen [2] we consider two elements of $O(d,1)$ denoted by T and S,

$$
\begin{aligned}
T &= \text{diag}(-1, 1, \cdots, 1) \\
S &= \text{diag}(1, -1, 1, \cdots, 1),
\end{aligned}
$$

which correspond to time reversal and space reflection, respectively. Next define

$$O_T \equiv \{g \cdot T |\; g \in SO(d,1)\}.$$

The subsets O_S and O_{TS} are defined similarly. Thus, we have decomposed

$O(d,1)$ into the subsets:

$$\begin{array}{rll} SO(d,1) & \text{for which} & \det g = 1, \;\; g^0{}_0 \geq 1 \\ O_T & \text{for which} & \det g = -1, \;\; g^0{}_0 \leq -1 \\ O_S & \text{for which} & \det g = -1, \;\; g^0{}_0 \geq 1 \\ O_{TS} & \text{for which} & \det g = 1, \;\; g^0{}_0 \leq -1. \end{array}$$

The element, A, defined by

$$A = \mathrm{diag}(-1, \cdots, -1),$$

which is contained in O_T, is called the antipodal transformation. Antipodal points are always separated by a horizon.

The d-dimensional spacetime point in dS_d, x^a, is positioned in $\mathcal{M}^{d+1}$ at the point $X^A(x)$ which satisfies the embedding equation. For the point, $\bar{x}$, antipodal to x we have $X^A(\bar{x}) = -X^A(x)$. The geodesic distance, D, between the spacetime points x and y is defined by

$$D(x,y) \equiv \int_x^y (\eta_{AB}\dot{X}^A\dot{X}^B)^{1/2} d\lambda$$

with λ the affine parameter of the geodesic joining x and y whose tangent is $\dot{X}^A$.

We introduce the function $Z(x,y)$ defined as

$$Z(x,y) = \frac{1}{l^2}\eta_{AB}X^A(x)X^B(y). \tag{1.29}$$

In complete analogy with the sphere we write[6]

$$\cos\frac{D(x,y)}{l} = Z(x,y). \tag{1.30}$$

The function Z and the associated geodesic distance d are both invariant under $O(d,1)$. Because of the indefiniteness of the metric, equation 1.30 is not defined for the whole range of Z values, but only for $-1 < Z < 1$; as we shall see below for $Z = \pm 1$ the point x is null separated from y and $\bar{y}$, respectively.

Consider the squared distance from $X^A(x)$ to $X^B(y)$

$$(X^A(x) - X^A(y))^2 = 2l^2(1 - Z).$$

Therefore Z is a de Sitter invariant quantity which measures the separation between two points in dS_d. It changes sign under the antipodal transformation,

$$Z(\bar{x}, y) = -Z(x,y).$$

[6]On a sphere we have for the geodesic distance, D, that $D = l\theta$, where θ is the angle between $X^A(x)$ and $X^B(y)$ each of which satisfies $\delta_{AB}X^AX^B = l^2$, so that $\delta_{AB}X^A(x)X^B(y) = l^2\cos\theta$.

We can therefore write

$$(X^A(x) - X^A(\bar{y}))^2 = 2l^2(1+Z).$$

The function Z provides us with the following information:

$$\begin{aligned} Z &= 0 \text{ if } x \text{ is halfway between } y \text{ and } \bar{y} \\ Z &< 1 \text{ if } x \text{ is spacelike separated from } y \\ Z &= 1 \text{ if } x \text{ and } y \text{ are null separated} \\ Z &> 1 \text{ if } x \text{ and } y \text{ are timelike separated} \\ Z &> -1 \text{ if } x \text{ is spacelike separated } \bar{y} \\ Z &= -1 \text{ if } x \text{ is null separated from } \bar{y} \\ Z &< -1 \text{ if } x \text{ is timelike separated from } \bar{y}. \end{aligned}$$

We can without loss of generality place the spacetime point y at the north pole at the instant $T/l = -\pi/2$ in the conformal coordinate system. We then obtain the following division of the conformal spacetime diagram, figure 1.7.

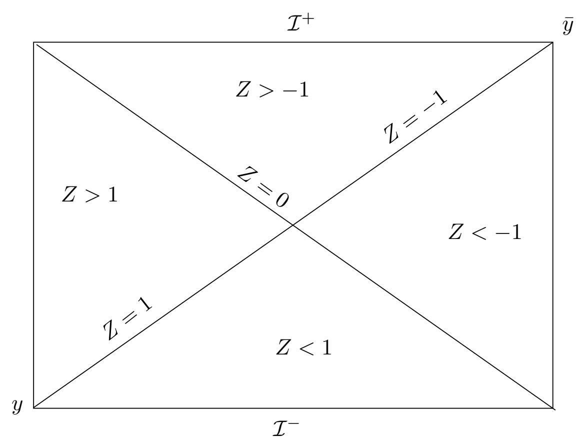

Figure 1.7: The values of the distance function Z for each of the four parts of the Penrose diagram of dS_d.

The function Z is, as is clear from its definition 1.29, invariant under $O(d,1)$, that is, it is time reversal invariant. However, when we will discuss Green functions of scalar fields on dS_d it will prove convenient to work with a quantity which is not time reversal invariant. We define

$$\tilde{Z} = \begin{cases} l^{-2}\eta_{AB}X^A(x)X^B(y) + i\epsilon & \text{if } x \text{ lies in the future light cone of } y \\ l^{-2}\eta_{AB}X^A(x)X^B(y) - i\epsilon & \text{if } x \text{ lies in the past light cone of } y \\ Z & \text{if } x \text{ and } y \text{ are spacelike separated,} \end{cases}$$

where ϵ is a positive real infinitesimal. We then define the so-called signed geodesic distance, $\tilde{D}$, as

$$\tilde{D}(x,y) = l\cos^{-1}\tilde{Z}(x,y).$$

The function $\cos^{-1}\tilde{Z}$ is defined on the complex Z plane and is analytic on the cut plane with branch cuts from $Z = 1$ to ∞ and from $Z = -1$ to $-\infty$. Just above and just below the right-hand branch cut the value of $\cos^{-1}\tilde{Z}$ is $i|\cos^{-1}\tilde{Z}|$ and $-i|\cos^{-1}\tilde{Z}|$, respectively. To prove this write $\cos^{-1}\tilde{Z} = w$ with $w = w_1 + iw_2$. Then

$$\begin{aligned} \tilde{Z} &= \cos(w_1 + iw_2) = \cos w_1 \cos iw_2 - \sin w_1 \sin iw_2 \\ &= \cos w_1 \cosh w_2 - i \sin w_1 \sinh w_2, \end{aligned}$$

so that with $\tilde{Z} = Z \pm i\epsilon$ we find

$$\begin{aligned} Z &= \cos w_1 \cosh w_2 \\ \epsilon &= \mp \sin w_1 \sinh w_2. \end{aligned}$$

We require $Z > 1$ and we know that $\epsilon > 0$. Then if we take $w_2 > 0$ we must have $w_1 \uparrow\downarrow 0$ in order that $\epsilon \downarrow 0$. We can write

$$\cos^{-1}\tilde{Z} = |\cos^{-1}\tilde{Z}|e^{-i\phi},$$

where $\phi = \tan^{-1}\frac{w_2}{w_1}$ with $-\pi \leq \phi < \pi$. We see that $\phi \to \mp\pi/2$, and finally $\cos^{-1}\tilde{Z} \to \pm i|\cos^{-1}\tilde{Z}|$. Since $\tilde{Z}$ changes sign when crossing the branch cut it is no longer invariant under O_T.

The physical events that take place beyond the horizon are not causally related to those which take place in the observer's region. From the conformal diagram fig. it is clear that it takes an infinite amount of time for y's future (past) light cone to cross $\bar{y}$'s future (past) light cone. Further there exist no timelike geodesics connecting x and y when $Z < -1$. Therefore the causal geodesics of an observer at the north pole cannot at any finite instant interact with those of an observer at the south pole.

1.6 Schwarzschild-de Sitter spacetime

Consider the static line element of dS_d. It is of the form

$$ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r^2 d\Omega_{d-2}^2. \tag{1.31}$$

The Schwarzschild-de Sitter spacetime (SdS) is a generalization of dS_d which corresponds to taking

$$V(r) = 1 - \frac{\omega_n M}{r^{n-1}} - \frac{r^2}{l^2},$$

where $n \geq 1$ is related to the dimension d through $d = n + 2$, and where

$$\omega_n = \frac{16\pi G}{n\mathrm{Vol}(S^n)}$$

with $\mathrm{Vol}(S^n)$ the volume of a unit n-sphere. In the limit $l \to \infty$ the function $V(r)$, sometimes referred to as the gravitational potential[7], becomes that of a Schwarzschild black hole in a space which is asymptotically Minkowskian. The integer $n = d - 2$ appears because black holes can only exist in $d \geq 4$ spacetimes. The case $n = 1$ is a special case. It does not describe a black hole but a point-like particle situated at $r = 0$ in the static coordinate system[8]. When $n = 2$, V will be

$$V(r) = 1 - \frac{2GM}{r} - \frac{r^2}{l^2}, \tag{1.32}$$

and when $n = 1$ it will be

$$V(r) = 1 - 8GM - r^2/l^2.$$

In fact, one may write for arbitrary dimensions

$$V(r) = 1 - \frac{\tilde{M}}{r^{d-3}} - \frac{r^2}{l^2},$$

where $\tilde{M}$ denotes the mass of the object were the space asymptotically Minkowskian. The number ω_n then appears as a dimension dependent redefinition of the mass parameter.

In chapter 3 we will consider the Schwarzschild-de Sitter spacetime (SdS) in 4 dimensions. Here some general properties of SdS_4 will be discussed. Most of these properties depend on the potential $V(r)$. For SdS_4 space it can be shown using an algebraic calculation programme that when $27G^2M^2/l^2 < 1$, V(r) has two zero points for positive values of r. This will be assumed to hold throughout. Zeros of the potential are the positions of the horizons for then the timelike Killing vector $\frac{\partial}{\partial t}$ becomes null. A generic potential is shown in figure 1.8.

The two positive zeros of $V(r)$ will be denoted by r_+ and r_{++}. These are the positions of the black hole and the cosmological event horizons, respectively. For $M > 0$ it follows from the form of $V(r)$ that the value of r_+ is larger than that of the Schwarzschild radius in Minkowski space and that r_{++} is smaller than the de Sitter radius of a pure dS_d space. Further as M increases they will approach each other until the two horizons touch. Such a black hole is called a Nariai black hole.

More explicitly, we set

$$1 - \frac{2GM}{r} - \frac{r^2}{l^2} = 0,$$

[7]This name makes sense in terms of the Newtonian approximation where only g_{00} plays a role.

[8]See [19] for a nice account of SdS_3.

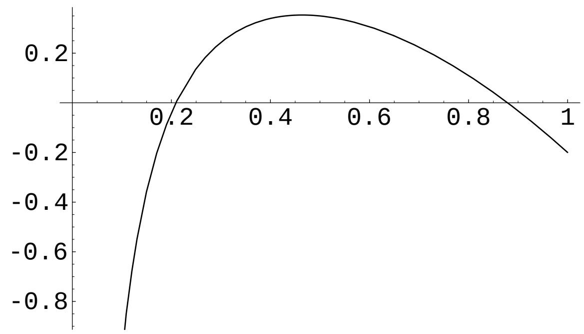

Figure 1.8: The potential $V(r)$ when $27G^2M^2/l^2 < 1$ for positive r values.

or what is the same, consider the zeros of

$$r^3 - l^2 r + 2GMl^2 = 0.$$

As stated for $27G^2M^2/l^2 < 1$ there will be two positive zeros. The third is at, say, $r = -r_{--}$. Then the above equation should factorize as

$$(r - r_+)(r - r_{++})(r + r_{--}) = 0$$

with $r_{--} = r_+ + r_{++}$ so that there are no terms quadratic in r. Setting $r_+ = r_{++}$ it follows that the corresponding value of M denoted by M_N is

$$M_N = \frac{l}{3\sqrt{3}G}.$$

For $M > M_N$ the two horizons disappear (the zeros become complex) since then $27G^2M^2/l^2 > 1$. The line element will in that case describe a naked conical singularity. To avoid the occurrence of such an event we must demand $27G^2M^2/l^2 < 1$ so that the parameter M is bounded, $0 \leq M \leq M_N$.

1.6.1 Penrose diagram of SdS_4 spacetime in static coordinates

The Penrose diagram for the SdS_4 spacetime can somehow be guessed using the Penrose diagram of the Schwarzschild spacetime and of the de Sitter spacetime. The Penrose diagram for the Schwarzschild spacetime contains four regions. The first region describes the asymptotically flat space outside the black hole, the second the interior of the black hole, then there is a third region describing a white hole to which there is an asymptotically flat spacetime, the fourth region. The time development of these four regions corresponds to the time development of the Einstein-Rosen bridge. But now for a black hole in de Sitter space things are a little different. Most importantly, due to the north-south symmetry, if there is a black hole at the north pole there will correspondingly be one at the south pole. Furthermore, we really must consider the equivalent of an Einstein-Rosen bridge at each of the poles, so that for example at the north pole a part of the diagram should describe the time evolution of a contracting dS_d space into a white hole. Hence we obtain the infinite sequence part of which is shown in figure 1.9.

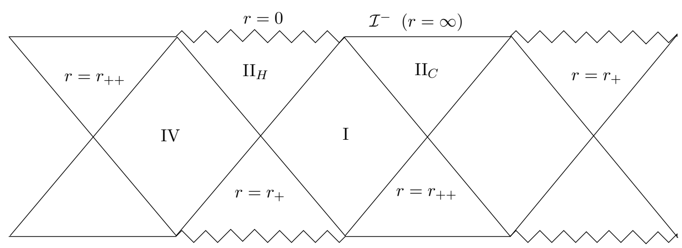

Figure 1.9: Penrose diagram for Schwarzschild-de Sitter spacetime.

Region I describes the spacetime as seen by a static observer who is in between the black hole and the cosmological horizon. Region IV describes the dS_d space that existed previous to the white hole, the region right below II_H. It is identical to the region to the right of II_C, the future triangle behind the cosmological horizon. In fact one can make a periodic identification following the periodicity of the spacelike singularity at $r = 0$ and $r = \infty$, so that this diagram can be wrapped infinitely many times around the cylinder of the Einstein static universe.

1.6.2 Asymptotically flat coordinates for SdS_4

For future reference the line element of the SdS_4 space in asymptotically flat coordinates is given. It reads

$$ds^2 = -\left(\frac{1 - \frac{M}{2r}e^{-t/l}}{1 + \frac{M}{2r}e^{-t/l}}\right)^2 dt^2 + \left(1 + \frac{M}{2r}e^{-t/l}\right)^4 e^{2t/l}(dx^2 + dy^2 + dz^2), \quad (1.33)$$

where $r^2 = x^2 + y^2 + z^2$ and with G set equal to one. The same symbol t is used to denote both the time parameter of the static coordinate system and of the flat coordinate system. Since they will never be used at the same time this should not lead to any confusion.

Chapter 2

Classical and quantum scalar field theory on a fixed de Sitter background

2.1 Introduction

In this chapter we discuss quantization of massive and massless scalar fields defined on a fixed de Sitter background. These are scalar fields with respect to the de Sitter group $SO(d, 1)$ (see section 2.5 for a definition of mass). By a fixed de Sitter background we mean that we ignore the energy-momentum tensor of the scalar field on the right-hand side of the Einstein equation 1.1. On the other hand we do allow the de Sitter background to couple to the scalar field, ϕ, through a term $\xi R\phi^2$ in the Lagrangian, where ξ is a coupling constant. Since the scalar field is otherwise treated as a free field this is the only possible coupling that has the correct dimension, that is, that ξ is dimensionless (the Ricci scalar has dimension mass^2).

We will consider the procedure of covariant quantization even though it is frustrated due to the absence of a global timelike Killing vector field with respect to which one defines, in Minkowski space, the positive and negative frequency modes of a free field. Still, as we shall see in section 2.2, as long as we can define a suitable scalar product with respect to which we can decompose the field ϕ, covariant quantization works. The particular decomposition of a quantum field in terms of its modes automatically determines the ground state or vacuum of that particular field. In Minkowski space the standard Lorentz invariant mode decomposition defines a Poincaré invariant vacuum, but one that is not unique. In fact one can never make a unique choice of vacuum because it is always possible to apply a Bogoliubov transformation that acts on the annihilation operators without mixing them with the creation operators. These new annihilation operators define equivalent vacuum states. Such transformations are called non-mixing or trivial.

Vacua are defined with respect to the inertial observers. In a de Sitter space the observers are freely falling. Since these observers move along timelike geodesics which form a set which is invariant under the $SO(d,1)$ isometry group (see 1.4.1) these vacua are required to be invariant under the de Sitter group.

The existence of a Hamiltonian in the Minkowski case leads to a preferred or natural choice of vacuum as the zero energy state. Then Poincaré invariance requires this state to be unique up to trivial Bogoliubov transformations. In a de Sitter space the requirement of invariance of a vacuum under the isometry group does not exclude the possibility of inequivalent vacua. To study the effects of a particular vacuum choice the properties of the correlation functions in these vacua will be investigated. The two-point functions fully determine a free field theory. It will be shown that there exists a one real parameter family of $O(d,1)$ invariant symmetric two-point functions, and a two real parameter family of $SO(d,1)$ invariant symmetric two-point functions; these functions are expectation values of inequivalent vacua. The inequivalence results from a mixing of the positive and negative frequency modes. If one imposes the restriction that all Green's and correlation functions must locally be of the usual Minkowski type then a vacuum is picked out which is unique up to trivial Bogoliubov transformations. This vacuum will be called the Euclidean vacuum, and it will be the goal of this chapter to argue that this is the right choice of vacuum.

2.2 The orthonormal mode decomposition and Bogoliubov transformations

We shall be working in Planck units, $G = 1/M_{pl}^2 = \hbar = k_B = c = 1$.

We start with the action of the massive scalar field, ϕ, coupled to the gravitational background through, $\xi R\phi^2$,

$$S = -\frac{1}{2}\int d^dx\sqrt{-g}((\nabla\phi)^2 + m^2\phi^2 + \xi R\phi^2).$$

Since the Ricci scalar is a constant we may absorb the coupling constant ξ into the mass m defining the effective mass[1],

$$\tilde{m}^2 = m^2 + \xi R.$$

Then, apart from overall scales the theory has as its only parameter the dimensionless quantity, $\tilde{m}l$, which is equal to the ratio of the de Sitter radius to the Compton wavelength of a particle of mass $\tilde{m}$.

[1]The word "mass" in this section only refers to the parameter in the Lagrangian that we are from Minkowskian physics used to interpret as the mass of a classical field. In section 2.5 we will define the mass of a classical "desitterian" field by considering group representations of the de Sitter group that somehow have a sensible limit, $H \to 0$, to a Minkowskian field.

In the following we will quantize the scalar field on a fixed background. This is permissable as long as the energy-momentum tensor for the scalar field does not couple to the gravitational background. In a quantum description this means that the gravitational quantum fluctuations must be many orders in magnitude smaller than the quantum fluctuations of the scalar field. This will be the case when $l \gg 1$, the Compton wavelength of a particle of mass M_{pl} being equal to unity.

If we vary the field ϕ by $\delta\phi$, which vanishes on spacelike S^{d-1} hypersurfaces at future and past infinity, then we find the Klein-Gordon equation of motion,

$$(\Box - \tilde{m}^2)\phi = 0,$$

where $\Box = g^{ab}\nabla_a\nabla_b$ with g_{ab} the induced metric 1.11. The case for which $\xi = 0$ is called the minimally coupled case. On various occasions the conformally coupled case will be discussed. It has

$$\tilde{m}^2 = \frac{1}{4}\frac{d-2}{d-1}R, \tag{2.1}$$

the only value of the effective mass for which the equation of motion is invariant under a conformal transformation[2]. It follows that then $m = 0$ and $\xi = \frac{1}{4}\frac{d-2}{d-1}$.

Now, a scalar product is introduced with respect to which the field ϕ can be orthonormally decomposed. Denoting the scalar product by $(\cdot, \cdot)$ we define

$$(\phi_1, \phi_2) = -i\int_\Sigma \phi_1 \overleftrightarrow{\partial}_a \phi_2^* d\Sigma^a,$$

where $d\Sigma^a = d\Sigma n^a$ with n^a a future-directed unit vector orthogonal to the Cauchy surface Σ which has the invariant volume element $d\Sigma = \sqrt{-h}d^{d-1}x$, where, finally, h_{ab} is the induced metric on the Cauchy surface. The scalar product is independent of the choice of the Cauchy surface Σ. Denoting the integrand of the above integral by f_a one shows that $\nabla^a f_a = 0$ since the fields satisfy the Klein-Gordon equation. Integrals like $\int_\Sigma f_a n^a d\Sigma$ are independent of Σ. This scalar product generalizes the one employed on a Minkowski background to an arbitrary globally hyperbolic spacetime. For a de Sitter space $\Sigma = S^{d-1}$.

We can now introduce a set of mode functions $\{u_n\}$ which satisfy the Klein-Gordon equation and have the property:

$$(u_n, u_m) = \delta_{nm}. \tag{2.2}$$

It then follows in addition that

$$\begin{aligned}(u_n^*, u_m^*) &= -\delta_{nm}\\ (u_n, u_m^*) &= 0,\end{aligned}$$

[2]The conformal weight of the scalar field ϕ is $1 - d/2$.

so that we may decompose[3] the field ϕ as

$$\phi(x) = \sum_n (a_n u_n(x) + a_n^\dagger u_n^*(x)).$$

Modes u_n are said to have positive frequency if they satisfy equation 2.2. Their complex conjugates have negative frequency. The covariant quantization procedure then proceeds by adopting the commutation relations[4]:

$$\begin{aligned} [a_n, a_m^\dagger] &= \delta_{nm} \\ [a_n, a_m] &= 0 \\ [a_n^\dagger, a_m^\dagger] &= 0. \end{aligned}$$

The vacuum state, denoted by $|0\rangle$, associated with the above mode decomposition is then defined by

$$a_n|0\rangle = 0 \;\; \forall n.$$

The operators a_n, $a_n^\dagger$ are the nth mode annihilation/creation operators with respect to the vacuum $|0\rangle$. They have no energy interpretation.

Let us consider a second complete set of mode functions $\{\bar{u}_n\}$. The field ϕ then decomposes as, say,

$$\phi(x) = \sum_m (\bar{a}_m \bar{u}_m(x) + \bar{a}_m^\dagger \bar{u}_m^*(x)),$$

which defines a new vacuum state, say, $|\bar{0}\rangle$. The two sets of modes $\{u_n\}$ and $\{\bar{u}_n\}$ are related to each other by a Bogoliubov transformation:

$$\bar{u}_n(x) = \sum_m (\alpha_{nm} u_m(x) + \beta_{nm} u_m^*(x)). \tag{2.3}$$

By equating the two mode decompositions of ϕ we find for the creation and annihilation operators

$$\begin{aligned} a_m &= \sum_n (\alpha_{nm} \bar{a}_n + \beta_{nm}^* \bar{a}_n^\dagger) \\ \bar{a}_m &= \sum_n (\alpha_{mn}^* a_n - \beta_{mn}^* a_n^\dagger). \end{aligned}$$

[3]The index n is a collective index. In the global coordinate system the mode functions are defined on the spatial S^{d-1} sections, so that n collectively refers to the quantum numbers of the spherical harmonics on S^{d-1}. In the planar coordinate system the modes are defined on spatial planes, so that n is a continuous index and refers to a plane wave $(d-1)$-momentum.

[4]These relations are adopted from the usual form of the creation and annihilation operators on $\mathcal{M}^d$. However, in this case there is no Hamiltonian. Still they have meaning in terms of creation and annihilation of modes with respect to the mode number operator $N_n = a_n^\dagger a_n$.

We define the matrices $A \equiv (\alpha_{nm})$ and $B \equiv (\beta_{nm})$. Then it can be shown [4] that they satisfy $AA^\dagger - BB^\dagger = I$ and that AB^T must be symmetric. The first property follows from the commutation relation for the creation and annihilation operators and the second property follows from the property of the scalar product: $(\phi_1, \phi_2) = -(\phi_1^*, \phi_2^*)$.

Consider the two Fock spaces build on the vacua $|0\rangle$ and $|\bar{0}\rangle$. Let a_n act on the vacuum $|\bar{0}\rangle$,

$$a_n|\bar{0}\rangle = \sum_m (\alpha_{mn}\bar{a}_m|\bar{0}\rangle + \beta^*_{mn}\bar{a}^\dagger_m|\bar{0}\rangle) = \sum_m \beta^*_{mn}|\bar{1}_m\rangle.$$

It follows that the expectation value of the operator $N_n = a^\dagger_n a_n$ for the number of u_n-mode particles in the vacuum state $|\bar{0}\rangle$ is

$$\langle\bar{0}|N_n|\bar{0}\rangle = \sum_m |\beta_{mn}|^2. \tag{2.4}$$

We conclude that the two Fock spaces are inequivalent as long as $\beta_{nm} \neq 0$.

