Gerbes & M-theory

Jetze Sikkema

November 8, 2004

Intern Rapportnummer: 338

Contents

Introduction

O Crassum Ingenium!
Susplicor Fuisse Batavum
- Erasmus -

Five-branes are mysterious objects. They are in essence non-perturbative and should contain a non-perturbative string theory on there world volumes. Conformal field theory hints that their worldvolumes contains a theory with conformal non-abelian two-forms. This is only the beginning since quantization using the anti-field formalism shows that these non-abelian two-forms are not the deformation of an abelian theory. Anomaly cancellation further shows that the theory contains a number of degrees of freedom which can't come from a Lie-group. So we suspect we have to change the structure of a group somehow. To make this possible we first have to think about the notion of a gauge field. As is probably well know gauge fields are described by by a split of a tangent bundle into a horizontal and vertical part. Or more intuitive if you displace a point a gauge field tells you what happens to a corresponding point in the gauge group. This description still contains to much non-relevant material. If we abstract gauge fields to absolutely as little as possible we end up with torsors. Torsors consist only of open sets and groups. Since we wanted to extend the structure we should make sheaves with something more general than a group. The new structure we want to introduce is called a groupoid, it is a group with a freedom in the multiplication. The torsor analogue of groupoids is called a gerbe. We can construct gauge fields and connections on a gerbe. To do physics we need the curvatures of the gauge fields. These can be calculated in two ways, we can either use methods from algebraic geometry, or as I demonstrate in this thesis, we can make an infinitesimal expansion around a closed loop. Having the gauge fields with corresponding curvatures we might try to find gauge transformations and try to obtain corresponding invariant forms that can be integrated to an action.

Though the thesis focusses on the application of gerbes in the construction of an action for several coincident M-5 branes it can also be read as a introduction to the literature of gerbes. Gerbes live in mathematics independent from the physical application to 2-form gauge fields, but physicists tend to make mistakes with gerbes easily. To cite the most common, for a 2-form gauge field we need just a gerbe and not a 2-gerbe. A connection is always a one-form even on a gerbe. The mathematical structure of a gerbe is a groupoid not a Lie 2-group,

moreover the theory of gerbes can be developed (almost)* entirely without the use of 2-groups and 2-categories . From these examples the need of a thorough introduction can be deduced, however the thorough introductions to gerbes that I know of require a good knowledge of algebraic geometry and algebraic topology (see for these introductions in order of toughness [55], [56], [18]), these articles all don't describe connections on gerbes and are novels compared to the article that does describe connections [17]. So I hope to give an introduction to gerbes and connections on gerbes readable for physicists. Understanding gerbes makes it possible to define the connection and the curvature as an expansion. It is not necessary to know all the definitions since they contain many highly technical points, it is necessary to understand them, look through them to know what they try to tell us about gerbes. Only after the theory of gerbes has been fully developed it is possible to apply them to physics without getting lost.
There are three ways to construct gerbes, principal bundles, exact sequences and sheaves. The emphasis will however be on the sheaf approach, though this is probably the most abstract it's also conceptually the clearest.
Since this thesis covers many areas from physics and mathematics, I necessarily had to make some serious omissions due to lack of space or for readability. For mathematics these omissions include include schemes, which can be found in the EGA [37], Grothendiek Topologies and Sites, which can be found in [52] chapter 3 or [2] and simplicial objects (And a treatise on loop spaces), which can be found in [81] and [36]. For physics I had to omit a thorough treatise on string theory (see for example [28], [45] or [63]) and it's compactifications [38], a thorough treatement of supergravity see [83] and a thorough intrduction M theory see amongst others [74].

The first chapter is an introduction to string theory and membranes. The second chapter will be introductory on principal bundles, with the last section devoted to principal bundles as an extension of lie groups. The third chapter is an introductory chapter on sheaves. In the fourth chapter non-perturbative objects called D-branes will be introduced. The fifth chapter is devoted to supergravity and how these theories hint at M-theory. In the sixth chapter the various aspects of constructing an action of a single M-5 brane are treated. The seventh chapter is highly technical and is devoted to the construction of a special sheaf of categories, called a stack. The eight chapter deals with the elementary theory of groupoids. Gerbes are not introduced until chapter nine, where also connections on gerbes are defined. The two other approaches i.e. groupoid extensions and taking products of principal bundles are discussed and lead to respectively bundle gerbes and circle bundle gerbes. Finally in the tenth chapter gauge transformations are found that leave the curvatures invariant. Then it'll be shown how they prohibit the construction of an invariant with a degree equal to the dimension of space-time. Finally ways to avoid this problem will mentioned. The appendices are very, very brief reviews of category theory and homological algebra. A thorough treatment of these subjects would

*Two categories are only needed for the definition of a stack, which has nothing to do with the gauge structure of the gerbe.

require many more pages. I don't give much examples, they can be found in the references cited.

Note on Notation

I have to apologize for the notation. Since this article combines several fields with different fundamental objects the notation is chapter dependent. Upper and lower indices for example refer to differential forms when the chapter deals with physics. Unfortunately the same notation turns out to be extremely useful to denote the open sets a sheaf depends on, thus in chapters on mathematics the indices will indicate the open sets instead of the differential form basis (To make matters even worse in Čech cohomology the basis for the differential forms are the open sets and the two notations denote the same thing). Furthermore it should be noted that a connection is denoted by A or μ, the same notation is used for a 1-form gauge field. A two form gauge field will be denoted by B. Curvatures are denoted by F except for two form curvatures which will also be denoted ν and three form curvatures which will be also denoted by ω.

Chapter 1

Perturbative String Theory

String theory is probably best introduced by generalizing the action of a point particle to worldsheets (Though it it's also possible to start by introducing cohomological field theories, see [28] chapter 8 and [46]).

There are lots of excellent text introducing string theory [63], [78] [45]. Given a spinless free point particle it's action is proportional to the length of the worldline:

$$S = -mc \int \sqrt{-\mathrm{d}x_\mu \mathrm{d}x^\mu} = -mc \int \mathrm{d}\tau \sqrt{-(\dot{x}(\tau))^2} \tag{1.1}$$

This action has two symmetries.

1. It is invariant under reparametrizations of the parameter on the worldline τ.

2. The action is (manifestly) invariant under the action of the Poincaré group.

This action however fails in the case of a massless particle. If we however introduce an additional variablie $a(\tau)$

$$S = \int \mathrm{d}\tau \frac{1}{2}[-\frac{\dot{x}^2}{a} + m^2c^2a] \tag{1.2}$$

The equation of motion for a is

$$\dot{x}^2 = -m^2c^2a^2 \tag{1.3}$$

A transformation of the worldline parameter induces the following transformations of the fields:

$$\dot{x}^\mu(\tau) \to \dot{f}(\tau)\frac{\mathrm{d}x^\mu(f(\tau))}{\mathrm{d}f(\tau)} \quad \& \quad a(\tau) \to \dot{f}(\tau)a(f(\tau)) \tag{1.4}$$

Equation 1.2 is also clearly manifestly invariant under the action of the Poincaré group. Moreover an easy calculation shows that the first action can be deduced from the second.

1.1 Worldsheets

Consider a worldsheet. Now the natural form on the worldvolume is the area element

$$\sigma_{\mu\nu} = \mathrm{d}x_\mu \wedge x_\nu \tag{1.5}$$

If the area element is nowhere vanishing, i.e. we are working on a smooth manifold with a symplectic structure. The action is now of course the total area of the string:

$$S = \aleph \int \sqrt{-\sigma_{\mu\nu}\sigma^{\mu\nu}} \tag{1.6}$$

Written down in a specific coordinate system on the worldsheet:

$$S = -cT \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-\det(g_{\alpha\beta})} \tag{1.7}$$

Variation and Neumann boundary conditions ($x^\mu(\tau, 0) = x^\mu(\tau, \pi)$) yield the equations of motion (where a dot indicates a derivative with respect to the first component and a prime a derivative with respect to the second component):

$$\ddot{x}_\mu - x''_\mu = 0 \tag{1.8}$$

Again we can introduce an auxiliary metric a. Such that

$$S = -\int \frac{T}{2} \mathrm{d}^2\sigma \sqrt{-a} a^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{1.9}$$

Since the number of symmetries equals the number of independent components of the metric. String theory is conformal. How to continue with quantization etc. can be found in [63] , [45] or any other book on string theory.

M-theory is a theory containing membranes, however this doesn't guarantee it is a membrane theory. For the moment let's assume it is a membrane theory and see how far we can get the naive way, see also [72]. Again the action can be written with auxiliary metric a in the form:

$$S = -\frac{T}{2} \int \mathrm{d}^3\sigma \sqrt{-a} (a^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - 1) \tag{1.10}$$

The equations of motion are

$$a^{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \quad \& \quad \partial_\alpha (\sqrt{a} a^{\alpha\beta} \partial_\beta X^\mu) = 0 \tag{1.11}$$

For respectively varying a and X^μ.
We again have three symmetries, unfortunately there are six independent components in the metric. A convenient choice is to use the symmetries to make a of the form:

$$a_{0\alpha} = 0 \quad \& \quad a_{00} = -\frac{4}{\kappa^2} \det(\partial_\alpha X_\mu \partial_\beta X^\mu) \stackrel{not.}{=} -\frac{4}{\kappa^2} h \tag{1.12}$$

Where κ is an arbitrary constant.
We can eliminate a:

$$S = -\frac{T\kappa}{4} \int d^3\sigma(\partial_0 X_\mu \partial_0 X^\mu - \frac{4}{\kappa^2} h) \tag{1.13}$$

Introduce a poisson bracket at equal $\tau : \{f, g\} = \epsilon^{\alpha\beta}\partial_\alpha f \partial_\beta g$.
The membrane action now becomes

$$S = \aleph \int d^3\sigma(\partial_0 X_\mu \partial_0 X^\mu - \frac{2}{\kappa^2}\{X_\mu, X_\nu\}\{X^\mu, X^\nu\}) \tag{1.14}$$

With $\aleph$ a constant depending on the string tension, κ and the normalization of the bracket.
The equations of motion are now:

$$\partial_0 \partial^0 X_\mu = \frac{4}{\kappa^2}\{\{X_\mu, X_\nu\}, X^\nu\} \tag{1.15}$$

With auxiliary constraints:

$$\partial_0 X_\mu \partial_0 X^\mu = \frac{2}{\kappa^2}\{X_\mu, X_\nu\}\{X^\mu, X^\nu\} \tag{1.16}$$

$$\partial_0 X_\mu \partial^\alpha X^\mu = 0 \tag{1.17}$$

This system of equations is difficult to quantize. The straightforward way is to rewrite this action using the Nambu bracket and quantizing the Nambu bracket, see for example [7]. A more clever solution is so called matrix regularization.
These equations can be quantized for $N \times N$ matrices. However to make contact with the original theory we have to sent N in a well defined way to infinity. This limit still has to be defined properly. This matrix regularization and the large N limits are discussed in paper [72] The full matrix lagrangian for a $U(N)$ supersymmetric theory is:

$$L = \mathrm{tr}\Big[\frac{1}{R} D_t X_i D_t X^i - \bar{\theta}\gamma_- D_t\theta - R\bar{\theta}\gamma_-\gamma_i[\theta, X^i] - \frac{1}{4}[X^i, X^j]^2\Big] \tag{1.18}$$

In Matrix theory also contains just a M-2 and a M-5 brane. For the five-brane an action is proposed however since it's not instructive to reproduce it here since non of the questions of the M-5 brane are addressed for the matrix five-brane the reader is referred to [13] for the lagrangian with discussion.

1.2 Mode Decomposition

Given the Nambu-Goto action

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{-a} a^{ab} \partial_a X^\mu \partial_b X^\nu \tag{1.19}$$

First gauge the auxiliary metric to the lorentz metric. The equations of motion now become

$$\Big(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\Big) X^\mu(\tau, \sigma) = 0 \tag{1.20}$$

We can either have an closed string with boundary conditions:

$$X^{\mu}(\tau, 0) = X^{\mu}(\tau, \pi) \tag{1.21}$$

Or we can have an open string with have to impose either Neumann boundary conditions

$$X'^{\mu}(\tau, 0) = 0, \quad X'^{\mu}(\tau, \pi) = 0 \tag{1.22}$$

Or Dirichlet boundary conditions

$$X'^{\mu}(\tau, 0) = X'^{\mu}(\tau, \pi) \tag{1.23}$$

If we introduce lightcone coordinates

$$\sigma^{\pm} = \tau \pm \sigma \tag{1.24}$$

the equation of motion becomes

$$\partial_{\sigma^+}\partial_{\sigma^-}X = 0 \tag{1.25}$$

In other words X can be written as the sum of a holomorphic and a anti-holomorphic part. Since the equation is a wave equation the solutions are given by an expansion in complex exponentials

$$X^{\mu} = x^{\mu} + 2\alpha' p^{\mu}\tau + i(2\alpha')^{\frac{1}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{\mu} e^{-in\tau} \cos n\sigma \tag{1.26}$$

For an open string and a similar result for the closed string

The easiest way to do quantization is by choosing the lightcone gauge and to make operators of the oscillators with appropriate commutation relations. The same states are obtained by BRST quantization, the interested reader can find the BRST quantization in for example [63].

Chan-Paton Factors

Without spoiling poincaré invariance or conformal invariance we can add an addition symmetry group to the endpoints of an open string. Moreover we demand that the hamilonian is zero so they are non dynamical and that there are N different states in the group. This restriction implies that after summing over the possible states we are left with the trace of the product of these so called Chan-Paton factors. The endpoints are now invariant under an additional $U(N)$ symmetry, since the labeling of the states the endpoints are in can be mixed, see [45].

1.3 Supersymmetry

Supersymmetry is the only non-trivial extension of the Poincaré group. It is the only exception to the so called Coleman-Mandula theorem [21]:

1.3.1 Theorem *(Coleman-Mandula) Given any Lie group containing the Poincaré group*

$$[P_a, P_b] = 0 \tag{1.27}$$
$$[P_a, J_{bc}] = \eta_{a[b}P_{c]} \tag{1.28}$$
$$[J_{ab}, J_{cd}] = -(\eta_{[ac}J_{bd]}) \tag{1.29}$$

and containing an internal symmetry group G generated by T_α such that

$$[T_\alpha, T_\beta] = f_{\alpha\beta\gamma}T_\gamma \tag{1.30}$$

must be a direct product of the poincaré group and G, i.e.

$$[P_a, T_\alpha] = 0 = [J_{ab}, T_\alpha] \tag{1.31}$$

A more extensive treatment can be found in the excellent text on supersymmetry by Argyres [4]. I wouldn't have mentioned the theorem unless there's some exception. The unique exception was found by Gelfand and Likhtman see [33], which is not the Gelfand from the Gelfand-Naimark theorem of non-commutative geometry. In this case we can extend the Poincaré algebra in exactly one non-trivial way. To make this extension we have to embed the Poincaré algebra in a $\mathbb{Z}_2$ graded algebra, i.e. an algebra that satisfies

$$[even, even] = even \tag{1.32}$$
$$\{odd, odd\} = even \tag{1.33}$$
$$[even, odd] = odd \tag{1.34}$$

In other words add to the Poincaré algebra the generators Q^i_α, they are called supercharges and satisfy

$$\{Q^i_\alpha, Q^j_\beta\} = \text{Some other generator} \tag{1.35}$$

For example if we have only one supercharge we obtain the so called $N = 1$ superalgebra, which is given by

$$\{Q_\alpha, Q_\beta\} = 2(\gamma_a C)_{\alpha\beta}P^a \tag{1.36}$$
$$[Q_\alpha, P_a] = 0 \tag{1.37}$$
$$[Q_\alpha, J_{cd}] = \frac{1}{2}\sigma^{\beta}_{cd\alpha}Q_\beta \tag{1.38}$$
$$[Q_\alpha, R] = i\gamma^\beta_\alpha Q_\beta \tag{1.39}$$

In these equations C is the charge conjugation matrix and R is an internal symmetry.

1.4 Super Strings

In order to describe fermions in stringtheory supersymmetry is exactly the extension we need. This leads to superstrings consistent only in ten space-time dimensions, whose actions I'll try to treat very briefly in this section. We can make

the extension manifest in two different ways. Either we make the supersymmetry manifest on the worldsheet, this is the so called Neveu-Schwarz-Ramond (NSR) string or we can make the supersymmetry manifest in the targetspace, this is the so called Green-Schwarz (GS) string. I'll try to introduce the superstring using the latter approach since the structure of the supersymmetry algebra is in this way more clear.
Given the algebras without central charges

$$\{Q_\alpha, Q_\beta\} = \begin{cases} (C\Gamma^\mu(1+\Gamma))_{\alpha\beta}P_\mu & N=1 \\ (C\Gamma^\mu)_{\alpha\beta}P_\mu & IIA \\ \delta^{ij}(C\Gamma^\mu(1+\Gamma))_{\alpha\beta}P_\mu & IIB \end{cases} \tag{1.40}$$

Introduce for the IIA, IIB and heterotic superalgebras the invariant 1-forms

$$\Pi^\mu = \begin{cases} \mathrm{d}X^\mu - i\bar{\theta}_+\Gamma^\mu \mathrm{d}\theta_+ & Heterotic \\ \mathrm{d}X^\mu - i\bar{\theta}\Gamma^\mu \mathrm{d}\theta & IIA \\ \mathrm{d}X^\mu - i\delta_{ij}\bar{\theta}^i_+\Gamma^\mu \mathrm{d}\theta^j_+ & IIB \end{cases} \tag{1.41}$$

They are invariant under translations in space-time ($\mathrm{d}X$) combined with translations in the supercharges (θ).
Add an index $i = 0\ldots 9$ such that for the heterotic string

$$\Pi^\mu_i = \partial_i X^\mu - i\bar{\theta}_+\Gamma^\mu\partial_i\theta_+ \tag{1.42}$$

and similar for the IIA and IIB string.

The Nambu-Goto action will be

$$S = -\int \mathrm{d}\tau\mathrm{d}\sigma\sqrt{-\det(\Pi_i\Pi_j)} \tag{1.43}$$

To make this action consistent it has to be supplemented with a Wess-Zumino term, which is an eight form

$$S_{WZ} = \frac{1}{2}\int \mathrm{d}\sigma\mathrm{d}\tau\epsilon^{ij\cdots}b_{ij} \tag{1.44}$$

Where $\mathrm{d}b_{ij} = h$ and h is invariant under the algebra, thus

$$h = \Pi^\mu \mathrm{d}\bar{\theta}_+\Gamma_\mu \mathrm{d}\bar{\theta}_+ \tag{1.45}$$

For the heterotic cases and similarly for the IIA and IIB cases.
The sum of the Nambu-Goto and the Wess-Zumino actions is invariant under κ-symmetry which is a symmetry on half of the components of the spinors. In other words the variation of the spinors is given by the projection operator

$$\delta\theta = (1+\Gamma)\kappa \tag{1.46}$$

and there is an induced variation in the bosonic variables. κ symmetry guarantees the equivalence between the GS and the NSR superstring.

Moreover h is only exact for the $N = 1, 2$ supersymmetry algebras.
For the heterotic string the action has to be supplemented with an term on 32 worldsheet chiral fermions. This term turns out to be consistent only for the groups $SO(32)$ and $E_8 \times E_8$.
Type I string-theory is obtain by projecting out the world sheet parity of type IIB string theory. These are all the consistent string theories in ten dimensions.

Chapter 2

Principal Bundles, Connections

We have introduced gauge field without knowing what kind of object it is. In this chapter I try to make clear that it is a way of making sections in a bundle. In the first section I'll show that the geometrical object behind quantum field theories and general relativity is the notion of a principal bundle. Since the most basic definition is extremely clumsy to generalize to more complicated gauge theories, I'll just give a brief introduction into principal bundles and connections. There are many excellent text books and lecture notes covering (part of) the definitions given here (see for example [19], [24] and [53]). The most straightforward to generalize definition of a connection turns out to be a connection on a sheaf. This will be treated in chapter 2 and remains the central object throughout this article.

2.1 Principal bundles

A principal bundle is a manifold with a group action. Before rushing into the abstract definitions it's worthwhile to ponder on the choices that are to be made. To start with a famous example to focus our minds consider the electromagnetic field on a four dimensional space time. The electromagnetic field tensor at any point can take any value. It is only after imposing the field equations and the bianchi identity that the field tensor gets a specific form. Our first task will be to describe the geometrical structure without the restriction of the field equations. Once we have done this we'll have a look at the geometrical meaning of the field equations. The most naive thing to describe the electric field on the four dimensional manifold would be to to take the direct product of the field degrees of freedom with the manifold. This is however far to restrictive. So we should be looking for a manifold with the degrees of freedom acting freely and which is locally trivial.

Formulated a bit more explicit. Given a (Topological, Lie) group G and a (base) manifold B we want to form a new manifold P with an action of G on P. First of all this action of G on P by: $G \times P \to P$ is defined by $(p, g) \to pg$*. Secondly There should be a map $\pi : P \to B$ that ignores the group. In other words the orbits induced by the group action should be projected to a single point in the base manifold. Moreover to make life easy we demand local triviality. If we glue these elements into a rigorous definition we're forced to the following (I'll stick to [53] for a moment)

2.1.1 Definition *A principal bundle $P(B, G, \pi)$ is a manifold, where B is a manifold, $\pi : P \to B$ is a smooth surjective map*

1. *G is a lie group*† *acting freely on P.*
2. *The fibres of π equal the orbits of G.*
3. *There is an open cover $\{U_i\}$ of B and smooth maps $\sigma_i : U_i \to P$ such that $\pi \circ \sigma_i = \mathrm{id}_{U_i}$*

For any two local sections $\psi_\alpha, \psi_\beta \in \Gamma(U_\alpha, P), \psi_\alpha : U_\alpha \to G$, we can find transition functions $g_{\alpha\beta}$ such that $\psi_\beta = \psi_\alpha g_{\alpha\beta}$. These functions satisfy a so called cocycle condition: $g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$. From these g's we can construct a principal bundle as will be proofed by the following theorem:

2.1.1 Theorem *Let M be a manifold, U_α an open cover of M and G a lie group. And given non-empty maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ that satisfy the cocycle condition. We can find a principal bundle $P(M, G)$ with $g_{\alpha\beta}$ as the transition functions.*

Since the proof is a bit lengthy but not hard and since a more elegant proof can be given in the case of torsors (section 2.3) I'll refer to [47], I. Prop 5.2 chapter 1 for the proof.

2.1.2 Definition *A morphism f between two principal bundles $P(M, G)$ and $P'(M', G')$ is given by:*

1. *A map $f' : P \to P'$*
2. *A homomorphism $f'' : G \to G''$ such that the group action is preserved. In other words $f'(pg) = f'(p)f''(g)$.*

2.1.1 Example *Take the principal bundle $P(M, G, \pi)$. The set of automorphisms are the gauge transformations on P. This structure is the so called gauge groupoid and will be treated more extensively in chapter 8.*

---

*I have chosen a right action for convenience, a left action could equally well have been used (as a matter of fact this appears to be a specialism dependent notation, in other words this choice seems to depend on the field people are working on)

†This is all differential geometry. This definition can be made more general using for example topological groups. I will not dwell on this since this extension is more or less straight forward and is excellently treated by [53]

2.1.3 Definition *Let $\pi : P \to M$ be a principal bundle a global section of P is a smooth map $s : M \to P$ such that*

$$\pi \circ s = \mathrm{id}_M \tag{2.1}$$

This equation defines the notion of a section for many other objects in mathematics, including sequences in homological algebra.

2.2 Connections

A gauge field keeps a theory invariant when we move the theory a bit away from a point. Stated differently a connection or gauge field tells us when we move a tiny bit over the base manifold what should happen in the symmetry group to keep the theory invariant. In other words it renders us a notion of horizontally in the 'symmetry fibres'.
I'll now try to make this notion of horizontally precise. However since there are many definitions of a connection it'll turn out to be useful to give several equivalent definitions, some of them will be more easy to generalize than others. As is intuitively clear horizontally is a way of making sections in the tangent bundle this way of making sections is formally defined by

2.2.1 Definition *An (Ehresmann) connection on a principal bundle P is a choice of a splitting $TP = V \bigoplus H$, where V is the subbundle of TP generated by the group G.*

It is an easy exercise to show that this is equivalent to the following definition (Using $H = \ker A$):

2.2.2 Definition *A connection on a principal bundle P is a one form $A = \sum_{i=1}^{k} A_i \otimes X_i$ such that:*

1. *A is G invariant with respect to the product action of G on $\Omega^1(P)$ and Ad_g (The adjoint representation of G), i.e. $r_g^* A = \mathrm{Ad}_{g^{-1}} A$.*
2. *A is vertical in the sense that $\iota_{X^\sharp} A = X \ \forall \ X \in \mathfrak{g}$, where $X_\sharp$ is the one parameter subgroup generated by X and $\mathfrak{g}$ the lie algebra of G.*

A connection is defined such that a path γ on the manifold M can be lifted to a unique path in in the bundle called the horizontal lift of γ and will be denoted by $\tilde{\gamma}$. This will be made a bit more precise by the following standard theorems:

A vector $X \in TP_p$ is called horizontal if it's in the kernel of A, i.e. $A(X) = 0$. The vectors induced by the group action are still horizontal since $A(r_{g*}X) = r_g^* A(X)$ (Since A is G invariant).

Furthermore we note that the projection operator $\pi : P \to M, \pi : p \mapsto x$ induces an isomorphism $H_p \tilde{\to} T_x M$ between the horizontal subspace and the manifold.

2.2.1 Theorem *Suppose we have given a principal bundle $P(M, G, \pi)$, with connection A. And a vector field X on M. There exists an unique horizontal lift $\tilde{X}$ of X. The lift is invariant under the action r_g of the group on the principal bundle. Moreover every G invariant vector field on P is the horizontal lift of a vector field on M.*

Proof. The existence and uniqueness follows directly from the fact that the tangent space of the manifold is isomorphic to the horizontal tangent space. To prove the differentiability, take an open neighborhood U of x and use the isomorphism $\pi^{-1}(U) \tilde{\to} U \times G$
To see that every G invariant vector field $\tilde{X}$ on P is the horizontal lift of a vector field on M. Take $X = \pi_*(\tilde{X})$ since $\tilde{X}$ is G invariant this vector field is well defined.

Horizontal lifts have the algebraic structure of a Lie-algebra induced by the lie-algebra structure of the vector fields on M.

2.2.2 Theorem *Take $\tilde{X}$ and $\tilde{Y}$ to be the horizontal lifts of X and Y, let f be a function on M with induced function on P : $\tilde{f} = f \circ \pi$, and let $[,]$ be a lie bracket. Then*

1. *$\tilde{X} + \tilde{Y}$ is the horizontal lift of $X + Y$.*
2. *$\tilde{f}\tilde{X}$ is the horizontal lift of fX.*
3. *The horizontal component of $[\tilde{X}, \tilde{Y}]$ is the horizontal lift of $[X, Y]$.*

A more useful definition comes from general relativity, where a connection tells how to parallel transport a tensor along a given curve. This turns out to be the definition that will be used to define a connection on sheafs and on gerbes. We define a connection as a prescription what happens to an element in the fibre if we go from a point x to an infinitesimally close point y.

2.2.3 Definition *A connection is a morphism of principal bundles. Such that given the infinitesimal neighborhood U of a point x (which defines trivialization if the neighborhood is small enough). We have an isomorphism for the bundle above U: $\epsilon : P_U \to P_U$ of principal bundles such that if we define $\epsilon_y : P_y \to P_y$ (the diagonal) then $\epsilon_y = \mathrm{id}\ \forall\ y\ \in\ U$*

Parallel transport

A connection renders a notion of horizontality in the fibres. It allows us to lift a path γ horizontally.

2.2.4 Definition *A path in P [‡] is called horizontal if the tangent vector to the curve at every point is horizontal.*

[‡]Of course this path should be in C^1 to define a vector field, explicit reference to the differentiability I'll omit, since they can be found extensively in textbooks like [47] and [24] and are not essential line of argument of this thesis. So it's enough to assume everything C^∞.

2.2.3 Theorem *Let $\gamma : [0,1] \to M$ be a path in M and let $p \in P$ be a point in P with and let $x = \pi(p) = \gamma(0)$. There there exists an unique horizonal lift $\tilde{\gamma} : [0,1] \to P$ of γ such that $\gamma(0) = p$.*

Proof.
Due to the local trivializations there exists a curve $\hat{\gamma} : [0,1] \to P$ such that $\hat{\gamma}(0) = \tilde{\gamma}(0)$ and $\pi(\hat{\gamma})(t) = \gamma(t)\ \forall\ t \in [0,1]$. Any other curve with the property that it's projected to γ must necessarily be of the form $\tilde{\gamma} = \hat{\gamma} g(t)$ where $g(t)$ is a curve in the group G such that $g(0) = \mathrm{id}$.
We obtain for the tangent vector:

$$\dot{\tilde{\gamma}}(t) = \dot{\hat{\gamma}}(t) g(t) + \hat{\gamma}(t)\dot{g}(t) \tag{2.2}$$

If we apply the connectionform A to this equation we obtain:

$$A(\dot{\tilde{\gamma}}(t)) = A(\dot{\hat{\gamma}}(t) g(t)) + A(\hat{\gamma}(t)\dot{g}(t)) \tag{2.3}$$

$$A(\dot{\tilde{\gamma}}(t)) = \mathrm{Ad}_{\mathrm{g}^{-1}} A(\dot{\hat{\gamma}}(t)) + g^{-1}\dot{g}(t) \tag{2.4}$$

This implies that for $\tilde{\gamma}$ to be horizontal we have to solve the equation:

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = -A(\dot{\hat{\gamma}})g(T). \tag{2.5}$$

If the local trivializations are given by $\phi : \pi^{-1}(U) \to U \times G$. We can find sections $s : U \to P$, given for example by $s(x) = \phi^{-1}(x, \mathrm{id})$, such that:

$$A(\dot{\hat{\gamma}}) = (s^* A)(\dot{\gamma}). \tag{2.6}$$

We end up with the equation

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = -(s^* A)(\dot{\gamma})g(T). \tag{2.7}$$

Just to temporarily clear up the notation, write this as:

$$f'(t) + B(t)f(t) = 0 \tag{2.8}$$

By using Picard iteration repeatedly (see [80]) we obtain as a solution:

$$
\begin{aligned}
f(t) = \Big(1 - \int_0^t \mathrm{d}t_1 B(t_1) + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 B(t_1)B(t_2) \\
- \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 B(t_1)B(t_2)B(t_3) + \dots\Big) f(0) \qquad (2.9) \\
= \Big(\sum_{n=0}^{\infty} (-1)^n \int_{t \geq t_1 \geq \dots \geq t_n \geq 0} B(t_1)\dots B(t_n)\mathrm{d}t_1 \dots \mathrm{d}t_n\Big) f(0) \qquad (2.10) \\
= \Big(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{[0,1]^n} T(B(t_1)\dots B(t_n))\mathrm{d}t_1 \dots \mathrm{d}t_n\Big) f(0) \qquad (2.11) \\
= P\exp\Big(-\int_0^t B(s)\mathrm{d}s\Big) f(0) \qquad (2.12)
\end{aligned}
$$

Where $T(\dots)$ is the time-ordered product.
Changing back the notation we obtain for the uniquely horizontal lifted curve:

$$
g(t) = P\exp\Big(-\int_0^t s^* A(\dot{\gamma(u)})\mathrm{d}u\Big) g(0) \qquad (2.13)
$$

The exponential is the *parallel transport operator*. Let's denote the operator by T_γ

The parallel transport operator gives rise to an algebraic structure. Let M be the manifold. If we have a curve γ from a to b. We have seen that the curve γ induces a horizontally lifted curve $\tilde{\gamma}$. This curve we said to be parallel transported. However by reversing t, we get the inverse path γ^{-1} on the manifold. This enables us by lifting γ^{-1} horizontally to construct the inverse parallel transport.
Moreover if we have two paths γ from a to b and δ from b to c we can form a new path $\gamma \circ \delta$ from a to c. This composition is associative.
State differently we have a set of objects M. And a set of arrows given all parallel transported curves between points of M, denoted by $\mathrm{Hol}(M)$.
Satisfying:

1. The multiplication is defined if the end point of the first path equals the begin point of the second path. If the multiplication is defined it is associative.

2. The starting point of two paths is the starting point of the first path, and similarly for the end point.

3. Every parallel transport can be inverted.

4. There is a unit, the constant path from x to itself.

This mathematical structure of the parallel transport is called a groupoid. This particular example of a groupoid is called the *Holonomy groupoid*. Groupoids turn out to be essential in generalizing principal bundles so an short introduction into the general theory of groupoids is given in chapter 8.

2.3 Curvature

In general parallel transport of an element along two 'infinitesimal curves' spanned by the vectors X and Y will not yield the same result as parallel transport along the curve spanned by $X+Y$. If we take the wilsonoperator to the infinitesimal loop given by X, Y and back along $X+Y$ will provide us with a two form called the *curvature of the connection* A, denoted by $\mathrm{d}_A A$ or more in agreement with physics literature by F_A.

The connection A defines a multiplication in the fibres as we have seen in the previous section. There is a special class of differential forms on the principal bundle. If the differential forms take values in a vector space V and suppose there is an representation of the group action denoted by ρ on the vector space. The set of equivariant forms is the set for which the induced right multiplication on the cotangent space of the bundle can be realised by the multiplication on the vector space of the groups representation. Stated less precise, these forms 'commute' with the group action. This means that the action induced by the connection at the bundle or tangent space level can be translated to a action on the vector space. A precise definition will be like:

2.3.1 Definition *Let $P(M,G,\pi)$ be a principal bundle and ρ representation of G on a finite dimensional vector space V.*

1. *An equivariant or pseudotensorial form ω of degree n on P and type (ρ, V) is a n-form on P with values in V such that*

$$(r_g)^*\omega = \rho(g^{-1})\omega \tag{2.14}$$

2. *This equivariant form is called tensorial if it vanishes for any vector in the vertical tangentspace.*

The curvature of a connection (or more general of a equivariant form) is just the infinitesimal change of the connection along horizonal vectors, this means that the curvature of an equivariant from ω is given by $\mathrm{d}\omega$ acting on the projection onto the horizontal subspace of vectorfields. This makes immediately the importance of the following theorem clear:

2.3.1 Theorem *If ω is an equivariant n form on P of type (ρ, V) and let π_{h*} be the projection on the horizontal subspace, the following statements hold:*

1. *The form $(\pi_h^*\omega)(X_1, \dots X_n) = \omega(\pi_{h*}X_1, \dots, \pi_{h*}X_n)$, with $X_i \in T_xP$ is a tensorial form of type (ρ, V)*

2. $\mathrm{d}\omega$ *is an equivariant* $n+1$*-form of type* (ρ, V)

3. *The* $n+1$ *form* $\mathrm{d}_A\omega$ *defined by* $\mathrm{d}_A\omega = \pi_h^*(d\omega)$ *is a tensorial form of type* (ρ, V).

The proof an easy calculation using that $[r_g^*, \pi_h^*] = 0 = [r_g^*, \mathrm{d}]$.

The map $\mathrm{d}_A : \Omega_{eq}^k \to \Omega_{eq}^{k+1}$ on the set of equivariant forms is not a differential operator, i.e. $\mathrm{d}_A^2 \neq 0$. $\mathrm{d}_A A$ is as mentioned above the theorem called the curvature of the connection A

2.3.2 Theorem *Let* A *be a connection and* $\mathrm{d}_A A = \mathrm{F}_A$ *it's curvature, then:*

$$\mathrm{d}A = -A \wedge A + \mathrm{d}_A A \tag{2.15}$$

This equation is called the structure equation of E. Cartan.

Proof:
There are three separate cases to be considered:

1. If both X and Y are in the horizontal subspace, then $A(X) = 0 = A(Y)$ so the equation reduces to the definition.

2. If both X and Y are vertical then $\mathrm{d}_A A = 0$ and:

$$\begin{aligned}\mathrm{d}A(X_\sharp, Y_\sharp) &= X_\sharp(A(Y_\sharp) - Y_\sharp(A(X_\sharp)) - A([X_\sharp, Y_\sharp]) = -[X, Y] \\ &= -[A(X_\sharp), A(Y_\sharp)] = -\frac{1}{2}[A, A](X_\sharp, Y_\sharp)\end{aligned}$$

3. If X is in the horizontal subspace an Y is vertical. We denote by X also the extension of the vector to a horizontal vector field on P. Then again $\mathrm{d}_A A = 0$ since it is tensorial. Since the commutator is zero. It remains to show that $\mathrm{d}A = 0$. Expansion gives again

$$\mathrm{d}A(X, Y_\sharp) = X(A(Y_\sharp)) - Y_\sharp(A(X)) - A([X, Y_\sharp]) \tag{2.16}$$

Which vanishes since $[X, Y_\sharp]$ is horizontal, which follows from a direct calculation using the definition of the Lie derivative.

An other way of calculating the Cartan formula is done by comparing connections along a infinitessimal small closed curve. This will be discussed when the connection on a sheaf is defined since it's the most naive way of translating a connection on a sheaf to a formula in differential forms.

2.4 The Atiyah Sequence

The differential geometric definition of a principal bundle is extremely difficult generalize to gerbes. Moreover a connection is described in much more elegantly using algebraic definitions.

To prepare for the 'easy' paths leading to Gerbes I'll try to write down principal bundles using exact sequences. The first hint that this is possible and that a connection is just a splitting of an exact sequence comes from the first definition in section 1.2. Readers not familiar with exact sequences I recommend to spent some time studying appendix B or (preferably) one of the excellent textbooks on homological algebra, like [41], [81] or chapter 20 and 21 of [49].
Before rushing into sequences it is important to note that a manifold can always be made into a lie group. Take for the group elements the points of the manifold and for the multiplication the trivial multiplication.

Equivalent definition of Principal bundles

2.4.1 Definition *A Principal bundle $P(M, G, \pi)$ over a manifold M, with (right) action of the group G, is an exact sequence of Lie groups:*

$$1 \longrightarrow G \longrightarrow P \xrightarrow{\pi} M \longrightarrow 1 \tag{2.17}$$

This definition makes in natural way clear that the obstruction for the principal bundle P to be the trivial bundle $G \times M$ is given by a homology class called the first čech cohomology class, $\check{H}^1(M, G)$, compare with the group extensions from section B.5 from the appendix. Čech cohomology is defined in a natural way using sheaves, this means I'll have to postpone a discussion of this cohomology class until the next chapter (see section 3.3).

The sequence of definition 1.4.1 induces an exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{g} \longrightarrow TP \longrightarrow \pi^{-1}(TM) \longrightarrow 0 \tag{2.18}$$

Which is called the Atiyah sequence [6].

Connections

2.4.2 Definition *A connection is a G-invariant splitting of the sequence 2.18*

Chapter 3

Sheaves

A gauge field tells us how to multiply in the gauge group when we take a path in a neighborhood of a point. This only depends on all paths, which depends on the neighborhood. Since we have to know exactly what we can do with gauge fields and (2-form) curvatures before we go to 3-form curvatures we would like to have a formulation of gauge fields that depends only on neighborhoods.

Many notions in mathematics depend only on neighborhoods, i.e. open sets containing a certain point. All these structures are unified in the notion of a sheaf which is of course a very fundamental one. In particularly the notion of a connection or gauge field on a principal bundle is just a sheaf theoretic notion. The generalization of a connection is done by combining the theory of sheaves and category theory, which results in the notions of stacks and gerbes. To introduce sheaves and their language I'll start with an extremely familiar example just to give an idea of the main concepts. There are many good books for a first introduction into sheaves, see for example [14] & [64], for more advanced introductions see [37] & [44]

3.1 Introduction and elementary definitions

As a motivating example we'll have a look at the continuous functions on $\mathbb{R}$ or an open subset U. Let $C(U)$ be the ring of continuous functions defined above U. We know that when $V \subset U$ then the continuous functions on U are also continuous functions on V, to state this formally, we say that can find maps $\rho^U_V : C(U) \to C(V)$ (which are in this case the inclusion maps.)
In topology the empty set is also open. We define $C(\emptyset)$ to consist of one element. This element is included in all the groups $C(U)$ and is a normalization of the system of rings. This system of rings on a topological space is called a presheaf*.
We are now able to give a precise definition of a presheaf These maps have the

*The term sheaf (faisceaux) comes from French agriculture. There is strong resemblance between sheaves in mathematics and in agriculture this will become clear when the stalk of a sheaf is introduced.

following properties:

3.1.1 Definition *Let X be a topological space*†. *Suppose that with every open set $U \subset X$ we have associated a set $F(U)$ and with any open sets $V \subset U$ a map $\rho^U_V{:}F(U) \to F(V)$. This system of sets and maps is called a presheaf of sets if the following conditions hold:*

1. *$F(\emptyset)$ consists of one element.*
2. *ρ^U_U is the identity map for any open set U.*
3. *For any open sets $W \subset V \subset U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$*

Similar definitions hold for groups, modules and rings. Obviously the continuous functions on the real line with the inclusion maps form a presheaf of rings. This immediately leads to the most important example, that of the presheaf of continuous functions.
The sets $F(U)$ can be viewed as the sections of F over U. The elements of $F(X)$ will be called global sections of F.

In additions to the fact that they form a presheaf, the continuous functions on $\mathbb{R}$ have an other very important property. If we have two functions $f \in C(U)$ and $g \in C(V)$ with the property that they agree on the intersection i.e. $f|_{U\cap V} = g|_{U\cap V}$ than there exists a *unique* function $h \in C(U \cup V)$ such that $h|_U = f$ and $h|_V = g$. This defines a sheaf.
With this in mind the precise statement of the definition of a sheaf is easy to give and will be:

3.1.2 Definition *A presheaf F on a topological space X is called a sheaf if for any open set $U \subset X$ and any open cover $U = \bigcup U_\alpha$ the following conditions hold:*

1. *If $s_1, s_2 \in F(U)$ and $\rho^U_{U_\alpha}(s_1) = \rho^U_{U_\alpha}(s_2)\ \forall\ U_\alpha$ then $s_1 = s_2$.*
2. *If $s_\alpha \in F(U)$ are such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)\ \forall\ U_\alpha, U_\beta$ then there exists an $s \in F(U)$ such that $s_\alpha = \rho^U_{U_\alpha}(s)\ \forall\ U_\alpha$.*

The second condition states that if two sections agree on the intersection of the open sets they are defined on, we can glue them together to a section s on the entire open set U. The first condition makes this section s unique.

†The condition 'space' can be weakened to a so called site. I won't deal with this since it will take to far away from physics. However it is a thing that is important to note since it makes connection theory independent of the manifold structure. This means that the formulas for connections defined in this chapter are valid as soon as there exists a topology. This means that we have the same formulas even for gauge theories on objects (sites) that have to little structure to be called a space. More on sites can be found in [2], [35] and [52]

Important remark:
It is usual to write $U_{\alpha\beta}$ instead of $U_\alpha \cap U_\beta$ to keep notation and readability under control. I'll use this notation below.

In the same way as manifolds can have a group action, sheaves can be equiped with a group action leading to so called torsors. Although torsors are essential in the theory of gerbes I'll have to pospone the discussion of them until I have introduced Čech cohomology.

The stalk of a sheaf

Again back to the continuous functions on $\mathbb{R}$. Though the definitions may seem a bit cumbersome in this case things will become less trivial for a general sheaf. Suppose we are given a point x [‡]. And two functions f and g. If there is a neighborhood of x such that the two functions are equal, then we can't distinguish between the two functions on basis of their local properties at the point x (We can always add for example a continuous bump at a point not in the neighborhood or it's boundary). We say that f and g define the same *germ* at x. This can be glued to the following definition.

3.1.3 Definition 1. *Let F be a sheaf on X and $U, V \subseteq X$. We say that two elements $f \in F(U)$ and $g \in F(V)$ define the same germ if there is an open set $W \subset U \cap V$ such that*

$$\rho^U_W f = \rho^V_W g \tag{3.1}$$

i.e. f and g agree on the subset W.

2. *Since 'defining the same germ' is an equivalence relation we call the equivalence class of f at x the germ of f at x, notated as $\mathrm{germ}_x f$*

3. *the stalk at x, denoted by stalk_x is the set of all germs at x.*

This definition is the same as taking the inductive limit over the open set containing x, which is treated in for example [37] and [64]. Stated informally we look at smaller and smaller neighborhoods around the point x and the elements of stalk_x are those elements contained in all the neighborhoods regardless of how close we approach x. Since the point x is in general not an open set and isn't in the sheaf, the stalk is the set we would associate to x if it was an open set. It is defined by looking at the neighborhoods close to x.
The 'bundle' of stalks resembles the notion of a agricultural sheaf.

3.2 Sheaves and Categories (1)

Objects like presheaf and sheaves can be defined in a categorical way. I'd like to show this for presheaf, in the way it is treated in the book by Iversen [44].

[‡]Note that a point is in general not an open set and thus doesn't define a group in the sheaf

To define a presheaf categorical, the first thing to be done is to make a category out of the topology. The objects are given by the open sets of X. The arrows are given by the inclusion maps. A presheaf is now a contravariant functor C from this category to the category of sets (or groups etc.).
A morphism of presheaves on X is simply a natural transformation of functors. In other words for every open set U we have given a linear map $f(U) : F(U) \to G(U)$ such that whenever $V \subseteq U$ are open subsets, the following diagram is commutative:

$$\begin{array}{ccc} F(U) & \xrightarrow{\rho_V^U} & F(V) \\ \downarrow{\scriptstyle f(U)} & & \downarrow{\scriptstyle f(V)} \\ G(U) & \xrightarrow[\rho_V^U]{} & G(V) \end{array} \tag{3.2}$$

Using this definition it is straightforward to write down what the composition of two morphisms of sheaf should look like:

$$(g \circ f)(U) = g(U) \circ f(U) \tag{3.3}$$

The unique gluing property to turn a presheaf into a sheaf is much more difficult to realize and since it's not necessary for an understanding of the construction of a gerbe, I'd like to refer the interested reader to the book by MacLane and Moerdijk [52] chapter 2 section 1.

3.3 Sheaves and Čech cohomology

Torsors are the 1-cocyles of a cohomology theory called Čech cohomology. Gerbes are the 2-cocycles of the same cohomology theory. An important step in understanding why we make certain definitions while generalizing the connections on sheaves it's useful to understand the cohomology side of the story. Unfortunately this will be restricted to a very brief introduction to čech cohomology. It'll follow [14] very closely.