2.3 The Wightman function and the Euclidean vacuum

Consider the Wightman function,

$$G^+_\lambda(x,y) \equiv \langle\lambda|\phi(x)\phi(y)|\lambda\rangle,$$

where λ labels the possible inequivalent vacuum choices; it contains the parameters of the Bogoliubov transformations. The labelling is explained in section 2.4. There it is shown that a general Bogoliubov transformation is described by two parameters α and β. Then $|\lambda\rangle$ means $|\alpha,\beta\rangle$. In the case of free fields the Wightman function is the basic building block of any other correlation or Green's function, and it can be ascribed physical meaning for it is an observable being part of the transition amplitude of an Unruh detector. We will discuss this highly theoretical device in section 2.7. It contains a symmetric and an antisymmetric part,

$$G^+_\lambda(x,y) = \frac{1}{2}(G^{(1)}_\lambda(x,y) + iD_\lambda(x,y)). \tag{2.5}$$

$G^{(1)}_\lambda$ is the Hadamard or symmetric two-point function and iD_λ is the commutator function;

$$\begin{aligned} G^{(1)}_\lambda(x,y) &= \langle\lambda|\{\phi(x),\phi(y)\}|\lambda\rangle \\ iD_\lambda(x,y) &= \langle\lambda|[\phi(x),\phi(y)]|\lambda\rangle. \end{aligned}$$

It follows immediately from its definition that the Wightman function is an $O(d,1)$ invariant quantity. It therefore only depends on Z. Let f be a function of Z that satisfies the Klein-Gordon equation,

$$(\Box - \tilde{m}^2)f(Z) = 0.$$

We have[5]

$$\Box f(Z) = \frac{d^2 f}{dZ^2}(\partial Z)^2 + \frac{df}{dZ}\nabla^a(\partial_a Z),$$

and

$$\begin{aligned}\partial_a Z &= \frac{\partial}{\partial x^a}(l^{-2}\eta_{AB}X^A(x)X^B(y)) = l^{-2}X_A(y)\frac{\partial}{\partial x^a}X^A(x)\\ &= l^{-2}(X_d(y)\frac{\partial}{\partial x^a}X^d(x) + X_a(y)) = -l^{-2}(\frac{X^d(y)}{X^d(x)}X_a(x) - X_a(y)).\end{aligned}$$

We evaluate

$$(\partial Z)^2 = g^{ab}(\partial_a Z)(\partial_b Z) = l^{-2}(1-Z^2),$$

and

$$\nabla^a(\partial_a Z) = -l^{-2}dZ.$$

The Klein-Gordon equation becomes the second order ordinary differential equation:

$$(1-Z^2)\frac{d^2 f}{dZ^2} - dZ\frac{df}{dZ} - \tilde{m}^2 l^2 f = 0. \tag{2.6}$$

Changing the variable Z to $z = \frac{1+Z}{2}$ we obtain the hypergeometric differential equation:

$$z(1-z)\frac{d^2 f}{dz^2} + (\frac{d}{2} - dz)\frac{df}{dz} - \tilde{m}^2 l^2 f = 0. \tag{2.7}$$

The hypergeometric function[6] ${}_2F_1(h, (d-1)-h, \frac{d}{2}; \frac{1+Z}{2})$, where h satisfies

$$h(h-(d-1)) + l^2\tilde{m}^2 = 0,$$

provides a solution of the differential equation 2.6 for Z. This solution is real for $Z < 1$ as it should since, by microcausality, the commutator function should vanish for spacelike separated points. A second independent solution follows by recognizing that equation 2.6 is invariant under $Z \to -Z$. It is given by ${}_2F_1(h, (d-1)-h, \frac{d}{2}; \frac{1-Z}{2})$. They are only linearly independent for

[5]See page 9 for a good many formulae that we use here.
[6]See appendix A for some basic results on hypergeometric functions.

$\tilde{m}^2 > 0$ since for $\tilde{m} = 0$ the parameter h is equal to zero so that the hypergeometric function becomes a constant. Thus, the most general solution of equation 2.6, and hence of the Klein-Gordon equation, is

$$af(Z) + bf(-Z) \tag{2.8}$$

with $f(Z) \propto {}_2F_1(h, (d-1) - h, \frac{d}{2}; \frac{1+Z}{2})$. This linear combination is the most general form of the Wightman function, $G^+_\lambda(x, y)$, for general λ and arbitrary real constants a and b.

Equation 2.8 tells us that in general the Wightman function has two singularities, one for $Z = 1$ and one for $Z = -1$. From section 1.5 we know that between positive and negative Z values one always finds a horizon, so that there is no way for an observer to find out about the presence of the singularity at $Z = -1$.

One defines the Euclidean vacuum as corresponding to the choice $b = 0$ in equation 2.8, that is, to a set of mode functions that give rise to a Wightman function which is only singular at $Z = 1$. It is de Sitter invariant because these modes only depend on Z. It is termed the Euclidean vacuum because dS_d can be analytically continued to S^d by introducing an imaginary time axis in $\mathcal{M}^{d+1}$. On S^d the Green's or inverse function of $\Box - \tilde{m}^2$ is unique and has one singularity: when y and x coincide. If we analytically continue this Green's function to dS_d we obtain the Feynman propagator for dS_d in the vacuum which has only one singularity, that is, with $b = 0$. The Euclidean vacuum will be denoted by $\lambda = 0$. This corresponds to the parameter value $\alpha = 0$ as explained in section 2.4.

The Wightman function in the Euclidean vacuum, $G^+_0(x, y)$, is defined to be

$$G^+_0(x, y) \equiv f(Z) \equiv C_{d,h}\, {}_2F_1(h, (d-1) - h, \frac{d}{2}; \frac{1+Z}{2}), \tag{2.9}$$

where $C_{d,h}$ is a normalization constant.

The function f is normalized such that sufficiently locally it has the same singular behavior as the Wightman function in $\mathcal{M}^d$ (see equation A.2),

$$\frac{\Gamma(\frac{d}{2})}{2(d-2)\pi^{d/2}} (\frac{1}{D})^{d-2}.$$

From equation A.1 it follows that the function f behaves at $Z = 1$ as

$$f \sim (1 - Z)^{1-\frac{d}{2}} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} - 1)}{\Gamma(h)\Gamma((d-1) - h)}.$$

The geodesic distance was defined as $D(x, y) = l \cos^{-1} Z(x, y)$. One shows by Taylor expansion that $\frac{D^2}{4l^2}$ around $Z = 1$ equals to $1 - Z$ to first order in Z. Therefore,

$$f \sim (\frac{4l^2}{D})^{d-2} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} - 1)}{\Gamma(h)\Gamma((d-1) - h)},$$

so that

$$C_{d,h} = \frac{\Gamma(h)\Gamma((d-1)-h)}{l^d(4\pi)^{d/2}\Gamma(\frac{d}{2})}.$$

The $i\epsilon$-prescription for propagators in the Euclidean vacuum is the same as in Minkowski space, and follows from regularization of the Wightman function. In terms of the mode functions $u_n(x)$ the Wightman function is

$$G_0^+(x,y) = \sum_n u_n(x)u_n^*(y) = f(Z).$$

Let x be timelike separated from y, then $Z > 1$. However, the hypergeometric function is not analytic for real $Z > 1$. In order to overcome this problem we need to consider f away from the branch cut. Note that for timelike separated x and y

$$G_0^+(x,y) \neq G_0^+(y,x) = G_0^-(x,y).$$

We must therefore distinguish the two cases: x lies to the future of y and x lies to the past of y.

It is shown in [4], equation (3.154), that for conformally coupled scalar fields in conformally flat spacetimes the Feynman propagator can be written as a product of the massless Minkowski space Feynman propagator with two conformal factors. Since the $i\epsilon$-prescription is not to depend on the mass it carries over to dS_d. Thus, we find that when x is to the future of y then the Wightman function G^+ is regularized[7] by $f(Z+i\epsilon)$ and when x is to the past of y then the Wightman function G^- is regularized by $f(Z-i\epsilon)$. Collectively we write

$$G_0^W = f(Z + i\epsilon\,\mathrm{sign}(x,y)).$$

Put another way, we take the fields in $\langle 0|\phi(x)\phi(y)|0\rangle$ to be time ordered breaking explicitly the time reversal invariance and consider the Wightman function as a function of the signed geodesic distance of section 1.5.

From 2.5 we have for the time ordered commutator and symmetric two-point function

$$D_0(x,y) = 2\mathrm{Im}\, f(Z + i\epsilon\,\mathrm{sign}(x,y)) \tag{2.10}$$

$$G_0^{(1)}(x,y) = 2\mathrm{Re}\, f(Z + i\epsilon\,\mathrm{sign}(x,y)). \tag{2.11}$$

Then the Feynman propagator is given by

$$iG_0^F(x,y) = \frac{1}{2}[G_0^{(1)}(x,y) + i\,\mathrm{sign}(x,y)D_0(x,y)].$$

It then follows that the Feynman propagator becomes

$$iG_0^F(x,y) = f(Z + i\epsilon). \tag{2.12}$$

[7] From what was said below equation A.2 we know that for x to the future of y we must take $D \to D + i\epsilon$. From section 1.5 we know that this means for Z that we must take $Z \to Z + i\epsilon$.

2.4 The Mottola-Allen vacua

Any other vacuum state can be obtained from the Euclidean vacuum by application of a Bogoliubov transformation provided the resulting vacuum is de Sitter invariant. A vacuum is de Sitter invariant if the symmetric two-point function in that vacuum is de Sitter invariant. One proves [2] that this requires the matrices A and B of the Bogoliubov transformation to be proportional to the identity matrix. Further we have the restriction that $AA^\dagger - BB^\dagger = I$. Thus, A and B can be parameterized by $A = I\cosh\alpha$ and $B = Ie^{i\beta}\sinh\alpha$, where $0 \leq \alpha < \infty$, $-\pi \leq \beta < \pi$. The Euclidean vacuum corresponds to $\alpha = 0$, hence the designation $\lambda = 0$. The new mode functions, $\bar{u}_n$, are then related to those corresponding to the Euclidean vacuum, u_n, by

$$\bar{u}_n(x) = \cosh\alpha\, u_n + e^{i\beta}\sinh\alpha\, u_n^*(x).$$

We will denote the MA vacua by $\lambda = (\alpha, \beta)$.

From the mode decomposition of the symmetric two-point function and the commutator function we find

$$G^{(1)}_{\alpha,\beta}(x,y) = \cosh 2\alpha\, G^{(1)}_0(Z) + \sinh 2\alpha\, [\cos\beta\, G^{(1)}_0(-Z) - \sin\beta\, D_0(\bar{x}, y)],$$

and

$$D_{\alpha,\beta}(x,y) = D_0(x,y).$$

In the calculation of these two correlation functions use has been made of the fact that we can choose the basis of mode functions for the Euclidean vacuum such that [2] $u_n^*(x) = u_n(\bar{x})$. The expression above for $G^{(1)}_{\alpha,\beta}(x,y)$, coming directly from the mode decomposition, is not regularized. This can be achieved by replacing $G^{(1)}_0$ and D_0 by their regularized versions, equations 2.10 and 2.11.

Applying the time reversal operator T of the de Sitter group we see that

$$G^{(1)}_{\alpha,\beta}(Tx, Ty) = G^{(1)}_{\alpha,-\beta}(x,y)$$

because the commutator function D_0 changes sign under T. We conclude that the MA vacua violate CPT invariance if $\beta \neq 0$. In this case we have a two real parameter family of $SO(d,1)$ invariant symmetric two-point functions. If $\beta = 0$ we have a one real parameter family of $O(d,1)$ invariant symmetric two-point functions.

From equation 2.4 we see that the total number of $\alpha \neq 0$ modes in the Euclidean vacuum or any other vacuum is infinite. Gibbons and Hawking [16] showed that this a necessary condition for any de Sitter invariant state, since it is required to have an equal number of particles of every momentum.

The commutator, advanced and retarded Green's functions are all only non-zero within the local light cone as is required by microcausality. One further shows that they do not depend on the choice of vacuum. Hence, the classical theory of the field ϕ is the same no matter what quantum vacuum has been chosen.

In order to see the effect an MA vacuum may have, the difference between an MA propagator and a Euclidean propagator is discussed for the special case of a conformally coupled field in 4 dimensions. For simplicity the parameter β is set equal to zero. The symmetric two-point function $G^{(1)}_{\alpha,0}(x,y)$ becomes

$$G^{(1)}_{\alpha,0}(x,y) = \cosh 2\alpha G^{(1)}_0(Z) + \sinh 2\alpha G^{(1)}_0(-Z).$$

Consequently, the Feynman propagator reads

$$iG^F_{\alpha,0}(x,y) = \frac{1}{2}[G^{(1)}_{\alpha,0}(x,y) + i\ \mathrm{sign}(x,y) D_0(x,y)] =$$

$$\cosh^2\alpha\ iG^F_0(Z) + \sinh^2\alpha (iG^F_0(Z))^* + \frac{1}{2}\sinh 2\alpha[iG^F_0(-Z) + (iG^F_0(-Z))^*].$$

Using equation 2.12 the regularized Feynman propagator for a general $(\alpha, \beta = 0)$ vacuum is

$$iG^F_{\alpha,0}(x,y) =$$

$$\cosh^2\alpha f(Z+i\epsilon) + \sinh^2\alpha f^*(Z+i\epsilon) + \frac{1}{2}\sinh 2\alpha (f(-Z-i\epsilon) + f^*(-Z-i\epsilon)).$$

In general the function $f(Z)$, the Wightman function in the Euclidean vacuum, is given by equation 2.9. For the special case of a conformally coupled field in 4 dimensions the parameter $h = 2$. It is easily shown that then

$$f(Z) = \frac{1}{8\pi^2 l^2}\frac{1}{1-Z}.$$

Substituting this in the expression for $iG^F_{\alpha,0}(x,y)$ we obtain

$$iG^F_{\alpha,0}(x,y) = \frac{1}{8\pi^2 l^2}[\cosh^2\alpha \frac{1}{1-Z-i\epsilon} + \sinh^2\alpha \frac{1}{1-Z+i\epsilon} + \sinh 2\alpha \frac{1}{1+Z}]. \tag{2.13}$$

where $1/(1+Z)$ is actually the principal value of $1/(1+Z)$, that is, it is the average of $(1+Z+i\epsilon)^{-1}$ and $(1+Z-i\epsilon)^{-1}$. It is recalled that the Euclidean vacuum corresponds to $\alpha = 0$.

In section 2.6 it will be discussed what it means to have such a free field propagator when self-interactions are introduced.

2.5 Scalar representations of the de Sitter group; on the concept of mass

Let us consider again the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}(g^{ab}(\nabla_a\phi)(\nabla_b\phi) + (m^2 + \xi R)\phi^2).$$

A free "massive" scalar field ϕ on dS_d satisfies the Klein-Gordon equation $(\Box - \tilde{m}^2)\phi(x) = 0$, where $\tilde{m}^2 = m^2 + \xi R$.

Not every classical scalar field on a de Sitter space can be ascribed a sensible notion of mass. Mass and also the spin of a field are essentially concepts of a Minkowski space. In group theoretical language: they label the unitary irreducible representations (UIR) of the Poincaré group. The mass squared is the eigenvalue of the Casimir operator $-P_aP^a$, where P^a is the total momentum operator. The spin s enters the eigenvalue of the Casimir operator $-W_aW^a$, whose eigenvalues are $m^2s(s+1)$. The vector W^a is the Pauli-Lubanski vector. For this group these mass eigenvalues form a continuous non-negative spectrum. It will turn out that this is not the case for the eigenvalues of the corresponding Casimir operator of the de Sitter group, $SO(d,1)$. Further not all of the latter eigenvalues can be interpreted as a mass.

If our universe really is asymptotically de Sitter then we should be able to define, at least in our local neighborhood, a notion of mass. From the induced metric 1.11 it is clear that dS_d approaches $\mathcal{M}^d$ in the limit $l \to \infty$ or equivalently as $H \to 0$. A heuristic argument will be presented that will lead to a definition of mass in terms of the above Minkowskian interpretation.

Assume de Sitter space to be 4-dimensional. There is one particular value of the parameter $\tilde{m}$ for which we have a priori notion of its physical meaning and that is the conformally coupled case, $\tilde{m}^2 = 2H^2$, see equation 2.1. The equation of motion becomes

$$(\Box - 2H^2)\phi = 0.$$

The additional geometric structure enables us to take the limit $H \to 0$ more or less straightforwardly. Transform to a conformal metric which can be taken to equal the Minkowski metric since the spacetime is conformally flat, and simply let H go to zero. It follows immediately that the Minkowskian mass of this scalar field is zero.

The generators of the lie algebra of $SO(d,1)$ are, in the index notation of chapter 1, J_{AB}, where $A, B = 0, 1, \cdots, 5$, simply because it is the homogeneous Lorentz group of $\mathcal{M}^{d+1}$. They satisfy the algebra

$$[J_{AB}, J_{CE}] = i(\eta_{AC}J_{BE} - \eta_{AE}J_{BC} + \eta_{BE}J_{AC} - \eta_{BC}J_{AE}).$$

The Casimir operators of $SO(d,1)$ are

$$Q^{(1)} = -\frac{1}{2}J_{AB}J^{AB} \quad \text{and}$$

$$Q^{(2)} = -W_A W^A,$$

where $W_A = -\frac{1}{8}\epsilon_{ABCEF}J^{BC}J^{EF}$. We write J_{AB} as

$$J_{AB} = L_{AB} + S_{AB},$$

where $L_{AB} = -i(X_A\partial_B - X_B\partial_A)$ is the orbital part and S_{AB} the spinorial part. Let $Q_0^{(1)} = -\frac{1}{2}L_{AB}L^{AB}$. It represents the scalar part of $Q^{(1)}$.

The generators J_{AB} are 5×5 matrices and from the 4-dimensional point of view of the embedded de Sitter space it can be said that it decomposes into the upper left 4×4 antisymmetric matrix J_{ab} and the vector J_{4a}. In the limit $H \to 0$ these operators should have a Poincaré group interpretation. The Casimir operator $Q_0^{(1)}$ for a scalar field decomposes as

$$Q_0^{(1)} = -L_{4a}L^{4a} - \frac{1}{2}L_{ab}L^{ab}.$$

To facilitate taking the limit $H \to 0$ we write

$$H^2 Q_0^{(1)} = -\frac{L_{4a}}{l}\frac{L^{4a}}{l} - \frac{1}{2l^2}L_{ab}L^{ab}.$$

Letting $l \to \infty$ the operator L_{ab} becomes the orbital angular momentum operator of the 4-dimensional Poincaré group, and

$$\frac{L_{4a}}{l} \to P_a.$$

These are rather non-trivial limits. The process is called group contraction. Thus, in order to define the Minkowskian mass of a scalar field we need to consider the eigenvalues of $H^2 Q_0^{(1)}$.

The d'Alembertian $\Box$ is defined on the tangent space T_{dS_4} and acts on $\phi(x)$ where $x \in dS_4$. In the ambient space description of dS_4 the tangent space lies in $\mathcal{M}^5$. In this embedding of T_{dS} the vectors that span T_{dS} are denoted by $\bar{\partial}_A$. They can be found by introducing the so-called transverse projection operator $\theta_{AB} \equiv \eta_{AB} - H^2 X_A X_B$. It satisfies $\theta_{AB}X^A = 0 = \theta_{AB}X^B$. Thus, using the ambient space construction the tangent space to dS_4 is spanned by the vectors $\bar{\partial}_A = \theta_{AB}\partial^B = \partial_A - H^2 X_A X^B \partial_B$. Hence $X^A\bar{\partial}_A = 0$. Therefore we have

$$\Box\phi(x) = \bar{\partial}^2\phi(X).$$

$H^2 Q_0^{(1)}$ is a differential operator acting on $\phi(X)$. For the Casimir operator $Q_0^{(1)}$ it must be that

$$H^2 Q_0^{(1)}\phi(X) = \bar{\partial}^2\phi(X)$$

which allows an easy Minkowskian interpretation provided we can give meaning to the group contraction procedure. Using the correspondence

$$H^2 Q_0^{(1)} \phi(X) = \Box \phi(x)$$

the equation for the conformally coupled scalar field can be written as

$$H^2 (Q_0^{(1)} - 2)\phi(X) = 0.$$

Returning to the general problem of interpreting the parameter $\tilde{m}$ from a Minkowskian point of view, we note that the mode functions with $\tilde{m} \neq 0$ of the operator $\Box - \tilde{m}^2$, as shown in section 2.3, may be labelled by the parameter h determining the hypergeometric function completely. In 4 dimensions it is $h = \frac{3}{2} + \frac{1}{2}\sqrt{9 - 48\xi - 4m^2 H^{-2}}$. One customarily writes $h = \frac{3}{2} + i\nu$, $\nu = \sqrt{m^2 H^{-2} + 12\xi - \frac{9}{4}}$.

Dixmier [9] has proven that the UIR's of the de Sitter group may be labelled by the pair of parameters (p, q) with $2p \in \mathbb{N}$ and $q \in \mathbb{C}$, that is, they label the eigenvalues of $Q^{(1)}$ and $Q^{(2)}$ which are diagonalized to

$$Q^{(1)} = [-p(p+1) - (q+1)(q-2)]Id,$$
$$Q^{(2)} = [-p(p+1)q(q-1)]Id.$$

Dixmier then continues to show that three series of inequivalent UIR's emerge corresponding to particular values of (p,q). They are called the principal, complementary and discrete series of representations.

The principal series of representations is denoted by $U_{p,\nu}$ with $(p, q) = (p, \frac{1}{2} + i\nu)$ where either

$$p = 0, 1, 2, \cdots \quad \text{and} \quad \nu \geq 0 \quad \text{or}$$
$$p = \frac{1}{2}, \frac{3}{2}, \cdots \quad \text{and} \quad \nu > 0.$$

The complementary series of representations is denoted by $V_{p,\nu}$ with $(p, q) = (p, \frac{1}{2} + i\nu)$ where either

$$p = 0 \quad \text{and} \quad i\nu \in \mathbb{R},\ 0 < |\nu| < \frac{3}{2} \quad \text{or}$$
$$p = 1, 2, \cdots \quad \text{and} \quad i\nu \in \mathbb{R},\ 0 < |\nu| < \frac{1}{2}.$$

The discrete series of representations is denoted by $\Pi_{p,0}$ and $\Pi^{\pm}_{p,q}$ with either

$$p = 1, 2, \cdots \quad \text{and} \quad q = p, p-1, \cdots, 1, 0 \quad \text{or}$$
$$p = \frac{1}{2}, \frac{3}{2}, \cdots \quad \text{and} \quad q = p, p-1, \cdots, \frac{1}{2}.$$

For the principal and complementary series $Q^{(1)}$ becomes

$$Q^{(1)} = [(\frac{9}{4} + \nu^2) - p(p+1)]I_d.$$

Therefore, since we want to know how the mass parameter is contained in $\tilde{m}^2 = H^2(\frac{9}{4} + \nu^2)$, the scalar fields which are of interest to us all have $p = 0$ and so do not belong to the discrete series. For the complementary series with $p > 1$ the eigenvalues of $Q_0^{(1)}$ are negative. From the correspondence $H^2 Q_0^{(1)} \phi(X) = \Box\phi(x)$ it follows that these representations may be qualitatively described as exponentially damped.

It follows that for general p and q the eigenvalues of $Q^{(1)}$ form a spectrum which contains both positive and negative values with parts being discrete and parts being continuous. Further there are regions in the range of eigenvalues of $Q^{(1)}$ for which no pair (p,q) exists. These regions are termed forbidden. For more details see [13].

For the conformally coupled case the label ν equals $\frac{i}{2}$ so that the eigenvalue of $H^2 Q_0^{(1)}$ in this case is 2. Thus, apparently, if we define the Minkowskian mass of the field ϕ as the eigenvalue of the expression

$$H^2(Q_0^{(1)} - 2)\phi$$

it comes out correctly for the conformally coupled case. Following the notation of Garidi [13], we denote the Minkowskian mass of ϕ by m_H^2. It reads

$$m_H^2 = (\frac{1}{4} + \nu^2)H^2. \tag{2.14}$$

To which values of ν does it apply?

In the limit $H \to 0$ the UIR label $q = h - 1 = \frac{1}{2} + i\nu$ behaves as: $q \sim \frac{im}{H}$. Therefore if we want to be able to take this limit for a UIR of the de Sitter group, we must use a representation for which q is unbounded. Thus, heuristically showing that only the principal series contracts towards UIR's of the Poincaré group. Note that the conformally coupled case belongs to the complementary series. This is not a problem. Group contraction is not needed in this special case. It was already argued that $m_H = 0$, and that the formula 2.14 also applies in this case. Denoting the generic UIR of $SO(4,1)$ by D_ν the group contraction goes according to the scheme

$$D_\nu \to \mathcal{P}(\pm m),$$

where $\mathcal{P}(\pm m)$ denotes a UIR of the 4-dimensional Poincaré group, and where $\nu \sim \frac{m}{H}$.

We conclude that the scalar fields for which we have a notion of mass, from a Minkowskian point of view, are the UIR's of the principal series. The scalar fields belonging to the complementary series have no Minkowski equivalent concept of mass except for the case $p = 0$ and $q = 0$, that is, for the conformally coupled case, $\nu = \frac{i}{2}$. For this class of scalar fields we have defined the mass as $m_H^2 = (\frac{1}{4} + \nu^2)H^2$.

An example of a pure desitterian scalar field is provided by the minimally coupled "massless" field whose equation of motion is $\Box\phi = 0$. This field corresponds to $(p, q) = (1, 0)$, the lowest term in the discrete series. Such a field will at any point of dS_4 move along the local light cone whereas the field with $m_H = 0$ will partially move along the local light cone and sometimes be scattered inside the light cone due to its coupling to the background.

Finally it is mentioned that only with respect to the Euclidean vacuum is it the case that the positive frequency modes contract toward the positive frequency modes as $H \to 0$, in any point of spacetime, [22].

2.6 Self-interacting scalar fields and the Euclidean vacuum

In this section it will be argued that a scalar field having self-interactions can if one uses perturbation theory only be described in the euclidean vacuum. In order not to make things unnecessarily complicated we will discuss ϕ^3 interactions of conformally coupled massless scalar fields whose vacua respect CPT invariance. When there are no interactions the Feynman propagator of such particles is given by equation 2.15. It will be repeated here for convenience

$$iG^F_{\alpha,0}(x,y) = \frac{1}{8\pi^2 l^2}[\cosh^2\alpha\frac{1}{1-Z-i\epsilon}+\sinh^2\alpha\frac{1}{1-Z+i\epsilon}+\sinh 2\alpha\frac{1}{1+Z}]. \tag{2.15}$$

where $1/(1+Z)$ is actually the principal value of $1/(1+Z)$, that is, it is the average of $(1+Z+i\epsilon)^{-1}$ and $(1+Z-i\epsilon)^{-1}$. The singularities are at $Z = \pm 1$ just above and below the real Z axis. The euclidean vacuum corresponds to $\alpha = 0$ in which case the above propagator reduces to

$$iG^F_0(x,y) = \frac{1}{8\pi^2 l^2}\frac{1}{1-Z-i\epsilon}. \tag{2.16}$$

This has as its only singularity a pole at $Z = 1$ in the limit $\epsilon \to 0$. In the euclidean vacuum the propagator 2.16 is the boundary value in the sense of ϵ going to zero of an analytic function. In the $\alpha \neq 0$ case the singularity at $Z = 1$ becomes "pinched", and away from the real axis at $Z = 1$ the propagator is no longer analytic. Further, in this case two more singularities at $Z = -1$ appear.

Consider the one loop correction to the free propagator. The coordinate space Feynman diagram for ϕ^3 theory is depicted in figure 2.1. The rules for finding the transition amplitude are in curved spacetime not essentially different from the usual rules in $\mathcal{M}_4$. If λ represents the coupling strength of the self-interactions then the amplitude in the $\alpha \neq 0$ vacua (with $\beta = 0$)

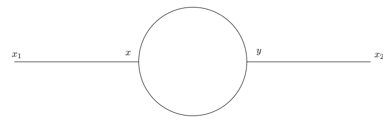

Figure 2.1: One-loop contribution to the self-energy.

is given by

$$-ig\int\sqrt{g(x)}dx\sqrt{g(y)}dy iG^F_{\alpha,0}(x_1,x)(iG^F_{\alpha,0}(x,y))^2 iG^F_{\alpha,0}(y,x_2).$$

The only difference with Minkowski space is the appearance of $\sqrt{g}$ in the integration measure.

In order to be able to compare with the Minkowski case the function Z will be expressed in the flat coordinates of section 1.3.3. The analogy will be strongest if we redefine the time parameter $t \to \eta(t)$ so that the metric becomes

$$ds^2 = \frac{1}{\eta^2}(-d\eta^2 + d\vec{x}^2).$$

This line element differs from a flat Minkowskian line element by a conformal factor. Then it can be shown that

$$1 - Z(x,y) = \frac{(\eta_x - \eta - y)^2 - (\vec{x}-\vec{y})^2}{2\eta_x\eta_y},$$

where $x^a = (\eta_x, \vec{x})$.

Now, we will consider the part of the amplitude of the one-loop diagram which has a singularity at $Z = 1$. This is the physically most relevant singularity. In fact it is not even clear how to interpret the other singularity at $Z = -1$. Consider the part of the external legs that are proportional to $(1 - Z - i\epsilon)^{-1}$, and of the terms comprising the square loop contribution, $(iG^F_{\alpha,0}(x,y))^2$, only the term proportional to $(1-Z-i\epsilon)^{-1}(1-Z+i\epsilon)^{-1}$ will be considered. The corresponding term that appears in the amplitude is proportional to

$$\int dxdy\sqrt{g(x)g(y)}\frac{1}{(x_1-x)^2 - i\epsilon}\frac{(2\eta_x\eta_y)^2}{((x-y)^2-i\epsilon)((x-y)^2+i\epsilon)}\frac{1}{(y-x_2)^2-i\epsilon}. \tag{2.17}$$

The middle term between the external legs contributes to the one-particle irreducible self-energy, and it has poles at

$$\eta_x - \eta_y = \pm(|\vec{x}-\vec{y}| \pm i\epsilon).$$

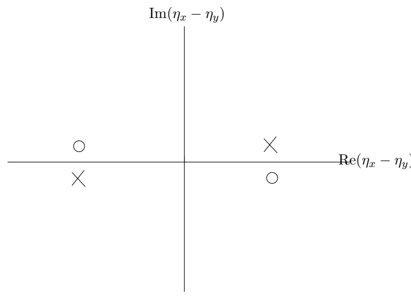

Figure 2.2: The pinched singularities of the one-loop contribution to the one-particle irreducible self-energy of a conformally invariant massless scalar field.

Figure 2.2 shows the position of these four poles in the complex $(\eta_x - \eta_y)$-plane. The poles denoted by a cross are the usual poles one finds in Minkowski space. The other two, denoted by a circle result from the term having an $i\epsilon$ prescription with the opposite sign, a plus sign. The combination of one cross and one circle is called a pinched singularity. In performing the integrations the contour cannot be deformed and thus runs over the singularities in the limit of ϵ going to zero. This renders the expression 2.17 ill-defined; it is mathematically not defined.

Thus, apart from the physically unclear singularity of the $\alpha \neq 0$ propagators at $Z = -1$ the one-loop contribution to the propagator cannot even be made mathematically precise. The only value of α for which this does not happen is 0, the euclidean vacuum. The euclidean vacuum will be regarded as the only possible vacuum on which a quantum scalar field theory can be build.

2.7 Thermal aspects of dS_d.

We will calculate the temperature of dS_d, that is, the temperature measured by an arbitrary observer, and discuss some elements of the theory of the thermodynamics of event horizons.

In this section we will frequently speak of a detector, and each time it will be assumed to be an Unruh detector. This is a small box containing a nonrelativistic particle, e.g. an atom, with internal energy levels, coupled via a monopole interaction with the scalar field.

It is known [4, 23] that an accelerating observer in Minkowski space equipped with some particle detector measures a finite nonzero temperature. An acceleration in Minkowski space does not come about naturally, that is, an external agent who supplies energy to the detector is needed. The detector's acceleration will, from the point of view of an inertial observer, be accompanied by inertial effects; it emits quanta through its coupling to the scalar field. Therefore, from the point of view of the detector upward transitions must have taken place. These come from the absorption of field quanta. Thus, when a field is in its ground state with respect to an inertial observer it will be in some excited state with respect to the detector. The reason for the apparent absorption/emission of particles by the detector is that the detector "sees" the modes of the field quanta with respect to its proper time. This results in a "detector sense" of positive frequency which is inequivalent to what an inertial observer would define as positive frequency.

However, the vacuum expectation value of the energy-momentum tensor of the scalar field vanishes in both the inertial and the detector's frame. We say that virtual particles are created due to the noninertial motion of the detector.

Accelerated observers possess an event horizon. It has been demonstrated in the seventies by Hawking, Gibbons, Bekenstein and others that there exists an intricate relationship between event horizons and thermodynamics.

In a de Sitter space the observers are freely falling and all possess an event horizon. We may thus expect that they too will measure a finite temperature. Yet, there is an important difference, namely that in this case there is no external agent who supplies the energy, but it is the spacetime curvature itself. We will now calculate[8] this temperature and show how it can be understood in terms of the horizon.

We shall be working in the global coordinate system. The detector-field coupling is described by the Hamiltonian

$$gm(\tau)\phi(x(\tau)),$$

where $x(\tau)$ is the observers world-line parameterized by the world or proper time τ, and g is the coupling strength which is assumed to be small. The states of the detector-field configuration form tensor product states. Assume the system to be initially in its ground state $|0\rangle|E_i\rangle$, where E_i is the detector's lowest energy level. The final state will be denoted by $|\beta\rangle|E_f\rangle$. Then to first order in perturbation theory the transition amplitude is

$$g\int_{-\infty}^{\infty} d\tau\langle E_f|\langle\beta|m(\tau)\phi(x(\tau))|0\rangle|E_i\rangle.$$

Let H denote the Hamiltonian of the detector; $H|E_i\rangle = E_i|E_i\rangle$. Then using the Heisenberg equation for the time evolution of the monopole operator $m(\tau)$ and defining $m_{fi} \equiv \langle E_j|m(0)|E_i\rangle$ we obtain for the transition amplitude

$$gm_{fi}\int_{-\infty}^{\infty} d\tau e^{i(E_f-E_i)\tau}\langle\beta|\phi(x(\tau))|0\rangle.$$

[8]The calculation is adopted from [19].

In order to find the probability, $P(E_i \to E_f)$, that the detector makes a transition form E_i to E_f not measuring the final state of the scalar field we must square the above amplitude and sum over all possible final states $|\beta\rangle$. We obtain

$$P(E_i \to E_f) = g^2|m_{fi}|^2 \int_{-\infty}^{\infty} d\tau d\tau' e^{i(E_f-E_i)(\tau-\tau')} G_0^+(x(\tau), x(\tau')).$$

The Wightman function depends only on Z. Assume that the observer is stationary with respect to the north pole. Then in the global coordinate system we have

$$\begin{aligned} X^0 &= l \sinh \frac{\tau}{l} \\ X^1 &= l \cosh \frac{\tau}{l}, \end{aligned}$$

with the $X^\alpha = 0$ for $\alpha = 2, \cdots, d$ so that we find for Z:

$$Z(x(\tau), x(\tau')) = \cosh \frac{\tau - \tau'}{l}.$$

Changing the integration variables from τ and τ' to $\tau - \tau' = \sigma$ and τ and considering the transition probability per unit proper time we obtain

$$\dot{P}(E_i \to E_f) = g^2|m_{fi}|^2 \int_{-\infty}^{\infty} d\tau e^{-i(E_f-E_i)\tau} G_0^+(\cosh \frac{\tau}{l}).$$

The integrand has singularities in the complex τ plane at $\tau = 2\pi lin$ for any integer n since for these values $\cosh \frac{\tau}{l} = 1$. The Wightman function G_0^+ is periodic in imaginary time, a property of all thermal correlation and Green's functions. Define the contour C which consists of two horizontal lines at $\tau = 0$ and at $\tau = -2\pi li$ which close at infinity, and which is traversed such that it passes under the singularity at $\tau = 0$ and above the singularity at $\tau = -2\pi li$. Integrating the above integrand along C we find

$$\int_{-\infty}^{\infty} d\tau e^{-i(E_f-E_i)\tau} G_0^+(\cosh \tau) + \int_{\infty - i\beta}^{-\infty - i\beta} d\tau e^{-i(E_f-E_i)\tau} G_0^+(\cosh \tau) = 0,$$

where $\beta = 2\pi l$. Performing the change of variable $\tau' = -\tau - i\beta$ in the second integral we obtain

$$\dot{P}(E_i \to E_f) = \dot{P}(E_f \to E_i) e^{-\beta(E_f - E_i)}.$$

Assume that the energy levels of the detector are thermally populated. Then

$$N_i = N e^{-\beta E_i},$$

where N is some normalization factor. The total transition rate R from E_i to E_f is equal to

$$R(E_i \to E_f) = N_i \dot{P}(E_i \to E_f).$$

Similarly, we have for the total transition rate from E_f to E_i

$$\begin{aligned} R(E_f \to E_i) &= N_f \dot{P}(E_f \to E_i) = N e^{-\beta E_f} \dot{P}(E_i \to E_f) e^{-\beta(E_i - E_f)} \\ &= R(E_i \to E_f). \end{aligned}$$

This is the principle of detailed balance. Thus, when the levels of the detector are thermally populated at the inverse temperature $\beta = 2\pi l$ then there is no change in the probability distribution for the energy levels with time. Hence, the observer which is stationary at the north pole measures a temperature T equal to

$$T = \frac{1}{2\pi l}.$$

Since all timelike geodesics are related to each other by transformations of the de Sitter isometry group $SO(d, 1)$, and because the Wightman function is de Sitter invariant, this temperature will be measured by any observer moving along a timelike geodesic.

We emphasize that the calculation has been performed in the Euclidean vacuum. For the MA vacua the principle of detailed balance is violated [21, 10].

If all observers which are moving relative to each other along timelike geodesics measure the same temperature, then they do not measure the same particles for the vacuum expectation value of the energy-momentum tensor is de Sitter invariant. Hence, the particle concept is essentially observer dependent.

To see how this temperature comes about in relation to the horizon we make use of the static coordinate system. According to an observer O situated at the north pole the timelike Killing vector $(\frac{\partial}{\partial t})^a$, the generator of the flow of energy, is past-directed in the southern causal diamond, figure 1.5. Therefore particle states in the southern causal diamond have negative energy with respect to O. This leads to an apparent energy difference, so that pair creation near the horizon is possible. In terms of the old-fashioned interpretation of particles described by wave functions the antiparticle is the charge-conjugate of the particle wave function which has a negative energy and propagates backward in time. Therefore, the antiparticles propagate into the region beyond the horizon where $(\frac{\partial}{\partial t})^a$ is spacelike (there are no timelike geodesics connecting points of the northern and southern causal diamonds). This would not be possible classically, but quantummechanically it is possible. The antiparticle has an intrinsic wavelength of the order of the Compton wavelength and can thus tunnel through the event horizon. Equivalently, because of the lack of information O has of what happens beyond the event horizon O may say that an antiparticle has scattered from the region beyond the event horizon into a particle in the northern causal diamond.

There are two main differences between Minkowski space at finite temperature and de Sitter space. First, in de Sitter space the heat bath does not break de Sitter invariance, whereas a heat bath on a Minkowski background does break Lorentz invariance (not all observers agree on the temperature). Second, the Euclidean vacuum in which we calculated the temperature is a pure vacuum state. Therefore, the temperature of de Sitter space is an inherent quantum property of the spacetime. One only needs to consider a mixed state description in terms of a thermal density matrix when one asks about the findings of a static observer. Such an observer is compelled to integrate over all modes beyond his horizon, [21]. Using the canonical ensemble, the thermal density matrix, ρ, will be given by $\rho = \text{Tr } e^{-\beta H}$, where H is the local Hamiltonian of the static observer.

The temperature associated with the de Sitter horizon in SI units is

$$T = \frac{H\hbar}{2\pi k_B}.$$

It is of the order 10^{-30}K. This temperature implies the existence of entropy in a de Sitter space. It has been shown [16] that the entropy of a cosmological event horizon like the one in a de Sitter space is given by the Bekenstein-Hawking entropy formula, in SI units

$$S = \frac{k_B c^3 A}{4G\hbar},$$

where A is the area of the event horizon. This entropy comes about from the totality of particle states that are beyond O's horizon. It is rather unclear how one should interpret these microstates. In chapter 3 we will say more about the thermodynamic properties of the cosmological event horizon.

2.8 Classical stability of scalar field fluctuations around the ground state of some positive definite potential

Consider the action[9]

$$S = \int d^d x \sqrt{-g}(-\frac{1}{2}g^{ab}(\nabla_a \phi)(\nabla_b \phi) - V(\phi)), \tag{2.18}$$

where the potential $V(\phi)$ is given by $V(\phi) = \frac{1}{2}(m^2 + \xi R)\phi^2 + \lambda\phi^4$. This is the action for a classical massive scalar field which couples to the background geometry through the term $\frac{1}{2}\xi R\phi^2$ and which has self-interactions described by $\lambda\phi^4$. The potential $V(\phi)$ is positive definite for it is only zero when $\phi = 0$ and otherwise positive. From the discussion of section 2.5 we know that we

[9]The discussion presented here is largely based on the article [1] by Abbott and Deser.

should be careful interpreting the parameter m as a mass. We avoid this discussion here by treating it as a part of the potential energy of the field.

In order to consider fluctuations of the field ϕ around the potential minimum at $\phi = 0$ we must introduce a coordinate system to describe the background geometry. We expect from the discussions of the thermality of de Sitter space that these fluctuations are unstable beyond the event horizon where negative energy is associated with the time translation Killing vector field. We thus need a coordinate system that adequately describes the spacetime region around the horizon. This rules out, for instance, the static coordinate system which is singular at the horizon. From the point of view of mathematical simplicity we choose the planar coordinate system, whose metric is given by

$$ds^2 = -dt^2 + a^2(t)((dx^1)^2 + \cdots (dx^{d-1})^2),$$

where $a(t) = e^{\lambda t}$ with $\lambda^2 = \frac{2\Lambda}{(d-2)(d-1)}$. In section 1.3.3 the time translation Killing vector field was obtained. It is

$$\xi^a = (1, -\lambda x^\alpha).$$

The norm is given by

$$\xi^2 = g_{ab}\xi^a\xi^b = -1 + a^2\lambda^2|\vec{x}|^2.$$

Hence the horizon is given by $a^2\lambda^2|\vec{x}|^2 = 1$, and ξ^a is timelike in the inner region.

The energy-momentum tensor of the field ϕ is defined by

$$-T_{ab} = 2\frac{\partial \mathcal{L}}{\partial g^{ab}} - g_{ab}\mathcal{L},$$

where

$$\mathcal{L} = -\frac{1}{2}g^{ab}(\nabla_a\phi)(\nabla_b\phi) - V(\phi).$$

Then in planar coordinates

$$-T_{0a} = -\dot{\phi}(\nabla_a\phi) + \frac{1}{2}g_{0a}g^{bc}(\nabla_b\phi)(\nabla_c\phi) + g_{0a}V(\phi).$$

Thus, the energy density is given by

$$T_{00} = \frac{1}{2}\pi^2 + \frac{1}{2}a^{-2}(\vec{\nabla}\phi)^2 + V(\phi)$$

and the momentum density by

$$T_{0\alpha} = \pi(\partial_\alpha\phi),$$

where $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$.

The total classical energy (potential plus dynamical) stored in the field ϕ in the spacelike region Σ is given by the Killing energy (see section 4.2 for more details)

$$E = \int_\Sigma d^{d-1}x\sqrt{-g}T_{0b}\xi^b = \int_\Sigma d^{d-1}x a^{d-1}T_{0a}\xi^a.$$

This energy expression is not conserved since T_{ab} does not go to zero on the boundary of the spacelike hypersurface $t =$ const. One shows that in planar coordinates (t, x^α)

$$\partial^0 E = -a^{d-1}\int_\Sigma d^{d-1}x\ \partial^\alpha(T_{\alpha b}\xi^b),$$

using

$$\nabla^a(\sqrt{-g}T_{ab}\xi^b) = \partial^a(\sqrt{-g}T_{ab}\xi^b) = 0$$

and the fact that a only depends on the time t. The spacelike region Σ is the infinite spatial plane given by $t =$ cst. Now, $\partial^0 E$ would vanish if it were true that $T_{\alpha b} \to 0$ as $|\vec{x}| \to \infty$. This is, however, not the case and so $\partial^0 E \neq 0$. The integrand is

$$a^{d-1}T_{0b}\xi^b = a^{d-1}(T_{00} - \lambda x^\alpha T_{\alpha 0}) =$$

$$a^{d-1}(\frac{1}{2}\pi^2 + \frac{1}{2}a^{-2}(\vec{\nabla}\phi)^2 + V(\phi) - \lambda\pi x^\alpha\partial_\alpha\phi).$$

For a non-trivial field configuration, $\phi \neq 0$, the energy is positive if and only if

$$a^2\pi^2 + (\vec{\nabla}\phi)^2 > 2\lambda a^2\pi x^\alpha\partial_\alpha\phi,$$

since $a > 0$ for all times and V is positive definite. Introduce the vectors

$$\vec{A} = a\pi\hat{x} \quad \text{and} \quad \vec{B} = \vec{\nabla}\phi.$$

Then we obtain the inequality

$$\vec{A}^2 + \vec{B}^2 > \lambda a|\vec{x}|2\vec{A}\cdot\vec{B}.$$

From the triangle inequality we then obtain that the Killing energy of the field ϕ is positive if and only if the region of integration is restricted to the spacetime region bounded by

$$\lambda a|\vec{x}| \leq 1.$$

Since the horizon at $\lambda a|\vec{x}| = 1$ forms a set of measure zero in the integral, we may conclude that only those fluctuations of ϕ around the potential minimum $\phi = 0$ which are inside the horizon have positive energy, and are thus stable. The vacuum state of the classical field ϕ is given by $\pi = \nabla\phi = \phi = 0$ since only in this case is the Killing energy identically zero. Non-zero field configurations outside the horizon thus have a negative energy associated with them. This is in agreement with what was said concerning the Hawking radiation.

We note that the results did not depend on the specific form of the potential $V(\phi)$, but in fact apply to any positive definite potential. The above result solely derived from the kinematical part of the action.

Chapter 3

Thermodynamic aspects of Schwarzschild-de Sitter spacetime

3.1 Introduction

In this chapter we will continue the thermodynamic discussion of section 2.7. However, this time we will no longer introduce scalar particles on a spacetime, but we will set up a formalism to extract the temperature and entropy of a particular horizon from the spacetime properties itself.

Everything is explicitly 4-dimensional.

3.2 Thermal gravitons

de Sitter space is at a temperature of $T = 1/2\pi l$. The best statistical approach toward finite temperature gravity would be a microcanonical description of all the quantum states that are beyond the horizon. However, one does not know how to do this there being no quantum theory for gravity. The canonical ensemble description has been shown to be inadequate when applied to black holes [15]. Still we will find it useful to pursue this approach, for one thing because the partition function can be written as a path integral of a classical expression. Besides it will not be applied to stationary final state black holes, but only to de Sitter spacetime and the thermally induced nucleation of black holes. The latter phenomenon will be discussed in the final chapter.

For finite temperature scalar field theory one shows that the partition function, Z, in the canonical ensemble defined by the trace of the operator

$e^{-\beta H}$, setting any chemical potential equal to zero, can be written as

$$Z = Tre^{-\beta H} = N \int_{\text{periodic}} \mathcal{D}\phi e^{-S_E[\phi]}.$$

S_E is the euclidean action of the field given by

$$S_E[\phi] = \int d\tau d^3x \sqrt{g}(\frac{1}{2}(\frac{\partial\phi}{\partial\tau})^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi))$$

obtained from the real time Lagrangian by replacing the time t with $-i\tau$, that is, by performing a Wick rotation. In real time every path contributes a phase $e^{iS[\phi]}$ to the above path integral. The prefactor N is just a normalization constant and the integral is taken over those field configurations which are periodic in the imaginary time τ, $\phi(\vec{x}, 0) = \phi(\vec{x}, \tau)$. This requirement follows from the trace operation; only those transitions in which the initial and final state are the same contribute to the partition function.

The fundamental quantity which relates statistics and thermodynamics in finite temperature scalar field theory can thus be approached from two rather different starting points: from the quantum mechanical Hamiltonian or from Wick rotation of the classical action. It will be assumed that the same is true for gravity and that the partition function for finite temperature gravity can be obtained from the euclidean version of the path integral

$$N \int \mathcal{D}g_{ab} e^{iS[g]}.$$

What was said above about the advantage of the canonical ensemble in that its partition function can be written as a classical object does not really avoid quantum gravity because the action should be such that when appropriately quantized it gives quantum gravity. We will avoid this issue by considering only the semi-classical limit. So the action $S[g]$ does not include any gauge-fixing terms.

The Einstein-Hilbert action only leads to the Einstein equation if both the normal derivative of the metric and the metric itself vanish on the space-time boundary at spatial infinity. In the path integral above the integral is over all metrics[1] including those with non-vanishing normal derivative at spatial infinity. The action which leads to the Einstein equation under the class of variations which vanish at infinity but whose normal derivative is not constrained is [25]

$$S = S_{EH} + \frac{1}{8\pi} \int_{\partial M} K - \frac{1}{8\pi} \int_{\partial M} K^0,$$

[1] There is considerable debate about exactly which metrics are integrated over (see [6] and references therein).

where S_{EH} is the Einstein-Hilbert action and K is the trace of the extrinsic curvature[2] of the boundary ∂M at spatial infinity of the spacetime M. The natural volume element of ∂M is assumed. The last term is a constant with K^0 the trace of the extrinsic curvature of the spacelike boundary of Minkowski spacetime, S^2. The term has been added so that the total action gives zero for the Minkowski metric.

Thus, the partition function for thermal gravity reads

$$Z = N \int_{\text{periodic}} \mathcal{D}g_{ab} e^{-S_E[g]}, \tag{3.1}$$

where S_E is defined by $S = iS_E$ after replacing t with $-i\tau$ and where τ is periodic. The euclidean action is (noting that one should replace n^b with $-in^b$ when ∂M is spacelike and that the same factor appears from the volume element of ∂M when it is timelike)

$$S_E[g] = -\frac{1}{16\pi}\int_M (R - 2\Lambda)\sqrt{g} d\tau d^3x - \frac{1}{8\pi}\int_{\partial M} (K - K^0)\sqrt{\gamma} d^3x, \tag{3.2}$$

where the metric g_{ab} here is positive definite.