Let (U_α) be an open cover of the topological space X. Take for the 0-cochains the functions which assign to every open set U_α an element of the presheaf of groups $F(U_\alpha)$. In other words

$$C^0((U_\alpha), F) = \prod_\alpha F(U_\alpha) \tag{3.4}$$

We can now define the 1-cochains in a similar way, they are all elements of:

$$C^1((U_\alpha), F) = \prod_{\alpha<\beta} F(U_\alpha \cap U_\beta) \tag{3.5}$$

The differential operator on the cohomology complex is naturally induced by the inclusion maps of the open sets:

$$U_\alpha \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U_{\alpha\beta} \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\; \cdots \tag{3.6}$$

The induced sequence of group homomorphism is:

$$\prod F(U_\alpha) \rightrightarrows \prod F(U_{\alpha\beta}) \rightrightarrows\!\!\!\!\!\!\rightarrow \cdots \tag{3.7}$$

We now take (as usual in these cases) the differentials δ of the chaincomplex to be the alternating sum of the $F(\delta)$'s, written down in formulae this means:

$$\delta^p : C^p((U_\alpha), F) \to C^{p+1}((U_\alpha), F) \tag{3.8}$$
$$\delta^p = F(\partial_0) - F(\partial_1) + \ldots + (-1)^{p+1} F(\partial_{p+1}) \tag{3.9}$$
$$\tag{3.10}$$

It is now easy to check that the differentials satisfy indeed $\delta^2 = 0$.

3.3.1 Definition *The cohomology of this complex is called the Čech cohomolgy and is denoted by $\check{H}^*((U_\alpha), F)$.*

Given a refinement $(V_\beta)_{\beta\in J}$ of the cover $(U_\alpha)_{\alpha\in I}$, i.e. given a map $\phi : J \to I$ such that $V_\beta \subseteq U_{\phi(\beta)}$. Then there is an obvious induced map

$$\phi^\sharp : C^p((U_\alpha), F) \to C^p((V_\beta), F) \tag{3.11}$$
$$\phi^\sharp \omega(V_{\beta_0 \ldots \beta_p}) = \omega(U_{\phi(\beta_0)\ldots\phi(\beta_p)}) \tag{3.12}$$

It's an easy exercise to show that $\phi^\sharp$ is a chain map by just writing out $\delta(\phi^{\sharp\omega})$ and $\phi^\sharp \delta\omega$.
Moreover given two refinements $\phi\psi : J \to I$ and using the chainmap $K : C^p((U_\alpha), F) \to C^{p-1}((V_\beta), F)$ defined by $(K\omega)(V_{\beta_0\ldots\beta_{p-1}}) = \sum (-1)^i \omega(U_{\phi(\beta_0)\ldots\phi(\beta_i)\psi(\beta_i)\ldots\psi(\beta_{p-1})})$ it is easy to show that chainmaps induced by the refinements $\phi^\sharp$ and $\psi^\sharp$ are homotopic and thus isomorphic in cohomology (A result that is proofed for homology in appendix B.2).

3.3.2 Definition *A direct system of groups is a collection of groups $\{G_i\}_{i\in I}$ indexed by a directed set I such that for any pair $a > b$ there is a group homomorphism $g_a^b : G_a \to G_b$ satisfying*

1. $g_a^a = \mathrm{id}$
2. $g_c^a = g_c^b \circ g_b^a$

On the disjoint union $\dot{\bigcup} G_i$ we can introduce an equivalence relation. Two elements $\gamma \in G_a$ and $\delta \in G_b$ are said to be equivalent if there exists a $c \in I$ such that $g_c^a(\gamma) = g_c^b(\delta)$ on G_c. We can now take the direct limit of the directed system G_i. It is the quotient of $\dot{\bigcup} G_i$ with respect to the equivalence relation.
Since a refinement map induces a well defined map in cohomology we find that the $\{\check{H}^*((U_\alpha), F)\}$ form a direct system of groups and the direct limit of this system is called the Čech cohomology of X with values in the presheaf F. i.e.

$$\check{H}^*(X, F) = \lim_{\to\,(U_\alpha)} \check{H}^*((U_\alpha), F) \tag{3.13}$$

3.3.1 Theorem *The is a bijective correspondence between isomorphism classes of F-torsors and cohomology classes in $\check{H}^1(X, F)$*

Torsors

3.3.3 Definition *Let G be a sheaf of groups on X and F be a sheaf on X. An action of G on F is a map of sheaves*

$$\mu : G \times F \to F \tag{3.14}$$

which satifies the usual conditions for an action (see chapter 2).

3.3.4 Definition *The sheaf F is called a G-torsor if*

1. *There exists a refinement such that there is a sections for every open set, ie $X = \bigcup\{U | F(U) \neq \emptyset\}$*
2. *For every open $U \subseteq X$ the action of $G(U)$ on $F(U)$ is free and transitive.*

The first condition states that every stalk is non-empty. This requirement is redundant since it's already in the sheaf axiom. However we don't require non-emptyness for fibred categories, which makes this condition non-trivial if we want to extend it to gerbes. The second condition guarantees that the group action reaches every point of the torsor. A torsor is a principal bundle without manifold structure. There is also a cocycle description of torsors as in the case with principal bundles:

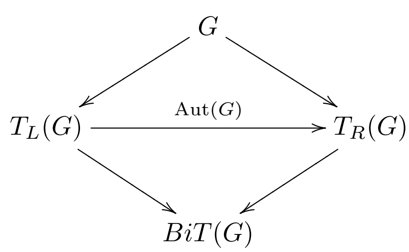
Proof of theorem 3.3.1. Let F be a G-torsor over X. By the non-emptyness we can choose a cover (U_α) of X and sections $f_\alpha \in F(U_\alpha)$. By the transitiveness we can vind $g_{\alpha\beta} \in G(U_{\alpha\beta})$ which are unique by the freeness and which satisfy on $U_{\alpha\beta}$:

$$g_{\alpha\beta} f_\beta = f_\alpha \tag{3.15}$$

Then $\{g_{\alpha\beta}\}$ is a cocycle in $\check{H}^1(X, G)$.
Take different sections $e_\alpha \in F(U_\alpha)$, this induces a cocycle $\{h_{\alpha\beta}\}$, with $h_{\alpha\beta} e_\beta = e_\alpha$ on $U_{\alpha\beta}$. Then there exists an unique element $a_\alpha \in G(U_\alpha)$ such that $a_\alpha f_\alpha = e_\alpha$ this element satisfies $a_\alpha g_{\alpha\beta} = h_{\alpha\beta} a_\beta$. Which shows that [g] = [h]. If we allow the cover (U_α) to vary we obtain a welldefined class $[F]$ in $\check{H}^1(X, G)$

We can construct from the cocycles a torsor.
Take a cocycle $\{g_{\alpha\beta}\} \in \check{H}^1(X, G)$. It defines a sheaf G and $g_{\alpha\beta}$ is a section of the sheaf over $U_{\alpha\beta}$. Form the union $F = \left(\dot{\bigcup\limits_\alpha} G(U_\alpha)\right) / \sim$. When we denote the points of $\dot{\bigcup\limits_\alpha} G(U_\alpha)$ as triples (x, α, g) with $g \in stalk_x G$ then $\sim$ is the equivalence relation given by $(x, \alpha, g) \sim (x, \beta, g g_{\alpha\beta})$. F is now obviously a G torsor. This proves the equivalence between cohomology classes and torsors.

Given a group G acting on the left on a torsor T_L. Intuitively, using the automorphism group we can construct an action on the right. For a $t \in \text{stalk}_x T_L$ we might be able to find a $\tilde{g} = f(g)$ such that $tg = \tilde{g}t$ and $f \in Aut(G)$. Now the torsor is also a right torsor T_R. This is an example of a bitorsor. Bitorsors can be a gerbe but in general a gerbe has more structure since a bitorsor is only transitive in each group. Gerbes are defined in chapter 5. For bitorors see [35] section III.1.5



(3.16)

3.4 Sheaves and Connections

The definition of a connection between sheaves is really abstract but when you think about it, on a manifold the definition agrees with the parallel transport definition of a connection. Why not do all the stuff with parallel transport then? All formulas hold over any 'thing[§]' with a topology. So there is only one way to go to a more general connection theory: replace groups by something more general! Even when this is done almost all the formula's remain unaltered. If we instead had done this in differential geometry things would have become a complete mess right from the start.
Denote the neighborhood of infinitesimal close points of x with D_x, let $\Delta^n = \underbrace{D_x \times D_x \times \ldots \times D_x}_{n+1}$ be the parameterization $(n+1)$-tuples of infinitesimally close points and π_0 the projection of Δ^n on the first factor D_x.

3.4.1 Definition *A connection is an isomorphism of torsors on* Δ^1
$\epsilon : \pi_1^* T \to \pi_0^* T$
such that $\Delta^* \epsilon = 1_T$

Δ^* is the diagonal imbedding. A connection can now be represented as an arrow:

$$\pi_1^* T \xrightarrow[\epsilon^{01}]{} \pi_0^* T \tag{3.17}$$

The connection can be viewed as a prescription for what happens when you go along all possible infinitesimal curves in the neighborhood D_x. And the diagonal embedding are all constant curves. To calculate curvature we try to

[§]Manifold, topological space, scheme, site

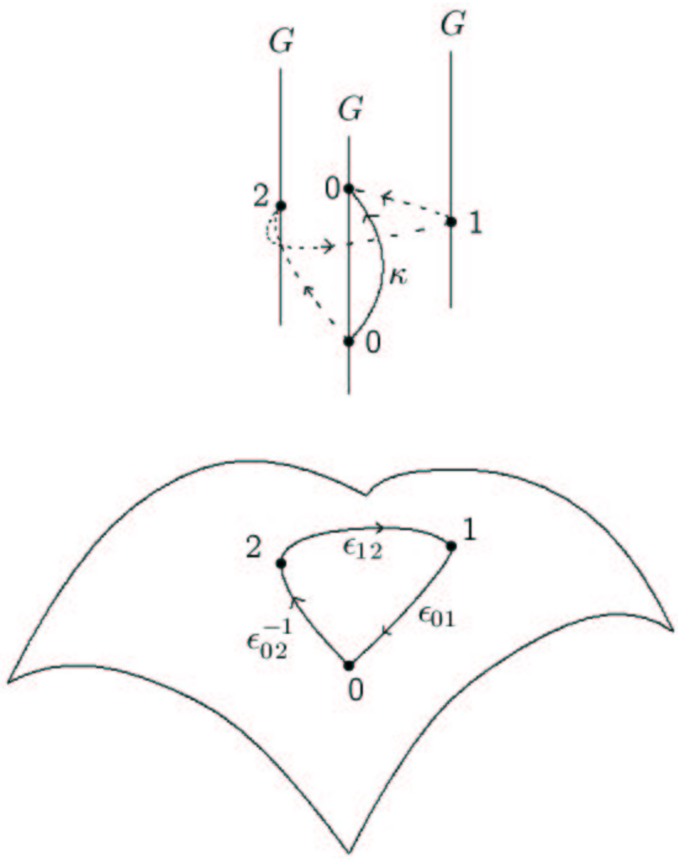


Figure 3.1: The intuitive picture of the curvature κ. It tells what happens if the ϵ_{ij} maps are used to walk around all infinitesimal closed loops. If we take a specific loop, for example $0 \to 2 \to 1 \to 0$ and then vary the points $0, 1$ and 2 we can calculate κ.

see what happens when we use ϵ around all inifintesimal closed loops:

$$\begin{array}{ccc} \pi_2^*T & \xrightarrow{\epsilon_{12}} & \pi_1^*T \\ \downarrow{\scriptstyle \epsilon_{02}} & & \downarrow{\scriptstyle \epsilon_{01}} \\ \pi_0^*T & \xrightarrow[\kappa]{} & \pi_0^*T \end{array} \tag{3.18}$$

The curvature κ is now simply what happens around all closed loops. We first express κ for all infinitessimal closed loops:

$$\kappa = \epsilon_{01}\epsilon_{12}(\epsilon_{02})^{-1} \tag{3.19}$$

To arrive at a formula in differential forms and to be able to tell whether this formula makes any sense we have now two ways to proceed. The first method is part of a area called 'synthetic differential geometry', it applies the methods of algebraic geometry in differential geometry. The area is established by A. Kock in a series of excellent texts in the 1980's. Since it uses a lot of advanced mathematics. Secondly we can follow a less rigorous but more naive approach which is suitable for physical applications.
As stressed earlier κ is the curvature for all closed infinitesimal loops. To be able to calcultate it we first take an arbitrary closed loop $x \to z \to y \to x$. To abuse notation we will denote the points by $0 \to 2 \to 1 \to 0$. After we have calculated the curvature for this loop we can vary the points $0, 1$ and 2 and we obtain the formula for the curvature around all closed loops in all neighborhood D_x for all x on the manifold. With the transition functions we can glue them finally together.
We'll start by writing down an expansion of the Wilsonloop for the ϵ for a single curve between two infinitesimal close points and make a first and second order expansion of formula (3.19). This will give us the correct formula in differential forms. Take X_{01} to be 'the vector from 0 to 1'

$$
\begin{aligned}
\epsilon_{01}(X_{01}) &= \exp[-\int_0^1 A_\mu(x)\mathrm{d}x^\mu] \\
&\approx 1 - A_\mu(0)X_{01}^\mu + \frac{1}{2}\partial_\nu A_\mu(0)X_{01}^\nu X_{01}^\mu + \frac{1}{2}A_\nu(0)A_\mu(0)X_{01}^\nu X_{01}^\mu \\
\kappa(X_{01},X_{12})(0) &\overset{X_{01}+X_{12}=X_{02}}{=} \epsilon_{01}(X_{01})\epsilon_{12}(X_{12})(\epsilon_{02}(X_{02}))^{-1} \\
&\approx (1 - A_\mu(0)X_{01}^\mu - \frac{1}{2}\partial_\nu A_\mu(0)X_{01}^\nu X_{01}^\mu + \frac{1}{2}A_\nu(0)A_\mu(0)X_{01}^\nu X_{01}^\mu)\cdot \\
&(1 - A_\mu(1)X_{12}^\mu - \frac{1}{2}\partial_\nu A_\mu(1)X_{12}^\nu X_{12}^\mu + \frac{1}{2}A_\nu(1)A_\mu(1)X_{12}^\nu X_{12}^\mu)\cdot \\
&(1 + A_\mu(0)X_{02}^\mu + \frac{1}{2}\partial_\nu A_\mu(0)X_{02}^\nu X_{02}^\mu + \frac{1}{2}A_\nu(0)A_\mu(0)X_{02}^\nu X_{02}^\mu) \\
&\approx (1 - A_\mu(0)X_{01}^\mu - \frac{1}{2}\partial_\nu A_\mu(0)X_{01}^\nu X_{01}^\mu + \frac{1}{2}A_\nu(0)A_\mu(0)X_{01}^\nu X_{01}^\mu)\cdot \\
&(1 - A_\mu(0)X_{12}^\mu - \partial_\nu A_\mu(0)X_{01}^\nu X_{12}^\mu - \frac{1}{2}\partial_\nu A_\mu(0)X_{12}^\nu X_{12}^\mu + \frac{1}{2}A_\nu(0)A_\mu(0)X_{12}^\nu X_{12}^\mu)\cdot \\
&(1 + A_\mu(0)X_{02}^\mu + \frac{1}{2}\partial_\nu A_\mu(0)X_{02}^\nu X_{02}^\mu + \frac{1}{2}A_\nu(0)A_\mu(0)X_{02}^\nu X_{02}^\mu) \\
&\approx 1 - A_\mu(0)(X_{01}^\mu + X_{12}^\mu - X_{02}^\mu) \\
&- \frac{1}{2}\partial_\nu A_\mu(0)(X_{01}^\nu X_{01}^\mu + X_{12}^\nu X_{12}^\mu - X_{02}^\nu X_{02}^\mu + 2X_{01}^\nu X_{12}^\mu) \\
&+ \frac{1}{2}A_\nu(0)A_\mu(0)(X_{01}^\nu X_{12}^\mu - X_{01}^\nu X_{02}^\mu - X_{12}^\nu X_{02}^\mu + X_{02}^\nu X_{02}^\mu) \\
&\overset{X_{01}+X_{12}=X_{02}}{=} 1 + 0 + \frac{1}{2}\partial_\nu A_\mu(0)(X_{01}^\nu X_{12}^\mu - X_{12}^\nu X_{01}^\mu) \\
&+ \frac{1}{2}A_\nu(0)A_\mu(0)(X_{01}^\nu X_{12}^\mu - X_{12}^\nu X_{01}^\mu) \\
\kappa(X_{01},X_{12})(0) &= 1 + (\partial_\nu A_\mu(0) - \partial_\mu A_\nu(0) + [A_\nu(0),A_\mu(0)])(X_{01}^\nu, X_{12}^\mu)
\end{aligned} \tag{3.20}
$$

We can now vary the points $0, 1$ and 2. This amounts to cancelling explicit dependens on these points, in other words making the formula coordinate independent and letting the one forms A not act on a vector. This implies for the curvature $K = 1 + \kappa$.

$$K = \mathrm{d}A + [A, A] \tag{3.21}$$

This curvature form describes trivial bundles which are used in Yang-Mills theory. So at present we can stop the derivation here if we want to do physics, just to describe the full theory for generalization I'll take the derivation a little bit futher in a minut.

But first note that the combinatorial formula for the curvature is independent of the algebraic structure we put on the sheaf and of the topological structure of the basis. The advantage of this cumbersome looking definition of curvature will become appearent when we start writing down curvatures of connections on Gerbes. Then we will change the algebraic structure and much of the formulae

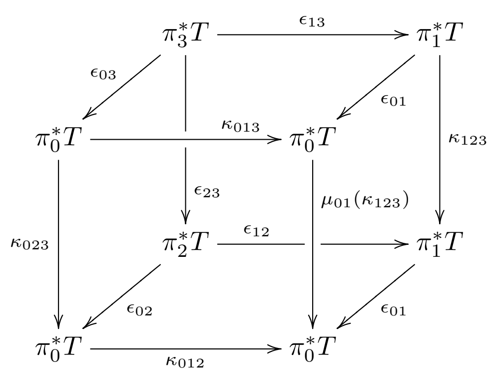
and the derivation will be the same.
The reason why we can give a combinatorial formula has it's origin in the theory of simplicial objects, see [48]. Since simplicial objects is quite a technical subject I won't introduce and use it here, the interested reader can find the basic material in [81] chapter 8 or [36].

The last step in the derivation is taking into account the transition functions or cocycles. Since these functions depend on just one open set we obtain for the curvature on non trivial bundles

$$K_i = \mathrm{d}A_i + [A_i, A_i] \tag{3.22}$$

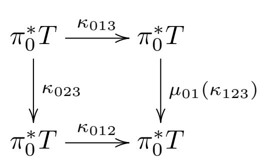
Where the index i indicates which local trivialization we should take.

When we commute once more we obtain the bianchi identity. We first obtain the commutative diagram:



(3.23)

This diagram induces the commutative square:



(3.24)

Which intuitively expresses the bianchi identity $\mathrm{d}_A\mathrm{d}_A A = 0$ by expressing $\mu_{01}(\kappa_{023})$ in infinitessimal close curvatures κ_{ijk}.

Chapter 4

Non-Perturbative String Theory

There are several ways to arrive at D-branes. We can either look for solitions [78] of string theory or do torodial compactification [63], [45]. Since torodial compactification is by far the easiest I'll show this approach in more detail.

4.1 T-Duality and Open Strings

Remember from chapter 1 the mode oscillator expansion for an open string i.e. (with $z = \sigma e^{\tau - i\sigma}$)

$$X^{\mu}(z,\bar{z}) = X^{\mu}(z) + X^{\mu}(\bar{z}) \tag{4.1}$$

Where

$$X^{\mu}(z) = \frac{x^{\mu}}{2} + \frac{x_T^{\mu}}{2} - i\alpha' p_0^{\mu}\log(z) + i\Big(\frac{a'}{2}\Big)^{\frac{1}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{\mu} z^{-n} \tag{4.2}$$

Where x_T is chosen such that term gets a minus sign for $X(\bar{z})$. It turns out that the coordinates x_T become the coordinates of space-time with one space direction less than the original space-time, where T stands for Torodial-compactified (i.e. compactification over a circle). reduced over a circle (or Tordially-compactified).
Compactification of one space direction, in the bosonic case call it the X^{25} direction. direction yields

$$X_T^{\mu}(z,\bar{z}) = X^{\mu}(z) - X^{\mu}(\bar{z}) \tag{4.3}$$

$$= x_T^{\mu} - i\alpha' p^{25}\log(\frac{z}{\bar{z}}) + i(2\alpha')^{\frac{1}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{25} e^{-in\tau}\sin n\sigma \tag{4.4}$$

$$= x_T^{\mu} 2\alpha' \frac{n}{R}\sigma + i(2\alpha')^{\frac{1}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{25} e^{-in\tau}\sin n\sigma \tag{4.5}$$

Since the oscillator term vanishes at $\sigma = 0, \pi$ the endpoints of the string are fixed at the planes $x^{25}(0,\tau)$ and $x^{25}(\pi,\tau)$. Or stated differently

$$X_T^{25}(\pi) - X_T^{25}(0) = \frac{2\pi\alpha' n}{R} \tag{4.6}$$

The endpoints of the string are still free to move in the 24 remaining spatial directions and in this way they span a hypersurface called a 24-brane (which is a 25 dimensional surface due to the time direction).

If we include chan-paton factors and we add a Wilson line due to the gauge field A_{25} taking values in $U(1)^N$. For a string in state $|ij>$ the endpoints of the strings are pick up an additional phase $e^{i(A_i - A_j)}$. Modifying the endpoints to be

$$X_T^{25}(\pi) - X^{25}(0) = (2\pi n + A_i^{25} - A_j^{25})R' \tag{4.7}$$

Or stated differently

$$X_T^{25} = 2\pi\alpha' A_i^{25} \tag{4.8}$$

This can be interpreted as N parallel D-branes.

4.2 D-Branes

What is the action for several parallel D-branes? General considerations give for a p dimensional brane (to lowest order) [63] chapter 8.7

$$S = -T_p \int \mathrm{d}^{p+1} x e^{-\phi}[G + F^2] \tag{4.9}$$

This is the dimensionally reduced action for a space filling ten dimensional brane.

If two branes coincide we might guess the that due to the Chan-Paton factors the symmetry group is enhanced from $U(1) \times U(1)$ to $U(2)$. To justify this we can have a look at the excitation spectrum. Strings stretching from a brane to the same brane induce an $U(1) \times U(1)$ symmetry, one U(1) for each endpoint. (Super) Strings stretching between the two branes also have a $U(1) \times U(1)$ symmetry with charges $(-1, 1)$ or $(1, -1)$. The bosons on the world volumes become massless as the branes coincide. giving an $U(2)$ symmetry.
There is more we can tell about this symmetry enhancement. Start with an space filling D-brane with a gauge field taking values in U(n). To simplify things a little just look at the bosonic string (for the general case see [85].
Take indices

$$\hat{\mu} = 0, \ldots, q \tag{4.10}$$

$$\mu = 0, \ldots, p \tag{4.11}$$

$$i = p, \ldots, q \tag{4.12}$$

Assume now that the fields depend only on the coordinates x^μ and not on the coordinates x^i, where $x^{\hat{\mu}}$ are the coordinates in the uncompactified space $\mathbb{R}^q$. This means that all the derivatives ∂_i vanish. Write the gauge field $A^{\hat{\mu}} = (A^\mu, X^i)$.
If we bring back to mind the formula for the curvature of a connection in a specific coordinate system:

$$F_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}} A_{\hat{\nu}} - \partial_{\hat{\nu}} A_{\hat{\mu}} + [A_{\hat{\mu}}, A_{\hat{\nu}}] \tag{4.13}$$

We can calculate the Yang-Mills term $\mathrm{YM} = F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}}$ in the compactified space.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{4.14}$$

$$F_{\mu i} = \partial_\mu X_i - \partial_i A_\mu + [A_\mu, X_i] = \partial_\mu X_i + [A_\mu, X_i] \tag{4.15}$$

$$F_{ij} = \partial_i X_j - \partial_j X_i + [X_i, X_j] = [X_i, X_j] \tag{4.16}$$

This implies for the Yang-Mills term:

$$\mathrm{YM} = \mathrm{tr} - \frac{1}{4}\Big(F_{\mu\nu}F^{\mu\nu} + 2F_{\mu i}F^{\mu i} + F_{ij}F^{ij}\Big) \tag{4.17}$$

$$= \mathrm{tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu X_i + [A_\mu, X_i])(\partial^\mu X^i + [A^\mu, X^i]) + -\frac{1}{4}[X_i, X_j][X^i, X^j]\Big) \tag{4.18}$$

The fields X_i behave as coordinate functions of the D-brane, however they take values in $U(N)$ proposed interpretations let the manifold structure of space time break down and instead use a non-commutative background space, see amongst others [68], [32], [31] or [23]. But essential for string theory is that it is defined on a manifold so no satisfactory interpretation has been found yet. This problem is of course relevant in the case of parallel M-5 branes. We'll only be looking at the case of coinciding M-5 branes since it's the only case in which there arises a non-abelian chiral theory. In this case the commutators of the coordinate functions just vanish.