The semi-classical approximation of Z will be treated in the next chapter.

Consider a (euclidean) background spacetime with metric $\bar{g}_{ab}$ and fluctuations h_{ab} about this background. Assuming that $\bar{g}_{ab}$ satisfies the euclidean Einstein equation the expansion of the euclidean action $S_E[g = \bar{g} + h]$ in h_{ab} will not contain terms of first order in h_{ab}. So that expanding to second order the action contains two terms:

$$S_E[g] = S_E[\bar{g}] + S_E^{(2)}[h],$$

where $S_E^{(2)}[h]$ is second order in h_{ab}. Variation of this term will lead to the linearized Einstein equation on the background manifold M with metric $\bar{g}_{ab}$. Hence, up two second order we have for the partition function

$$-\ln Z = S_E[\bar{g}] - \ln \int \mathcal{D}h_{ab} e^{-S_E^{(2)}[h]} = FT^{-1},$$

where F is the free energy and T the temperature. The term $S_E[\bar{g}]$ can be considered as the contribution of the background to the free energy F, and the second term as the contribution arising from thermal gravitons. Thus, we have an interesting thermodynamic result: The product FT^{-1} associated with a background spacetime or more generally any spacetime whose metric does not undergo fluctuations is equal to the euclidean action, that is,

$$FT^{-1} = S_E[\bar{g}] = -\frac{1}{16\pi}\int_M (R - 2\Lambda)\sqrt{\bar{g}} d\tau d^3x - \frac{1}{8\pi}\int_{\partial M} (K - K^0)\sqrt{\gamma} d^3x.$$

[2] It is defined by $K = \gamma^a{}_b \nabla_a n^b$ where γ_{ab} is the metric induced on and n^b the unit normal to the boundary ∂M.

It follows that the free energy of Minkowski spacetime is zero. However, it will be nonzero for de Sitter spacetime. In that case the manifold that is integrated over in evaluating the euclidean action is the euclidean manifold S^4. This can be understood by considering the line element of the global coordinate system, equation 1.13, and replacing the cosmic time parameter τ by $-i\tau$. Since this is a compact manifold without boundary S_E becomes

$$S_E[\bar{g}] = -\frac{1}{16\pi}\int_{S^4}(R-2\Lambda)\sqrt{\bar{g}}d\tau d^3x.$$

In 4 dimensions the Ricci scalar equals 4Λ, equation 1.8. Hence,

$$FT^{-1} = -\frac{\Lambda}{8\pi}\int_{S^4}\sqrt{\bar{g}}d\tau d^3x.$$

The integral over S^4 is just the volume of a 4-dimensional sphere of radius $l=\sqrt{\frac{3}{\Lambda}}$ which is $\frac{8\pi^2}{3}(\frac{3}{\Lambda})^2$. So

$$FT^{-1} = UT^{-1} - S = -3\pi\Lambda^{-1} = -\pi l^2,$$

where the thermodynamic potential $F = U - TS$ has been used with S the entropy and U the internal energy of a pure de Sitter space. The internal energy is related to all the mass which has a non-gravitational origin. For a Schwarzschild-de Sitter space U would be related to the mass parameter M that appears in equation 1.32. For a pure de Sitter space $U = 0$. Therefore it follows that the entropy S is equal to $S = \pi l^2$. If A represents the area of a closed S^2 surface which lies inside the cosmological event horizon, so that $A = 4\pi l^2$, then the entropy of de Sitter space is given by

$$S = \frac{1}{4}A,$$

which is in agreement with the Bekenstein-Hawking entropy formula.

We will come back to this partition function approach in the next chapter. Since the temperature of a de Sitter space depends only on the parameter l it might be possible to find the temperature from geometry. In fact what we will find is that the horizon temperature is directly proportional to the so-called surface gravity of the horizon.

3.3 Surface gravity

In the static coordinate system of SdS_4 the horizons are located at $r = r_+,\ r_{++}$ (see section 1.6). The generator of time translations $\frac{\partial}{\partial t}$ of the line element of equation 1.31 is timelike for $r_+ < r < r_{++}$, null at $r = r_+,\ r = r_{++}$ and spacelike otherwise. It defines the Killing vector of time translations ξ^a through $\frac{\partial}{\partial t} = \xi^a\partial_a$, which is obviously orthogonal to the surfaces $r = \text{const}$. In the limit $r \to r_+,\ r_{++}$ the hypersurface orthogonal Killing vector ξ^a becomes null and orthogonal to the horizons[3]. A horizon is thus a null hypersurface. We will replace ξ^a with l^a because the theory presented here also applies to the more general case of stationary black holes (see the footnote).

A null hypersurface is a degenerate surface in the sense that the normal vector is also tangent to the surface. Since l^a is hypersurface orthogonal it satisfies the Frobenius orthogonality condition[4]

$$l_{[a}\nabla_b l_{c]} = 0. \tag{3.3}$$

Further, on the horizon $l^a l_a = \text{const} = 0$ so that the gradient $\partial_b(l^a l_a)$ must be orthogonal to the horizon. Since the horizon forms a hypersurface, that is, can only have one normal vector the gradient $\partial_b(l^a l_a)$ must be proportional to l^a. This must be true at any point on the horizon. One defines a function κ such that

$$\partial_b(l^a l_a) = -2\kappa l_b. \tag{3.4}$$

Since $\nabla_b l_a = \nabla_{[b} l_{a]}$ this is equivalent to writing

$$l^b \nabla_b l_a = \kappa l_a, \tag{3.5}$$

so that the vector l^a satisfies a non-affinely parameterized geodesic equation. From equation 3.4 it follows that if $\kappa = 0$ then the horizon is degenerate in the sense that the gradient of $l^a l_a$ vanishes.

Using the Frobenius orthogonality condition together with $\nabla_a l_b = -\nabla_b l_a$ one finds

$$l_c \nabla_a l_b = -2 l_{[a} \nabla_{b]} l_c.$$

Contracting this with $\nabla^a l^b$ the following result is obtained

$$l_c(\nabla^a l^b)(\nabla_a l_b) = -2(\nabla^a l^b)(l_a \nabla_b l_c) = -2(l_a \nabla^a l^b)(\nabla_b l_c) = -2\kappa l^b \nabla_b l_c = -2\kappa^2 l_c.$$

The value of κ is thus given by

$$\kappa^2 = -\frac{1}{2}(\nabla^a l^b)(\nabla_a l_b). \tag{3.6}$$

The vector l^a which is orthogonal to the horizon is also tangent to the horizon and satisfies a geodesic equation. It may thus be said that the orbits

[3] In the more general case of a stationary black hole the Killing vector of time translations is not orthogonal to the horizon. But there always exists a vector l^a which is null on and orthogonal to the horizon [5, 25].

[4] See [8] for a simple account of this mathematical result.

of l^a generate the horizon. Differentiating κ^2 along the geodesic of l^a it is found that

$$l^c \nabla_c \kappa^2 = -l^c (\nabla_c \nabla_a l_b)(\nabla^a l^b) = -l^c R_{abcd} l^d \nabla^a l^b = 0.$$

This shows that κ is constant along each geodesic generating the horizon. It can be shown that κ is in fact constant over the horizon. This means that its gradient must be orthogonal to the horizon, that is[5], $l_{[a} \nabla_{b]} \kappa = 0$.

It will next be shown that κ must be strictly positive. Let v be the parameter of the geodesic generated by l^a which satisfies the equation $l^b \nabla_b l^a = \kappa l^a$, that is $l^a = \frac{dx^a}{dv}$ or $l^a \nabla_a v = 1$. Let λ be an affine parameter. Define $l^a = \frac{d\lambda}{dv} k^a$ with k^a satisfying $k^b \nabla_b k^a = 0$. The affine parameter λ and the Killing parameter v are related through

$$\frac{d}{dv}(\ln \frac{d\lambda}{dv}) = \kappa$$

which can be integrated since κ is constant. This gives for $\kappa \neq 0$

$$\lambda = c_2 e^{\kappa(v+c_1)},$$

where c_1 and c_2 are integration constants, and for $\kappa = 0$ it gives

$$\lambda = v + c_3.$$

For the integration constants the following values will be taken: $c_1 = 0 = c_3$ and $c_2 = 1$. For the case $\kappa \neq 0$ the affine parameter λ can only assume values on the interval $(0, \infty)$ whereas for the degenerate case λ can assume all values the Killing parameter being unrestricted. Then, if the horizon is degenerate it contains geodesics which can be extended to arbitrary affine distance not only towards the future but also towards the past. A Horizon forms the boundary of the observable region of spacetime. When $\kappa = 0$ any null geodesic in the horizon has points on $\partial\mathcal{I}^-$, whereas all null geodesics originate on $\mathcal{I}^-$. Thus, a horizon which has $\kappa = 0$ is not physically attainable.

The quantity κ can be given a physical interpretation. At any point on and outside the horizon we have

$$3(l^{[a} \nabla^b l^{c]})(l_{[a} \nabla_b l_{c]}) = l^a l_a (\nabla^b l^c)(\nabla_b l_c) - 2(l^a \nabla^b l^c)(l_b \nabla_a l_c).$$

On the horizon the left-hand side vanishes and so does $l^a l_a$. However if one considers the gradient of both of these expressions then it is seen that for nonzero κ only the gradient of the left-hand side vanishes but not of $l^a l_a$.

[5]For a general proof of this see [5], and for a simpler proof that applies only to static horizons see [25].

Hence by l'Hôpital's rule, the limit as one approaches the horizon of the expression

$$\frac{3(l^{[a}\nabla^b l^{c]})(l_{[a}\nabla_b l_{c]})}{l^a l_a} = (\nabla^b l^c)(\nabla_b l_c) - 2\frac{(l^a\nabla^b l^c)(l_b\nabla_a l_c)}{l^a l_a}$$

exists and vanishes. Using equation 3.6 it follows that

$$\kappa^2 = \lim[-\frac{(l^a\nabla^b l^c)(l_b\nabla_a l_c)}{l^a l_a}].$$

The limit here is of course towards the horizon. Away from the horizon the unit tangent of just any orbit of l^a, which is timelike outside the horizon, is $l^a/(-l^a l_a)^{1/2}$. The acceleration as experienced by an observer moving along such an orbit is then given by $a^c = l^b\nabla_b l^c/(-l^a l_a)$. Therefore

$$\kappa^2 = \lim[(-l^a l_a)^{1/2}(a^c a_c)^{1/2}]^2.$$

Since κ is strictly positive the square root of both sides can be taken giving

$$\kappa = \lim[(-l^a l_a)^{1/2} a],$$

where a is the magnitude of the acceleration experienced by the geodesic observer. This acceleration becomes infinite as the observer moves closer to the horizon. In a static spacetime the product $(-l^a l_a)^{1/2}a$ has a nice physical interpretation (see appendix B) as the acceleration of a test particle moving along the geodesic as measured by a static observer or put another way $(-l^a l_a)^{1/2}a$ is the force that must be exerted by a static observer to keep in place unit test mass. Then κ is the limiting value of this force at the horizon. Hence the name surface gravity.

Concluding, we have a quantity κ, the surface gravity, which is constant over the horizon and cannot approach zero. In the case of a static spacetime it can be interpreted as the force that a static observer must apply to keep in place unit mass on the horizon.

For a static observer in SdS_4 the line element is of the form

$$ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2 d\Omega_2^2.$$

The surface gravity κ will be related to the potential $V(r)$. Using formula 3.6 we have

$$\kappa^2 = -\frac{1}{2}(\partial_a\xi_b - \Gamma^c_{ab}\xi_c)g^{ad}(\partial_d\xi^b + \Gamma^b_{de}\xi^e).$$

The Killing vector of time translations is $\xi^a = (1,0,0,0)$, so that $\xi_a = (-V(r),0,0,0)$. Substituting this in the above equation for κ^2 the following result is obtained

$$\kappa^2 = -\frac{1}{2}(\partial_a\xi_b - \Gamma^c_{ab}\xi_c)g^{ad}\Gamma^b_{d0} = \frac{1}{2}g^{rr}\Gamma^0_{r0}\partial_r V - \frac{1}{2}V g^{ad}\Gamma^0_{ab}\Gamma^b_{d0}.$$

In the static coordinate system the only nonzero connection coefficients are

$$\begin{aligned}\Gamma^0_{r0} &= \frac{1}{2}V^{-1}\partial_r V \\ \Gamma^r_{00} &= \frac{1}{2}V\partial_r V.\end{aligned}$$

Using this κ^2 becomes

$$\kappa^2 = \frac{1}{4}(\partial_r V)^2,$$

so that the surface gravity can be related to the potential V through

$$\kappa = \frac{1}{2}|\partial_r V|,$$

which of course must be evaluated on the horizon.

For a de Sitter space the potential V is given by $V(r) = 1 - \frac{r^2}{l^2}$, equation 1.14. In this case

$$\kappa = \frac{1}{2}|-\frac{2l}{l^2}| = \frac{1}{l}.$$

The temperature $T = \frac{1}{2\pi l}$ can thus be written as

$$T = \frac{\kappa}{2\pi}. \tag{3.7}$$

It will be shown that the same is true for the horizon temperatures at $r = r_+,\ r_{++}$.

Finally the surface gravities at $r = r_+,\ r_{++}$ of SdS_4 will be calculated. For this space the potential of the static coordinate system is given by $V(r) = 1 - \frac{2M}{r} - \frac{r^2}{l^2}$. Under the assumption that $27G^2M^2/l^2 < 1$ this potential has two zeros at the positive $r_+,\ r_{++}$ values and one zero at $r = -r_{--}$, section 3.10. The potential can thus be written as

$$V(r) = -\frac{1}{l^2 r}(r - r_+)(r - r_{++})(r + r_{--}),$$

where $r_{--} = r_+ + r_{++}$, so that the factorization does not lead to terms quadratic in r. It follows that the surface gravities κ_H and κ_C of the black hole horizon and cosmological event horizon, respectively, are given by

$$\kappa_H = \frac{1}{2l^2}(r_{++} - r_+)(2 + \frac{r_{++}}{r_+}) \tag{3.8}$$

$$\kappa_C = \frac{1}{2l^2}(r_{++} - r_+)(2 + \frac{r_+}{r_{++}}). \tag{3.9}$$

Thus, $\kappa_H > \kappa_C$ and so a test particle placed at a certain $r_+ < r < r_{++}$ will fall into the black hole. A note of caution: the limit $r_+ \to r_{++}$ does not exist. This will be discussed in the next section where we will find a geometric interpretation of the proposed temperature formula, equation 3.7.

3.4 Euclidean sections and conical singularities

In section 3.2 it was argued that one needs the euclidean action 3.2 to obtain the partition function. The metric g_{ab} which appears in 3.2 is a positive definite euclidean (or Riemannian) metric. Let us consider the static line element of de Sitter and euclideanize it by replacing t with $-i\tau$. Then this line element is

$$ds^2 = V(r)d\tau^2 + V^{-1}(r)dr^2 + r^2 d\Omega_2^2 \tag{3.10}$$

with $V(r) = 1 - \frac{r^2}{l^2}$. Formally dS_4 may be analytically continued into a complexified 8-dimensional de Sitter space. One may then consider various euclidean sections of this complexified spacetime. For example at the end of section 3.2 it was pointed out that the euclidean section of the complexified global coordinate system is S^4. The euclidean section of the static coordinate system is (it being understood to be complexified) is the region for which $r < l$. This metric has a coordinate singularity at $r = l$. In order to find what kind singularity this is the line element will be considered in a neighborhood of $r = l$. Expanding the potential V to first order around $r = l$ we have

$$V(r) = V'(r = l)(r - l). \tag{3.11}$$

A new radial coordinate ρ defined by

$$d\rho = \frac{dr}{\sqrt{V(r)}}$$

is introduced. To first order around $r = l$ it reads

$$d\rho = \frac{dr}{\sqrt{V'(r = l)}}(r - l)^{-1/2}$$

which can be integrated to give

$$\rho = \frac{2}{\sqrt{V'(r = l)}}(r - l)^{1/2}. \tag{3.12}$$

Then the potential 3.11 can be written as

$$V(r) = V'(r = l)(r - l) = \frac{(V'(r = l))^2}{4}\rho^2.$$

Then the line element of the euclidean section of the static coordinate system can near $r = l$ be written as

$$ds^2 \simeq \frac{(V'(r = l))^2}{4}\rho^2 d\tau^2 + d\rho^2 + r^2(\rho)d\Omega_2^2, \tag{3.13}$$

where the dependence $r(\rho)$ is given through 3.12.

In finite temperature gravity the imaginary time parameter τ is periodic and the period is equal to the inverse temperature $\beta = T^{-1}$ by analogy with finite temperature field theory. A new imaginary time parameter ϕ is defined which has period 2π. It is defined by $\tau = \phi\frac{\beta}{2\pi}$. In terms of this new parameter the line element 3.13 becomes

$$\begin{aligned} ds^2 &\simeq \frac{(V'(r=l))^2}{4}(\frac{\beta}{2\pi})^2\rho^2 d\phi^2 + d\rho^2 + r^2(\rho)d\Omega_2^2 = \\ & (\frac{\kappa}{2\pi}\frac{1}{T})^2\rho^2 d\phi^2 + d\rho^2 + r^2(\rho)d\Omega_2^2, \end{aligned} \tag{3.14}$$

where $\kappa = \frac{1}{2}|V'(r=l)|$ has been used. This line element can be seen to describe a conical singularity.

In order to see that the above line element describes a conical singularity let us consider the line element of a cone embedded in $\mathbb{R}^3$. Such a cone can be formed as shown in figure 3.1. The angle δ is called the deficit angle. In

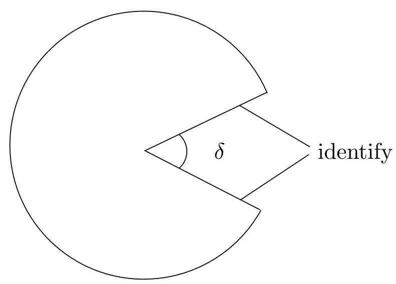

Figure 3.1: A cone may be formed by identifying the lines drawn to the center of the circle which has a deficit angle δ.

$\mathbb{R}^3$ a cone can be parameterized as

$$\begin{aligned} x &= \alpha\rho\cos\varphi \\ y &= \alpha\rho\sin\varphi \\ z &= \rho\sqrt{1-\alpha^2}, \end{aligned}$$

where $0 < \alpha < 1$. For $\alpha = 1$ a circle of radius ρ is obtained and for $\alpha = 0$ a line. The relation between α and the deficit angle δ is given by $\alpha = (2\pi - \delta)/2\pi$. The line element on the cone is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = \alpha^2\rho^2 d\varphi^2 + d\rho^2.$$

Comparing this with equation 3.14 it is seen that the (ϕ, ρ) part describes a conical singularity with $\alpha = \frac{\kappa}{2\pi}\frac{1}{T}$. Apparently if the temperature is equal to $\kappa/2\pi$ then $\alpha = 1$ or $\delta = 0$ in which case the singularity disappears. Thus, removal of the conical singularity of the euclidean section of a static coordinate patch whose line element is of the form 3.10 is equivalent to setting the temperature equal to $T = \kappa/2\pi$. Furthermore the surface gravity is given by $\kappa = \frac{1}{2}|V'|$ evaluated on the horizon.

In terms of the static (t, r) coordinates the line element 3.10 is singular at the horizon. In Kruskal-type coordinates these singularities would disappear. In section 1.3.5 Kruskal coordinates for de Sitter spacetime were constructed. The starting point in obtaining them was equation 1.18 which is repeated here for convenience,

$$dx^{\pm} = dt \pm \frac{dr}{1 - (\frac{r}{l})^2}.$$

The point is that the Eddington-Finkelstein coordinates (x^+, x^-) can be defined similarly for any static line element of the form 3.10 as

$$dx^{\pm} = dt \pm \frac{dr}{V(r)}.$$

Near some horizon at $r = r_H$ the Eddington-Finkelstein coordinates can be found by integrating the above equation. Using $V(r) = V'(r = r_H)(r - r_H)$ it is found that

$$x^{\pm} = t \pm \frac{1}{2\kappa} \ln |r - r_H|.$$

Then the Kruskal coordinates (U, V) near $r = r_H$ are defined through

$$\begin{aligned} U &= -e^{\kappa x^-} \\ V &= e^{-\kappa x^+}. \end{aligned}$$

Hence, all Kruskal-type coordinates have the property that

$$VU^{-1} = -e^{-\kappa t}.$$

Consider Kruskal-type coordinates at $r = r_+$ and at $r = r_{++}$ of SdS_4. The Kruskal coordinates near $r = r_+$ will be denoted by (U_1, V_1) and the Kruskal coordinates near $r = r_{++}$ will be denoted by (U_2, V_2). Then near the black hole horizon we have $V_1 U_1^{-1} = -e^{-\kappa_H t}$ and near the cosmological event horizon we have $V_2 U_2^{-1} = -e^{-\kappa_C t}$. Then in the overlap region of the two coordinate systems the following equality holds

$$-V_2 U_2^{-1} = (-V_1 U_1^{-1})^{\kappa_C/\kappa_H}.$$

Thus, in the complexified Schwarzschild-de Sitter spacetime there is a branch cut in the relation between the two coordinate systems. There is not one coordinate patch that covers both of the horizons.

Consider again the surface gravities, equations 3.8 and 3.9. The range of r values in the euclidean section of the static coordinate system is limited to $r_+ < r < r_{++}$. Hence, the limit in which the black hole horizon approaches the cosmological event horizon can only be taken using the above two Kruskal coordinate patches. Simultaneous removal of both of the conical singularities in the euclidean section of the static coordinate system would require the same value of κ at the two horizons, that is, it would only be possible when $\kappa_H = \kappa_C \neq 0$, and in this case the two horizons overlap. The limit corresponds to a black hole approaching the Nariai black hole, section 1.6. This will be of importance to the description of the process of black hole nucleation discussed in chapter 4.

3.5 The thermodynamics of black hole and cosmological event horizons

In this last section the mass of a black hole in SdS_4 space will be calculated. Using this it will finally be possible to show that the temperature T is equal to $\kappa/2\pi$.

The mass of 4-dimensional Schwarzschild spacetime can be calculated using equation B.1. Let Σ denote the spacelike hypersurface between the integration surface S at infinity and the black hole horizon H, so $\partial\Sigma = S \cup H$. Using the generalized Stokes' theorem it follows that

$$\frac{1}{4\pi}\oint_S \nabla^a \xi^b dS_{ab} = \frac{1}{4\pi}\int_\Sigma \nabla_a \nabla^a \xi^b d\Sigma_b + \frac{1}{4\pi}\oint_H \nabla^a \xi^b dS_{ab}.$$

Since ξ^a is a Killing vector the first term on the right vanishes because

$$\nabla_a \nabla^a \xi^b = R^b{}_a \xi^a = 0$$

in Σ since the spacetime is vacuum outside the black hole. Hence, the mass of a Schwarzschild black hole can be calculated through a surface integral over H,

$$M = \frac{1}{4\pi}\oint_H \nabla^a \xi^b dS_{ab} = \frac{1}{4\pi}\oint_H (\nabla^a \xi^b)\xi_{[a}n_{b]} dS$$

(see appendix **??**). The integrand can be written as

$$(\nabla^a \xi^b)\xi_{[a}n_{b]} = (\xi_a \nabla^a \xi^b)n_b = \kappa \xi^b n_b = \kappa,$$

where the first equality follows because ξ^a is a Killing vector and the second equality because the integral is evaluated on the horizon so that expression 3.5 with $l^a = \xi^a$ can be used. The last equality is due to the normalization $\xi^b n_b = 1$. Therefore the mass of the Schwarzschild black hole is given by

$$M = \frac{\kappa}{4\pi}\oint_H dS = \frac{\kappa}{4\pi}A,$$

where A is the area of an S^2 surface that lies within H.

What about the black hole in SdS_4? The spacetime region outside the black hole will apart form a contribution of the cosmological constant be vacuum; in 4 dimensions the Ricci tensor outside the black hole is given by $R_{ab} = \Lambda g_{ab}$. This follows from equations 1.2 and 1.3. So all the mass that should be attributed to the black hole resides in the finite region of spacetime that is bounded by the black hole horizon. In the static coordinate system of SdS_4 let Σ be the observable region of a static observer which is bounded by the black hole and cosmological event horizons. The total mass contained in Σ is given by

$$\frac{1}{4\pi}\int_\Sigma R_{ab}\xi^b d\Sigma^a = \frac{1}{4\pi}\int_\Sigma \nabla_b \nabla^b \xi_a d\Sigma^a = \frac{1}{4\pi}\oint_{C\cup H} \nabla^b \xi^a dS_{ab}.$$

Therefore, the mass contained in the spacetime region bounded by the horizon C is

$$-\frac{1}{4\pi}\oint_C \nabla^a \xi^b dS_{ab} = \frac{1}{4\pi}\oint_H \nabla^a \xi^b dS_{ab} + \frac{1}{4\pi}\int_\Sigma R_{ab}\xi^b d\Sigma^a.$$

The integrals over H and C can be calculated in the same way as for the Schwarzschild case giving

$$-\frac{\kappa_C}{4\pi}A_C = M_H + \frac{1}{4\pi}\int_\Sigma \Lambda \xi_a d\Sigma^a, \tag{3.15}$$

where M_H is the black hole mass

$$M_H = \frac{\kappa_H}{4\pi}A_H \tag{3.16}$$

and A_H and A_C are the areas of the black hole and cosmological event horizons, respectively. The total mass enclosed by C will be denoted by $M_C = -\kappa_C A_C (4\pi)^{-1}$, which is equal in magnitude but opposite in sign to the mass associated with the surface gravity of the cosmological event horizon. This makes sense since positive mass was assigned to the surface gravity of a black which attracts matter in a direction opposite to which the matter is attracted by the cosmological event horizon. The Λ contribution to M_C, the second term on the right of equation 3.15 can be seen to be negative for $\xi_a d\Sigma^a = \xi_a n^a d\Sigma = g_{00} d\Sigma = -V(r) d\Sigma$ and $V(r)$ is positive in Σ.

Taking the total differential of formula 3.15 assuming that $\delta\xi_a$ vanishes, which can always be made possible by choosing a particular gauge, we have

$$\delta M_C = \delta M_H.$$

The total differential of M_H is

$$\delta M_H = \frac{1}{4\pi}\kappa_H \delta A_H + \frac{1}{4\pi}A_H \delta\kappa_H$$

It has been shown [16, 17] that also

$$\delta M_H = -\frac{1}{4\pi} A_H \delta\kappa_H.$$

Adding the last two expressions one finds that

$$\delta M_H = \frac{1}{8\pi}\kappa_H \delta A_H. \tag{3.17}$$

Similarly one has

$$\delta M_C = -\frac{1}{8\pi}\kappa_C \delta A_C,$$

so that the equality $\delta M_H = \delta M_C$ leads to the important result

$$\kappa_H \delta A_H + \kappa_C \delta A_C = 0. \tag{3.18}$$

We now have enough information to state that the temperature T must be proportional to κ. Temperature is formulated in the zeroth law of thermodynamics is uniform over a body in thermal equilibrium. The surface gravity κ is constant over the horizon of a stationary black hole. The third law of thermodynamics states that it is impossible to reach $T = 0$ by any physical process. Similarly, it was shown that κ cannot become zero. The first law of thermodynamics reads

$$dE = TdS + \text{work terms}.$$

In the case of static black holes we have

$$dM_H = \frac{1}{8\pi}\kappa_H dA_H.$$

Such black holes do not rotate so changing their mass by dM can only affect the black hole radius in a fashion described by the above formula. If the mass increases/decreases the radius increases/derceases. For the case of a cosmological event horizon the first law differs from the above formula by a minus sign

$$dM_C = -\frac{1}{8\pi}\kappa_C dA_C.$$

This sign difference can be understood as follows. If the mass inside the horizon C increases then the mass beyond C must decrease, and it is exactly this that corresponds to a positive dA_C. The analogous thermodynamic formula would then have to be

$$d(-E) = TdS,$$

where $-E$ is the total positive energy of a microcanonical ensemble of states beyond the cosmological event horizon. Hence, not only the particle states discussed in section 2.7 have negative energy beyond the horizon with respect to some observer situated in the interior region of C, but the same must be true for thermal gravitons. In [1] it is proven that any gravitational perturbation beyond the horizon carries negative energy. The underlying fundamental result is equation 3.18 and may be termed the first law of SdS_4 thermodynamics.

Now the proportionality between T and κ has been fully established for black hole and cosmological event horizons the de Sitter space result $T = 1/2\pi l$ with $\kappa = 1/l$ may be used to set the constant of proportionality equal to $1/2\pi$. Then the entropy is given by $S = A/4$. The temperatures of the two horizons in SdS_4 are thus given by

$$T_H = \frac{\kappa_H}{2\pi} = \frac{1}{4\pi l^2}(r_{++} - r_+)(2 + \frac{r_{++}}{r_+}) \tag{3.19}$$

$$T_C = \frac{\kappa_C}{2\pi} = \frac{1}{4\pi l^2}(r_{++} - r_+)(2 + \frac{r_+}{r_{++}}), \tag{3.20}$$

where equations 3.8 and 3.9 have been used. It is seen that $T_H > T_C$ because $r_+ < r_{++}$.

There is also an analogue of the second law of thermodynamics which states that in any physical process $\delta A \geq 0$.

Chapter 4

Stability of de Sitter spacetime

4.1 Introduction

Processes in which the gravitational interaction is dominant are often unstable. Well known examples are: stellar formation out of a cloud of dust and the possible subsequent collapse into a black hole. The reason for this is simple: gravity always attracts. In fact one might say that in order to have a stable gravitational system either non-trivial boundary conditions are required or the interplay of another force causing some form of balance is needed. An all too familiar example of the first possibility is our solar system, and the various stages in a star's evolution provide examples for the second possibility. On a more fundamental level one might say that gravity anti-screens its source, that is, the stress-energy-momentum content of matter, whereas for example the electromagnetic interaction screens its source: charge and its state of motion.

In addressing the question of stability it is always extremely useful to know the energy balance of the process under consideration. However, in the case of gravity this is in general not possible because general relativity does not by itself provide us with a notion of gravitational energy neither locally nor globally. In any attempt to supply the theory with such a notion one is forced to introduce additional structure, for example certain conditions are required to hold at infinity or one treats the dynamic part of gravity as fluctuations with respect to some fixed background geometry.

It is not all too surprising that general relativity does not have a well-defined notion of energy. Due to the equivalence principle one can always, at an arbitrary spacetime point, choose a coordinate system in which the connection coefficients vanish. On the other hand gravity interacts with itself. Thus, since gravity is of a purely geometric nature, these self-interactions do not depend on any choice of coordinate system and must therefore reside in the second and higher order derivatives of the metric. This makes the gravitational energy essentially non-local, and disables in general the definition of a gravitational energy-momentum tensor. Further, because the spacetime structure comes about as a solution of Einstein's equation there is a priori no notion of the global geometric aspects of a particular spacetime. This makes it impossible to generally define energy globally.

Still, it is not all completely hopeless. On the contrary, for the class of spacetimes which are asymptotically flat one has been able to define the total gravitational energy stored in the spacetime and even more, there has been formulated the celebrated positive energy theorem[1]:

> The total energy of an asymptotically flat spacetime, including the energy of the matter and also the energy of the field, is always positive, if the matter contribution is positive, that is, if the weak energy condition[2] holds, and zero only for Minkowski spacetime.

The energy spoken of here is the ADM energy which is thus positive semi-definite. We will briefly discuss it here for future reference.

A non-stationary space has no notion of gravitational force. All that we could use is the asymptotic time translation symmetry. In order to define this one considers the asymptotic symmetry group of an asymptotically flat space, the Bondi-Metzner-Sachs (BMS) group[3]. The Arnowitt-Deser-Misner (ADM) energy gives the total energy available in the space and is defined at spatial infinity as follows. Let $x^1, \cdots, x^{d-1}$ be asymptotically Euclidean coordinates for a spacelike hypersurface Σ then E_{ADM} is defined by

$$E_{ADM} = \frac{1}{16\pi} \oint_S (\partial_\beta g_{\alpha\beta} - \partial_\alpha g_{\beta\beta}) dS^\alpha, \tag{4.1}$$

where $\alpha, \beta = 1, \cdots, d-1$ and S is a closed bounding (d-2)-surface of Σ, that is $S = \partial\Sigma$ at spatial infinity and $dS^\alpha = n^\alpha dA$, where n^α is the unit normal to S which has the "surface" element dA. The expression must be evaluated in the limit that $r \to \infty$ with $r^2 = (x^1)^2 + \cdots + (x^{d-1})^2$, and the metric components are evaluated on Σ.

Returning to the positive energy theorem. For the class of asymptotically flat spacetimes it is thus justified to say that Minkowski space forms the ground state of the gravitational interaction in the sense that it is the unique lowest energy solution of the Einstein equation (without cosmological constant).

[1] This was first proven by Schoen and Yau and later by Witten, see [27] and references therein.

[2] The weak energy condition states that for any timelike vector u^a the quantity $T_{ab}u^a u^b$ must be non-negative. If u^a is tangent to some timelike geodesic then $T_{ab}u^a u^b$ represents the local energy density of matter.

[3] For definitions and discussions see [3]

Can we say something similar for spacetimes which have a positive cosmological constant or, excluding the possibility of the formation of a future cosmological singularity, does there exist a positive energy theorem for the class of spacetimes which are asymptotically de Sitter so that de Sitter spacetime forms the unique lowest energy state? So far such a theorem has not been found. In fact it cannot even be formulated precisely for it is not clear what the energy should be. Besides that, how are we to treat the cosmological constant in this? Should it be considered part of the energy-momentum tensor of the matter content? It could for example be the ground state value of the potential of some scalar field. In that case the above stated positive energy theorem does not apply because there it is essential that the energy-momentum tensor of the matter vanishes sufficiently rapidly at infinity, whether spacelike or null. This is obviously not the case when there exists a scalar field potential whose ground state value is strictly positive. Or should it perhaps be considered as a fundamental new constant which like G is taken up in the definition of the gravitational energy?

For an asymptotically de Sitter spacetime the Killing vector of time translational symmetries is spacelike at future null infinity and so far all known gravitational energy definitions assume that it always be timelike[4].

Suppose that de Sitter space really is the ground state of all asymptotically de Sitter spaces. Then that would mean that there exists no gravitational process so that the spacetime geometry changes to a lower energy state than that of de Sitter spacetime. This would be quite a non-trivial result, certainly because the geometry of a de Sitter space itself is non-trivial. Might it be that since a de Sitter space is maximally symmetric any deformation of it can only take place when more energy is put into the system or, perhaps that any gravitational fluctuation will due to the anti-screening effect reduce the rate of expansion, which potentially could reduce the energy of a de Sitter space?[5].

In this chapter we will study deformations of de Sitter space and discuss its stability. We will not find any result that is against a positive energy theorem for the class of asymptotically de Sitter spacetimes.

The first approach is the one due to Abbott and Deser [1] in which the metric of an asymptotically de Sitter space will be written as

$$g_{ab} = \bar{g}_{ab} + h_{ab},$$

where $\bar{g}_{ab}$ is the metric of a pure de Sitter space. All gravitational dynamics is then assumed to reside in h_{ab}. Hence, it will not be possible to say anything

[4] There is one important exception to this. Recently, Balasubramanian, de Boer and Minic [24] proposed an energy definition for asymptotically de Sitter spaces based on the quasi-local energy description of Brown and York that is able to deal with this problem.

[5] In [24] it is shown that according to the choices of minus signs in their approach the total energy of a de Sitter spacetime is either the lowest or the highest in the class of asymptotically de Sitter spacetimes.

about the total mass of a de Sitter space, but only of deformations thereof. For these perturbations, however large, an energy will be defined that gives the total energy contained in some spacelike hypersurface of constant time.

In the Abbott and Deser approach it is not specified what was the source of the field h_{ab}. This is not necessary if one wishes only to define the total energy of the field h_{ab} and if it can be proven to always be positive then it would not at all be relevant. However, it can only be shown that it is positive for fluctuations inside the event horizon indicating that de Sitter space is classically stable to whatever happens inside the event horizon. Still one might be worried that it is unstable with respect to global large wavelength gravitational perturbations. The reason for this concern comes from the fact that Minkowski space at finite temperature is unstable precisely under such perturbations [6]. This is generally termed a Jeans instability. Now, a de Sitter space is also at finite temperature and therefore also might display this mode of instability. To prove that de Sitter space does not possess a Jeans instability we will discuss global perturbations using the approach due to Lifshitz and Khalatnikov [18, 14]. A perfect fluid is introduced on de Sitter space with a small density, pressure and velocity field, and the subsequent changes in the metric, described again by a field h_{ab}, are 'monitored' in the global coordinate system (which is not plagued by the presence of a cosmological horizon for it has no timelike Killing vector field). It will turn out that all such perturbations are exponentially damped at late times except for pure gravitational radiation which approaches a constant value, and so as will be made clear de Sitter space does not have a Jeans instability and is classically stable.

Finally, it will be argued to be likely that de Sitter space can withstand the process of black hole formation. A full proof of this cannot be given yet because it lies in the regime of semi-classical gravity and somewhat beyond, but one can come a long way.

4.2 Killing energy and the pseudo energy-momentum tensor

Consider an arbitrary spacetime manifold which possesses a timelike Killing vector field ξ^a, and let T_{ab} denote the total energy-momentum tensor of the matter content. The vector $T_{ab}\xi^b$ is divergenceless, that is, $\nabla^a(T_{ab}\xi^b) = 0$, because $\nabla^a T_{ab} = 0$ and $\nabla^{(a}\xi^{b)} = 0$. Multiply this by $(-g)^{1/2}$ it then becomes an ordinary divergence

$$\nabla^a((-g)^{1/2}T_{ab}\xi^b) = \partial^a((-g)^{1/2}T_{ab}\xi^b) =$$
$$\partial^0((-g)^{1/2}T_{0b}\xi^b) + \partial^\alpha((-g)^{1/2}T_{\alpha b}\xi^b) = 0,$$

where $\alpha = 1, \cdots, d-1$. Next we integrate this over the spacelike hypersurface Σ which is, in a coordinate system that covers Σ, determined by

$x^0 = \text{cst}$. Assuming that T_{ib} vanishes sufficiently rapidly at spatial infinity we obtain that the quantity, E, defined by

$$E(\xi) = \int_\Sigma (-g)^{1/2} T_{0b} \xi^b d^{d-1}x$$

is conserved. It is called the Killing energy.

It was argued in the introduction that there exists no notion of local gravitational energy density or put another way no energy-momentum tensor can be found. Still, one can construct a so-called pseudo energy-momentum tensor. We will briefly comment on this object because the same object for the field h_{ab} will turn out to be tensorial, expression 4.4.

The two main features of an energy-momentum tensor are the Bianchi identity $\nabla_a T^{ab} = 0$ and symmetry $T^{ab} = T^{ba}$. A local quantity can be constructed which has both of these properties but which is non-tensorial. It is called the pseudo energy-momentum tensor[6], $\hat{t}^{ab}$,

$$-16\pi \hat{t}^{ab} = \partial_l \partial_m \hat{U}^{ambl}, \tag{4.2}$$

where $\hat{U}^{ambl} = (-g)(g^{ab}g^{ml} - g^{mb}g^{al})$ is the Landau-Lifshitz superpotential[7], a non-tensorial quantity. It satisfies the continuity equation $\partial_a[(-g)\hat{t}^{ab}] = 0$ and it is symmetric. The superpotential has the same structure as the Riemann tensor of a maximally symmetric space and they therefore have the same index structure. We note that equation 4.2 assumes that the cosmological constant is zero[7].

We now take up the discussion of gravitational fluctuations with respect to a fixed background geometry. As already mentioned an expression very similar in form to 4.2 will be found. However, this time it will be generally covariant.

4.3 Gravitational fluctuations with respect to a fixed background geometry

In this section we follow the approach due to Abbott and Deser [1]. In their article they discuss gravitational fluctuations with respect to some fixed background geometry.

[6]By pseudo we mean here an object which transforms as a tensor only under a restricted class of transformations. In this case these are linear transformations.

[7]The Landau-Lifshitz superpotential can be used to define total energy, momentum and angular momentum only if the space is asymptotically flat.

4.3.1 The Killing energy associated with gravitational self-interactions acting on first order fluctuations

Assume that the Einstein equation reads

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 0, \tag{4.3}$$

and write the decomposition as $g_{ab} = \bar{g}_{ab} + h_{ab}$. We will obtain the linearized vacuum Einstein equation for a metric perturbation about the exact solution $\bar{g}_{ab}$. This problem is generally approached as follows [25], section 7.5. Let $g_{ab}(\lambda)$ denote a one-parameter family of exact solutions of equation 4.3 such that g_{ab} depends differentiably on λ and equals $\bar{g}_{ab}$ when $\lambda = 0$. Then the linearized solution comes about by differentiating the equation 4.3 expressed in terms of $g_{ab}(\lambda)$ with respect to λ and setting $\lambda = 0$ at the end. The validity of this approach depends on the accurateness of the decomposition $g_{ab} = \bar{g}_{ab} + \lambda h_{ab}$.

Let ${}^{\lambda}\nabla_a$ denote the derivative operator associated with $g_{ab}(\lambda)$, and let $\bar{\nabla}_a$ denote the derivative operator associated with the background part $\bar{g}_{ab}$. They are related through the connection coefficients, $C^c_{ab}(\lambda)$[8],

$$C^c_{ab}(\lambda) = \frac{1}{2} g^{cd}(\lambda)(\bar{\nabla}_a g_{bd}(\lambda) + \bar{\nabla}_b g_{ad}(\lambda) - \bar{\nabla}_d g_{ab}(\lambda)).$$

One then finds for the Ricci tensor associated with $g_{ab}(\lambda)$

$$R_{ab}(\lambda) = \bar{R}_{ab} - 2\bar{\nabla}_{[a} C^d_{d]b} + 2C^e_{b[a} C^d_{d]e},$$

where $\bar{R}_{ac}$ is the Ricci tensor associated with the background. We note that $C^c_{ab}(\lambda = 0) = 0$. Differentiating the Ricci tensor $R_{ac}(\lambda)$ with respect to λ and setting $\lambda = 0$ gives,

$$\delta R_{ab} \equiv \frac{dR_{ab}(\lambda)}{d\lambda}|_{\lambda=0} =$$

$$-\frac{1}{2}\bar{g}^{cd}\bar{\nabla}_a\bar{\nabla}_b h_{cd} - \frac{1}{2}\bar{g}^{cd}\bar{\nabla}_c\bar{\nabla}_d h_{ab} + \bar{g}^{cd}\bar{\nabla}_c\bar{\nabla}_{(a} h_{b)d}.$$

Differentiating equation 4.3 we obtain

$$\delta R_{ab} - \frac{1}{2}\bar{g}_{ab}\delta R - \frac{1}{2} h_{ab}\bar{R} + \Lambda h_{ab} = 0,$$

where

$$\delta R = \bar{g}^{ab}\delta R_{ab} - h^{ab}\bar{R}_{ab}.$$

It is noted that $\frac{dg^{ab}(\lambda)}{d\lambda}|_{\lambda=0} = -h^{ab}$.

[8] The connection coefficients C^c_{ab} are defined by $({}^{\lambda}\nabla_a - \bar{\nabla}_a)A^b = C^b_{ad}A^d$.

The background space satisfies $\bar{R}_{ab} = \frac{2}{d-2}\Lambda\bar{g}_{ab}$. Substituting this into the linearized Einstein equation we obtain

$$\delta R_{ab} - \frac{1}{2}\bar{g}_{ab}\bar{g}^{cd}\delta R_{cd} - \frac{2\Lambda}{d-2}H_{ab} = 0,$$

where $H_{ab} = h_{ab} - \frac{1}{2}\bar{g}_{ab}h$ with $h = h^a{}_a$. The indices of h_{ab} are raised and lowered by $\bar{g}_{ab}$. It is now straightforward to show that the LHS of the linearized Einstein equation is equal to

$$\bar{\nabla}^e\bar{\nabla}^f K_{aebf} + \frac{1}{2}[\bar{\nabla}^e, \bar{\nabla}_b]H_{ae} - \frac{2\Lambda}{d-2}H_{ab},$$

where $K_{aebf} = \frac{1}{2}(\bar{g}_{af}H_{be} + \bar{g}_{be}H_{af} - \bar{g}_{ab}H_{ef} - \bar{g}_{ef}H_{ab})$ is called the superpotential in analogy with the Landau-Lifshitz superpotential, $\hat{U}_{aebf}$. It has the symmetry properties

$$K_{aebf} = K_{bfae} = -K_{eabf} = -K_{aefb}.$$

Consider again the full Einstein equation 4.3 then if we substitute $g_{ab} = \bar{g}_{ab} + h_{ab}$ and separate the part linear in h_{ab} from all the terms which are of second and higher order in h_{ab} we can write

$$\bar{\nabla}_e\bar{\nabla}_f K^{aebf} + \frac{1}{2}[\bar{\nabla}_e, \bar{\nabla}^b]H^{ae} - \frac{2\Lambda}{d-2}H^{ab} = (-\bar{g})^{-1/2}T^{ab}, \tag{4.4}$$

where T^{ab} is by construction a symmetric second rank tensor density of weight +1. It represents the effective gravitational self-interaction energy density for the linearized field.

From the contracted Bianchi identity $\nabla_a(R^{ab} - \frac{1}{2}Rg^{ab}) = 0$ and the fact that the metric is covariantly constant one finds that to first order in h_{ab} the linearized Einstein equation,

$$(R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab})_L = \bar{\nabla}_e\bar{\nabla}_f K^{aebf} + \frac{1}{2}[\bar{\nabla}_e, \bar{\nabla}^b]H^{ae} - \frac{2\Lambda}{d-2}H^{ab},$$

satisfies the identity (we need not expand ∇_a to first order because the Einstein equation for $\bar{g}_{ab}$ vanishes identically).

$$\bar{\nabla}_a(R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab})_L = 0,$$

so that

$$\bar{\nabla}_a T^{ab} = 0.$$

Let $\bar{\xi}_a$ be a timelike Killing vector of the background geometry, that is, it satisfies $\bar{\nabla}_{(a}\bar{\xi}_{b)} = 0$. Because T^{ab} is a symmetric tensor density we have the important continuity equation

$$\bar{\nabla}_a(T^{ab}\bar{\xi}_b) = \partial_a(T^{ab}\bar{\xi}_b) = 0.$$

From it one constructs the following conserved charge

$$\int_\Sigma d^{d-1}xT^{0b}\bar{\xi}_b,$$

where Σ is a spacelike hypersurface of the background spacetime manifold. It is, in a coordinate system that covers Σ, determined by $x^0 = \text{cst}$. Assuming that h_{ab} vanishes sufficiently rapidly at spatial infinity, the quantity,

$$E(\bar{\xi}) = \frac{1}{8\pi}\int_\Sigma d^{d-1}xT^{0b}\bar{\xi}_b, \tag{4.5}$$

we called the Killing energy, will be conserved. It represents the energy associated with the gravitational self-interactions that act on linear fluctuations with respect to some background geometry.

Let us consider again equation 4.4, specifically the term $\frac{1}{2}[\bar{\nabla}_e, \bar{\nabla}^b]H^{ae}$. It can be rewritten as

$$\begin{aligned}\frac{1}{2}[\bar{\nabla}_e, \bar{\nabla}^b]H^{ae} &= \frac{1}{2}\bar{g}^{bf}[\bar{\nabla}_e, \bar{\nabla}_f]H^{ae} = \frac{1}{2}\bar{g}^{bf}(\bar{R}_{ef}{}^a{}_cH^{ce} + \bar{R}_{ef}{}^e{}_cH^{ca}) = \\ &\frac{1}{2}\bar{R}_c{}^{ab}{}_eH^{ce} + \frac{\Lambda}{d-2}H^{ab}.\end{aligned}$$

Next we define X^{ab} as

$$X^{ab} = \frac{1}{2}[\bar{\nabla}_e, \bar{\nabla}^b]H^{ae} - \frac{2\Lambda}{d-2}H^{ab} = \frac{1}{2}(\bar{R}_c{}^{ab}{}_eH^{ce} - \frac{2\Lambda}{d-2}H^{ab}), \tag{4.6}$$

which is manifestly symmetric in a and b. Now, contract both sides of equation 4.4 with $\bar{\xi}_b$

$$\begin{aligned}(-\bar{g})^{-1/2}T^{ab}\bar{\xi}_b &= (\bar{\nabla}_e\bar{\nabla}_fK^{aebf})\bar{\xi}_b + X^{ab}\bar{\xi}_b = \bar{\nabla}_e[(\bar{\nabla}_fK^{aebf})\bar{\xi}_b - K^{afbe}\bar{\nabla}_f\bar{\xi}_b] + \\ &[K^{aebf}\bar{\nabla}_f\bar{\nabla}_e + X^{ab}]\bar{\xi}_b.\end{aligned}$$

The second term can be shown to vanish for we can write

$$\begin{aligned}K^{aebf}\bar{\nabla}_f\bar{\nabla}_e\bar{\xi}_b &= \frac{1}{2}(\bar{g}^{af}H^{be} + \bar{g}^{be}H^{af} - \bar{g}^{ab}H^{ef} - \bar{g}^{ef}H^{ab})\bar{R}^c{}_{feb}\bar{\xi}_c = \\ &-\frac{1}{2}(\bar{R}^c{}_{ef}{}^aH^{ef} - \frac{2\Lambda}{d-2}H^{ac})\bar{\xi}_c = -X^{ac}\bar{\xi}_c,\end{aligned}$$

where we used

$$\bar{\nabla}_f\bar{\nabla}_e\bar{\xi}_b = \bar{R}^c{}_{feb}\bar{\xi}_c$$

in the first equality and equation 4.6 in the second equality. Thus, we have that

$$(-\bar{g})^{-1/2}T^{ab}\bar{\xi}_b = \bar{\nabla}_e[(\bar{\nabla}_fK^{aebf})\bar{\xi}_b - K^{afbe}\bar{\nabla}_f\bar{\xi}_b].$$

Let us introduce the antisymmetric tensor F^{ae} defined by

$$F^{ae} = (\bar{\nabla}_f K^{aebf})\bar{\xi}_b - K^{afbe}\bar{\nabla}_f\bar{\xi}_b.$$

Then we can write

$$T^{ab}\bar{\xi}_b = \bar{\nabla}_e((-\bar{g})^{1/2}F^{ae}) = \partial_e((-\bar{g})^{1/2}F^{ae}),$$

where the last equality follows from the fact that $(-\bar{g})^{1/2}F^{ae}$ is an antisymmetric tensor density of weight +1. In conclusion, we have been able to express $T^{ab}\bar{\xi}_b$ as an ordinary divergence. Therefore it has become possible to express the Killing energy, also termed the Abbott and Deser (AD) energy, equation 4.5, as a flux integral

$$\begin{aligned} E(\bar{\xi}) &= \frac{1}{8\pi}\int_\Sigma d^{d-1}xT^{0b}\bar{\xi}_b = \frac{1}{8\pi}\int_\Sigma d^{d-1}x\partial_b((-\bar{g})^{1/2}F^{0b}) = \\ & \frac{1}{8\pi}\oint_S(-\bar{g})^{1/2}F^{0\beta}dS_\beta, \end{aligned} \tag{4.7}$$

where S is the closed boundary of Σ and where $F^{0\beta}$ is given by

$$F^{0\beta} = (\bar{\nabla}_e K^{0\beta fe} - K^{0\alpha f\beta}\bar{\nabla}_\alpha)\bar{\xi}_f.$$

The indices α and β range from 1 to $d-1$.