Chapter 5

M-Theory

5.1 Supergravity

N = 1 D = 4 Supergravity

We want to construct a theory which includes a graviton and to make live as simple as possible we assume N=1 supersymmetry in four dimensional space time. Given we have a graviton in our theory the only possible particles that can be included have spin 3/2 or spin 5/2. Since the spin 5/2 fermion gives rise to considerable difficulties we content ourselves with just construction a theory for the spin 3/2 fermion.
The graviton is represented with a symmetric second rank tensor $B_{\mu\nu}$ and the spin 3/2 particle is a majorana vector spinor $\Psi_{\mu\alpha}$. They have the following infinitesimal gauge transformations:

$$\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \tag{5.1}$$

$$\delta \Psi_{\mu\alpha} = \partial_\mu \xi_\alpha \tag{5.2}$$

The free field equations are

$$0 = R^L_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \tag{5.3}$$

$$0 = i\epsilon^{\mu\nu\rho\kappa}\gamma_5\gamma_\nu\partial_\rho\Psi_\kappa \tag{5.4}$$

Where the linearized Riemann tensor is given by

$$R^{L\rho\kappa}_{\mu\nu} = -\partial_\rho\partial_\mu B_{\kappa\nu} + \partial_\kappa\partial_\mu B_{\rho\nu} + \partial_\rho\partial_\nu B_{\kappa\mu} - \partial_\kappa\partial_\nu B_{\rho\mu} \tag{5.5}$$

Supersymmetry transformations can be found that represent the N=1 supersymmetry algebra, i.e. the algebra from section 1.3. It turns out see [83] that the action that produces the equations of motion and that is invariant under the supersymmetry transformations is

$$S = -\frac{1}{2}\int \mathrm{d}^4x \Big[B^{\mu\nu}(R^L_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^L) + \bar{\Psi}_\mu(i\epsilon^{\mu\nu\rho\kappa}\gamma_5\gamma_\nu\partial_\rho\Psi_\kappa)\Big] \tag{5.6}$$

We now have to introduce interactions between the two fields. There are (at least) three ways to construct this non-linear theory. Firstly we could make the abelian invariance local. The supersymmetry algebra fails to be closed. This can be repaired order for order and the abelian gauge symmetry mixes with the supersymmetry. This procedure is called the Noether method. Secondly we can introduce vierbeins and a spin connection. Given the transformation properties of a majorana vector spinor and a graviton we calculate the restrictions due to the (graded) bianchi-identities and dimensional analysis. This turns out to be enough to yield the non-linear equations of motion. Thirdly we can start with the super Poincaré group, introduce gauge fields and gauge the field strength of the translations to zero. Then the most general first-order action that is invariant under the super-poincaré group is the supergravity action. Which of the methods is used, the action that results is

$$S = \int \mathrm{d}^4 x \Big[\frac{e}{2\kappa^2} R - \frac{1}{2} \bar{\Psi}_\mu i \epsilon^{\mu\nu\rho\kappa} \gamma_5 \gamma_\nu D_\rho \Psi_\kappa \Big] \tag{5.7}$$

Where the Riemann tensor is given by

$$R^{\rho\kappa}_{\mu\nu} \frac{\sigma_{\rho\kappa}}{4} = [D_\mu, D_\nu] \tag{5.8}$$

With covariant derivative

$$D_\mu = \partial_\mu + \omega_{\mu\kappa\rho} \frac{\sigma^{\rho\kappa}}{4} \tag{5.9}$$

Where ω is the connection form derived from the vielbeins induced by the graviton and the majorana vector spinor.

5.2 Supergravity in eleven dimensions

Apart from one dimensional representations there are two inequivalent 32 dimensional representations of the clifford algebra in 11 space times dimensions (see [83] chapter 1). Since the highest spin state is the spin 2 graviton we know from the representations of the algebra that we can have at maximum N = 8 supersymmetry (op. cit. chapter 3). In four dimensions we have eight majorana charges with four real components. This gives a maximal number of 32 components. Given a supergravity theory in any dimension n we can do a trivial dimensional reduction by taking all the fields independent of all but four directions. This induces that the spinor representation in the dimension n should have at most 32 components. For twelve dimensions the number of components is already 64 so the maximal dimension is eleven. Thus N = 1, D = 11 is the unique supergravity theory with the largest dimension (unless a way is invented to couple a spin 5/2 to a graviton).
The action can again be derived using the Noether procedure.

The Supersymmetry Algebra

Denote the 11-momentum by P_i and the supersymmetry charges by Q_α which are 32-component Majorana spinors. Translational invariance implies that

$$[P, P] = 0 \tag{5.10}$$

$$[P, Q] = 0 \tag{5.11}$$

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^i)_{\alpha\beta} P_i \tag{5.12}$$

With the usual Dirac matrices Γ^i and the real, antisymmetric charge conjugation matrix C. Suppose that the product of all Dirac matrices is 1 (Since there are two inequivalent representations there is also a representation for which the product has a minus sign.). Let $\Gamma^{ij} \overset{\text{not.}}{=} \Gamma^i\Gamma^j$.

Are there any states that preserve part of the supersymmetry of the vacuum? These states are called BPS states and a lot is known about these states see for example [28] Chapter 11 and 13.

Suppose the state preserves a fraction b of the supersymmetry. The expectation value of $\{Q, Q\}$ is a real symmetric positive semi-definite matrix with $b \cdot 32$ zero eigenvalues. It's determinant will vanish, this implies

$$0 = \det(\Gamma \cdot P) = (P^2)^{16} \tag{5.13}$$

Thus the momentum is null and there is a frame in which

$$P = \frac{1}{2}(-1, \pm 1, 0, \dots 0) \tag{5.14}$$

Choose the Majorana representation of the Dirac matrices, i.e. $C = \Gamma^0$. Then the supersymmetry algebra reduces to (ignoring the vanishing components)

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}(1 \mp \Gamma^{01})_{\alpha\beta} \tag{5.15}$$

So the BPS states satisfy

$$\Gamma^{01}|BPS >= \pm|BPS > \tag{5.16}$$

This implies that the eigenvalues are all ± 1. Tracelessness implies that expactly half are $+1$ and half are -1. Thus the space of solutions is sixteen dimensional and the amount of preserved supersymmetry $b = \frac{1}{2}$

Since the momentum is a null vector this algebra is naturally identified with a supersymmetric massless particle in eleven dimensional spacetime.

The algebra can be extended by central charges. The best way is just try to add all central charges and see whether we can deduce any constraints.

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^i)_{\alpha\beta} P_i + \sum_p (C\Gamma^{i_1 \dots i_p})_{\alpha\beta} Z^{i_1 \dots i_p} \tag{5.17}$$

In this equation C is the charge conjugation matrix. Since $(\Gamma^{i_1 \cdots i_p} C)_{\alpha\beta}$ should be symmetric under the interchange of α and β we can have central charges only in $p = 1, 2 \mod 4$ [83]. Using the hodge dual for $p > 5$ we see that there are only two possible central extensions of the algebra: A two form and a five form central charge. The modified algebra becomes:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^i)_{\alpha\beta} P_i + \frac{1}{2}(C\Gamma_{ij})_{\alpha\beta} Z^{ij} + \frac{1}{5!}(C\Gamma_{ijklm})_{\alpha\beta} Y^{ijklm} \tag{5.18}$$

We have taken the representation with the product of all Γ matrices to be $+1$. This yields us group of all symmetric matrices. This group is however reducible on $SO(1, 10)$. The decomposition of the symmetric group coincides with the decomposition in P, Z and Y, for the proof see [76].

The five-form and two-form charge again give rise to BPS states that preserve half of the supersymmetries. These BPS states are called M-5 branes and M2-branes respectively.

The ten dimensional algebra of IIA supergravity is given by

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^i)_{\alpha\beta} P_i + \frac{1}{2}(C\Gamma_{ij})_{\alpha\beta} Z^{ij} + \frac{1}{5!}(C\Gamma_{ijklm})_{\alpha\beta} Z^{ijklm} \tag{5.19}$$

$$+ \frac{1}{4!}(C\Gamma_{ijkl}\tilde{\Gamma})_{\alpha\beta} Z^{ijkl} + (C\Gamma_i\tilde{\Gamma})_{\alpha\beta} Z^i + (C\tilde{\Gamma})_{\alpha\beta} Z \tag{5.20}$$

Since the central charges are anti-symmetric this suggests we can obtain the ten dimensional algebra by a dimensional reduction on one of the space directions. A more thorough study done by reducing the fields in the action confirms this. However as will be explained in the next section supergravity theories are the low energy limits of string theories. So there could be a kind of string theory with low energy effective action eleven dimensional supergravity which upon compactification gives IIA string theory.

5.3 Dual theories

Superstring theory has been quite successfully in explaining dynamics of quantumgravity. However as indicated in chapter 1, there are five consistent superstring theories in ten dimensions.

1. II A
2. II B
3. $E_8 \times E_8$ heterotic
4. $SO(32)$ heterotic
5. Type I

If the string tension is sent off to infinity the theories are approximated by four supergravity theories (The fifth ten dimensional supergravity theory is not the limit of any string theory). The other theories are called:

1. Non-chiral N=2 sugra (IIA)

2. Chiral N=2 sugra (IIB)

3. N=1 sugra/YM with $E_8 \times E_8$ gauge group

4. N=1 sugra/YM with $SO(32)$ gauge group

Type I and heteorotic string theory have coincident effective field theories. This is however at a non-perturbative level since the heterotic string is closed and oriented and the type I string theory contains open and closed unoriented strings.

We know that there is a unique eleven dimensional supergravity theory from which all other supergravity theories can be derived, except for IIB, which must be torodially compactified befored it can be derived from the eleven dimensional theory, as indicated very briefly in the previous section. The two form potential of string theory is naturally associated with strings. Or better, a two-form is the highest form on a world sheet, together with the one-forms induced by the coordinate functions they saturate all the possible physical fields. Eleven dimensional supergravity has a three-forms potential. This is naturally identified with a membrane [11] and [12]. So there is an eleven dimensional theory containing membranes. Unfortunately this doesn't imply that it is a membrane theory (but as always we can just try and see how far we can get with the membranes). But there is more, it can be shown that IIA string theory is a theory containing membranes compactified on a circle and IIB theory compactified on a circle equals this membranes theory compactified on a torus. The heterotic string theories can be obtained from the eleven dimensional theory by compactification on the orbifold $\mathbb{S}/\mathbb{Z}_2$. For detailed acounts of these duality see [74]and [83], and for the 11 dimensional origin of D-branes see [75].

The fact that string theory is only consistent in ten dimensions means that perturbative string theory (or more specifically, anomaly cancellation in string theory) only sees ten dimensions. There could be more dimensions invisible to perturbation theory.

Chapter 6

M-5 Branes

. . . , il primo, "Ragazzi che tagliatelle vi farei mangiare!", un vero slancio d'amore generale, dando inizio nello stesso momento al concetto di spazio, e allo spazio propriamente detto, e al tempo, e alla gravitazione universale, . . .
- Italo Calvino, Le Cosmicomiche -

Membranes can end on an M-5 brane as strings on D-branes ([71]). We know that an M-2 brane ending on an M5-brane reduces to a string ending on a D4-brane under double dimensional reduction. This double dimensional reduction removes one degree or no degree of all the forms describing the interaction. We know that a string ending on a D4-brane is described by a one form field on the brane. So the M2-M5 brane system is described by a two form and possibly one forms. Demanding self duality should remove the one-forms.
Supersymmetry gives a more rigorous derivation of this, I'll closely follow the argument by Sezgin in [67]. We know that the superspace of M-theory is given by $\mathbb{R}^{11|32}$ (we have eleven spatial and time directions and 32 spinor components). We know from section 5.2 that the two-brane and the five-brane are BPS states preserving half of the number of supersymmetry components. In other words the 2 brane has superspace $\mathbb{R}^{3|16}$ and the M-5 brane is given by $\mathbb{R}^{6|16}$. In 5 + 1 dimensions the minimum spinor has 8 components. So an M-2 brane ending on an M-5 brane is described by $N = 2$ supersymmetry. The eight scalars are of course the embedding coordinates. For the M-5 brane we have five embedding scalars. This leaves us six independent components. The only form that can satisfy this is a self dual three form curvature.

So A description of several (coincident) five-branes with M2-branes exchanged between them is obstructed by two problems. The first problem consist of finding the curvature of a non-abelian two form field, this will be dealt with in chapter 9. The second and by far the most difficult problem is finding an action which describes the theory. The naive approach of just writing down a Lagrangian is prohibited by the fact that the gauge transformations should take only chiral fields to chiral fields and by the fact that we can't find an invariant form that can be integrated to an action (see chapter 10). Chirality is at the classical level means that we must have a Lagrangian with a constraint ensuring the equations of motion are such that a chiral field remains chiral for all time. We can add a term, that has as an equation of motion the self-duality condition, by using a Lagrange multiplier. Which in the classical case already is enough.

$$L = \int F^2 + \lambda(F - *F)^2 \tag{6.1}$$

However if this Lagrangian is to be quantized the constraint has to be first class. There are several ways to proceed. Firstly we can content ourselves with just the equations of motion. For a single M5-brane this is done amongst others (see the references in op. cit.) in [42] [43] and [69], this method will not be treated here. The main reason for this is that the equations of motion are difficult to generalize since the underlying structure of the gauge theory is difficult to find. Secondly we can try to rewrite the secondary constraint $\lambda(F - *F)$ as a first class constraint this is done in [61] and [9], this method will be treated in the next section though there is some crucial proof missing. Thirdly we can break manifest covariance and try to write down an action that has as a primary constraint the self-duality condition right from the start this was done by [40]. This paper treats the time direction different from the space direction. According to General relativity we could equally well break covariance in one of the space directions (as is done in [66] and [3]), however since quantization 'prefers' time directions the self-duality constraint is only first class if we break covariance in the time direction. The details will be treated in the second section below. Fourthly we could start with the Lagrangian for an arbitrary two-form field and try to find a method to take the chiral gauge transformations apart. Then we can construct a Lagrangian with only the chiral gauge transformations which can be quantized properly. This method is due to Witten [84]. Since it is very easy to obscure the key issues we'll focus on the bosonic case in this chapter.

The same problem that appears for the M-5 brane in six dimensions appears for IIB theory in ten dimensions. It is again the problem to construct a chiral (2p + 1) curvature. Since gerbes describe only three form curvatures I won't mention this problem again. But if gerbes yield anything useful for five-branes, might try to construct 3-gerbes and see if they describe anything sensible in IIB theory.

6.1 Chiral Fields Using Auxiliary Fields

We start by writing down the action for an ordinary p form in $2n = 2(p+1)$ dimensions (p = even).

$$S = \int F \wedge *F = -\frac{1}{2n!}\int \mathrm{d}^{2n}x F_{[\alpha_1\ldots\alpha_n]}F^{[\alpha_1\ldots\alpha_n]} \tag{6.2}$$

As we have seen the self-duality condition can be included as a secondary constraint

$$S = \int \mathrm{d}^{2n}x F_{[\alpha_1\ldots\alpha_n]}F^{[\alpha_1\ldots\alpha_n]} - \Lambda(F_{[\alpha_1\ldots\alpha_n]} - F_{[\alpha_1\ldots\alpha_n]})(F^{[\alpha_1\ldots\alpha_n]} - F^{[\alpha_1\ldots\alpha_n]})$$

(6.3)

The term $\Lambda(F - *F)^2$ gives rise to a gauge invariance that is not present in the original gauge theory. The constraint should be first class. However it is the squared of a second class constraint. We could try to absorb the gauge freedom that is present in the Lagrangian but isn't physical for a chiral theory in an auxiliary field Λ.

Since the square of a second class constraint is not even regular. We'd best start with another second class constraint even though this breaks lorentz invariance

$$S = \int \mathrm{d}^{2n}x F_{[\alpha_1 \ldots \alpha_n]} F^{[\alpha_1 \ldots \alpha_n]} - \Lambda(F_{[\alpha_1 \ldots \alpha_n]} - F_{[\alpha_1 \ldots \alpha_n]}) \tag{6.4}$$

We now introduce an infinite set of auxiliary three form fields that make the action again covariant see also [54] or [9]

$$S = \int \mathrm{d}^6 x \Big(-\frac{1}{4} F_{\alpha\beta\gamma} F^{\alpha\beta\gamma} - \Lambda_1^{\alpha\beta\gamma}(F_{\alpha\beta\gamma} - F_{\alpha\beta\gamma}) + \sum_{n=0}^{\infty} (-1)^n \Lambda_{(n)}^{\alpha\beta\gamma} \Lambda_{\alpha\beta\gamma}^{(n+1)} \Big) \tag{6.5}$$

Where the Lagrange multipliers Λ are alternating self-dual and anti-self dual:

$$*\Lambda^{(n)}_{\alpha\beta\gamma} = (-1)^n \Lambda^{(n)}_{\alpha\beta\gamma} \tag{6.6}$$

These Lagrange multipliers span the whole space of three forms. Moreover (and often omitted) the series has to converge, in other words $|\Lambda^{(n)}| \sim \frac{f(x)}{n}$ see [9]. However as far as I know we're not allowed to make this assumption at the level of the Lagrangian. Only after variation this condition might turn out to be satisfied.
There is a serious problem aroused since this convergence restriction assures that we can describe the five-brane using only one auxiliary field. This is due to the fact that we can resum the Lagrange multipliers series to a single auxiliary field. In [60] the way to proof this was indicated, however they assumed that the series in the Lagrange multipliers always converges, which is definitely not the case. The best way to proceed is probably like in the case with ordinary convergent series, first do all the manipulations and afterwards check whether they were allowed.

If the five-brane is described by only one auxiliary field $a(x)$ we get the lagrangian from op. cit.

$$S = \int \mathrm{d}x \big[\frac{1}{4} F_{\mu\nu\rho} F^{\mu\nu\rho} - \frac{1}{(\partial a)^2} \partial^\mu a (F - *F)_{\mu\nu\rho} (F - *F)^{\nu\rho\lambda} \partial_\lambda a \big] \tag{6.7}$$

Let's have a look at the primary constraints. The constriant for $\partial_0 A_{\mu\nu}$ gives us no useful constraints, which was to be expected, however the canonical momentum for a is

$$\pi^a = \frac{\delta L}{\delta \partial_0 a} = \frac{g^{0\xi}\partial_\xi a + g^{\xi 0}\partial_\xi a}{(\partial a)^4}\partial^\mu a(F - *F)_{\mu\nu\rho}(F - *F)^{\nu\rho\lambda}\partial_\lambda a \tag{6.8}$$

$$- \frac{1}{(\partial a)^2} g^{\mu 0}(F - *F)_{\mu\nu\rho}(F - *F)^{\nu\rho\lambda}\partial_\lambda a \tag{6.9}$$

$$- \frac{1}{(\partial a)^2} g^{\mu\xi}\partial_\xi a(F - *F)_{\mu\nu\rho}(F - *F)^{\nu\rho 0} \tag{6.10}$$

If the fields are abelian this result can be simplified however one can already figure out that this equation is non-vanishing and thus we can take as a primary constraint

$$\pi^a = 0 \tag{6.11}$$

Which constraints the system to be self-dual.

6.2 Chiral Fields Breaking Covariance

Following the article of Schwartz [66]. We can introduce a split in the coordinates on the five-brane by writing $x^{\hat{\mu}} = (x^\mu, x^5)$, where $\mu = 0, \ldots, 4$. And a similar split for all the fields on the brane. Denote the two form field by A, with The self-duality condition now amounts to (Given that $F = \mathrm{d}A$ or in coordinates $F_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3\partial_{[\hat{\mu}}A_{\hat{\nu}\hat{\rho}]}$):

$$F_{\hat{\mu}\hat{\nu}\hat{\rho}} = \frac{1}{6\sqrt{-G}} G_{\hat{\mu}\hat{\alpha}} G_{\hat{\nu}\hat{\beta}} G_{\hat{\rho}\hat{\gamma}} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}\hat{\epsilon}\hat{\zeta}} F_{\hat{\delta}\hat{\epsilon}\hat{\zeta}} \tag{6.12}$$

There exists a solution:

$$F_{\mu\nu 5} = K_{\mu\nu}(G, F) \tag{6.13}$$

Taking the derivative yields a field equation:

$$\frac{1}{2}\epsilon^{\mu\nu\delta\epsilon\zeta}\partial_\delta K_{\epsilon\zeta} = \frac{1}{2}\epsilon^{\mu\nu\delta\epsilon\zeta}\partial_5\partial_\delta A_{\epsilon\zeta} = \partial_5 \tilde{F}^{\mu\nu} \tag{6.14}$$

This equation is induced by the lagrangian:

$$L = \frac{\sqrt{-G}}{-2G_5}\mathrm{tr}(G\tilde{F}G\tilde{F}) + \frac{1}{2}\tilde{F}^{\mu\nu}\partial_5 A_{\mu\nu} - \frac{1}{4}\epsilon_{\mu\nu\delta\epsilon\zeta}\frac{G^{5\delta}}{G^{55}}\tilde{F}^{\mu\nu}\tilde{F}^{\epsilon\zeta} \tag{6.15}$$

It would be convenient to make a lowest order approximation. This is easily done and yields:

$$S = \int \mathrm{d}t\mathrm{d}^5x(\partial_5 A^{ij} - F_{ij5})\frac{\epsilon^{ij5klm}}{6}F_{klm} \tag{6.16}$$

We still didn't use the fact that x^5 was a spatial direction. However we run in to troubles if it is spatial, since we need to impose the self-duality as a primary constraint. A short calculation shows that this condition is only imposed if we break manifest covariance in the time direction. For a proof use as an anzats the action

$$S = \int \epsilon^{0ijklm} F_{klm} (\partial_0 A_{ij} - \epsilon_{0ijabc} \epsilon^{abcklm} F_{klm}) \tag{6.17}$$

This induces for the canonical momemta:

$$\pi^{ij} = \frac{\partial L}{\partial(\partial_0 A_{ij})} = \epsilon^{0ijklm} F_{klm} \tag{6.18}$$

We can now take the constraint

$$0 = \pi^{ij} = \epsilon^{0ijdef} \partial_d A_{ef} = \epsilon^{aijdef} \partial_d A_{ef} - \partial_0 A_{ij} + \partial_i A_{j0} - \partial_j A_{0i} = *F - F \tag{6.19}$$

So $F - *F$ is a primary constraint.