Assume for the moment that $\Lambda = 0$ and that the background metric is the Minkowski metric η_{ab}. In a Cartesian coordinate system of the background the timelike Killing vector reads $\bar{\xi}_f = \delta^0_f$. The integrand in the expression for the Killing energy reduces to $\partial_\gamma K^{0\beta 0\gamma}$, where $K^{0\beta 0\gamma}$ is given by

$$K^{0\beta 0\gamma} = \frac{1}{2}(H^{\beta\gamma} - \delta^{\beta\gamma}H^{00}) = \frac{1}{2}(h^{\beta\gamma} + \delta^{\beta\gamma}h^0{}_0) = \frac{1}{2}(h^{\beta\gamma} - \delta^{\beta\gamma}h^\alpha{}_\alpha).$$

The integrand becomes

$$\partial_\gamma K^{0\beta 0\gamma} = \frac{1}{2}(\partial_\gamma h^{\beta\gamma} - \partial^\beta h^\alpha{}_\alpha) = \frac{1}{2}(\partial_\gamma h_{\beta\gamma} - \partial_\beta h_{\alpha\alpha}).$$

Thus, the Killing energy for the interaction energy of the field h_{ab} describing first order fluctuations with respect to a Minkowski background is

$$E = \frac{1}{16\pi}\oint_S(\partial_\gamma h_{\beta\gamma} - \partial_\beta h_{\alpha\alpha})dS_\beta.$$

It has the same form as the ADM energy expression for an asymptotically flat space at spatial infinity, equation 4.1, since we can replace h_{ab} by the full metric $g_{ab} = \eta_{ab} + h_{ab}$ in the above integrand.

Formula 4.7 gives the total gravitational energy associated with the fluctuations h_{ab} that is contained at a particular instant in the spacelike hypersurface Σ. If for instance ξ^a is a global timelike Killing vector the expression gives the total energy defined with respect to no fluctuations, the ground state say, on the boundary of a spacelike section of the whole spacetime. For this to be conserved it must be that T_{ab} vanishes sufficiently rapidly as h_{ab} approaches the boundary S. For de Sitter space such an interpretation is not possible, because it lacks a global timelike Killing vector field, that is, it is not stationary. Hence the applicability of formula 4.7 is limited. In fact, the surface S must lie inside the event horizon.

4.3.2 The linearized Einstein equation on a maximally symmetric background

If we ignore the self-interactions by setting $T_{ab} = 0$ we are left with the linearized Einstein equation,

$$\bar{\nabla}^e \bar{\nabla}^f K_{aebf} + \frac{1}{2}[\bar{\nabla}^e, \bar{\nabla}_b] H_{ae} - \frac{2\Lambda}{d-2} H_{ab} = 0.$$

Further, one proves that the field h_{ab} to this order possesses the gauge invariance,

$$h_{ab} \to h_{ab} - 2\bar{\nabla}_{(a}\Lambda_{b)}.$$

This invariance can be used to set the gauge condition

$$\bar{\nabla}^a H_{ab} = \bar{\nabla}^a (h_{ab} - \frac{1}{2}\bar{g}_{ab} h) = 0.$$

It is straightforward to show that in this gauge the linearized Einstein equation reduces to

$$\Box H_{ab} - 2\bar{R}^c{}_{ab}{}^e H_{ce} = 0,$$

where the box denotes the d'Alembertian of the background geometry, $\Box = \bar{g}^{ab}\bar{\nabla}_a \bar{\nabla}_b$. The remaining gauge freedom can be used to set the gauge conditions,

$$\bar{\nabla}^a h_{ab} = 0 \quad \text{and} \quad h = 0,$$

called the transverse traceless gauge, in which case the linearized equation reduces further to

$$\Box h_{ab} - 2\bar{R}^c{}_{ab}{}^e h_{ce} = 0. \tag{4.8}$$

For a maximally symmetric space with cosmological constant Λ we find

$$(\Box - \frac{2\bar{R}}{d(d-1)}) h_{ab} = (\Box - \frac{4\Lambda}{(d-1)(d-2)}) h_{ab} = 0. \tag{4.9}$$

We note that the equation of motion for h_{ab} is conformally invariant. Due to the conformal invariance the field h_{ab} may be termed Minkowskian massless in the spirit of section 2.5. The field h_{ab} corresponds to the discrete representation with $p = q = 2$ of the group $SO(d,1)$[9].

4.3.3 The AD mass of SdS_4

In order to test the applicability and usefulness of the Abbott and Deser approach to asymptotically de Sitter spaces the specific case of a Schwarzschild-de Sitter space will be discussed. The space SdS_4 can be treated as a deformation of dS_4 in the sense that

$$g_{ab}^{SdS} = \bar{g}_{ab} + h_{ab},$$

where now $\bar{g}_{ab}$ denotes the metric of dS_4. The Killing or Abbott and Deser (AD) energy of h_{ab} will be calculated using expression 4.7. The expression has meaning only when $\bar{\xi}^a$ is timelike. Hence, the integration surface S appearing in the AD energy expression must lie inside the event horizon of dS_4. The best suited coordinates will then be asymptotically flat coordinates because they are analytic at the horizon. This system of coordinates was discussed in section 1.3.3. It is straightforward to show that in terms of these coordinates the AD energy expression reads

$$E(\bar{\xi}) = \frac{1}{8\pi} \oint_S f^3 (\bar{\nabla}_e K^{0\beta fe} - K^{0\alpha f\beta} \bar{\nabla}_\alpha) \bar{\xi}_f dS_\beta, \tag{4.10}$$

where $f = e^{t/l}$, $\bar{\xi}_f = (1, f^2 \frac{x_\tau}{l})$ and where $S = \partial\Sigma$ with Σ an infinite spatial plane, so that the integration surface is an S^2 surface. In order to evaluate this integral we need to find $h_{ab} = g_{ab}^{SdS} - \bar{g}_{ab}$ in these asymptotically flat coordinates. The asymptotically flat metric of SdS_4, equation 1.33, was discussed in section 1.6.2. Introducing the function, Ψ, defined by

$$\Psi = \frac{M}{2rf}$$

the deformation metric h_{ab} can be written as

$$\begin{aligned} h_{00} &= 1 - \left(\frac{1-\Psi}{1+\Psi}\right)^2 \\ h_{\alpha\beta} &= f^2 \delta_{\alpha\beta}((1+\Psi)^4 - 1). \end{aligned}$$

[9] See [13] for a concise presentation of the group theoretical aspects of linear gravity on a de Sitter space. There conformally invariant fields are termed strictly massless and their gauge invariance can be used to reduce the number of degrees of freedom to two.

The integrand of the AD energy can be evaluated in the asymptotically flat coordinates to be

$$(\bar{\nabla}_e K^{0\beta fe} - K^{0\alpha f\beta}\bar{\nabla}_\alpha)\bar{\xi}_f = \frac{1}{2}(-\partial_\beta h^{\gamma\gamma} + \partial_\alpha h^{\alpha\beta} + \frac{2}{l}h^{0\beta}) + \frac{1}{2l}f^2 x_\tau(-\partial_0 h^{\beta\tau} + f^{-2}\delta^{\beta\tau}\partial_0(f^2 h^{\gamma\gamma}) + f^{-2}(\delta^{\beta\tau}\partial_\alpha h^{0\alpha} - \partial_\beta h^{0\tau})) + \frac{1}{l^2}(h^{00}x_\beta - f^2 h^{\beta\tau}x_\tau),$$

where $h^{\gamma\gamma} = h^{\alpha\gamma}\delta_{\alpha\gamma}$. The indices of h_{ab} are raised with $\bar{g}^{ab}$, and h^{ab} has the components

$$\begin{aligned} h^{00} &= 1 - (\frac{1-\Psi}{1+\Psi})^2 \\ h^{\alpha\beta} &= f^{-2}\delta^{\alpha\beta}((1+\Psi)^4 - 1). \end{aligned}$$

It will be assumed that the integration surface S is inside the cosmological event horizon which is at lf^{-1}, but still at a distance R far away from the horizon of the black hole whose radius is $r_S f^{-1}$ with $r_S = 2M$, the Schwarzschild radius. In this case $\Psi \ll 1$. Then the metric component h^{00} can be expanded in a Taylor series around $\Psi = 0$, so that a series expansion for $E(\bar{\xi})$ will be obtained. Expanding around $\Psi = 0$ we find

$$\begin{aligned} h^{00} &= 1 - (\frac{1-\Psi}{1+\Psi})^2 = 4\Psi - 8\Psi^2 + 12\Psi^3 - 16\Psi^4 + \cdots + (-1)^{n+1}4n\Psi^n + \cdots \\ h^{\alpha\beta} &= \delta^{\alpha\beta}f^{-2}(4\Psi + 6\Psi^2 + 4\Psi^3 + \Psi^4). \end{aligned}$$

At each order h^{00} and $h^{\alpha\beta}$ are of the form $h^{00} = B$ and $h^{\alpha\beta} = A\delta^{\alpha\beta}$, where A and B are functions of t and r. Then at each order the AD energy is given by

$$E(\xi) = \frac{f^3}{8\pi}\oint_S dS_\alpha(-\partial_\alpha A + x_\alpha(\frac{1}{l}f^2\partial_0 A + \frac{1}{l^2}(2f^2 A + B))). \tag{4.11}$$

For example the zeroth order contribution to the AD energy, $E^{(0)}$, say, is obtained with $B = 4\Psi$ and $A = f^{-2}4\Psi$. Substituting this into expression 4.11 we have

$$E^{(0)}(\bar{\xi}) = \frac{1}{8\pi}\oint_S dS_\alpha \frac{2M}{r^3}x_\alpha$$

with $dS_\alpha = n_\alpha dS$ and $x_\alpha n_\alpha = r$. This gives

$$E^{(0)}(\bar{\xi}) = \frac{1}{4\pi}\frac{M}{R^2}\oint_S dS\frac{M}{R^2} = M.$$

Proceeding in this way one eventually obtains

$$\begin{aligned} E(\bar{\xi}) &= M + 3M\frac{f^{-1}r_0}{R} - 5M\frac{r_0}{l^2}Rf + 3M(\frac{f^{-1}r_0}{R})^2 + \\ &\quad M(\frac{f^{-1}r_0}{R})^3 - 5M\frac{r_0^3}{l^2 Rf} + \frac{M}{l^2}\sum_{n=5}^{\infty}\frac{(-1)^{n+1}nr_0^{n-1}}{(fR)^{n-3}}, \end{aligned} \tag{4.12}$$

where $r_0 = r_S/4$. It represents the Killing or AD energy contained in the region bounded by S of deformations of $\bar{g}_{ab}$ describing a black hole. The zeroth order term is equal to the mass of a Schwarzschild black hole. The higher order terms may be considered as corrections to the mass M which are due to the cosmological expansion of the space. All the higher order terms are proportional to at least $1/R$ except for the third term on the right-hand side of equation 4.12 which is proportional to R. In this system of coordinates the horizon is at $(\frac{f|\vec{x}|}{l})^2 = 1$. Since $R < |\vec{x}|$ it follows that $fR/l^2 < (fR)^{-1}$, so that inside the cosmological event horizon the term $5M\frac{r_0}{l^2}Rf$ is smaller than the second term, $3M\frac{f^{-1}r_0}{R}$. However, it also shows that the limit $R \to \infty$ does not exist.

The assumption of the black hole radius $f^{-1}r_S$ to be much smaller than R not only enabled the expansion made above, but it also ensures that all of the gravitational energy associated with the black hole has been accounted for. This is because if the surface of integration would be close to the black hole horizon whose value is time dependent a fixed volume element like dS_i would be inadequate, and integration with respect to it would not account for all of the gravitational energy.

This section started with the question whether or not the AD formalism is applicable to SdS_4 and if so whether or not it is a useful approach. Well, it is applicable, though in a limited fashion. The expression 4.12 painfully depends on R. This also makes it doubtful if it can be deemed useful. In fact all it tells us is that the AD formalism is a physically sensible one for it is able to produce sensible corrections to the mass M of a Schwarzschild black hole and these corrections relate to the cosmological expansion of the spacetime. Still, it is not able to truly give the mass of a black hole in SdS_4. It merely indicates what possible modifications there could be. The above AD expression can be contrasted with expression 3.16 which does give the mass of a black hole in SdS_4 as measured by a static observer.

It can be proven as was already mentioned in the introduction to this chapter that the Killing or AD energy of gravitational fluctuations with respect to a de Sitter background are always positive inside the cosmological event horizon and negative outside. Therefore the energy expression 4.7 cannot be used to establish globally the classical stability of de Sitter spacetime. In the next section this stability will be established. Fluctuations of the metric in the global coordinate system of dS_4 will be discussed. These fluctuations will be generated by the presence of a perfect fluid with a small density, pressure and velocity field. In a way this is in contrast to the Abbott and Deser approach where the fluctuations were not required to be small. Then again it is not necessary to treat large fluctuations because if it can be shown that de Sitter space is globally stable against any small fluctuation they are allowed to grow arbitrarily large, still de Sitter space would remain stable.

4.4 Global perturbations of de Sitter space induced by a perfect fluid

We shall be working in the global coordinate system of dS_4 in which the metric is given by

$$ds^2 = -d\tau^2 + a^2(\tau)d\Omega_3^2,$$

where $a(t) = l\cosh t/l$. The cosmic time parameter τ is redefined by $d\tau = ad\eta$, so that the line element becomes

$$ds^2 = a^2(\eta)(-d\eta^2 + \gamma_{\alpha\beta}dx^\alpha dx^\beta), \tag{4.13}$$

where $\gamma_{\alpha\beta}$ is the metric of a unit S^3 sphere, and $a(\eta) = l\sec\eta$ with $-\pi/2 \leq \eta \leq \pi/2$. The parameter η equals T/l where T is the conformal time of section 1.3.4. The Ricci tensor components are

$$R_{00} = -3a^{-2}(aa'' - (a')^2) \tag{4.14}$$

$$R_{\alpha 0} = 0 \tag{4.15}$$

$$R_{\alpha\beta} = a^{-2}(2a^2 + aa'' + (a')^2)\gamma_{\alpha\beta}, \tag{4.16}$$

where the prime denotes differentiation with respect to η. The Ricci scalar is given by

$$R = 6a^{-3}(a + a'').$$

As always the perturbations and the de Sitter background will be denoted by h_{ab} and $\bar{g}_{ab}$, respectively, so that the full metric is $g_{ab} = \bar{g}_{ab} + h_{ab}$. In discussing these fluctuations in the global (conformal) coordinate system it is convenient to choose a gauge for h_{ab} at the start. This gauge choice will be to set h_{00} and $h_{\alpha 0}$ equal to zero, and it is called the synchronous gauge. The fluctuations induce a change in the Einstein tensor, $\delta G_{ab} = G^{(1)}_{ab}(\bar{g}+h) - \bar{G}_{ab}$, where $G^{(1)}_{ab}(\bar{g}+h)$ is the Einstein tensor of the full metric $\bar{g}_{ab} + h_{ab}$ up to first order in h_{ab} and where $\bar{G}_{ab}$ is the Einstein tensor of the background de Sitter space. The components of this tensor are calculated in appendix C, equations C.1, C.2 and C.3. The result is of this calculation is

$$\delta G^0{}_0 = \delta R^0{}_0 - \frac{1}{2}\delta R = \frac{h}{a^2} - \frac{a'h'}{a^3} - \frac{1}{2a^2}(\bar{\nabla}^\gamma\bar{\nabla}_\delta h^\delta{}_\gamma - \Box h) \tag{4.17}$$

$$\delta G^0{}_\beta = \delta R^0{}_\beta = \frac{1}{2a^2}(\bar{\nabla}_\beta h' - \bar{\nabla}_\gamma h^\gamma{}_\beta{}') \tag{4.18}$$

$$\begin{aligned}\delta G^\alpha{}_\beta = \delta R^\alpha{}_\beta - \frac{1}{2}\delta^\alpha_\beta\delta R = \frac{1}{2a^2}[&\frac{2a'}{a}h^\alpha{}_\beta{}' + h^\alpha{}_\beta{}'' - 4h^\alpha{}_\beta + \\ &\delta^\alpha_\beta(\Box h - \bar{\nabla}^\gamma\bar{\nabla}_\delta h^\delta{}_\gamma + 2h - h'' - \frac{2a'h'}{a}) + \\ &\bar{\nabla}_\gamma\bar{\nabla}^\alpha h^\gamma{}_\beta + \bar{\nabla}^\gamma\bar{\nabla}_\beta h^\alpha{}_\gamma - \bar{\nabla}^\alpha\bar{\nabla}_\beta h - \Box h^\alpha{}_\beta],\end{aligned} \tag{4.19}$$

where h is the trace of h_{ab} and where $\bar{\nabla}_\alpha$ is the covariant derivative of unit S^3, so that $\bar{\nabla}^\alpha = \gamma^{\alpha\beta}\bar{\nabla}_\beta$. The components are written with one upper and one lower index because then the equations are somewhat compacter. This is because the following relation holds

$$\delta G_{ab} + \Lambda h_{ab} = \bar{g}_{ac}\delta G^c{}_b.$$

The fluctuations h_{ab} are assumed to result from the presence of a perfect fluid with energy-momentum tensor T_{ab}. The full Einstein equation is

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}.$$

To first order in h_{ab} this becomes

$$\bar{G}_{ab} + \delta G_{ab} + \Lambda(\bar{g}_{ab} + h_{ab}) = 8\pi\delta T_{ab},$$

and because $\bar{g}_{ab}$ satisfies the vacuum Einstein equation this reduces to

$$\delta G_{ab} + \Lambda h_{ab} = 8\pi\delta T_{ab} = \bar{g}_{ac}\delta G^c{}_b,$$

so that

$$\delta G^a{}_b = 8\pi\delta T^a{}_b.$$

The energy-momentum tensor $\delta T^a{}_b$ has the perfect fluid form

$$\delta T^a{}_b = (\delta p + \delta\rho)u^a u_b + \delta p\delta^a_b,$$

where δp is the small pressure, $\delta\rho$ the small density and u^a is the velocity field of the fluid particles. The real 3-velocity v^α of the fluid is $v^\alpha = au^\alpha$. This will be assumed to be small. In fact much smaller than the speed of light, so that $u^\alpha \ll 1/a$. The timelike 4-vector u^a is normalized to $u^a u_a = -1$. Then to first order we can put $u^0 = 1/a$. Further, because δp and $\delta\rho$ are small they are interrelated through

$$\delta p = \frac{dp}{d\rho}\delta\rho = v_s^2\delta\rho,$$

where $v_s \equiv \sqrt{\frac{dp}{d\rho}}$ is the speed of sound in units of the speed of light. The Einstein equations relating the perturbation h_{ab} to the presence of a perfect fluid are then

$$\delta G^0{}_0 = -8\pi\delta\rho \tag{4.20}$$

$$\delta G^0{}_\beta = 8\pi\frac{1}{a}\delta\rho(1 + v_s^2)u_\beta \tag{4.21}$$

$$\delta G^\alpha{}_\beta = 8\pi\delta\rho v_s^2\delta^\alpha_\beta. \tag{4.22}$$

Comparing the vacuum Einstein equation 4.21 with equation 4.15, we see that the velocity field u_α makes $R_{\alpha 0}$ nonzero. The component $R_{\alpha 0}$ was zero because the line element of the global coordinate system, expression 4.13, consists of comoving coordinates, that is, the timelike geodesics of test particles have $x^\alpha = \text{const}$. When the perturbations h_{ab} in the synchronous gauge are included the line element changes to

$$ds^2 = a^2(\eta)(-d\eta^2 + (\gamma_{\alpha\beta} + a^{-2} h_{\alpha\beta}(\eta, \vec{x}))dx^\alpha dx^\beta) \tag{4.23}$$

preserving the synchronity. However, the coordinates x^α are no longer comoving coordinates.

The new line element 4.23 shows that the perturbations $h_{\alpha\beta}$ describe time dependent changes of the S^3 spacelike sections of dS_4. In section A.3 it is shown that the most general perturbation can be written as, equation A.3,

$$h_{\alpha\beta} = \lambda(\eta)P_{\alpha\beta} + \mu(\eta)Q_{\alpha\beta} + \sigma(\eta)S_{\alpha\beta} + \nu(\eta)H_{\alpha\beta}. \tag{4.24}$$

The quantities appearing here are defined in section A.3.

It will turn out convenient to consider the off-diagonal components of equation 4.22 and to take the trace of the same equation and eliminate from it $\delta\rho$ using equation 4.20. This gives

$$\delta G^\alpha{}_\beta = 0 \quad \text{for } \alpha \neq \beta \tag{4.25}$$

$$\delta G^\alpha{}_\alpha = -3v_s^2 \delta G^0{}_0. \tag{4.26}$$

Plugging equation 4.24 into equations 4.25 and 4.26 the following set of differential equations is obtained

$$\mu'' + \mu'(2 + 3v_s^2)\tan\eta + \frac{1}{3}(1 + 3v_s^2)(n^2 - 4)(\mu + \lambda) = 0 \tag{4.27}$$

$$\lambda'' + 2\lambda' \tan\eta - \frac{1}{3}(n^2 - 1)(\mu + \nu) = 0 \tag{4.28}$$

$$\sigma'' + 2\sigma' \tan\eta = 0 \tag{4.29}$$

$$\nu'' + 2\nu' \tan\eta + (n^2 - 1)\eta = 0, \tag{4.30}$$

where $a(\eta) = l \sec\eta$ has been used together with equations 4.20 and 4.22. Further, if one substitutes equation 4.24 into equations 4.20 and 4.21 one obtains

$$\delta\rho = \frac{\cos^2\eta}{8\pi l^2}(\frac{1}{3}(n^2 - 4)(\mu + \lambda) + \mu' \tan\eta)Q \tag{4.31}$$

$$v_\alpha \delta\rho = \frac{1}{8\pi(1 + v_s^2)}((\frac{1}{3}(n^2 - 4)\lambda' + \frac{1}{3}(n^2 - 1)\mu')\frac{1}{n^2 - 1}\nabla_\alpha Q + \frac{1}{2}\sigma'(n^2 - 4)S_\alpha). \tag{4.32}$$

Not every change described by $h_{\alpha\beta}$ will correspond to a physical fluctuation. This is because there is some gauge freedom left. It can be shown [18, 14] that the remaining gauge freedom after setting $h_{00} = 0 = h_{0\alpha}$ is

$$h_{\alpha\beta} \to h_{\alpha\beta} + \sin\eta \bar{\nabla}_\alpha \bar{\nabla}_\beta f_0 + f_0 \gamma_{\alpha\beta} \sin\eta + \bar{\nabla}_\alpha f_\beta + \bar{\nabla}_\beta f_\alpha, \tag{4.33}$$

where f_0 and f_α are arbitrary functions defined on S^3.

We may consider the perturbations described by the scalar, vector and tensor spherical harmonics separately. We will start with the perturbations described by the tensor harmonics. In this case it is only the metric which changes leaving the density and velocity of the fluid unaffected. The tensor harmonic $H_{\alpha\beta}$ is transverse and traceless, so it describes gravitational radiation. The functions f_0 and f_α appearing in the remaining gauge freedom cannot be constructed since nor scalars or vectors can be made out of H_{ab}. The dynamics is described by equation 4.30. In order to solve this differential equation substitute for ν the function $e^{ik\eta}$. It then follows from inspection that the solution is

$$\nu(\eta) = c((n+1)e^{i(n-1)\eta} + (n-1)e^{i(n+1)\eta}),$$

where c is a constant. The real solution is then

$$\nu(\eta) c_1(\sin(n-1)\eta + (n-1)\cos\eta \sin n\eta) + c_2(\cos(n-1)\eta + (n-1)\cos\eta \cos n\eta)$$

with c_1 and c_2 constant. The parameter η and the time parameter τ of the global coordinate system are related by $\cos\eta = \frac{1}{\cosh \tau/l}$, see the beginning of this section and section 1.3.4. In the limit $\tau \to \infty$, the parameter η approaches $\pi/2$. In this limit we have $\cos\eta \simeq 2e^{-\tau/l}$, so that asymptotically in the infinite future

$$\mu(t) \sim A_1 + A_2 e^{-\tau/l}.$$

Therefore the perturbations do not grow in time. Thus, gravitational radiation does not lead to an instability.