6.2.1 Theorem *In the abelian case we can impose self-duality if we break manifest covariance in the time direction, i.e. if the action is for the lowest order*

$$S = \int \epsilon^{0ijklm} F_{klm} (\partial_0 A_{ij} - \epsilon_{0ijabc} \epsilon^{abcklm} F_{klm}) \tag{6.20}$$

6.3 Chiral Fields Using Line-Bundles

Only one article has appeared on this method [84], by Witten. I'll point at the main issues of this construction following closely the article of Witten. At least one attempt has been made to apply it to five-branes [20], but except for the fact that we can project the non-chiral parts of the partition function the construction of the partition function, which is the crucial point of this approach, was done in a totally different way.

For a free fermion on a Riemann surface of genus g the partition function can be written as:

$$Z = \sum_\alpha \left| \frac{\theta_\alpha}{\eta} \right|^2 \tag{6.21}$$

Where η is de Dedekind eta function. And θ_α is a theta function. The problem is how to pick out only the θ_α that belong to the chiral fields.

Chiral Scalars

Start with the action of a free scalar field taking values in U(1)

$$L = \frac{1}{8\pi} \int \mathrm{d}^2 x \sqrt{g} g^{ij} \partial_i \phi \partial_j \phi \tag{6.22}$$

Introduce a gauge field also taking values in U(1)

$$L = \frac{1}{8\pi} \int \mathrm{d}^2 x \sqrt{g} g^{ij} (\partial_i \phi + A_i)(\partial_j \phi + A_j) + \frac{i}{4\pi} \int \phi \epsilon^{ij} \partial_i A_j \tag{6.23}$$

With this lagrangian the gauge field A only couples to the chiral part of ϕ which can be seen by changing to complex coordinates, taking $\epsilon_{z\bar{z}} = i$

$$L = \frac{1}{4\pi} \int \mathrm{d}z \wedge \mathrm{d}\bar{z} (\partial_z \phi \partial \bar{z} \phi + 2\partial_z \phi A_{\bar{z}} + A_z A_{\bar{z}}) \tag{6.24}$$

Define the partition function

$$Z(A) = \int D\phi e^{-L} \tag{6.25}$$

A holomorphic line-bundle $\mathbb{L}$ over the space of gauge fields is defined by taking the trivial line-bundle with a connection such that we have covariant derivatives

$$D_{A_z} = \frac{\partial}{\partial A_z} + \frac{A_{\bar{z}}}{4\pi} \tag{6.26}$$

This connective structure is holomorphic since

$$[D_{A_z}, D_{A_z}] = 0 \tag{6.27}$$

It is easy to see that $D_{A_z} e^{-L} = 0$. This implies that the partition function is a holomorphic section of $\mathbb{L}$, i.e. $D_{A_z} Z = 0$.

If the chiral boson is defined on a Riemann surface R we can interpret the partition function as a section of a line bundle over $H^1(R, \mathbb{R})/H^1(R, \mathbb{Z})$, which is called the intermediate Jacobian of R.
The Lagrangian still contains fields of the wrong chirality, the fields of different chirality are decoupled however. Holomorphic factorization reduces the partition function to a sum of independent terms.

Chiral (2p +1)-forms

In the case of a chiral boson in p dimensions the same argument holds and we get that the partition function is a line bundle over the jacobian $H^{2p+1}(R, \mathbb{R})/H^{2p+1}(R, \mathbb{Z})$
Once we find a line bundle over the jacobian the partition function of the chiral boson is naturally defined.

After introducing the gauge field A the partition function depends on $\left|\frac{\theta(A,\tau)}{\tilde{\eta}(\tau)}\right|^2$.
The $\theta(A, \tau)$ are sections of all different line bundles on the jacobian over Σ. Moreover every line-bundle has one and only one holomorphic section (up to complex multiples). On the Jacobian there is a symplectic form which gives upon integration over the whole space 1. It also induces an kähler metric ω. This metric determines the chiral form partition function.
The problem reduces to finding a U(1) gauge field on the jacobian with the restriction that the curvature is $F = 2\pi\omega$. Since the Riemann surface is an identification the holonomy should be invariant under the symplectic group acting on the lattice vectors of the indentification. If l is a lattice vector we would like to choose $H(l) = 1$. This is however impossible. The maximal restriction we can demand is $H(l)^2 = 1$. Instead of the line-bundle $\mathbb{L}$ we are constructing the line bundle $\mathbb{L}^2$ which has 2ω as the first Chern class instead of the desired ω.

To construct the line bundle over the jacobian on a two dimensional surface take the Chern-Simons functional on a three dimensional manifold. Let M be a closed oriented three dimensional manifold and A a connection on a $U(1)$ bundle $\mathbb{B}$ over M. If the bundle is topological trivial the Chern-Simons functional is

$$CS(A) = \frac{1}{2\pi}\int_M \epsilon^{ijk}A_i\partial_j A_k \tag{6.28}$$

If the bundle is topologically non-trivial we pick an oriented four manifold with boundary M such that there is an extension of A and $\mathbb{B}$. We pick such an extension and take the first chern class (or instaton number)

$$CS_X(A) = \frac{1}{2\pi}\int_X \epsilon^{ijkl}\partial_i A_j \partial_k A_l \tag{6.29}$$

This integral is independent of the manifold X over which we do the extension up to an ambiguity of $2\pi\mathbb{Z}$. Since we take the exponential of i times the chern simons functional, the partition function will be well defined.
With this well defined Chern Simons functional we are able to define the holonomy on the desired line-bundle by

$$H(\gamma) = e^{iCS(A_\gamma)} \tag{6.30}$$

Where γ is a curve in the space of gauge fields on M and A_γ are the associated gauge fields. This bundle has first Chern class 2ω however we need to find a bundle with class ω.
It would just suffice to take $\frac{CS}{2}$ if this is defined. It turns out that in the case in which there is a spin structure on the manifold it is well defined.

This construction, though the cleanest, is difficult to work with. Also the extension to non-abelian fields is highly non-trivial, especially if we replace the line-bundles by (special) gerbes we have to check of every statement whether it still holds. That's why I'd like to follow the more naive approach of breaking manifest covariance.

6.4 Non-abelian Chiral fields (1)

Given that we have found a abelian two form chiral theory can we construct a non-abelian theory out of it by a deformation? This important question has been answered in [8].

Start with a free action

$$S_0^{(0)}[\phi^i] \tag{6.31}$$

with gauge symmetries

$$\delta_z\phi^i = R_\alpha^{(0)i} z^\alpha \tag{6.32}$$

With the Noether identities

$$\frac{\delta S^{(0)}}{\delta\phi^i} R_\alpha^{(0)i} = 0 \tag{6.33}$$

We can perturb the free action by a series in the coupling constant ϵ

$$S_0 = S_0^{(0)} + \epsilon S_0^{(1)} + \epsilon^2 S_0^{(2)} + \dots \tag{6.34}$$

We want the deformed action to be gauge invariant under the transformations:

$$R_\alpha^i = R_\alpha^{(0)i} + \epsilon R_\alpha^{(1)i} + \epsilon^2 R_\alpha^{(2)i} + \dots \tag{6.35}$$

It should be consistent in the sense that the Noether identities should still hold:

$$\frac{\delta S}{\delta \phi^i} R_\alpha^i = 0 \tag{6.36}$$

If the theory is reducible as in the case with chiral two-forms we have an additional constraint coming from the non-independent gauge transformations in the free theory

$$R_\alpha^{(0)i} z_A^{(0\alpha)} = 0 \tag{6.37}$$

The deformations fall apart into three classes

1. The added deformations are not gauge invariant.
2. The added deformations are gauge invariant under the original gauge transformations.
3. The added deformations are entirely gauge invariant.

Let's assume that there is a solution to the master equation denoted by $S^{(0)}$, i.e. $(S^{(0)}, S^{(0)}) = 0$, where $(,)$ is the anti-field bracket. There exists a deformation

$$S = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + \dots \tag{6.38}$$

The master equation for S, i.e. $(S, S) = 0$ can be expanded in powers of ϵ, this yields:

$$(S^{(0)}, S^{(0)}) = 0 \tag{6.39}$$

$$(S^{(0)}, S^{(1)}) = 0 \tag{6.40}$$

$$2(S^{(0)}, S^{(2)}) + (S^{(1)}, S^{(1)}) = 0 \tag{6.41}$$

$$\vdots$$

The first equation holds of course by assumption. Using the initial assumption translated in $(S^{(0)}, (S^{(0)}, .)) = 0$ we find after application to the second formula a cocycle condition for $S^{(0)}$. This in turn implies that $S^{(0)}$ is a cocycle of the diffential $s^0 = (S^{(0)}, .)$. If $S^{(1)}$ is also coboundary we can conclude that $S^{(1)}$ is a trivial deformation.
This implies that the third equation reduces to $(S^{(0)}, S^{(2)}) = 0$. This in turn implies that the deformations up to second order in ϵ are also independent of the anti-fields. Continuing we can show that the deformations are anti-field

independent for all orders in the coupling constant. Since only the anti-field dependent terms perturb the gauge transformations we have to draw the conclusion that non-abelian gauge fields can't be described by a deformation of the abelian theory. The proof of the many claims made can be found in [8].

After the deformations have been translated to local functionals. The cohomology groups of the double complex spanned by s^n and d^n, i.e. $H^{i,j}$ are the possible deformations. The procedure above tells us however that we only have to look at the groups $H^{0,D}$ where we denote the dimensionality of space-time by D. Fortunately these cohomology groups can be computed in the case of a five-brane and shown to vanish, see op. cit.
This result puts enormous restrictions on the things to be done. The immediate result is that we can't construct a non-abelian chiral theory using sheaves (or principal bundles). There is however a loophole in this argument. Clearly changing the mathematical structure of the theory is not a deformation. So if define our fields as connections taking value in some other structure the argument may not hold. To be more specific we have seen that sheaves that assign to every open set a group give rise to gauge fields or connections. What we could try to do is to assign a structure different from a group. Adding additional restriction to the structure of a group won't make a difference, for example a sheaf of rings will give (up to some additional restrictions) the same physics as a sheaf of groups. However we could try to relax the axioms of a group, most straightforwardly we could try to take categories as our structure. This is indeed a good choice for the structure and in chapter 7 we will see that we can define objects that assign to open sets a category which is a variation of the definition of a sheaf and it will be called a stack. We can define connections on stacks however a connection on a stack is far to general for our purposes. We can restrict the definition of a category a little to a so called groupoid, which will be introduced in chapter 8. A groupoid is a bit more general object than a group but a bit more restrictive then a category. We can restrict stacks to groupoids with some additional 'non-nastiness' requirements we end up with objects called gerbes. In the same way as on torsors we can introduce connections on gerbes. A connection turns out to be described by two gauge fields.

6.5 Anomaly Cancellation

The lie-group in which the fields take values can normally be determined by anomaly cancellation. However anomaly cancellation raises some severe questions. The dimension of the gauge group should scale as n^3 with n the number of coincident M5-branes. There are indications it should be of the form $4n^3-3n$ [77]. The n^3 behavior can't be explained from group theory, so we should be looking at other objects than sheaves. Though the scaling law puts enormous restrictions on the objects that can be used I'll ignore anomaly cancellation in the rest of this thesis and instead I'll try to work out the gauge theory for gerbes.

Chapter 7

Fibred Categories and Stacks

This chapter is highly technical in nature. It can be skipped if the reader is willing to see a gerbe as some magical kind of sheaf of groupoids or if the reader only want to see the physical applications and doesn't care about a strict definition of a gerbe. This introduction to stack closely follows chapter 1 of the book by Breen [16]. An introduction which also follows this book closely but uses a bit more categorical approach is chapter 3 of [55]. The standard references are [35] or [1]

For sets, groups, rings and module a sheaf is a natural object. The natural object combining groups, rings etc. is a category. So why haven't we defined a sheaf of categories? We can just give the same definition for a sheaf of category as we have given for a sheaf of groups etc., but this turns out to be to naive and not natural.

7.1 Sheaves and categories (2)

We begin by constructing a natural 'presheaf of categories', it is called a fibered category. This is a presheaf up to isomorphisms. These additional isomorphisms are just the space we need to make a natural 'presheaf of categories"

7.1.1 Definition *Let X be a topological space. Suppose that with every open set $U \subset X$ we have associated a category $F(U)$, for any inclusion map $i : V \rightarrowtail U$ there is a functor $i^* : F(U) \to F(V)$ and for any $W \overset{i}{\rightarrowtail} V \overset{j}{\rightarrowtail} U$ there is a natural isomorphism $\tau_{i,j} : (ij)^* \to j^* i^*$. This is called a fibered category if the following diagram commutes:*

$$\begin{array}{ccc} (ijk)^* & \xrightarrow{\tau_{ij,k}} & k^*(ij)^* \\ \big\downarrow{\scriptstyle \tau_{i,jk}} & & \big\downarrow{\scriptstyle k^*\tau_{i,j}} \\ (jk)^*i^* & \xrightarrow[\tau_{j,k}i^*]{} & k^*j^*i^* \end{array} \tag{7.1}$$

A functor between fibred categories F and G consists of functors $F(U) \overset{\Phi_U}{\to} G(U)$ such that for every morphism $V \overset{\phi}{\to} U$ there is a natural isomorphism $\phi^* \circ F(U) \overset{\alpha_\phi}{\to} F(V) \circ \phi^*$. Such that these transformations are compatible with respect to the τ's. In other words for inclusions

$$W \overset{\phi}{\rightarrowtail} V \overset{\psi}{\rightarrowtail} U \tag{7.2}$$

The diagram

$$\begin{array}{ccccc} & \Phi_W(\psi\phi)^* & \xrightarrow{\alpha_{\psi\phi}} & (\psi\phi)^*\Phi_U & \\ \swarrow{\scriptstyle \Phi_W\tau} & & & & \searrow{\scriptstyle \tau\Phi_U} \\ \Phi_W\phi^*\psi^* & \xrightarrow{(\alpha_\phi)\psi^*} & \phi^*\Phi_V\psi^* & \xrightarrow{\phi^*\alpha_\psi} & \phi^*\psi^*\Phi_U \end{array} \tag{7.3}$$

should commute.

Fibred categories with their morphisms and so called fibred transformations form a 2-category. Since more details would only distract the reader I'd like to refer to [16] and [55] for more details.

If every Φ_U is an equivalence of categories we call the morphism (Φ, α) a strong equivalence. If every Φ_U is fully faithful and 'locally surjective' on object (in other words for every object $g \in G(U)$ and every $x \in U$ there exists a neighborhood V, with $x \in V \overset{\psi}{\rightarrowtail} U$ and an object $b \in F(V)$ such that $\Phi_V(b)$ is isomorphic in $G(V)$ to $\psi^*(a)$) we call (Φ, α) a weak equivalence.
The reader should for a better understanding compare these definitions with the definitions of equivalences of groupoids in chapter 8.

A fibred category naturally defines a presheaf. Since if F is a fibred category over X and $a, b \in F(U)$ and if $V \overset{\phi}{\rightarrowtail} U$ then the assignment

$$V \mapsto \mathrm{Hom}_{F(V)}(\phi^*a, \phi^*b) \tag{7.4}$$

defines a presheaf on U. This definition strongly hints at the definition of a lien functor to be given in chapter 9. It is denoted by $\underline{\mathrm{Hom}}_F(a, b)$.
Any morphism of fibred categories $\Phi : F \to G$ induces a morphism of presheaves on U:

$$\Phi_{a,b} : \underline{\mathrm{Hom}}_F(a, b) \to \underline{\mathrm{Hom}}_G(\Phi_U(a), \Phi_U(b)) \tag{7.5}$$

To check this statement it's important to note that the restriction maps of the presheaf are given for $W \overset{\psi}{\rightarrowtail} V \overset{\phi}{\rightarrowtail} U$ by:

$$\underline{\mathrm{Hom}}_{F(V)}(\phi^* a, \phi^* b) \xrightarrow{\psi^*} \underline{\mathrm{Hom}}_{F(W)}(\psi^*\phi^* a, \psi^*\phi^* b) \xrightarrow{\tau^*} \underline{\mathrm{Hom}}_{F(W)}((\phi\psi)^* a, (\phi\psi)^*) \tag{7.6}$$

Where τ^* is the usual conjugation by τ.
For the presheaf morphisms we can use the fact that the component of $\Phi_{a,b}$ at $\phi V \rightarrowtail U$ is the composition:

$$\underline{\mathrm{Hom}}_{F(V)}(\phi^* a, \phi^* b) \xrightarrow{\Phi_V} \underline{\mathrm{Hom}}_{G(V)}(\Phi_V \phi^* a, \Phi_V \phi^* b) \xrightarrow{(\alpha_\phi)_*} \underline{\mathrm{Hom}}_{G(V)}(\phi^* \Phi_U a, \phi^* \Phi_U b) \tag{7.7}$$

Where $(\alpha_\phi)_*$ is conjugation by α_ϕ

7.2 Stacks

Essential in defining gerbes is the notion of a stack. This is a fully sheafified version of the more naive notion of a 'sheaf of categories'. I'll now treat both stacks and sheafs of categories using the so called effective descent condition. This approach exactly coincides with defining a the category of descent data as it is done in [55].

Consider the following gluing law on objects in a fibred category, it is known as the effective descent condition. Let U_α be an open cover of an open set $U \subseteq X$ and suppose there are given a family of objects $a_\alpha \in F(U_\alpha)$ and a family of isomorphisms

$$\phi_{\alpha\beta} : a_{\beta|U_{\alpha\beta}} \to a_{\alpha|U_{\alpha\beta}} \tag{7.8}$$

in $F(U_{\alpha\beta})$ satisfying a cocycle identity $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ also called the descent condition. This descent condition is said to be effective if there exists an object $a \in F(U)$ together with a family of isomorphims $\psi_\alpha : a_{|U_\alpha} \to a_\alpha$ such that the restrictions to the various $U_{\alpha\beta}$ are compatible with the $\phi_{\alpha\beta}$'s and the τ's from the definition of a fibred category. The effectiveness of the descent condition may be viewed as a gluing condition on the objects of F a separate gluing condition on the morphisms in F finishes the definition of a stack.
For the gluing condition we require that for any pair of objects $a, b \in \mathrm{Obj}(F(U))$ and any open cover (U_α) of U the ordinary sheaf axioms for the presheaf $\underline{\mathrm{Hom}}(a, b)$ given by the exactness of the sequence

$$\underline{\mathrm{Hom}}(a, b) \longrightarrow \underline{\mathrm{Hom}}(a_\alpha, b_\alpha) \rightrightarrows \underline{\mathrm{Hom}}(a_{\alpha\beta}, b_{\alpha\beta}) \tag{7.9}$$

hold as soon as the identification with τ has been performed.

A more restrictive condition on the on the objects yields a sheaf of categories. If we maintain the gluing condition for the arrows and as a gluing condition on the objects we take as gluing condition

$$a_{\beta|U_{\alpha\beta}} = a_{\alpha|U_{\alpha\beta}} \tag{7.10}$$

7.2.1 Definition *A fibred category having these gluing conditions is called a sheaf of categories.*

See [35] chapter 2.2 for more details.

7.2.2 Definition *A fibred category F over X is called a prestack if for any objects $a, b \in \mathrm{Obj}(F(U))$ the presheaf $\underline{\mathrm{Hom}}_F(a, b)$ is a sheaf.*

Chapter 8

Groupoids

We have already seen that connection theory can't be altered by changing the manifold structure of principal bundles but we have to change the algebraic structure, i.e. the sheaf of groups. You could try to start with assigning an arbitrary category to every open set, but there is a simpler generalization of groups that does the trick. This object is called a groupoid.
There are several ways to arrive at groupoids and although the application of groupoids started in the 1970's and the 80's they were already introduced in 1926 by H. Brandt. I will give some motivating examples before starting with a definition.

8.1 Introduction and definition

Several ways lead to the definition of a groupoid. I'll first treat an intuitive way based on the thing we already now from elementary group theory. At the end I'll briefly give the categorical way, which is less cumbersome but more abstract. I'll first give the following motivating example. This example based approach is due to weinstein [82]. Other sources on groupoids are [22], [53] and [57]. The

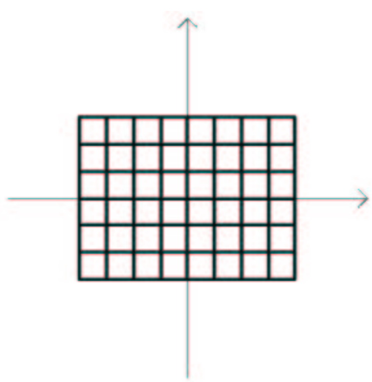


Figure 8.1: A set with automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

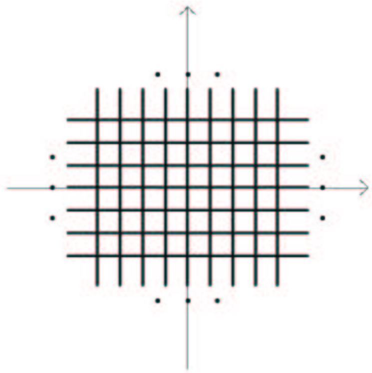


Figure 8.2: An other set with automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

set from figure 8.1 has the automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This is however the same automorphism group as the set from figure 8.2. The underlying set is on the contrary totally different. Although the automorphism group is a good structure to discern different spaces (Given a covering map $\pi X \to Y$. The set of automorphism of the covering or deck transformations (These are automorphisms of X that leave Y invariant) is isomorphic to $\pi_1(Y, y_0)/\pi_\sharp\pi_1(X, x_0)$), it fails here completely (Where $\pi_\sharp$ is the induced map in homotopy, its image are the classes of loops starting at x_0 in X that lift to loops starting at y_0 in Y). So the thing to do is to form a structure similar to a group but also taking into account the underlying set of object on which the group acts. We could try to loosen the restrictions of a group, we could consider defining a less restrictive multiplication. Two sets are needed. One set of objects G_0 and one set of arrows G_1. Moreover we need something the tell us at which point an arrow begins and at which point an arrow ends. We have two maps doing this for us, called the *source* and the *target maps* denoted by α and β. Moreover we need a composition $\circ$ of arrows (a multiplication). For a group the arrows have to satisfy the following well known properties in a unfamiliar appearence.

1. The starting point of two composed arrows is just the starting point of the first arrow (and similarly for the endpoint).
2. The multiplication is associative and is always defined, this means that all the begin and end points are the same, i.e. G_0 consists of one point.
3. Every arrow has an inverse.
4. There is a unit element.

The only requirement we can reasonably drop is the fact that the multiplication is always defined. The set of object G_0 can now consists of more than one element. We could write down the exact definition of a groupoid at once. However I'll give a categorical approach first:

A category has to little structure to be of much use. The most straightforward restriction is to demand every arrow invertible. We can now copy the definition of a category from appendix A, add the condition on the inverses. Then it is easily seen that this category coincides with the definition of a groupoid given above. Writing everything down more explicitly:

8.1.1 Definition *A Groupoid consists of two sets G and G_0, the groupoid and the base, two maps $\alpha, \beta : G \to G_0$ called the source and target projections, a map $\epsilon : G_0 \to G, x \mapsto \tilde{x}$, called the object inclusion map, and a multiplication $\circ : G \times_{G_0} G \to G$, where $G \times_{G_0} G = \{(\eta, \epsilon) \in G \times G | \alpha(\eta) = \beta(\epsilon)\}$ is a fibred product. The maps are subject to the conditions:*

1. $\alpha(\eta\epsilon) = \alpha(\epsilon)$ & $\beta(\eta\epsilon) = \beta(\eta)$ $\forall$ $(\eta, \epsilon) \in G \times_{G_0} G$.
2. $\zeta(\eta\epsilon) = (\zeta\eta)\epsilon$ $\forall$ $\zeta, \eta, \epsilon \in G$ *such that the multiplication is defined.*
3. $\alpha(\tilde{x}) = \beta(\tilde{x}) = x$ $\forall$ $x \in G_0$.
4. $\epsilon\widetilde{\alpha(\epsilon)} = \epsilon$ & $\widetilde{\beta(\epsilon)} = \epsilon$ $\forall$ $\epsilon \in G$.
5. *Every $\epsilon \in G$ has an inverse ϵ^{-1} such that $\alpha(\epsilon^{-1}) = \beta(\epsilon)$, $\beta(\epsilon^{-1}) = \alpha(\epsilon)$ and $\epsilon^{-1}\epsilon = \widetilde{\alpha(\epsilon)}$, $\epsilon\epsilon^{-1} = \widetilde{\beta(\epsilon)}$.*

Given a point x, the set of arrows given by $\alpha^{-1}(x)$ is called the α-fibre of x, it is the set of arrows that map from x and will be denoted by G_x. The definition of the β-fibre is obvious and will be denoted by G^x. The set G_x^x is a group and is called the vertex group or the isotropy group at x.