The perturbations described by the spherical harmonics have $h_{\alpha\beta} = \sigma(\eta) S_{\alpha\beta}$, so that $Q = 0$, and therefore $\delta\rho = 0$. The function $\sigma(\eta)$ satisfies equation 4.29, which is a first order differential equation. It is solved by $\sigma' = a\cos^2\eta$. This integrates to give

$$\sigma(\eta) = c_3 + c_4(2\eta + \sin 2\eta).$$

With $Q = 0$ and $\delta\rho = 0$ it must be that $\sigma' = 0$ as follows from equation 4.32. Therefore we must have that $c_4 = 0$ so that σ is constant. However, metric perturbations like $h_{\alpha\beta} = c_3 S_{\alpha\beta} = c_3(\bar{\nabla}_\alpha S_\beta + \bar{\nabla}_\beta S_\alpha)$ can be transformed away by a gauge transformation with $f_\alpha = -c_3 S_\alpha$. Thus, the vector harmonic perturbations are completely irrelevant.

Finally, we consider the perturbations generated by the scalar spherical harmonics, that is, we set $h_{\alpha\beta} = \lambda P_{\alpha\beta} + \mu Q_{\alpha\beta}$. First, we exhaust the remaining gauge freedom. Choose $f_0 = CQ$ and $f_\alpha = \frac{C'}{2(n^2-1)}\bar{\nabla}_\alpha Q$. Then by comparing equation 4.33 with $h_{\alpha\beta}$ it can be shown that in $h_{\alpha\beta} = \lambda P_{\alpha\beta} + \mu Q_{\alpha\beta}$ the functions λ and μ can be shifted by

$$\lambda(\eta) = C(n^2-1)\sin\eta + C' \tag{4.34}$$

$$\mu(\eta) = -C(n^2-4)\sin\eta - \frac{1}{3}C'. \tag{4.35}$$

This can be used to show that for the cases $n = 1$ and $n = 2$ no physical changes in the metric appear. The equations governing the time dependence of λ and μ are equations 4.28 and 4.27.

The larger the value of the speed of sound, v_s, the larger the pressure, and the larger the resistance against gravitational collapse. So if there is an instability it will most likely occur for the case of pressureless dust ($v_s = 0$). Adding and subtracting equations 4.28 and 4.27 and setting $v_s = 0$ we obtain

$$(\mu+\lambda)'' + 2(\mu+\lambda)'\tan\eta - (\mu+\lambda) = 0.$$

Subtracting equation 4.28 from equation 4.27 a second differential equation is found

$$(\mu-\lambda)'' + 2(\mu-\lambda)'\tan\eta + (\frac{2}{3}n^2 - \frac{5}{3})(\mu+\lambda) = 0.$$

The first can be solved using the method of reduction of order. Substituting $\mu + \lambda = \sin\eta w$ leads to a first order differential equation for w' which can be integrated. The second differential equation is inhomogeneous. It can be solved by first finding a solution to $(\mu-\lambda)'' + 2(\mu-\lambda)'\tan\eta = 0$ and then adding a particular solution to it. The solutions one finds are

$$\mu(\eta) = c_5(\frac{1}{2}\sin 2\eta + (\eta - \frac{\pi}{2})) + c_6(n^2-4)(\cos\eta + (\eta - \frac{\pi}{2})\sin\eta)$$

$$\lambda(\eta) = -c_5(\frac{1}{2}\sin 2\eta + (\eta - \frac{\pi}{2})) + c_6(n^2-2)(\cos\eta + (\eta - \frac{\pi}{2})\sin\eta),$$

where the gauge freedom equations 4.34 and 4.35 has been used to set $\mu(\pi/2) = 0 = \lambda(\pi/2)$. Differentiating these two equations it is also seen that $\mu' = \lambda' = \mu'' = \lambda'' = 0$ at $\eta = \pi/2$. Thus, the energy momentum tensor associated with the field $h_{\alpha\beta}$ defined as in equation 4.4 vanishes at late time infinity. Further, also the density $\delta\rho$ and $\delta\rho v_\alpha$ can be shown by evaluating equations 4.31 and 4.32 for the case at hand to vanish fast as $\eta \to \pi/2$ or as $\tau \to \infty$. Once again the perturbations generated by scalar spherical harmonics do not lead to any instability. Using $\cos\eta \simeq 2e^{-\tau/l}$ it follows that the scalar generated gravitational perturbations are exponentially damped.

Hence, all gravitational perturbations are exponentially damped with the exception of gravitational radiation which approaches a constant value, so that we conclude that dS_4 is classically stable.

4.5 Black hole nucleation and subsequent evaporation in de Sitter space

In this last section an outline will be given of the description of black hole nucleation[10] in a de Sitter space. Because de Sitter space is at finite temperature it might be possible for thermal gravitons to induce a local change of the spacetime metric that eventually leads to a global topological change. These are thus thermally induced metric fluctuations or better topological deformations.

Null infinity of Schwarzschild-de Sitter space is spacelike and has the topology of $S^2 \times S^1$. Hence, in the complexified SdS_4 space there must be a euclidean section which has the topology of $S^2 \times S^2$ assuming that the imaginary time is periodic, that is, S^1, since $S^2 = S^1 \times S^1$. In fact in [14] it is shown that such euclidean sections exist. Therefore the $S^2 \times S^2$ euclidean section of the complexified SdS_4 satisfies the euclidean Einstein equation. Further, it has no intrinsic singularities and it has finite action because it is compact. Such spaces are called gravitational instantons.

Let us consider again the partition function 3.1,

$$Z = N \int_{\text{periodic}} \mathcal{D} g_{ab} e^{-S_E[g]}.$$

This path integral is over a set of metrics which includes instanton metrics and deviations thereof. In a semi-classical approximation to this integral one would replace it with a sum over the stationary metrics, that is, those which satisfy the euclidean Einstein equation. The partition function will be considered in the neighborhood of the saddle point corresponding to the $S^2 \times S^2$ instanton. Then including fluctuations about the $S^2 \times S^2$ metric a part of the above partition function is

$$e^{-S_E[\tilde{g}]} \int \mathcal{D} h_{ab} e^{-S_E^{(2)}[h]},$$

where $S_E^{(2)}[h]$ is the euclidean action of the fluctuations h_{ab} about the metric $\tilde{g}_{ab} \equiv g^{S^2 \times S^2}$ of $S^2 \times S^2$ evaluated up to second order. So variation of $S_E^{(2)}[h]$ will lead to the linearized Einstein equation on $S^2 \times S^2$. This Gaussian integral will give the larger part of the action of the fluctuations h_{ab} around the stationary metric $g_{ab}^{S^2 \times S^2}$.

The fluctuations h_{ab} can be decomposed into a transverse traceless part, a trace part, and a traceless longitudinal part as follows

$$h_{ab} = h_{ab}^{TT} + \frac{1}{4} \tilde{g}_{ab} h^c{}_c + (\tilde{\nabla}_a \xi_b + \tilde{\nabla}_b \xi_a - \frac{1}{2} \tilde{g}_{ab} \tilde{\nabla}_c \xi^c), \tag{4.36}$$

[10]The term nucleation is borrowed from statistical physics where nucleation processes describe phase transitions. For example, the nucleation of bubbles of water vapor at the boiling point initiate the phase transition from the liquid to the gaseous phase.

where $\tilde{\nabla}_a$ is the covariant derivative of $S^2 \times S^2$, h^{TT}_{ab} is the transverse traceless part, $h^c{}_c = h$ is the trace part and the last term in parenthesis is the traceless longitudinal part where ξ^a describes the longitudinal excitations of h_{ab}.

The action $S^{(2)}_E[h^{TT}, h, \xi]$ contains the full gauge freedom of h_{ab}. A proper way to deal with this in evaluating the action is to add gauge fixing terms to it. This method is described in[11] [12, 6]. Only the result will be stated here: for the Gaussian fluctuations described by h and ξ^a it can be proven that their action is positive (semi-)definite, but this need not be the case for the h^{TT}_{ab} fluctuations. Therefore if one is only interested in perturbations which may cause an instability, the part of the partition function considered above, expression 4.36 can be written as

$$e^{-S_E[\tilde{g}]} \int \mathcal{D}h^{TT}_{ab} e^{-S^{(2)}_E[h^{TT}]}(\text{real-valued contributions of stable Gaussian fluctuations}). \tag{4.37}$$

The action $S^{(2)}_E[h^{TT}]$ can be found using equation 4.8 which is the linearized einstein equation on a background spacetime in the transverse traceless gauge. The corresponding Lagrangian is $\mathcal{L} = -\frac{1}{16\pi}\frac{1}{4}h^{TT}_{ab}G^{abcd}h^{TT}_{cd}$, where $G_{abcd} = -\tilde{g}_{ac}\tilde{g}_{bd}\Box - 2\tilde{R}_{acbd}$. Next we will decompose h^{TT}_{ab} in terms of the eigenfunctions of the operator G^{abcd} which will be denoted by ϕ^n_{ab}, and are defined by $G_{abcd}\phi^{cd}_n = \lambda_n \phi^n_{ab}$. The transverse traceless fluctuations can be decomposed orthonormally in terms of the eigenfunctions of G_{abcd} as $h^{TT}_{ab} = \sum_n a_n \phi^n_{ab}$, where the inner product of ϕ^n_{ab} and ϕ^m_{ab} is defined as

$$\int \sqrt{g} d\tau d^3x \phi^n_{ab}\phi^{ab}_m = \delta^n_m.$$

With the help of this orthonormal decomposition of h^{TT}_{ab} the integration measure $\mathcal{D}h^{TT}_{ab}$ can be replaced by $\frac{\mu}{(32\pi)^{1/2}}da_n$ for each eigenfunction where μ is a parameter of dimension mass. Then the path integral appearing in expression 4.37 can be written as

$$\int \mathcal{D}h^{TT}_{ab} e^{-S^{(2)}_E[h^{TT}]} = \int \prod_n \frac{\mu}{(32\pi)^{1/2}} da_n e^{-\frac{1}{16\pi}\frac{1}{4}\lambda_n a_n^2}.$$

In [14] it is shown that there exist one fluctuation about $S^2 \times S^2$ which has a negative eigenvalue λ with respect to the operator G_{abcd}. In order to regularize the integral for this particular λ the variable a must be considered in the complex plane. The integrand can be analytically continued to the complex a-plane. Then by taking the contour along the imaginary axis, that is, by replacing a with ia the integral is convergent and gives an

[11] Actually in [12, 6] one considers the case of zero cosmological constant but the results generalize to the case of nonzero cosmological constant.

imaginary contribution to the partition function. Hence, it follows that the expression 4.37 becomes

$$ie^{-S_E[\tilde{g}]}(\text{real-valued contributions of Gaussian fluctuations}).$$

Therefore there is an imaginary contribution to the partition function and hence to the free energy coming from an unstable fluctuation about the $S^2 \times S^2$ instanton.

A de Sitter space thus has an instability. Its partition function has an imaginary part due to an unstable fluctuation around the $S^2 \times S^2$ instanton. If this is continued to Minkowskian metric it is seen that this is a thermal process in which the entire de Sitter space decays into a Schwarzschild-de Sitter space. The semi-classical analysis requires the black hole horizon to fall on top of the cosmological event horizon, because only in that case the two horizon temperatures are equal. It is believed that quantum fluctuations will bring the black hole horizon inside the cosmological event horizon. Then the black hole is at a higher temperature than the surrounding space, see equations 3.19 and 3.20. Therefore the black hole on average emits more particles than it absorbs and starts to evaporate. In this way the black hole mass reduces leading to a reduction of the horizon radius which in turn leads to an increase of the black hole horizon temperature (a black hole has negative specific heat). In the end the black hole will have evaporated completely and a pure de Sitter space is left behind. Thus, even though de Sitter has an instability, it does survive. It evaporates much faster than the time required for another nucleation to take place.

Conclusions

The introduction of a positive cosmological constant in the Einstein equation has some rather strong effects on the global spacetime properties. Null infinity will be spacelike leading to the presence of a cosmological event horizon, and if it forms a compact hypersurface then there will be no spatial infinity. In describing physical processes in a de Sitter space observers are introduced. For observers comoving with the expansion the radial coordinate has only a finite range of values. For static observers horizons appear, and for an observer who experiences no spatial curvature, the space is geodesically singular (incomplete) in the infinite past and his future light cone is bounded by a horizon which moves along with the observer.

These observers all have in common that they measure the same temperature. This temperature is thus generally covariant and derives from the geometric properties of the spacetime itself. In fact it is directly proportional to the surface gravity of the cosmological event horizon. This intrinsic thermality of de Sitter space first became apparent from the properties of the Green's functions of scalar particles which are all periodic in imaginary time. The quantum state of these particles, which was argued to be the euclidean vacuum, is a pure quantum state. This is another confirmation of the fact that the temperature really is an intrinsic property of the space.

The scalar particles that were introduced on the spacetime divide roughly into two classes: those whose mass parameter has a Minkowskian interpretation in the sense that these fields approach irreps of the Poincaré group in the limit in which de Sitter space approaches Minkowski space, and those which have no Minkowskian interpretation. The physical status of these latter particles has not been brought to light yet.

In quantizing these scalar fields using the covariant procedure different inequivalent vacuum choices can be made. Then if self- interactions are introduced the euclidean vacuum remains as the only possible choice. This vacuum leads to Green's functions which can be analytically continued to the euclidean de Sitter space, S^4, and back. This explains why the Green's functions defined with respect to a pure vacuum are thermal.

The thermal properties of de Sitter space are also cause for concern. It might be that just as is the case for hot flat space that it is unstable and decays. In order to study the stability of de Sitter a perfect fluid was introduced which generates fluctuations in the de Sitter space metric. No instability was found. However, analysis of the partition function shows that the whole de Sitter space can and therefore will decay into a Schwarzschild-de Sitter space. Still, this space does not live long and due to the rapid black hole evaporation a pure de Sitter space is found again.

So far no instability has been found that goes against the existence of a positive energy theorem for de Sitter space. An attempt to define a notion of gravitational energy has been made by Abbott and Deser. But, however, physically sensible it is not sufficiently general to apply to the whole space; it can only be used in that region where the Killing vector of time translations is in fact timelike. But at null infinity this Killing vector is spacelike which makes it difficult to define an energy expression that gives the total energy of the whole spacetime.

Putting everything together it must be concluded that if it is really true that our universe is presently undergoing large scale expansion driven by a cosmological constant or that such a phase has occurred in the past then there is still a lot of work to do.

Appendix A

Special functions

A.1 The hypergeometric function

Consider the hypergeometric differential equation in the complex z plane,

$$z(z-1)\frac{d^2\Psi}{dz^2} + (c - (a+b+1)z))\frac{d\Psi}{dz} - ab\Psi = 0,$$

which has regular-singular points at $z = 0, 1, \infty$. Comparing with 2.7 we read off

$$c = \frac{d}{2}, \quad a + b + 1 = d, \quad ab = l^2\tilde{m}^2.$$

Both a and b satisfy

$$h(h - (d-1)) + l^2\tilde{m}^2 = 0.$$

The hypergeometric function ${}_2F_1(a, b, c; z)$ defined through the series expansion

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

with $c \neq 0, -1, -2, \cdots$ provides a solution for Ψ which is regular at $z = 0$, real for real values of z, and defined on $|z| < 1$.

If we solve for h we find

$$h = \frac{1}{2}((d-1) + \sqrt{(d-1)^2 - 4l^2\tilde{m}^2}).$$

We put $a = h$ and $b = (d-1) - h$. One proves that for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(c - a - b) < 0$,

$$\lim_{z \to 1-0} (1-z)^{a+b-c} \, {}_2F_1(a, b, c; z) = \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}. \quad \text{(A.1)}$$

The parameter b has the value $\frac{1}{2}((d-1)-\sqrt{(d-1)^2-4l^2\tilde{m}^2})$ so that for $\tilde{m}^2>0$ the function ${}_2F_1(a,b,c;z)$ behaves near $z=1$ as $(1-z)^{1-d/2}$. We conclude that the hypergeometric function ${}_2F_1(h,(d-1)-h,\frac{d}{2};z)$ has a singularity at $z=1$, which becomes a simple pole in $d=4$ dimensions.

Now, we consider the analytic continuation of ${}_2F_1(a,b,c;z)$ to $|z|>1$. One proves that

$$\begin{aligned} {}_2F_1(a,b,c;z) &= c_1 z^{-a}\ {}_2F_1(a,a-c+1,a-b+1;\frac{1}{z})+ \\ & \quad c_2 z^{-b}\ {}_2F_1(b,b-c+1,b-a+1;\frac{1}{z}). \end{aligned}$$

The precise value of c_1 and c_2 is irrelevant for our purposes. It follows immediately that unless a and b are integers ${}_2F_1(a,b,c;z)$ has a branch cut from 1 to ∞. In the conformally coupled case, $\tilde{m}^2=\frac{d(d-2)}{4l^2}$ and $h=\frac{d}{2}$, so that $b=\frac{d-2}{2}$ which means that for even dimensions ${}_2F_1(a,b,c;z)$ has no branch point at $z=1$.

A.2 Singular behavior of the Wightman function in $\mathcal{M}^d$

The Wightman function in $\mathcal{M}^d$ is given by

$$\begin{aligned} G^+(x,y) &= \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{(2\pi)^{d-1}}\int d^dk\delta(k^2+m^2)\theta(k^0)e^{ik(x-y)} = \\ & \frac{1}{(2\pi)^{d-1}}\int d^{d-1}k\frac{e^{-ik^0(x^0-y^0)+i\vec{k}\cdot(\vec{x}-\vec{y})}}{2k^0}|_{k^0=\omega_{\vec{k}}}, \end{aligned}$$

where $\omega_{\vec{k}}=\sqrt{\vec{k}^2+m^2}$. Introduce spherical coordinates

$$d^{d-1}k = k^{d-2}dkd\Omega_{d-1},$$

where we define $k\equiv\sqrt{\vec{k}^2}$ and $r=\sqrt{(\vec{x}-\vec{y})^2}$. We orient the axes such that $\vec{k}\cdot(\vec{x}-\vec{y})=kr\cos\theta_{d-3}$ where θ_{d-3} is the angle for which $d\Omega_{d-1}=d\Omega_{d-2}\sin^{d-3}\theta_{d-3}d\theta_{d-3}$. Using

$$\int d\Omega_{d-2} = \frac{2\pi^{\frac{1}{2}(d-2)}}{\Gamma(\frac{d-2}{2})},$$

and

$$\int_0^\pi d\theta_{d-3}\sin^{d-3}\theta_{d-3}e^{ikr\cos\theta_{d-3}} = \int_{-1}^1 dt(1-t^2)^{(n-4)/2}e^{ikrt} =$$

$$\frac{\pi^{1/2}}{\Gamma(\nu+\frac{1}{2})}(\frac{2}{kr})^\nu J_\nu(kr),$$

where $t = \cos\theta_{d-3}$ and J_ν is the νth Bessel function with $\nu = \frac{d-3}{2}$, we obtain

$$G^+(x,y) = \sqrt{\frac{\pi}{2}} \frac{1}{(2\pi)^{d/2}} \frac{1}{r^\nu} \int_0^\infty \frac{dk}{k^0} k^{(d-1)/2} e^{-ik^0 t^0} J_\nu(kr),$$

where $t^0 = x^0 - y^0$. If $d =$ odd so that ν is an integer we use the formula

$$J_\nu(kr) = (\frac{r}{k})^\nu (-\frac{1}{r}\frac{d}{dr})^\nu J_0(kr).$$

When $d =$ even we write $\nu = l + 1/2$ with $l = (n-4)/2$ integer and work with spherical Bessel functions

$$J_{l+1/2}(kr) = \sqrt{\frac{2}{\pi}}(kr)^{1/2} j_l(kr) = \sqrt{\frac{2}{\pi}}(kr)^{1/2}(\frac{r}{k})^l(-\frac{1}{r}\frac{d}{dr})^l j_0(kr),$$

where $j_0(kr) = \frac{\sin kr}{kr}$. Since we do not expect any dependence of the final result on the even or oddness of the dimension we will continue the calculation for $d =$ even. We obtain for the Wightman function

$$G^+(x,y) = \frac{1}{(2\pi)^{d/2}}(-\frac{1}{r}\frac{d}{dr})^{(d-2)/2} \int_0^\infty \frac{dk}{k^0} e^{-ik^0 t^0} \cos kr.$$

Define the function, $f(t^0, r)$,

$$f(t^0, r) \equiv \int_0^\infty \frac{dk}{k^0} e^{-ik^0 t^0} \cos kr.$$

The mass shell condition $(k^0)^2 - k^2 = m^2$ with $k^0 > 0$ can be expressed by putting $k^0 = m\cosh\varphi$ and $k = m\sinh\varphi$ with $0 < \varphi < \infty$. Next, define $\lambda = (t^0)^2 - r^2$, so that we may set

$$\begin{aligned} t^0 &= \pm\sqrt{|\lambda|}\cosh\varphi_0 \\ r &= \sqrt{|\lambda|}\sinh\varphi_0. \end{aligned}$$

We distinguish four different cases:

1. $t^0 > 0$, $\lambda > 0$ with $t^0 = \sqrt{\lambda}\cosh\varphi_0$ and $r = \sqrt{\lambda}\sinh\varphi_0$
2. $t^0 > 0$, $\lambda < 0$ with $t^0 = \sqrt{-\lambda}\sinh\varphi_0$ and $r = \sqrt{-\lambda}\cosh\varphi_0$
3. $t^0 < 0$, $\lambda > 0$ with $t^0 = -\sqrt{\lambda}\cosh\varphi_0$ and $r = \sqrt{\lambda}\sinh\varphi_0$
4. $t^0 < 0$, $\lambda < 0$ with $t^0 = -\sqrt{-\lambda}\sinh\varphi_0$ and $r = \sqrt{-\lambda}\cosh\varphi_0$.

One then shows, using

$$\int_0^\infty d\phi e^{im\sqrt{\lambda}\cosh\phi} = \frac{\pi i}{2} H_0^{(1)}(m\sqrt{\lambda}),$$

where $H_0^{(1)}$ is the zeroth order Hankel function of the first kind, that

1. $f(t^0, r) = \frac{\pi}{2i} H_0^{(2)}(m\sqrt{\lambda})$
2. $f(t^0, r) = K_0(m\sqrt{-\lambda})$
3. $f(t^0, r) = \frac{\pi i}{2} H_0^{(1)}(m\sqrt{\lambda})$
4. $f(t^0, r) = K_0(m\sqrt{-\lambda})$,

where $H_0^{(2)}$ is the zeroth order Hankel function of the second kind, and

$$K_0(m\sqrt{-\lambda}) = \frac{\pi i}{2} H_0^{(1)}(mi\sqrt{\lambda}).$$

We work out the first case explicitly. Introduce $w = m\sqrt{\lambda}$. Then

$$-\frac{1}{r}\frac{d}{dr} = 2m^2 \frac{d}{dw^2},$$

so that

$$G^+(x, y) = \frac{\pi}{2i} \frac{1}{(2\pi)^{d/2}} (2m^2)^{(d-2)/2} (\frac{d}{dw^2})^{(d-2)/2} H_0^{(2)}(w).$$

Finally, employing the formula,

$$(\frac{d}{dw^2})^{(d-2)/2} H_0^{(2)}(w) = (-1)^{(d-2)/2} (\frac{1}{2})^{(d-2)/2} w^{-(d-2)/2} H_{\frac{d-2}{2}}^{(2)}(w),$$

we obtain for the case with $t^0 > 0$ and $\lambda > 0$:

$$G^+(x, y) = \frac{i(-1)^{d/2}\pi}{(4\pi)^{d/2}} (\frac{2m}{\sqrt{\lambda}})^{(d-2)/2} H_{\frac{d-2}{2}}^{(2)}(w).$$

Performing similar calculations for the other three cases we find in the end

$$G^+(x, y) = \frac{-i}{4\pi^{(d-2)/2}} \epsilon(t^0) \delta^{(d-4)/2}(\lambda) -$$

$$\frac{2(-1)^{d/2}}{(4\pi)^{d/2}} \theta(-\lambda) (\frac{2m}{\sqrt{-\lambda}})^{(d-2)/2} K_{\frac{d-2}{2}}(m\sqrt{-\lambda}) +$$

$$\frac{(-1)^{d/2}\pi}{(4\pi)^{d/2}} \theta(\lambda) (\frac{2m}{\sqrt{-\lambda}})^{(d-2)/2} (N_{\frac{d-2}{2}}(m\sqrt{\lambda}) + i \,\mathrm{sign}(t^0) J_{\frac{d-2}{2}}(m\sqrt{\lambda})),$$

where $\mathrm{sign}(t^0) = \theta(t^0) - \theta(-t^0)$. The delta function appears because the function $f(t^0, r)$ has a finite discontinuity at $\lambda = 0$. More precisely,

$$(\lim_{\lambda \to 0^+} - \lim_{\lambda \to 0^-}) f(t^0, r) = \frac{\pi}{2i}.$$

G^+ has four different singularities: a discontinuity, delta function and logarithmic singularities, and poles of various orders. The strongest singular behavior is exhibited by the highest pole

$$-\frac{1}{\pi}(\frac{2}{m\sqrt{\lambda}})^{(d-2)/2}(\frac{d-4}{2})!$$

which is contained in the Neumann function, $N_{\frac{d-2}{2}}(m\sqrt{\lambda})$. Then if we approach the light cone from the inside G^+ behaves like

$$G^+ \sim \frac{-1(-1^{d/2})}{\pi^{d/2}}\frac{\Gamma(\frac{d}{2})}{2(d-2)}(\frac{1}{\lambda})^{(d-2)/2}.$$

The geodesic distance, D, is related to λ by $D^2 = -\lambda$, so that we have the short-distance behavior of the Wightman function in d-dimensional Minkowski spacetime:

$$G^+ \sim \frac{\Gamma(\frac{d}{2})}{2(d-2)\pi^{d/2}}(\frac{1}{D})^{(d-2)}. \tag{A.2}$$

The regulating $i\epsilon$-prescription can be found from $t^0 \to t^0 - i\epsilon$. Then, to the same order in ϵ, $D \to D + i\tilde{\epsilon}$, for $t^0 > 0$ and $\lambda > 0$, that is, for x to the future of y, where $\tilde{\epsilon}$ is proportional to ϵ. Similarly, $D \to D - i\tilde{\epsilon}$ for x to the past of y.

A.3 Scalar, vector and tensor spherical harmonics on S^3

The perturbations $h_{\alpha\beta}$ describing changes of S^3 can be decomposed into irreducible representations (irreps) of the group of motion of S^3. These are continuous single-valued representations of the rotation group. There exists a standard method of obtaining them[1]. Consider the Laplace equation $\nabla^2\Psi = 0$. The Laplacian is invariant under the rotation group. Let Ψ be a homogeneous polynomial satisfying the Laplace equation, then due to the rotational invariance of ∇^2 and since rotations carry one homogeneous polynomial into another, the space of homogeneous polynomials of, say, degree $n-1$ forms a representation space for the rotation group. Let x^a, $a = 1, 2, 3, 4$, be cartesian coordinates for $\mathbb{R}^4$. A scalar spherical harmonic of degree $n-1$, $Q^{(n)}$, on S^3 can be defined by

$$r^{n-1}Q^{(n)} = A^{(n)}_{abc\cdots}x^a x^b x^c \cdots ,$$

where $A^{(n)}_{abc\cdots}$ is a certain (in cartesian coordinates) constant euclidean tensor of rank $n-1$ which is symmetric with respect to all its indices, and which

[1] This section is heavily based on [18], appendix J.

gives zero when contracted over any pair of indices. The latter property ensures that $r^{n-1}Q^{(n)}$ satisfies the Laplace equation. Thus

$$\nabla^2(r^{n-1}Q^{(n)}) = (\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^4)^2})A^{(n)}_{abc\cdots}x^a x^b x^c \cdots = 0.$$

In spherical coordinates the Laplacian acting on a any function f can be written as

$$\nabla^2 f = \frac{1}{r^2}\bar{\nabla}^2 f + \frac{1}{r^3}\frac{\partial}{\partial r}(r^3\frac{\partial f}{\partial r}),$$

where $\bar{\nabla}_\alpha$ is the covariant derivative of unit S^3. Then it follows from the definition of $Q^{(n)}$ that it satisfies

$$\bar{\nabla}^2 Q^{(n)} = -(n^2-1)Q^{(n)}.$$

Therefore the $Q^{(n)}$ are the scalar eigenfunctions of the Laplacian operator on S^3.

The vector spherical harmonics can be defined as follows. Let $B_{ab,cd\cdots}$ be a constant tensor of rank n ($n = 2, 3, \cdots$), antisymmetric in a and b, symmetric with respect to all other indices and satisfying the following two conditions: it gives zero when contracted over any pair of indices, and secondly, it gives zero when contracted with the Levi-Civita symbol ϵ^{abe}, where e is an arbitrary index not being a or b. Then the vector spherical harmonics, $S^{(n)}_a$, are defined as

$$r^n S_a = B_{ab,cd\cdots}x^b x^c x^d \cdots .$$

From now on the label n on S_a will be dropped. In $\mathbb{R}^4$ they form a vector perpendicular to the radius vector x^a since $B_{ab,cd\cdots}$ is antisymmetric in a and b. The components of S_a are homogeneous polynomials of degree $n-1$ lying on S^3 and satisfying the Laplace equation

$$\nabla^2(r^n S_a) = 0.$$

The second condition in the definition of $B_{ab,cd\cdots}$ ensures that the dual of $B_{ab,cd\cdots}$ does not exist, which if it would exist could be used, since the dual map is an isomorphism, to reduce the rank of tensor B. Transforming to spherical coordinates the vector $S_\alpha = \frac{\partial x^a}{\partial x^\alpha}S_a$ is obtained. Since S_a is proportional to r^{-1}, S_α does not depend on r and is thus a vector on S^3. It really is a 3-component vector for $S_r = \frac{\partial x^a}{\partial r}S_a = \frac{1}{r}x^a S_a = 0$. It is straightforward to show that in spherical coordinates the Laplacian acting on any vector defined on S^3 is

$$\nabla^2 f_\alpha = \frac{1}{r^2}\bar{\nabla}^2 f_\alpha + \frac{\partial^2 f_\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial f_\alpha}{\partial r} - \frac{2}{r^2}f_\alpha.$$

It then follows that the vector spherical harmonics satisfy the eigenvalue equation

$$\bar{\nabla}^2 S_\alpha = -(n^2-2)S_\alpha.$$

If we let ∂^a act on $(r^n S_a)$ we obtain

$$\partial^a(r^n S_a) = nr^{n-1}\frac{\partial r}{\partial x_a}S_a + r^n\partial^a S_a = nr^{n-2}x^a S_a + r^n \partial^a S_a = 0,$$

so that $\partial^a S_a = 0$. This is the differential analogue of $S^a x_a = 0$. In spherical coordinates this becomes

$$\bar{\nabla}^\alpha S_\alpha = 0.$$

The vector spherical harmonics are divergenceless.

A straightforward generalization of the definition of the vector spherical harmonics leads to a definition of the tensor spherical harmonics of second rank. These tensors will be denoted by H_{ab}. In order for these not to reduce to either scalar or vector spherical harmonics we must demand that $H^a{}_a = 0 = H_{ab}x^a x^b$ and $H_{ab}x^b = 0$. Let $C_{ac,bd,ef\cdots}$ be a constant tensor of rank $n+1$ $(n = 3, 4, \cdots)$ which is antisymmetric in (a, c) and (b, d), symmetric with respect to the interchange of (a, c) and (b, d) (so that $H_{ab} = H_{ba}$), gives zero when contracted over any pair of indices and gives zero when one takes a cyclic sum over the pair (a, c) with any other index or the pair (b, d) with any other index. Then the tensor spherical harmonics are defined by

$$r^{n-1}H_{ab} = C_{ac,bd,ef\cdots}x^c x^d x^e x^f \cdots .$$

Then in spherical coordinates $H_\alpha{}^\beta = \frac{\partial x^a}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^b}H_a{}^b$ does not depend on r. It can be shown that they satisfy the eigenvalue equation

$$\bar{\nabla}^2 H_\alpha{}^\beta = -(n^2-3)H_\alpha{}^\beta.$$

Finally, it can be proven that $\bar{\nabla}_\beta H_\alpha{}^\beta = 0$.

Using the scalar harmonics the following second rank tensors may be constructed

$$\begin{aligned} Q_\alpha{}^\beta &= \frac{1}{3}\delta_\alpha^\beta Q \\ P_\alpha{}^\beta &= \frac{1}{n^2-1}\bar{\nabla}_\alpha\bar{\nabla}^\beta Q + Q_\alpha{}^\beta, \end{aligned}$$

which are defined so that $Q_\alpha{}^\alpha = Q$ and $P_\alpha{}^\alpha = 0$. Using the vector harmonics the tensor $S_\alpha{}^\beta$ defined by

$$S_\alpha{}^\beta = \bar{\nabla}^\beta S_\alpha + \bar{\nabla}_\alpha S^\beta$$

may be constructed. From it no scalar can be constructed because $S_\alpha{}^\alpha = 0$. In terms of these the most general perturbation $h_{\alpha\beta}$ of S^3 is

$$h_{\alpha\beta} = \lambda(\eta)P_{\alpha\beta} + \mu(\eta)Q_{\alpha\beta} + \sigma(\eta)S_{\alpha\beta} + \nu(\eta)H_{\alpha\beta}. \tag{A.3}$$

Appendix B

Gravitational energy of stationary spacetimes

An asymptotically flat space represents an ideally isolated gravitational system. A coordinate dependent designation of an asymptotically flat space can be made as follows: a space is called asymptotically flat if there exists any coordinate system with spatial coordinates, $(x^1, \cdots, x^{d-1})$, such that the metric reads $g_{ab} = \eta_{ab} + o(\frac{1}{r})$ as $r \to \infty$, where $r^2 = (x^1)^2 + \cdots + (x^{d-1})^2$, along either spatial or null directions.

Let us consider a static asymptotically flat space, that is, an asymptotically flat space which possesses a hypersurface orthogonal timelike Killing vector field ξ^a. It is assumed that $V \equiv (-\xi_a \xi^a)^{1/2}$ approaches unity at spatial infinity. In such a spacetime there exists a notion of a rest frame, namely an orbit of ξ^a. Let $\gamma(\tau)$ be such an orbit, where τ is the proper time. Then in the tangent space to the curve γ the tangent vector ξ^a is defined through $\frac{d\gamma}{d\tau} = \xi^a \partial_a$. The unit tangent to this curve is ξ^a/V. Hence, the acceleration of a test particle moving along γ with respect to the proper time is given by

$$\frac{\xi^b}{V}\nabla_b \frac{\xi^a}{V} = \frac{1}{V^2}\xi^b \nabla_b \xi^a.$$

The equality follows from the fact that V is constant along γ, that is, $\xi^b \nabla_b V = 0$. Now, the proper time parameter τ satisfies $\xi^b \nabla_b \tau = 1$ or equivalently $\frac{\partial}{\partial \tau} = \xi^b \nabla_b$. So, transforming to coordinate time t, say, using

$$\frac{\partial}{\partial t} = \sqrt{g_{00}}\frac{\partial}{\partial \tau} = V\frac{\partial}{\partial \tau},$$

which holds because ξ^a is hypersurface orthogonal, it follows that the acceleration in coordinate time is

$$\frac{1}{V}\xi^b \nabla_b \xi^a.$$

Consider the spacelike hypersurface, Σ, to which ξ^a is orthogonal. Let $S \subset \Sigma$ be a closed (d-2)-surface whose unit normal n^b lies in Σ and satisfies $n^a \xi_a = 1$. Then the following quantity

$$F \equiv \oint_S n^b \frac{\xi^a}{V} \nabla_a \xi_b dS,$$

represents the total force exerted by a distant observer to keep in place a uniform distribution over S of unit mass test particles. Introduce

$$N^{ab} = V^{-1} n^{[b} \xi^{a]}$$

the normal bivector to S; $\nabla_a \xi_b$ is antisymmetric in a and b. Then F becomes

$$F = \oint_S \nabla^a \xi^b dS_{ab},$$

where $dS_{ab} = N_{ab} dS = \frac{1}{(d-2)!} \epsilon_{r_1 \cdots r_{d-2} ab} dx^{r_1} \wedge \cdots \wedge dx^{r_{d-2}}$. One proves [25] that this integral is independent of the choice of closed $(d-2)$-surface. Therefore if we divide by $\mathrm{Vol}(S^{d-2})$, the volume of a unit $d-2$-sphere, we obtain the mass, M, contained in the spacetime region bounded by S. If the integration surface S is at infinity then it gives the total mass of an asymptotically flat static spacetime. The independence of the integral on the choice of S depends only on the fact that ξ^a is a global timelike Killing vector field. So the expression

$$M = \frac{1}{\mathrm{Vol}(S^{d-2})} \oint_S \nabla^a \xi^b dS_{ab} \tag{B.1}$$

is well-formed for any stationary spacetime. The expression is due to Komar.

It is noted that when the surface S forms the boundary of a spacelike hypersurface Σ such that the union $\Sigma \cup S$ is a compact manifold with boundary then the total mass M can be written as

$$M = \frac{1}{\mathrm{Vol}(S^{d-2})} \int_\Sigma R_{ab} n^a \xi^b d\Sigma = \frac{1}{\mathrm{Vol}(S^{d-2})} \int_\Sigma (-g)^{1/2} R_{0b} \xi^b d^{d-1} x,$$

where n^a is the unit future pointing normal to Σ at spatial infinity and $d\Sigma$ the volume element on Σ.

Appendix C

Einstein equation for fluctuations on dS_4 in the synchronous gauge

In this appendix the change in the Einstein tensor will be calculated due to the fluctuations h_{ab}. This change is given by

$$\delta G_{ab} = G^{(1)}_{ab}(\bar{g}+h) - \bar{G}_{ab},$$

where $G^{(1)}_{ab}(\bar{g}+h)$ is the Einstein tensor of the full metric $g_{ab} = \bar{g}_{ab} + h_{ab}$ calculated to first order in h_{ab} and where $\bar{G}_{ab}$ is the Einstein tensor of the background de Sitter space.

The connection coefficients Γ^c_{ab} associated with g_{ab} up to first order in h_{ab} are

$$\Gamma^c_{ab} = \bar{\Gamma}^c_{ab} + \frac{1}{2}\bar{g}^{cd}(\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}) - \frac{1}{2}h^{cd}(\partial_a \bar{g}_{bd} + \partial_b \bar{g}_{ad} - \partial_d \bar{g}_{ab}).$$

In the synchronous gauge, $h_{00} = 0 = h_{\alpha 0}$, the following components of Γ^c_{ab} can be differentiated

$$\begin{aligned}
\Gamma^\gamma_{\alpha\beta} &= \bar{\Gamma}^\gamma_{\alpha\beta} + \frac{1}{2}\bar{g}^{\gamma\delta}(\partial_\alpha h_{\beta\delta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta}) - \\
&\quad \frac{1}{2}h^{\gamma\delta}(\partial_\alpha \bar{g}_{\beta\delta} + \partial_\beta \bar{g}_{\alpha\delta} - \partial_\delta \bar{g}_{\alpha\beta}) \\
\Gamma^\gamma_{\alpha 0} &= \bar{\Gamma}^\gamma_{\alpha 0} + \frac{1}{2}{h^\gamma{}_\alpha}' \\
\Gamma^\gamma_{00} &= 0 \\
\Gamma^0_{\alpha\beta} &= \bar{\Gamma}^0_{\alpha\beta} + \frac{1}{2}a^{-2}h'_{\alpha\beta} \\
\Gamma^0_{\alpha 0} &= 0 \\
\Gamma^0_{00} &= \bar{\Gamma}^0_{00}.
\end{aligned}$$

The inequivalent components of the Riemann tensor are $R^0{}_{\beta 0\delta}$, $R^0{}_{\beta\sigma\delta}$ and $R^\alpha{}_{\beta\sigma\delta}$. To first order in h_{ab} they are given by

$$\begin{aligned}
R^0{}_{\beta 0\delta} &= \bar{R}^0{}_{\beta 0\delta} + \frac{1}{2}a^{-2}h''_{\beta\delta} - \frac{3}{2}\frac{a'}{a^3}h'_{\beta\delta} + \frac{(a')^2}{a^4}h_{\beta\delta} \\
R^0{}_{\beta\sigma\delta} &= \bar{R}^0{}_{\beta\sigma\delta} + \frac{1}{2a^2}(\bar{\nabla}_\sigma h'_{\beta\delta} - \bar{\nabla}_\delta h'_{\beta\sigma}) + \frac{a'}{a^3}(\bar{\nabla}_\delta h_{\beta\sigma} - \bar{\nabla}_\sigma h_{\beta\delta}) \\
R^\alpha{}_{\beta\sigma\delta} &= \bar{R}^\alpha{}_{\beta\sigma\delta} + \frac{1}{2}\frac{a'}{a^3}\delta^\alpha_\sigma h'_{\beta\delta} - \frac{1}{2}\frac{a'}{a^3}\delta^\alpha_\delta h'_{\beta\sigma} + \frac{1}{2}\frac{a'}{a}\gamma_{\beta\delta}h^\alpha{}_\sigma{}' - \frac{1}{2}\frac{a'}{a}\gamma_{\beta\sigma}h^\alpha{}_\delta{}' + \\
&\quad \frac{1}{2}\bar{g}^{\alpha\tau}[\bar{\nabla}_\sigma\bar{\nabla}_\beta h_{\delta\tau} - \bar{\nabla}_\delta\bar{\nabla}_\beta h_{\sigma\tau} - \bar{\nabla}_\sigma\bar{\nabla}_\tau h_{\beta\delta} + \bar{\nabla}_\delta\bar{\nabla}_\tau h_{\beta\sigma} + [\bar{\nabla}_\sigma, \bar{\nabla}_\delta]h_{\beta\tau}],
\end{aligned}$$

where $\bar{\nabla}_\alpha$ is the covariant derivative of unit S^3, so that $\nabla^\alpha = \gamma^{\alpha\beta}\nabla_\beta$ and $\bar{R}^0{}_{\beta\sigma\delta} = 0$. Then the change in the Ricci tensor δR^a_b has the components

$$\begin{aligned}
\delta R^0{}_0 &= \bar{g}^{\beta\delta}R^0{}_{\beta 0\delta} - h^{\beta\delta}\bar{R}^0{}_{\beta 0\delta} - \bar{R}^0{}_0 = \frac{1}{2a^2}(h'' + \frac{a'}{a}h') \\
\delta R^0{}_\sigma &= \bar{g}^{\beta\delta}R^0{}_{\beta\sigma\delta} = \frac{1}{2a^2}(\bar{\nabla}_\sigma h' - \bar{\nabla}_\delta h^\delta{}_\sigma{}') \\
\delta R^\alpha{}_\sigma &= \bar{g}^{\beta\delta}R^\alpha{}_{\beta\sigma\delta} - h^{\beta\delta}\bar{R}^\alpha{}_{\beta\sigma\delta} + \bar{g}^{00}R^\alpha{}_{0\sigma 0} - \bar{R}^\alpha{}_\sigma = \\
&\quad \frac{a'}{a^3}h^\alpha{}_\sigma{}' + \frac{1}{2a^2}h^\alpha{}_\sigma{}'' - \frac{2}{a^2}h^\alpha{}_\sigma + \frac{a'h'}{2a^3}\delta^\alpha_\sigma + \\
&\quad \frac{1}{2a^2}[\bar{\nabla}_\gamma\bar{\nabla}^\alpha h^\gamma{}_\sigma + \bar{\nabla}^\gamma\bar{\nabla}_\sigma h^\alpha{}_\gamma - \bar{\nabla}^\alpha\bar{\nabla}_\sigma h - \Box h^\alpha{}_\sigma],
\end{aligned}$$

where $h = h^\gamma{}_\gamma$ is the trace of h_{ab}. From this it follows that the change in the Ricci scalar is

$$\delta R = \delta R^0{}_0 + \delta R^\gamma{}_\gamma = \frac{1}{a^2}h'' + \frac{3a'h'}{a^3} - \frac{2}{a^2}h + \frac{1}{a^2}(\bar{\nabla}^\gamma\bar{\nabla}_\alpha h^\alpha{}_\gamma - \Box h).$$

Then finally the change in the Einstein tensor becomes

$$\delta G^0{}_0 = \delta R^0{}_0 - \frac{1}{2}\delta R = \frac{h}{a^2} - \frac{a'h'}{a^3} - \frac{1}{2a^2}(\bar{\nabla}^\gamma\bar{\nabla}_\delta h^\delta{}_\gamma - \Box h) \qquad \text{(C.1)}$$

$$\delta G^0{}_\sigma = \delta R^0{}_\sigma = \frac{1}{2a^2}(\bar{\nabla}_\sigma h' - \bar{\nabla}_\gamma h^\gamma{}_\sigma{}') \qquad \text{(C.2)}$$

$$\begin{aligned}
\delta G^\alpha{}_\sigma &= \delta R^\alpha{}_\sigma - \frac{1}{2}\delta^\alpha_\sigma\delta R = \frac{1}{2a^2}[\frac{2a'}{a}h^\alpha{}_\sigma{}' + h^\alpha{}_\sigma{}'' - 4h^\alpha{}_\sigma + \\
&\quad \delta^\alpha_\sigma(\Box h - \bar{\nabla}^\gamma\bar{\nabla}_\delta h^\delta{}_\gamma + 2h - h'' - \frac{2a'h'}{a}) + \\
&\quad \bar{\nabla}_\gamma\bar{\nabla}^\alpha h^\gamma{}_\sigma + \bar{\nabla}^\gamma\bar{\nabla}_\sigma h^\alpha{}_\gamma - \bar{\nabla}^\alpha\bar{\nabla}_\sigma h - \Box h^\alpha{}_\sigma]. \qquad \text{(C.3)}
\end{aligned}$$

Bibliography

[1] L. F. Abbott and S. Deser. Stability of gravity with a cosmological constant. *Nucl. Phys.*, B195:76–96, 1982.

[2] B. Allen. Vacuum states in de sitter space. *Phys. Rev. D*, 32(12):3136–3149, December 1985.

[3] A. Ashtekar and R. O. Hansen. A unified treatment of null and spatial infinity in general relativity. i. universal structure, asymptotic symmetries and conserved quantities at spatial infinity. *J. Math. Phys.*, 19:1542–1566, 1978.

[4] N. D. Birrel and P.C.W. Davies. *Quantum fields in curved space*. Cambridge University Press, 1982.

[5] B. Carter. *Black Holes, les Houches 1972*, chapter Black Hole Equilibrium States, pages 59–214. Gordon and Breach Science Publisher, Inc., 1973.

[6] M. J. Perry D. J. Gross and L. G. Yaffe. Instability of flat space at finite temperature. *Phys. Rev. D*, 25(2):330–355, 1982.

[7] F. de Felice and C. J. S. Clarke. *Relativity on curved manifolds*. Cambridge University Press, 1990.

[8] R. d'Inverno. *Introducing Einstein's Relativity*. Oxford University Press, 1992.

[9] J. Dixmier. Représentations intégrables du groupe de de sitter. *Bull. Soc. Math. France*, 89:9–41, 1961.

[10] M. B. Einhorn and F. Larsen. Interacting quantum field theory in de sitter vacua. *arXiv:hep-th/0209159*, 2002.

[11] A. G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.*, 116:1009, 1998.

[12] S.W. Hawking G. W. Gibbons and M. J. Perry. Path integrals and the indefiniteness of the gravitational action. *Nucl. Phys.*, B138:141–150, 1978.

[13] T. Garidi. What is mass in desitterian physics? *arXiv:hep-th/0309104*, 2003.

[14] P. Ginsparg and M. J. Perry. Semiclassical perdurance of de sitter space. *Nucl. Phys.*, B222:245–268, 1983.

[15] S. W. Hawking. Black holes and thermodynamics. *Phys. Rev. D*, 13(2):191–197, 1976.

[16] S. W. Hawking and G. W. Gibbons. Cosmological event horizons, thermodynamics, and particle creation. *Phys. Rev. D*, 15:2738–2751, 1977.

[17] B. Carter J. M. Bardeen and S. W. Hawking. The four laws of black hole mechanics. *Commun. Math. Phys.*, 31:161, 1973.

[18] E. M. Lifshitz and I. M. Khalatnikov. Relativistic cosmology. *Adv. in Phys.*, 12:185, 1963.

[19] A. Strominger M. Spradlin and A. Volovich. Les houches lectures on de sitter space. *arXiv:hep-th/0110007*, 2001.

[20] R. Penrose. *Relativity, Groups and Topology*, chapter Conformal Treatment of Infinity, pages 565–584. Blackie and Son Limited, 1964.

[21] A. Maloney R. Bousso and A. Strominger. Conformal vacua and entropy in de sitter space. *Phys. Rev. D*, 65:104039, 2002.

[22] E. Huguet T. Garidi and J. Renaud. de sitter waves and the zero curvature limit. *Phys. Rev. D*, 67:124028, 2003.

[23] W. Unruh. Notes on black-hole evaporation. *Phys. Rev. D*, 14:870–892, 1976.

[24] J. de Boer V. Balasubramanian and D. Minic. Mass, entropy and holography in asymptotically de sitter spaces. *arXiv:hep-th/0110108*, 2002.

[25] R. Wald. *General Relativity.* University of Chicago Press, 1984.

[26] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* Wiley, New York, 1972.

[27] E. Witten. A new proof of the positive energy theorem. *Commun. Math. Phys.*, 80:381–402, 1981.

[28] C. Y. Oh Y. Kim and N. Park. Classical geometry of de sitter spacetime: An introductory review. *arXiv:hep-th/0212326*, 2002.