A *Lie groupoid* is a groupoid G together with the structure of a smooth, Hausdorff manifold on the base G_0 and the structure of a smooth manifold on G (non-Hausdorff, non-second-countable) such that all structure maps are smooth, the source projection is also a submersion and the fibres are Hausdorff.

8.1.1 Example *Any set is a groupoid if all the arrows are units. The set of arrows is now the base. It is called the unit groupoid.*

8.1.2 Example *Any group is a groupoid by taking for the base one point. The set of arrows are the group elements and the multiplication of the arrows is the group multiplication. Moreover any group is the vertex group of a groupoid.*

8.1.3 Example *Any manifold M gives, apart from the unit groupoid, rise to a Lie groupoid called the pair groupoid of M. The arrows are given by the set $M \times M$. The source and the target projections are projections on the first and the second factor.*

8.1.4 Example *Let $G \times G_0 \to G_0$ be an action of a group G on a set G_0. We can give $G \times G_0$ the structure of a groupoid with base G_0 in the following way. The source map is the first projection and the target projection is the action of the group on G_0. The multiplication is defined by $(g', x')(g, x) = (g'g, x)$*

8.1.5 Example *The holonomy groupoid introduced in section 2.2 is indeed a groupoid. Moreover it is even a Lie-groupoid as the reader can check for himself. The vertex groups are the holonomy groups of the points in the manifold as can be found in for example [24], [47] and [59]*

8.1.6 Example *Given a principal bundle $G \rightarrowtail P \twoheadrightarrow M$ we define the gauge groupoid ($\mathrm{Gauge}(P)$)associated to P to be the Lie groupoid for which the manifold of arrows is the orbit space of the diagonal action of G on $P \times P$. The source and the target map are obviously the projections on the first and the second component. The multiplication is induced by the multiplication from the pair groupoid, in other words the morphism $P \times P \to \mathrm{Gauge}(P)$ is a homomorphism from the pair groupoid.*

8.2 Morphisms of Groupoids

8.2.1 Definition *A morphism $G \to G'$ between groupoids is a pair of maps $\gimel : G \to G'$ and $\gimel_0 : G_0 \to G'_0$ such that:*

$$\begin{array}{ccc} G & \xrightarrow{\gimel} & G' \\ \downarrow{\scriptstyle \alpha/\beta} & & \downarrow{\scriptstyle \alpha'/\beta'} \\ G_0 & \xrightarrow{\gimel_0} & G'_0 \end{array} \qquad \gimel(gh) = \gimel(g)\gimel(h) \tag{8.1}$$

Commutes seperately for α, α' and β, β'. A homomorphism between Lie groupoids G and H is a functor $\gimel : G \to H$ that is smooth both on objects and arrows.

If $\gimel_0 = \mathrm{id}_{G_0}$ we say that $\gimel$ is a base-preserving morphism.

8.2.2 Definition *To fix notation we'll write $\gimel_g$ for the restriction of $\gimel$ to $G_g \to G'_{\gimel_0(g)}$ and similar definitions for $\gimel^g$ and $\gimel_g^h$.*

1. *A morphism of groupoids $\gimel : G \to G'$ is called piecewise-surjective (respectively -injective, -bijective) if the morphism is fiberwise surjective (injective or bijective). Stated more precise, if $\gimel_g^h : G_g^h \to G'^{\gimel_0(h)}_{\gimel_0(g)}$ is surjective (respectively injective, bijective).*

2. *$\gimel$ is base-surjective (respectively -injective, -bijective) if $\gimel_0$ is sujective (respectively injective, bijective).*

3. *Two Groupoids are called isomorphic if there are two homomorphisms $\gimel : G \to H$ and $\aleph : H \to G$ such that $\gimel \circ \aleph = \mathrm{id}_H$ and $\aleph \circ \gimel = \mathrm{id}_G$.*

8.2.3 Definition *A subgroupoid of a groupoid G is a pair of subsets $G' \subseteq G$ and $G'_0 \subseteq G_0$ such that $\alpha(G') \subseteq G'_0$, $\beta(G') \subseteq G'_0$, $\tilde{a} \in G'\ \forall\ a \in G'_0$ and G' is of course closed under the multiplication and inversion in G. A subgroupoid is called wide if $G'_0 = G_0$ and full if $G'^h_g = G^h_g$*

8.2.1 Example *The inner subgroupoid is the subgroupoid of all the vertex-groups* $\mathrm{V}(G) = \bigcup_{g \in G_0} G_g^g$

8.2.4 Definition *A normal subgroupoid of a groupoid G is a wide subgroupoid H such that for any $h \in \mathrm{V}(H)$ and $g \in G$ with $\alpha g = \alpha h = \beta h$ we have $ghg^{-1} \in H$.*

8.2.5 Definition *Let $\gimel : G \to G'$ be a morphism of groupoids. The kernel of $\gimel$ is the set $\{g \in G \mid \exists\ g_0 \in G'_0,\ \gimel(g) = \tilde{g_0}\}$*

Just as in group theory a subgroupoid is normal if and only if it is the kernel of a morphism.

8.2.6 Definition *For any groupoid G, the map $[\beta, \alpha] : G \to G_0 \times G_0, g \mapsto (\beta(g), \alpha(g))$ is a morphism over the base. It is called the anchor of G.*

8.2.7 Definition *Let G be a groupoid with base G_0, it is called connected or transitive if its anchor $[\beta, \alpha]$ is surjective. It is called totally disconnected or totally intransitive if the image of $[\beta, \alpha]$ is the diagonal imbedding Δ_{G_0} of $G_0 \times G_0$.*

The anchor defines an equivalence relation on the base. The equivalence class containing $g \in G_0$ is called the transitivity component of G containing a.

8.2.8 Definition *A homomorphism $\gimel : G \to H$ between two groupoids is called a strong equivalence if there exists a homomorphism $\aleph : H \to G$ and transformations $T : \gimel \circ \aleph \to \mathrm{id}_H$ and $S : \aleph \circ \gimel \to \mathrm{id}_G$.*

8.2.9 Definition *A homomorphism $\gimel : G \to H$ between two Lie groupoids is called a weak equivalence if it satisfies the following conditions:*

1. *The map $\beta \circ \mathrm{pr}_1 : H \times_{H_0} G_0 \to H_0$ defined by sending a pair (h, x) in the fibred product ($\alpha(h) = g(x)$) to $\beta(h)$ is a surjective submersion*

2. *The square*

$$\begin{array}{ccc} G & \xrightarrow{\gimel} & H \\ {\scriptstyle (\alpha,\beta)}\downarrow & & \downarrow{\scriptstyle (\alpha,\beta)} \\ G_0 \times G_0 & \xrightarrow{\gimel \times \gimel} & H_0 \times H_0 \end{array} \tag{8.2}$$

   is a fibred product of manifolds.

8.2.10 Definition *Two groupoids are called Morita equivalent if there is a third groupoid which is weakly equivalent to the two groupoids.*

8.2.2 Example *If G is a transitive groupoid. All the vertex groups are isomorphic and the groupoid is weakly equivalent to this group. So Morita equivalent transitive groupoids have the same vertex groups. This implies that two Morita equivalent gerbes will have the same lien functor. The lien functor and this proposition (Proposition (IV 5.2.5 from [35]) will be treated in the next chapter. The proof of this statement can be found in [57], proposition 5.14. Or can be done an exercise using that the target map $G_x \to G_0$ is a surjective submersion for a transitive groupoid. And the inclusion $G_x \to G$ is a weak equivalence (Proof!).*

8.3 Lie Algebroids

Just as in the theory of Lie groups there is associated a tangent space to a Lie groupoid. A different characterization of a Lie groupoid is given by differential structures on the base and the arrows and moreover the condition that the groupoid is locally trivial.

Let G be a groupoid. If $h : x \to y$ is an arrow in G. We can make an diffeomorphism of alpha fibres by $R_h : \alpha^{-1}y \to \alpha^{-1}x$, where $R_h(g) = g \circ h$. If we look at the tangent-space of the alpha fibre:

$$T^\alpha(G) = \ker(\mathrm{d}\alpha) \subset T(G) \tag{8.3}$$

If we take the alpha fibre tangent space at point $g \in \alpha^{-1}x$. Our diffeomorphism induces an map of tangent spaces:

$$\mathrm{d}R_h : T^\alpha_g \to T^\alpha_{g \circ h} \tag{8.4}$$

The sections of the tangent space T^α are the vector fields on G_1 tangent to the alpha or source fibres.
This means that it is an involutive subbundle of $T(G)$ and in particular the vector fields form a Lie-subalgebra of all vector fields on G_1. We can now define right invariant vector fields. Let $g,\ h \in G_1$ such that the can be composed (i.e. $\beta(h) = \alpha(g)$). Then an invariant vector field $X \in \Gamma(T^\alpha(G))$ is a vector field such that

$$X_{gh} = X_g h \qquad \forall\ g,\ h \in G_1 \times_{G_0} G_1 \tag{8.5}$$

The invariant vector fields define again a Lie subalgebra. An invariant vector field is uniquely determined by it's restriction to the unit maps $\{\tilde{x} \mid x \in G_0\}$, in the same way an invariant vector field in the tangent space of a lie group is determined by it's restriction to the unit. Denote the tangent space of the alpha fibre at arrow $\gamma : x \to y$ by $T^\alpha_{(x,\gamma)}(G)$. It's a notation that will become useful when we try to interpret the formalism of connections on gerbes.

Not all Lie algebroids arise in this way. There is a more natural definition of Lie algebroids independent of groupoids. When one uses this defintion one

can prove that Lie's third theorem (to every Lie algebra there is an associated Group) doesn't hold for Lie algebroids and Lie groupoids.
The basic theory on Lie algebroids which includes the what is used in this thesis, can be found in [53] and [57].

Chapter 9

Gerbes

As a sheaf is the natural object for 2-form curvatures a gerbe turns out to be the natural object for 3-form curvatures. All three definitions of a gerbe that are know at this moment are complicated at the technical level. To understand the main idea of connections on gerbes it is enough to assume that a gerbe is a manifold with a groupoid. Then we can define the holonomy along an infinitesimal curve to be a multiplication in the groupoid. With this in mind one can already read section 9.4 which deals with connections on trivial gerbes and derives the formulas that are likely to appear in physics.

It's time to have a look at the three ways of defining gerbes.

9.1 Definition of a gerbe

A gerbe is the geometric realization of the second čech Cohomology class. The definition of a gerbe is by now straightforward, though a cocyclic description will still take some time and requires a lot of technical knowledge. The definition closely follows [55].

9.1.1 Definition *A gerbe on X is a stack of groupoids G on X with the properties:*

1. *It is non empty: $X = \bigcup\{U \mid G(U) \neq \emptyset\}$*
2. *It is transitive:*

   Given objects $a, b \in G(U)$, any point $x \in U$ has a neighborhood $V \subseteq U$ for which there is at least one arrow $a_{|V} \to b_{|V}$ in $G(V)$.

The first requirement is essential since we didn't require a fibred category to be non-empty.

9.1.2 Definition *A gerbe is called neutral if the fiber $G(X)$ is non-empty, in other words, there is a global section.*

9.1.1 Example *The stack $Tors(G)$ on X associated to a sheaf of groups G on X is a gerbe. By definition it is non-empty. The base of the groupoid reduces to a point and the group action, which is now just composition of arrows is of course transitive due to property 2.*

Cocycle description

Also gerbes can be described using cocycles. Before this can be done some technical choices will have to be made. We first have to define a local trivialization. This subsection follows [16] very closely.

9.1.3 Definition *To define a labelling take an object $x_i \in G(U_i)$. Since the group $\mathrm{Aut}(x_i)$ is non-empty we can find groups G_i such that we can define a isomorphism of sheaves*

$$g_i : G_i \to \underline{\mathrm{Aut}}(x_i) \tag{9.1}$$

The gerbe G is now called relevant to the family of groups $\{G_i\}_{i\in I}$. The choice of the x_i is called a labelling of the gerbe.

We still have to make a choice for the arrows, called a decomposition of the gerbe. For any $i, j \in I$ take an arrow

$$\phi_{ij} : x_j \to x_i|_{U_{ij}} \tag{9.2}$$

Provided that

$$\phi_{ii} = \mathrm{id}_{x_i} \tag{9.3}$$

Since the gerbe is transitive there is always a neighborhood $V \in U_{ij}$ such that we can find such a ϕ on V unfortunately this doesn't guarantee that this arrow exists on U_{ij}, assuming it exists will result in constructing a geometric realization of something less than the second čech cohomology group (actually it is the cohomology group of to the nerve of the cover of X).

The first property in definition 9.1.1 simply states that any gerbe is relevant to some family of groups. A $G-gerbe$ a gerbe for which the G_i's are simply the restrictions to the U_i of some sheaf of groups G.

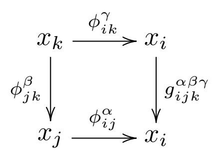
Let $(U_i)_{i\in I}$ be a cover of X and $(U_{ij}^{\alpha})_{\alpha\in I_{ij}}$ an open cover of U_{ij}. A labelling x_i and a decomposition ϕ_{ij}^{α} with respect to the same x_i is called a labelled decomposition.

We can now assign G_i valued cocycles to a labelled decomposition of a gerbe relevant to the family $\{G_i\}$. The morphisms ϕ_{ij}^{α} induce isomophisms λ_{ij}^{α} in the usual way by:

$$\lambda_{ij}^{\alpha} = (\phi_{ij}^{\alpha})_* : \underline{\mathrm{Aut}}(x_j)|_{U_{ij}^{\alpha}} \to \underline{\mathrm{Aut}}(x_i)|_{U_{ij}^{\alpha}} \tag{9.4}$$

$$g_j \mapsto (\phi_{ij}^{\alpha})^{-1} \circ g_j \circ (\phi_{ij}^{\alpha}) \tag{9.5}$$

There is no compatibility relation between the ϕ_{ij}^{α}'s. The obstruction to a compatibility is measured by $g_{ijk}^{\alpha\beta\gamma}$, it is defined such that it makes the following diagram commute:

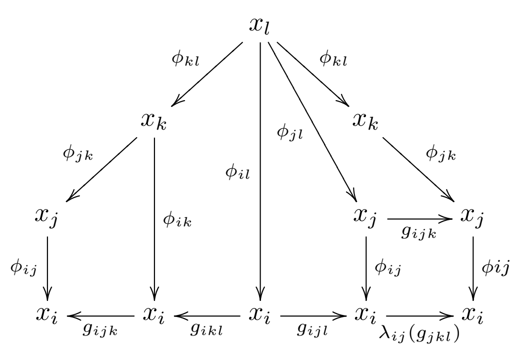


(9.6)

If the gerbe is relevant the arrows $g_{ijk}^{\alpha\beta\gamma}$ can be viewed as a section of the sheaf G_i on the open set $U_{ijk}^{\alpha\beta\gamma}$. The arrows λ_{ij}^{α} can be viewed a sections of the sheaf of isomorphisms $\underline{\mathrm{Iso}}(G_i, G_j)$.
The family of pairs $(\lambda_{ij}^{\alpha}, g_{ijk}^{\alpha\beta\gamma})$ satisfy cocycle conditions which make it a 2-cochain. A derivation can be found in op. cit. I'll merely state the results for completeness.
The first cocycle condition states that conjugation of $g_{ijk}^{\alpha\beta\gamma}$ is the same as composing with the composed arrow of diagram 9.6 and it's inverse, ie:

$$\lambda_{ij}^{\alpha} \circ \lambda_{jk}^{\beta} = i_{g_{ijk}^{\alpha\beta\gamma}} \circ \lambda_{ik}^{\gamma} \tag{9.7}$$

Where $i_g(h) = ghg^{-1}$ is the conjugation map.
The second cocycle condition follows from the compatibility of diagram 9.6 which is a of repeated application of the ϕ maps:



(9.8)

Since the $x_l \to x_i$ along the diagonal edges coincide, the maps $x_l \to x_i$ along the base also must coincide. This means that the second cocycle identity can be written down as:

$$\lambda_{ij}(g_{jkl})g_{ijl} = g_{ijk}g_{ikl} \tag{9.9}$$

Where the greek indices are omitted for readability.

Due to equation 9.3 we have the following normalization conditions on $(\lambda_{ij}^{\alpha}, g_{ijk}^{\alpha\beta\gamma})$.

$$\lambda_{ii} = \mathrm{id}_{G_i} \tag{9.10}$$
$$g_{iij}^{\alpha\alpha} = \mathrm{id}_{x_i} \tag{9.11}$$
$$g_{ijj}^{\alpha\alpha} = \mathrm{id}_{x_i} \tag{9.12}$$

Of course there could be a second labelled decomposition if the gerbe is also relevant to the family $\{G'_i\}$. We can find a common refinement renaming this refinement we can without loss of generality use the same cover (U^{α}_{ij}) as the original decomposition (x_i, ϕ_{ij}). So we have a second set of objects and morphisms $(x'_i, \phi'_{ij}), \phi'^{\alpha}_{ij} : x'_j \to x'_i$ in G^{α}_{ij}. If necessary a futher refinement will allow us to choose isomorphisms in $G(U_i)$

$$\xi_i : x_i \to x'_i \tag{9.13}$$

This induces an isomorphism $\epsilon_i : G_i \to G'_i$ such that the following diagram commutes:

$$\begin{array}{ccc} G_i & \xrightarrow{\epsilon_i} & G'_i \\ \downarrow{\scriptstyle a_i} & & \downarrow{\scriptstyle a'_i} \\ \underline{\mathrm{Aut}}(x_i) & \xrightarrow{(\xi_i)_*} & \underline{\mathrm{Aut}}(x'_i) \end{array} \tag{9.14}$$

This diagram induces equivalence relations on the cocycles, the so called coboundary relations, after these equivalence relations are modded out we obtain $\check{H}^2$ (This is done explicitly in [16] and [55]).

In much the same way as in the case of torsors we can make the inverse construction. This is done explicitly in op. cit. So we can end this section with the theorem

9.1.1 Theorem *Let L be an lien on X, then there is a bijective correspondence*

$$\mathrm{Gerbes}(X, L) \cong \check{H}^2(X, L) \tag{9.15}$$

Lien of a Gerbe

A transitive groupoid is described up to weak equivalence by any of it's vertex groups. In the case of gerbes the notion of a vertex group is replaced by the so called lien functor.

We first construct a family of sheaves of groups $lien(G_i)$. Let $(U_i)_{i \in I}$ be an open cover of X and let $lien(G_i)$ be a family of sheaves of groups above this cover. At the moment they are just an odd notation for the family of sheaves of groups G_i. We can glue the sheaves $lien(G_i)$ and $lien(G_j)$ on open sets U_{ij} by a section γ_{ij} of the quotient sheaf $\underline{\mathrm{Out}}(G_j, G_i) = \underline{\mathrm{Iso}}(G_j, G_i)/G_i$ (Where the equivalence relation is defined by conjugation with the groups elements). The lien L is thus determined by a family $\gamma_{ij} \in \Gamma(\underline{\mathrm{Out}}(G_j, G_i))$ of sections satisfying the cocycle identity in $\underline{\mathrm{Out}}(G_k, G_i)$:

$$\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik} \tag{9.16}$$

Together with a normalization condition

$$\gamma_{ii} = \mathrm{id} \tag{9.17}$$

When are two liens isomorphic? Suppose we are given a pair of liens L and L' defined locally by the families of groups (G_a) respectively (G_b). L is isomorphic to L' if there exists a refinement (V_i) and a family of isomorphisms $\xi : lien(G_i) \to lien(G'_i)$ of liens on the open sets V_i that are compatible with the gluing data. The associated sections (γ_{ij}) and (γ'_{ij}) have to satisfy the coboundary relation

$$\gamma'_{ij} = \xi_i \gamma_{ij} \xi_j^{-1} \tag{9.18}$$

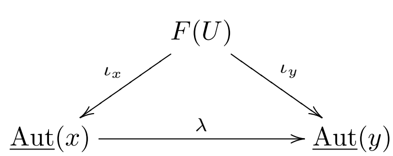
By this definition it is guaranteed that the lien of a refinement of the cover still gives an isomorphic lien. To every gerbe G on X is associated in a functorial way a lien on X, denoted by $lien(G)$. To understand what happens at the cocycle level let (λ_{ij}, g_{ijk}) be the 2-cocycle associated to a labelled decomposition of the gerbe G relevant to the family of groups $\{G_i\}$. The section λ of the sheaf $\underline{\mathrm{Iso}}(G_j, G_i)$ induces a section $\gamma_{ij} = [\lambda_{ij}]$ of the sheaf $\underline{\mathrm{Out}}(G_j, G_i)$.

Some special cases

Abelian gerbes

9.1.4 Definition *Let G be a $\{G_i\}$-gerbe on X. And F a sheaf of groups on X. If for all $g_i \in \mathrm{Obj}(G_i)$ and for all $x \in \mathrm{Obj}(G_{g_i})$ the groups $\mathrm{Aut}_{g_i}(x)$ are commutative then the lien functor is abelian.*

9.1.5 Definition *Let G be a gerbe on X and F a sheaf of groups on X. If for every oject $x \in G(U)$ there exists an isomorphism of sheaves of groups $\iota_x : F(U) \to \underline{\mathrm{Aut}}(x)$ and for any morphism $f : x \to y$ in $G(U)$ the corresponding diagram of sheaves on U commutes*



(9.19)

Where λ is the morphism induced by f. Then the gerbe G is called an abelian $G - gerbe$ on X.

9.1.2 Theorem *A gerbe G is an abelian G-gerbe if and only if it's lien functor is abelian.*

9.2 Definition of a bundle gerbe

A sheaf and a principal bundle are related in the same way as a gerbe and a bundle gerbe. It could in a sense be defined as a gerbe over a manifold.
A precise definition of a bundle gerbe can be done using the 'exact sequence' definition of a principal bundle. The definition of a Gerbe as an extension of groupoids is mainly due to Moerdijk [56].

9.2.1 Definition *Let L be a fixed Lie group, M and X be manifolds. An L-bundle gerbe G over X is*

1. *A surjective submersion $\pi : M \twoheadrightarrow X$ and*
2. *An exact sequence*

$$\begin{array}{ccccc} L \times M & \overset{i}{\rightarrowtail} & G & \overset{\phi}{\twoheadrightarrow} & M \times_X M \\ \downarrow & & & & \\ M & & & & \end{array} \tag{9.20}$$

   $L \times M \to M$ is the trivial bundle of Lie groups over M. $M \times_X M$ has been given the structure of a Lie-groupoid.

This definition resembles closely the definition of a principal bundle from section 1.4. It is not surprising that the bundlegerbes are therefore classified by the first čech cohomology class, $\check{H}^1(M, G)$. But now taking values in a set of groupoids instead of a set of groups. Gerbes, when the groupoid is a semidirect product $G \to \mathrm{Aut}(G)$ take values in the cohomology group $\check{H}^2(M, G)$. As we will soon see a bundle gerbe is nothing more than a gerbe over a manifold, so we can conclude that there is an isomorphism $\check{H}^1(M, G \to \mathrm{Aut}(G)) \xrightarrow{\sim} \check{H}^2(M, G)$ An equivalent definition of an L-bundle gerbe is more close to the sheaf definition of a gerbe.

9.2.2 Definition *An L-bundle gerbe is a family of groupoids over X:*

$$G \underset{t}{\overset{s}{\rightrightarrows}} M \xrightarrow{\pi} X \tag{9.21}$$

Such that

1. *$(s, t) : G \to M \times_X M$ is a surjective submersion*
2. *There is an isomorphism of Lie-groupoids $L \xrightarrow{l} \mathrm{Aut}_G(m)$*

9.2.1 Theorem *A L-bundle gerbe is a gerbe with lien L.*

We first construct the gerbe corresponding to the bundle gerbe.
The construction starts with defining a sheaf of groupoids G over X. For the base we take the sheaf of sections of the map $\pi : M \to X$. The sheaf of arrows is defined in the natural way. i.e. Given two sections $g, h \in G_0(U)$. An arrow $G \ni \daleth : g \to h$ is a section $\gamma : U \to G$ with $s\gamma = g$ and $t\gamma = h$. The stalk of G is non-empty and connected because of the surjective submersioness of π and (s, t). Thus the associated stack is a Gerbe.
We'll now calculate it's band. Let $(U_i)_{i \in I}$ be an open cover of X. With sections $m_i : U_i \to M$ of π. Due to the surjective submersion property of the second definition each point $x \in U_{ij}$ has a neighborhood V on which there exists an arrow $m_j \to m_i$ in $G(V)$. If we choose a cover for which ($\underline{L}$ is the sheaf of

sections of the trivial bundle of L on M) $H^1(U_{ij}, \underline{L})$ is trivial (Take for example a cover for which every intersection U_{ij} is contractible) then we can find a section $g_{ij} \in G(U_{ij})$ such that $g_{ij} : m_j \to m_i$ on U_{ij}. The inverse of the map l induces an isomorphism φ_i of sheaves of groups on U_i: $\varphi_i : \underline{\mathrm{Aut}}(m_i) \to \underline{L}_{|U_i}$. These maps induce the following square on U_{ij} and define in this way the λ_{ij}.

$$\begin{array}{ccc} \underline{\mathrm{Aut}}(m_i) & \xrightarrow{\varphi_i} & \underline{L} \\ {\scriptstyle (g_{ij})_*}\uparrow & & \uparrow{\scriptstyle \lambda_{ij}} \\ \underline{\mathrm{Aut}}(m_j) & \xrightarrow{m_j} & \underline{L} \end{array} \tag{9.22}$$

The set $\{\underline{L}|U_i, \lambda_{ij}\}$ is the band of G.

We have seen in section 2.4 that we can define connections using exact sequences. Bundle gerbes are the natural formulation of gerbes for which this Atiyah sequence can most easily be constructed. However as far as I know this construction still has to be carried out.

Circle Bundle Gerbes

A more naive approach of bundle gerbes is given by Murray [58] (or from a more algebraical point of view [55]), I'll briefly review his work and how it fits in the general scheme of gerbes.

Given principal bundles P and Q over a manifold M we can define a new principal bundle $P \otimes Q$. Only a very special class of groupoids is used in the construction. They are the $\mathbb{C}^\times$ groupoids, where $\mathbb{C}^\times$ is of course the group of complex numbers. The product $P \otimes Q$ is defined by $P \otimes Q = P \times Q/$ where the equivalence relation is given by identifying the skew diagonal (just as in the case of the Bear sum). Since $\mathbb{C}^\times$ is abelian this is well defined.
Define a product $circ$ on $M^2 \times M^2$ by $M^2 \circ M^2 = \{(x, y), (y, z) | x, y, z \in X\}$.
For a bundle define $P \circ P$ to be the restriction of $P \otimes P$ to $M^2 \circ M^2$.

9.2.3 Definition *A $\mathbb{C}^\times$ groupoid is now defined to be a principal bundle P over $X \times X$ with the product given by the multiplication of points on the principal bundle if they are in the same fibre.*

9.2.4 Definition *A circle bundle gerbe over M is defined to be a choice of a fibration $Y \to M$ and a bundle $P \to Y \times_M Y$ with a product $P \circ P \to P$ covering $(y_1, y_2) \circ (y_2, y_3) \to (y_1, y_3)$.*

In other words circle bundle gerbes are bundle gerbes given by the central extension

$$\mathbb{S}^1 \rightarrowtail G \twoheadrightarrow M \times_X M \tag{9.23}$$

By theorem 9.2.1 the lien of a circlebundle gerbe is $\mathbb{S}^1 \times M$ and this implies immediately that circlebundle gerbes are always abelian.

9.3 Connections on Gerbes(1)

In the section I'll assume that the morphism set and the object set of a groupoid are groups.

9.3.1 Definition *A connection on a gerbe P is an equivalence* of gerbes:*

$$\pi_1^* P \xrightarrow{\epsilon} \pi_0^* \tag{9.24}$$

On Δ^1 together with a natural equivalence η

$$\begin{array}{ccc} P & \xrightarrow{1_P} & P \\ \downarrow & \Downarrow\eta & \downarrow \\ \Delta^*(\pi_1^* P) & \xrightarrow{\Delta^*\epsilon} & \Delta^*(\pi_0^* P) \end{array} \tag{9.25}$$

on X. The vertical arrows are canonical equivalences induced by the identities† $\pi_1 \Delta = \pi_0 \Delta = 1_X$ on Δ^*

Why all this fuss about a tiny change. The difference is in finding an inverse to arrow ϵ_{02}. If we have a quick glance at the semi-direct product $\mathrm{Aut}(G) \ltimes G$. This groupoid consists of pairs (h, g) with multiplication $(h, g) \circ (f, e) = (hf, hgeh^{-1})$ It follows that we can only find an inverse up to an isomorphism. Called a quasi-inverse. More formally it can be stated that there is a morphism $\epsilon' : \pi_0^* P \to \pi_1^* P$ together with a natural equivalence $\epsilon'\epsilon \to 1$.
This defines a morphism of stacks

$$\kappa = \pi_{01}^* \epsilon \pi_{12}^* \epsilon \pi_{02}^* \epsilon' \tag{9.26}$$

And a natural equivalence K.

We obtain a diagram similar to 3.18

$$\begin{array}{ccccc} & & \pi_0^* P & & \\ & \overset{\pi_{02}^*\epsilon}{\nearrow} & \Downarrow K & \overset{\kappa}{\searrow} & \\ \pi_2^* P & \xrightarrow[\pi_{12}^*\epsilon]{} & \pi_1^* P & \xrightarrow[\pi_{01}^*\epsilon]{} & \pi_0^* P \end{array} \tag{9.27}$$

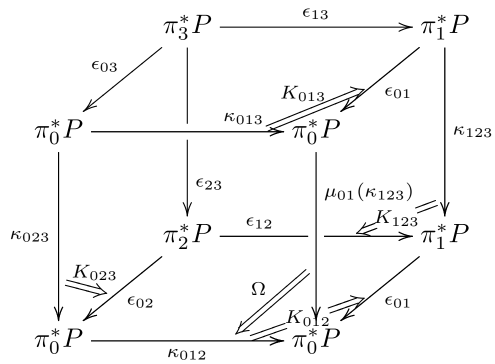
Equivalently:

$$\begin{array}{ccc} \pi_2^* P & \xrightarrow{\epsilon_{12}} & \pi_1^* P \\ \downarrow \epsilon_{02} & \overset{K}{\nearrow\!\!\!\nearrow} & \downarrow \epsilon_{01} \\ \pi_0^* P & \xrightarrow[\kappa]{} & \pi_0^* P \end{array} \tag{9.28}$$

*Since we are working with a sheaf of categories the natural notion is not isomorphism but equivalence. For more on morphisms in categories see appendix A

†The reader familiar with the theory of simplicial sets will recognize these as being simplicial identities.

If we write down the commutative cube for the curvatures.



(9.29)

9.4 Connections on Trivial Gerbes

Precise formulas for the curvature κ and the arrow Ω using differential forms can be given. I'll now give an intuitive derivation. However again as in the case of torsors the intuitive approach is much more restrictive than the synthetic differential geometry approach. In the article by Breen and Messing [17] the formulas for the curvature and the arrow mentioned are derived by rigorous means. In section 9.5 I'll give a brief excerpt from the article with the connection and curvature formulas in the most general case for groupoids consisting of the semi-direct product.

There are two crucial points to be remarked now.

1. **A connection is always a one form**. Intuitively this can be made clear by realizing that given a vector on a manifold, the connection tells what the group (or groupoid) multiplication is if you want to move an element from the tail of the vector to the head of the vector. Since at least one and only one vector is involved the connection must be given by one-forms.

2. **The morphism group works on the object group**. This means that the connection morphism ϵ_{01} can be written down by a one-form $\gamma \in \mathrm{Obj}(G)$ that gives the multiplication in the object groups and two one-forms $\mu_1, \mu_2 \in \mathrm{Mor}(G)$ that respectively give the multiplication in the morphism group and the action of the morphism group on the object group. Written down in formulae this means that ϵ is given by:

$$\epsilon_{01} : (h, g) \in (\mathrm{Mor}(G), Obj(G)) \mapsto (\mu_1(X_{01})h, \mu_2(X_{01})(\gamma(X_{01})g)) \quad (9.30)$$

A redefinition of μ_2 and using an additional conjugation gives accordingly:

$$\epsilon_{01} : (h, g) \in (\mathrm{Mor}(G), Obj(G)) \mapsto (h, \mu(X_{01})(g)) \quad (9.31)$$

The inverse of epsilon is now well defined and in the same way as for the curvature on sheaves we obtain the formula

$$\tilde{\nu} = \mathrm{d}\mu + [\mu, \mu] \quad (9.32)$$

However since a conjugation of ν by a group element defines again an automorphism we have an additional degree of freedom and the formula for the so called fake curvature is:

$$\nu = K^{-1}(\mathrm{d}\mu + [\mu, \mu])K \tag{9.33}$$

Where K takes values in $\mathrm{Obj}(G)$ since it can be different for any choice of the two vector that ν acts on it is a two-form.
If we expand K and ϵ again as in section 2.4 we can calculate diagram 9.29. First this gives us for B and μ:

$$\epsilon(X) = 1 + \mu_\alpha X^\alpha + \partial_\alpha \mu_\beta X^\alpha X^\beta + \mu_\alpha \mu_\beta X^\alpha X^\beta + \dots \tag{9.34}$$

$$K_(X, Y) = 1 + B_{\alpha\beta} X^\alpha Y^\beta + \partial_\gamma B_{\alpha\beta}(X^\gamma + Y^\gamma) X^\alpha Y^\beta + \dots \tag{9.35}$$

$$\tag{9.36}$$

Note in the above formula that since B is a two form is is anti-symmetry.

The formula for the three form curvature Ω is obtained by commuting the K arrows. It should be noted that every arrow gives a factor B so we obtain for the curvature

$$\Omega^{-1} = K_{023}\epsilon_{01}(K_{123}^{-1})K_{012}K_{013}^{-1} \tag{9.37}$$

Every K arrow can be viewed as consisting of all infinitesimal closed loops. We abuse notation as we have done previously and denote by K_{012} the inverse image of just the special infinitesimal closed loop, the loop that leaves the point 0 with vector X_{01} and arrives again at the point 0 with vector X_{02}. So what we basically try to do is taking loops around a closed loop and compare it with the loop that has not been displaced.

If we furthermore note that $X_{12} = X_{02} - X_{01}$, similarly $X_{13} = X_{03} - X_{01}$ and $K_{012} = K(X_{01}, X_{02})$. We can expand the equation for commuting loops to formulas entirely in differential forms. And we can obtain an expression for the differential form expression ω of the curvature arrow.
We have three contributions to consider. First of all we should look at the expansion up to two vectors. Then equation 9.37 yields us just a sum of B's. For this term we can derive

$$-\omega = B_{\alpha\beta}(X_{02}^\alpha X_{03}^\beta - X_{12}^\alpha X_{13}^\beta + X_{01}^\alpha X_{02}^\beta) - X_{01}^\alpha X_{03}^\beta \tag{9.38}$$

$$= B_{\alpha\beta}(X_{02}^\alpha X_{03}^\beta - (X_{12}^\alpha - X_{01}^\alpha)(X_{13}^\beta - X_{01}^\alpha) + X_{01}^\alpha X_{02}^\beta - X_{01}^\alpha X_{03}^\beta) \tag{9.39}$$

$$= B_{\alpha\beta}(X_{02}^\alpha X_{03}^\beta - X_{02}^\alpha X_{03}^\beta + X_{02}^\alpha X_{01}^\beta + X_{01}^\alpha X_{03}^\beta - X_{01}^\alpha X_{01}^\beta + X_{01}^\alpha X_{02}^\beta - X_{01}^\alpha X_{03}^\beta) \tag{9.40}$$

$$= B_{\alpha\beta} X_{01}^\alpha X_{01}^\beta \overset{\text{B anti-symmetric}}{=} 0 \tag{9.41}$$

The expansion up to three vectors with μ

$$-\omega = B_{\alpha\beta}\mu_\gamma X^\gamma_{01}X^\alpha_{02}X^\beta_{03} \tag{9.42}$$
$$- \mu_\gamma B_{\alpha\beta}(X^\gamma_{01}X^\alpha_{12}X^\beta_{13} - X^\gamma_{01}X^\alpha_{01}X^\beta_{02} + X^\gamma_{01}X^\alpha_{01}X^\beta_{03}) \tag{9.43}$$
$$= B_{\alpha\beta}\mu_\gamma X^\gamma_{01}X^\alpha_{02}X^\beta_{03} \tag{9.44}$$
$$- \mu_\gamma B_{\alpha\beta}(X^\gamma_{01}(X^\alpha_{02} - X^\alpha_{01})(X^\beta_{03} - X^\alpha_{01}) - X^\gamma_{01}X^\alpha_{01}X^\beta_{02} + X^\gamma_{01}X^\alpha_{01}X^\beta_{03}) \tag{9.45}$$
$$= B_{\alpha\beta}\mu_\gamma X^\gamma_{01}X^\alpha_{02}X^\beta_{03} \tag{9.46}$$
$$- \mu_\gamma B_{\alpha\beta}(X^\gamma_{01}X^\alpha_{02}X^\beta_{03} - X^\gamma_{01}X^\alpha_{01}X^\beta_{03} - X^\gamma_{01}X^\alpha_{02}X^\beta_{01} \tag{9.47}$$
$$+ X^\gamma_{01}X^\alpha_{01}X^\beta_{01} - X^\gamma_{01}X^\alpha_{01}X^\beta_{02} + X^\gamma_{01}X^\alpha_{01}X^\beta_{03}) \tag{9.48}$$
$$= B_{\alpha\beta}\mu_\gamma X^\gamma_{01}X^\alpha_{02}X^\beta_{03} - \mu_\gamma B_{\alpha\beta}X^\gamma_{01}X^\alpha_{02}X^\beta_{03} \tag{9.49}$$

Which gives us the term $[B, \mu]$. Finally we have to make an expansion for the first derivatives of B. We have to take into account that $B(X_{12}X_{13})$ starts in point 1 instead of point 0 this results in the subtraction of a term $-3\partial_\gamma B_{\alpha\beta}X^\gamma_{01}X^\alpha_{12}X^\beta_{13}$

$$\begin{aligned}
-\omega = \partial_\gamma B_{\alpha\beta}\Big[&(X^\gamma_{02} + X^\gamma_{03})X^\alpha_{02}X^\beta_{03} - (X^\gamma_{12} + X^\gamma_{13})X^\alpha_{12}X^\beta_{13} \\
&- 3X^\gamma_{01}X^\alpha_{12}X^\beta_{13} + (X^\gamma_{01} + X^\gamma_{02})X^\alpha_{01}X^\beta_{02} - (X^\gamma_{01} + X^\gamma_{03})X^\alpha_{01}X^\beta_{03}\Big] \\
= \partial_\gamma B_{\alpha\beta}\Big[&(X^\gamma_{02} + X^\gamma_{03})X^\alpha_{02}X^\beta_{03} \\
&- (X^\gamma_{02} + X^\gamma_{03} - 2X^\gamma_{01})(X^\alpha_{02} - X^\alpha_{01})(X^\beta_{03} - X^\beta_{01}) \\
&- 3X^\gamma_{01}(X^\alpha_{02} - X^\alpha_{01})(X^\beta_{03} - X^\beta_{01}) \\
&+ (X^\gamma_{01} + X^\gamma_{02})X^\alpha_{01}X^\beta_{02} - (X^\gamma_{01} + X^\gamma_{03})X^\alpha_{01}X^\beta_{03}\Big] \\
= \partial_\gamma B_{\alpha\beta}\Big[&X^\gamma_{02}X^\alpha_{02}X^\beta_{03} + X^\gamma_{03}X^\alpha_{02}X^\beta_{03} - X^\gamma_{02}X^\alpha_{02}X^\beta_{03} + X^\gamma_{02}X^\alpha_{01}X^\beta_{03} \\
&+ X^\gamma_{02}X^\alpha_{02}X^\beta_{01} - X^\gamma_{02}X^\alpha_{01}X^\beta_{01} - X^\gamma_{03}X^\alpha_{02}X^\beta_{03} + X^\gamma_{03}X^\alpha_{02}X^\beta_{01} \\
&+ X^\gamma_{03}X^\alpha_{01}X^\beta_{03} - X^\gamma_{03}X^\alpha_{01}X^\beta_{01} + 2X^\gamma_{01}X^\alpha_{02}X^\beta_{03} - 2X^\gamma_{01}X^\alpha_{02}X^\beta_{01} \\
&- 2X^\gamma_{01}X^\alpha_{01}X^\beta_{03} + 2X^\gamma_{01}X^\alpha_{01}X^\beta_{01} - 3X^\gamma_{01}X^\alpha_{02}X^\beta_{03} + 3X^\gamma_{01}X^\alpha_{02}X^\beta_{01} \\
&+ 3X^\gamma_{01}X^\alpha_{01}X^\beta_{03} - X^\gamma_{01}X^\alpha_{01}X^\beta_{01} + X^\gamma_{01}X^\alpha_{01}X^\beta_{02} + X^\gamma_{02}X^\alpha_{01}X^\beta_{02} \\
&- X^\gamma_{01}X^\alpha_{01}X^\beta_{03} - X^\gamma_{03}X^\alpha_{01}X^\beta_{03}\Big] \\
= \partial_\gamma B_{\alpha\beta}\Big[&X^\gamma_{02}X^\alpha_{01}X^\beta_{03} + X^\gamma_{02}X^\alpha_{02}X^\beta_{01} + X^\gamma_{03}X^\alpha_{02}X^\beta_{01} + 2X^\gamma_{01}X^\alpha_{02}X^\beta_{03} \\
&- 2X^\gamma_{01}X^\alpha_{02}X^\beta_{01} - 2X^\gamma_{01}X^\alpha_{01}X^\beta_{03} - 3X^\gamma_{01}X^\alpha_{02}X^\beta_{03} + 3X^\gamma_{01}X^\alpha_{02}X^\beta_{01} \\
&+ 3X^\gamma_{01}X^\alpha_{01}X^\beta_{03} + X^\gamma_{01}X^\alpha_{01}X^\beta_{02} + X^\gamma_{02}X^\alpha_{01}X^\beta_{02} - X^\gamma_{01}X^\alpha_{01}X^\beta_{03}\Big] \\
= \partial_\gamma B_{\alpha\beta}\Big[&-\frac{1}{2}X^{[\gamma}_{01}X^\alpha_{02}X^{\beta]}_{03} + X^\gamma_{02}X^\alpha_{02}X^\beta_{01} + X^\gamma_{01}X^\alpha_{02}X^\beta_{01} + X^\gamma_{01}X^\alpha_{01}X^\beta_{03} \\
&- X^\gamma_{01}X^\alpha_{02}X^\beta_{01} - X^\gamma_{02}X^\alpha_{02}X^\beta_{01} - X^\gamma_{01}X^\alpha_{01}X^\beta_{03}\Big] \\
= \partial_\gamma B_{\alpha\beta}\Big[&-\frac{1}{2}X^{[\gamma}_{01}X^\alpha_{02}X^{\beta]}_{03}\Big]
\end{aligned} \tag{9.50}$$

Thus an expansion of the terms with ∂B gives us a term $\mathrm{d}B$. So if we add these results we obtain for the three-form curvature

$$\omega = \mathrm{d}B + [\mu, B] \tag{9.51}$$

The bianchi identities are calculated along the same lines (we have to compute in this case a commutative hypercube.

There is a special class of gerbes called trivial gerbes for which formulas formulas 9.33 and 9.51 describe the curvatures completely. In the case of trivial gerbes all transition functions vanish or equivalently all cocycles are trivial. For all other gerbes these two equations are modified and we get additional cocycle conditions. They are described in the last section of this chapter. However since all gerbes are locally trivial we can single out a neighborhood small enough that the gerbe on it is trivial. We focus on the construction of gauge theory on this trivial gerbe.

Interpretation

We have constructed two formulae for the curvature, equation 9.33 and 9.51, but it's still vague what kind of objects appear in them. In order to be able to interpret what the objects are exactly let's go back to our trivial gerbe. The structure of the gerbe is $M \times G \rtimes \mathrm{Aut}(G)$. Here M is the manifold we are working on (space-time). Since we assumed the gerbe to be trivial, it's a direct product with the groupoid, which we assumed to be a semi-direct product. First of all we like to know what the Lie algebroid looks like.

Since the groupoid $G \rtimes \mathrm{Aut}(G)$ is transitive, the tangent fibres of the inverse image of the source map $T^{\alpha}_{(x,\gamma)}(G \rtimes \mathrm{Aut}(G))$ are equal to the tangent fibres of $\mathrm{Aut}(G)$. The invariant vector fields are equal. And thus the Lie algebra of $T^{\alpha}_{(x)}(G \rtimes \mathrm{Aut}(G))$ is equal to the Lie algebra of $\mathrm{Aut}(G)$. Thus suggests an infinitesimal structure of $G \times \mathfrak{Aut}(G)$. Since The construction of the Lie algebroid leaves the manifold structure of the group invariant.
However we still have a group structure. Take a group element $g \in G_0$ and suppose $g : x \to y = xg$ for some $x, y \in G_0$. The conjugation map (i_g) is now an isomorphism of vectorspaces

$$i_g : T^{\alpha}_{(y,\gamma)} \to T_{(x,g^1\gamma g)} \tag{9.52}$$

We can define a conjugation of a group element on a algebra element by means of the exponential map.
Take an element $g \in \mathrm{Aut}(G)$. We conjugate it with an element $B \in G$. We know that there is an element $\tilde{g}$ in the lie-algebra $\mathfrak{Aut}(G)$ such that g is in the one-parameter family generated by $\tilde{g}$ in other words

$$B^{-1}gB = B^{-1}\exp(t\tilde{g})B \tag{9.53}$$

For some $t \in \mathbb{R}$. This implies

$$B^{-1}gB = B^{-1}\exp(t\tilde{g})B = B^{-1}\Big(1 + t\tilde{g} + \frac{t^2}{2}\tilde{g}^2 + \dots\Big)B \tag{9.54}$$

$$= 1 + tB^{-1}\tilde{g}B + \frac{t^2}{2}B^{-1}\tilde{g}BB^{-1}\tilde{g}B + \dots \tag{9.55}$$

$$= \exp(tB^{-1}\tilde{g}B) \tag{9.56}$$

In this way the conjugation of B on the Lie algebra is defined by the conjugation of the group elements. Note that this cannot be used to define a multiplication.

Remember that in deriving formula 9.33 we made precisely this approximation. In other words the two from curvature

$$\nu = K^{-1}(\mathrm{d}\mu + [\mu, \mu])K \tag{9.57}$$

Is given by a one form μ taking values in the Lie algebra which at every point $g \in G$ is isomorphic to the Lie algebra of $\mathrm{Aut}(G)$. The full algebra is $\Gamma_{\mathrm{inv}}(T^{\alpha}(G \rtimes \mathrm{Aut}(G)))$. And there is a two form taking values in the group G. This conjugation is defined by means of the exponential map. ν itself takes values in the Lie algebra of the automorphism group.

We still have to interpret formula 9.51. We now take the tangent space of the group G. This can be done in the usual way, we only have to think about what happens with the conjugation map

$$B^{-1}gB = \exp(-tb)\exp(t\tilde{g})\exp(tb) \tag{9.58}$$

$$= 1 + t(-b + \tilde{g}b) + \frac{t^2}{2}(b^2 - \tilde{b}\tilde{g} + \tilde{g}^2 + \tilde{g}b + b^2 + \dots) \tag{9.59}$$

$$= 1 + \tilde{g} + \dots \tag{9.60}$$

If we look at the tangent space of the group we need to take the derivative of this formula with respect to t and then let t approach zero. This shows that the conjugation map is just the identity if we approximate once again. So finally our lie groupoid is approximated by two lie-algebras. With no non-trival maps between the Lie-algebras. We arrive at the interpretation that B in 9.51 is a two form taking values in the Lie algebra of G. And ω is a three form taking values in the same Lie algebra.

Curvatures

Summarized, the connections on trivial gerbes are given by the formulae (with μ and ν respectively a one- and a two-form taking values in the lie algebra $\Gamma_{\mathrm{inv}}(T^{\alpha}(G \rtimes \mathrm{Aut}(G)))$, K a two form taking values in the group G and it generates the element B which is a two form taking values in the Lie algebra of

G. ω is a three-form taking values in the lie algebra of G)

$$\nu = K^{-1}(\mathrm{d}\mu + [\mu, \mu])K \tag{9.61}$$

$$\omega = \mathrm{d}B + [\mu, B] \tag{9.62}$$

With bianchi identities

$$0 = \mathrm{d}\nu + [\mu, \nu]) \tag{9.63}$$

$$\mathrm{d}\omega + [\mu, \omega] = [\nu, B] \tag{9.64}$$

The first bianchi identity states that the curvature of the curvature of the connection vanishes. The second identity also amounts to taking twice the curvature of a gauge field. It can be noted that only μ behaves as a connection since it's the only field that appears in commutators with the field from which the curvature is taken.

9.5 Connections on Gerbes (2)

We can now have a look at the article by Breen and Messing [17] and see what the connection looks like in the general case.

Fully decomposed gerbes are described by

	functions	1-forms	2-forms	3-forms
G_i valued	g_{ijk}	γ_{ij}	δ_{ij}, B_i	ω_i
$\mathrm{Aut}(G_i)$ valued	λ_{ij}	μ_i	ν_i	

g_{ijk} and λ_{ij} describe the decomposition of the gerbe using cocycles. The connection is given by the one forms (γ, μ). An arrow x between infinitesimally close points is associated with a multiplication in the groupoid by $(\gamma(x), \mu(x))$. The inverse connection is defined up to the two form B. The fake curvature of the connection is given by the two forms (δ, ν). The three form curvature is given by ω.

They satisfy the equation:

$$\begin{aligned}
&\text{Cocycle conditions}\\
\lambda_{ij}(g_{jkl})g_{ijl} &= g_{ijk}g_{ikl} && (9.65)\\
\lambda_{ij}\lambda_{jk} &= g_{ijk}\lambda_{ik}g_{ijk}^{-1} && (9.66)\\
&\text{Coboundary equations}\\
{}^{\lambda_{ij}*}\mu_j &= \gamma_{ij}\mu_i\gamma_{ij}^{-1} && (9.67)\\
\mathrm{d}^1_{\lambda_{ij}}(\gamma_{ij}) &=^g \delta^0_{\mu_i}(g_{ijk}) && (9.68)\\
&\text{Transformation of the curving datum}\\
\lambda_{ij}(B_j) &= B_i + \delta_{ij} + \delta^1_{\mu_i}(-\gamma_{ij}) && (9.69)\\
&\text{Conditions for fake curvature}\\
{}^{\lambda_{ij}}\nu_j &= \nu_i - i_{\delta_{ij}} && (9.70)\\
\mathrm{d}^1_{\lambda_{ij}}(\delta_{ij}) &= [\nu_i, g_{ijk}] && (9.71)\\
&\text{Conditions for the curvature}\\
\lambda_{ij}(\omega_j) &= \omega_i + \delta^2_{\mu_i}(\delta_{ij}) + [\gamma_{ij}, {}^{\lambda}_{ij}\,\nu_j] && (9.72)\\
\delta^3_{\mu_i}(\omega_i) &= [\nu_i, B_i]_{\mu_i} && (9.73)\\
&\text{Curvatures}\\
\nu_i &= \kappa_{\mu_i} - i_{B_i} && (9.74)\\
\omega_i &= \delta^2_{\mu_i}(B_i) && (9.75)\\
i_{\omega_i} &= -\delta^2_{\mu_i}\nu_i && (9.76)
\end{aligned}$$

The δ_i's are combinatorial differential forms. ${}^{\lambda_{ij}}$ is defined conjugation with $p_0^*\lambda_{ij}$. ${}^{\lambda_{ij}*}$ is a twisted conjugation, i.e. ${}^{\lambda_{ij}*}\mu = (p_0^*\lambda_{ij})\mu(p_1^*\lambda_{ij})^{-1}$. For more on combinatorial differential forms see [18].

When we again look at trivial gerbes (This means that all the structures defined on one or more intersections of open sets vanish). This set of equations reduce to the equations obtained in the intuitive (and naive) way.

Chapter 10

M-5 brane Lagrangian

10.0.1 Remark *Indices of forms are again used indicate the decomposition of a form written down in a specific bases of the cotangent bundle, i.e. an object with two indices is in this chapter a two form.*

10.1 Gauge Transformations

An additional assumption

If we return to the equations for the curvature on a gerbe

$$\nu = K^{-1}(\mathrm{d}\mu + [\mu, \mu])K \tag{10.1}$$

$$\omega = \mathrm{d}B + [\mu, B] \tag{10.2}$$

Where $\mu \in \Omega^1 \otimes \Gamma_{\mathtt{inv}}(T^\alpha(G \rtimes \mathrm{Aut}(G)))$ and $\nu \in \Omega^2 \otimes \Gamma_{\mathtt{inv}}(T^\alpha(G \rtimes \mathrm{Aut}(G)))$. $K \in \Omega^2 \otimes G$, B is the associated element in $\Omega^2 \otimes \mathfrak{G}$ and finally $\omega \in \Omega^3 \otimes \mathfrak{G}$. At every point in the group the algebra $\Gamma_{\mathtt{inv}}(T^\alpha(G \rtimes \mathrm{Aut}(G)))$ is isomorphic to the Lie algebra of $\mathrm{Aut}(G)$.
In order to get more insight in how the system of curvatures works take as simplifying assumption one infinitesimal structure, with base the algebra of G and fibres the algebra of $\mathrm{Aut}(G)$. Stated differently take both infinitesimal approximations of the groupoid at the same time. We have seen in section 9.4 that the conjugation map is now the identity.
This assumption yields us new equations for the curvatures

$$F_\mu = \mathrm{d}\mu + [\mu, \mu] \tag{10.3}$$

$$\omega = \mathrm{d}B + [\mu, B] \tag{10.4}$$

Where F_μ is the two form curvature taking values in the Lie algebra of $\mathrm{Aut}(G)$. μ is a one form taking values in the Lie algebra of $\mathrm{Aut}(G)$. B is a two form taking values in the Lie algebra of G. And ω is a three form taking values in the Lie algebra of G.

10.2 Infinitesimal transformations

Equation 10.3 suggests that the gauge transformations of μ should be just the transformations of ordinary Yang-Mills theory. For the gauge transformations of B we can try to write down all the possible commutators with two indices.

$$\delta\mu = -\alpha \mathrm{d}\Lambda^{(0)} + \beta[\Lambda^{(0)}, \mu] \tag{10.5}$$

$$\delta B = a\mathrm{d}\Lambda^{(1)} + b[\Lambda^{(0)}, B] + c[\Lambda^{(1)}, \mu] \tag{10.6}$$

Where $\Lambda_{(0)} \in \Omega^0 \otimes \mathfrak{Aut}(G)$. In other words $\Lambda_{(0)}$ is a function that takes values in the Lie algebra of the automorphism group. $\Lambda^{(1)} \in \Omega^{(1)} \otimes \mathfrak{G}$ is a one form that takes values in the Lie algebra of G.

If we add space-time indices and write the commutators on the generators of the algebra we obtain

$$\delta\mu^I_\mu = \alpha \mathrm{d}_\mu \Lambda^I + \beta g f^{IJK} \Lambda^J \mu^K_\mu \tag{10.7}$$

$$\delta B^I_{\mu\nu} = a\mathrm{d}_\mu \Lambda^I_\nu + bhf^{IJK}\Lambda^J B^K_{\mu\nu} + chf^{IJK}\Lambda^J_\mu \mu^K_\nu \tag{10.8}$$

Where g and h are the couplings of the commutators with respect to the derivatives. And $\mathrm{d}\Lambda_\nu = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$.

We now have to check gauge invariance under these transformations:

$$\begin{aligned}
\nu &= \mathrm{d}_\mu \mu^I_\nu + hf^{IJK}\mu^J_\mu \mu^K_\nu \\
&\rightarrow \mathrm{d}_\mu \mu^I_\mu + \alpha \mathrm{d}_\mu \mathrm{d}_\nu \Lambda^I + \beta g f^{IJK} \mathrm{d}_\mu \Lambda^J \mu^K_\nu + \beta g f^{IJK}\Lambda^J \mathrm{d}_\mu \mu^K_\nu \\
&+ gf^{IJK}\mu^J_\mu \mu^K_\nu + \alpha g f^{IJK} \mu^J_\mu \mathrm{d}_\nu \Lambda^K + \beta g^2 f^{IJK} f^{KLM} \mu^J_\mu \Lambda^L \mu^M_\nu \\
&+ \alpha g f^{IJK} \mathrm{d}_\mu \Lambda^J \mu^K_\nu + \beta g^2 f^{IJK} f^{JLM} \Lambda^L \mu^M_\mu \mu^K_\nu \\
\delta\nu &= (\beta + 2\alpha) g f^{IJK} \mathrm{d}_\mu \Lambda^J \mu^K_\nu + \beta g f^{IJK} \Lambda^J \mathrm{d}_\mu \mu^K_\nu \\
&+ \beta g^2 f^{IJK} f^{KLM} \mu^J_\mu \Lambda^L \mu^M_\nu + \beta g^2 f^{IJK} f^{JLM} \Lambda^L \mu^M_\mu \mu^K_\nu \\
&= (\beta + 2\alpha) g f^{IJK} \mathrm{d}_\mu \Lambda^J \mu^K_\nu + \beta g f^{IJK} \Lambda^J \mathrm{d}_\mu \mu^K_\nu \\
&- \beta g^2 f^{IJK} f^{KLM} \mu^L_\mu \Lambda^M \mu^J_\nu - \beta g^2 f^{IJK} f^{KLM} \mu^M_\mu \Lambda^J \mu^L_\nu + \beta g^2 f^{IJK} f^{JLM} \Lambda^L \mu^M_\mu \mu^K_\nu \\
&= (\beta + 2\alpha) g f^{IJK} \mathrm{d}_\mu \Lambda^J \mu^K_\nu + \beta g f^{IJK} \Lambda^J \mathrm{d}_\mu \mu^K_\nu \\
&+ \beta g^2 f^{IKJ} f^{JLM} \mu^M_\mu \Lambda^L \mu^K_\nu - \beta g^2 f^{IJK} f^{KLM} \mu^M_\mu \Lambda^J \mu^L_\nu + \beta g^2 f^{IJK} f^{JLM} \Lambda^L \mu^M_\mu \mu^K_\nu \\
&= (\beta + 2\alpha) g f^{IJK} \mathrm{d}_\mu \Lambda^J \mu^K_\nu + \beta g f^{IJK} \Lambda^J \mathrm{d}_\mu \mu^K_\nu \\
&- \beta g^2 f^{IJK} f^{JLM} \mu^M_\mu \Lambda^L \mu^K_\nu - \beta g^2 f^{IJK} f^{KLM} \mu^M_\mu \Lambda^J \mu^L_\nu + \beta g^2 f^{IJK} f^{JLM} \Lambda^L \mu^M_\mu \mu^K_\nu \\
&= (\beta + 2\alpha) g f^{IJK} \mathrm{d}_\mu \Lambda^J \mu^K_\nu + \beta g f^{IJK} \Lambda^J \mathrm{d}_\mu \mu^K_\nu - \beta g^2 f^{IJK} f^{KLM} \mu^M_\mu \Lambda^J \mu^L_\nu \\
&\overset{2\alpha = -\beta}{=} \beta f^{IJK} \Lambda^J \nu^K_\nu
\end{aligned} \tag{10.9}$$

This gives us for the gauge transformation of the curvature

$$\nu \rightarrow \nu - 2\alpha[\Lambda_{(0)}, \nu] \tag{10.10}$$

Equating the gauge transformations for ω

$$
\begin{aligned}
\omega &= \mathrm{d}_\mu B_{\nu\lambda} + hf^{IJK}\mu^J_\mu B^K_{\nu\lambda} \\
&\to \mathrm{d}_\mu B_{\nu\lambda} + a\mathrm{d}_\mu \mathrm{d}_\nu \Lambda^I_\lambda + bhf^{IJK}\mathrm{d}_\mu \Lambda^J B^K_{\nu\lambda} + bhf^{IJK}\Lambda^J \mathrm{d}_\mu B^K_{\nu\lambda} + chf^{IJK}\mathrm{d}_\mu \Lambda^J_\nu \mu^K_\lambda \\
&+ chf^{IJK}\Lambda^J_\nu \mathrm{d}_\mu \mu^K_\lambda + hf^{IJK}\mu^J_\mu B^K_{\nu\lambda} + ahf^{IJK}\mu^J_\mu \mathrm{d}_\nu \Lambda^K_\lambda + bh^2 f^{IJK} f^{KLM}\mu^J_\mu \Lambda^L B^M_{\nu\lambda} \\
&+ ch^2 f^{IJK} f^{KLM}\mu^J_\mu \Lambda^L_\nu \mu^M_\lambda + \alpha hf^{IJK}\mathrm{d}_\mu \Lambda^J B^K_{\nu\lambda} - 2\alpha hg f^{IJK} f^{JLM}\Lambda^L \mu^M_\mu B^K_{\nu\lambda} \\
\delta\omega &= (b+\alpha)hf^{IJK}\mathrm{d}_\mu \Lambda^J B^K_{\nu\lambda} + bhf^{IJK}\Lambda^J \mathrm{d}_\mu B^K_{\nu\lambda} + (c-a)hf^{IJK}\mathrm{d}_\mu \Lambda^J_\nu \mu^K_\lambda \\
&+ chf^{IJK}\Lambda^J_\nu \mathrm{d}_\mu \mu^K_\lambda + bh^2 f^{IJK} f^{KLM}\mu^J_\mu \Lambda^L B^M_{\nu\lambda} \\
&+ ch^2 f^{IJK} f^{KLM}\mu^J_\mu \Lambda^L_\nu \mu^M_\lambda - 2\alpha gh f^{IJK} f^{JLM}\Lambda^L \mu^M_\mu B^K_{\nu\lambda} \\
&\overset{b=-\alpha, c=a}{=} bhf^{IJK}\Lambda^J \mathrm{d}_\mu B^K_{\nu\lambda} + ahf^{IJK}\Lambda^J_\nu \mathrm{d}_\mu \mu^K_\lambda + bh^2 f^{IJK} f^{KLM}\mu^J_\mu \Lambda^L B^M_{\nu\lambda} \\
&+ ah^2 f^{IJK} f^{KLM}\mu^J_\mu \Lambda^L_\nu \mu^M_\lambda + 2bgh f^{IJK} f^{JLM}\Lambda^L \mu^M_\mu B^K_{\nu\lambda} \\
&= bhf^{IJK}\Lambda^J \mathrm{d}_\mu B^K_{\nu\lambda} + ahf^{IJK}\Lambda^J_\nu \mathrm{d}_\mu \mu^K_\lambda - bh^2 f^{IJK} f^{KLM}\mu^L_\mu \Lambda^M B^J_{\nu\lambda} \\
&- bh^2 f^{IJK} f^{KLM}\mu^M_\mu \Lambda^J B^L_{\nu\lambda} + \frac{a}{2}h^2 f^{IJK} f^{KLM}\mu^J_\mu \Lambda^L_\nu \mu^M_\lambda \\
&- \frac{a}{2}h^2 f^{IJK} f^{KLM}\mu^M_\mu \Lambda^L_\nu \mu^J_\lambda + 2bgh f^{IJK} f^{JLM}\Lambda^L \mu^M_\mu B^K_{\nu\lambda} \\
&= bhf^{IJK}\Lambda^J \mathrm{d}_\mu B^K_{\nu\lambda} + bh^2 f^{IJK} f^{KLM}\Lambda^J \mu^L_\mu B^M_{\nu\lambda} + ahf^{IJK}\Lambda^J_\nu \mathrm{d}_\mu \mu^K_\lambda \\
&- \frac{a}{2}h^2 f^{IJK} f^{KLM}\mu^M_\mu \Lambda^J_\nu \mu^L_\lambda + (2bgh - bh^2) f^{IJK} f^{JLM}\Lambda^L \mu^M_\mu B^K_{\nu\lambda} \\
&\overset{a=\frac{ah}{2g}, 2bg=bh}{=} bhf^{IJK}\Lambda^J \omega^K_{\mu\nu\lambda} + ahf^{IJK}\Lambda^J_\mu \nu^K_{\nu\lambda}
\end{aligned}
$$

These transformations are consistent if

$$
\alpha = -b,\ \beta = 2b,\ c = a,\ g = \frac{1}{2}h \tag{10.11}
$$

The full gauge transformations are thus

$$
\delta\mu^I_\mu = -b\mathrm{d}_\mu \Lambda^I + bhf^{IJK}\Lambda^J \mu^K_\mu \tag{10.12}
$$

$$
\delta B^I_{\mu\nu} = a\mathrm{d}_\mu \Lambda^I_\nu + bf^{IJK}\Lambda^J B^K_{\mu\nu} + ahf^{IJK}\Lambda^J_\mu \mu^K_\nu \tag{10.13}
$$

This yields us the transformation of the curvature

$$
\nu \to \nu + b[\Lambda_{(0)}, \mu] \tag{10.14}
$$

$$
\omega \to \omega + b[\Lambda_{(0)}, B] + a[\Lambda_{(1)}, \mu] \tag{10.15}
$$

This shows that there's something terribly wrong. Upon sending h to zero all transformations and the equations for the curvature reduce to the abelian case. Due to the no-go theorem of section 6.4 this can't describe a non-abelian 2-form field. Presently nothing prohibits the construction. But in the next section we'll run into serious problems if we want to construct invariants using these gauge transformations.

10.3 Construction of the Theory

To find a consistent theory we have to find invariant elements under the gauge transformations just discussed. For ordinary Yang-Mills theory, or our μ we can find the invariant element $\mathrm{tr}(\nu^2)$.

$$\begin{aligned}\mathrm{tr}(\nu)^2 \to \mathrm{tr}(\nu+\delta\nu)^2 &= \mathrm{tr}(\nu)^2 + \mathrm{tr}(\nu[\Lambda_{(0)},\nu] + [\Lambda_{(0)},\nu]\nu) \\ &= \mathrm{tr}(\nu)^2 + \mathrm{tr}(\nu\Lambda_{(0)}\nu - \nu\nu\Lambda_{(0)} + \Lambda_{(0)}\nu\nu - \nu\Lambda_{(0)}\nu) \\ &= \mathrm{tr}(\nu)^2 + \mathrm{tr}(\Lambda_{(0)}\nu^2 - \nu^2\Lambda_{(0)}) = \mathrm{tr}(\nu)^2 + \mathrm{tr}([\Lambda_{(0)},\nu^2]) = \mathrm{tr}(\nu^2)\end{aligned} \tag{10.16}$$

Since the action is defined by an integration over space-time ν^2 should be a form with a number of indices equal to the dimension of space-time. Mathematically this invariant is defined by the product of ν with it's hodge-dual ([24] and the chapter on classical field theory in [25]) *. In formulae this amounts to

$$\nu \wedge *\nu \tag{10.17}$$

This equation doesn't say much when we actually want to use the invariant we write it in indices

$$\epsilon^{\mu\nu\sigma\tau}\nu^I_{\mu\nu}\epsilon_{\sigma\tau\rho\lambda}\nu^{\rho\lambda I} \tag{10.18}$$

However for ω things get more involved. Remember the gauge transformation

$$\omega \to \omega + [\Lambda_{(0)},\omega] + [\Lambda_{(1)},\nu] \tag{10.19}$$

The obvious invariant to construct is $\mathrm{tr}(\omega^2)$ however if calculate the variation we obtain

$$\begin{aligned}\mathrm{tr}(\omega^2) \to \mathrm{tr}((\omega &+ [\Lambda_{(0)},\omega] + [\Lambda_{(1)},\mu])(\omega + [\Lambda_{(0)},\omega] + [\Lambda_{(1)},\nu])) \\ &= \mathrm{tr}(\omega^2 + \omega[\Lambda_{(0)},\omega] + [\Lambda_{(0)},\omega]\omega + \omega[\Lambda_{(1)},\nu] + [\Lambda_{(1)},\nu]\omega) \\ &= \mathrm{tr}(\omega^2 + [\Lambda_{(0)},\omega^2]) + \mathrm{tr}(\omega\Lambda_{(1)}\nu - \omega\nu\Lambda_{(1)} + \Lambda_{(1)}\nu\omega - \nu\Lambda_{(1)}\omega)\end{aligned} \tag{10.20}$$

With this result we run into serious problems, due to the last term $\mathrm{tr}(B^2)$ is not invariant and it can obviously not be made invariant using ν. The same problem and the same equations we would have obtained if we didn't use gerbes at all, but just started to calculate gauge transformations with arbitrary 1-form and 2-form fields.

In a nutshell, we want to construct a 3-form from 1-form and 2-form fields. We only want to use exterior derivatives and commutators. The only three forms that we can obtain are the exterior derivative of the 2-form field, the commutator of a 2-form with a 1-form field and the double commutator of 1-form field. The double commutator is zero due to the bianchi identities. So the only possible three form curvature is our ω. The same argument yields the Yang-Mills equations for the 1-form field.

If we write down all possible gauge transformations of the right degree, we arrive at the results obtained earlier in this chapter. Had a non-abelian 2-form

*A speedy course on hodge theory can be found in [19] gauge theory be obtainable in this way it would have been found years ago by just trial and error.

We might however have missed some gauge transformations. It is natural to include gauge transformations of the form $[\Lambda^{(1)}_\alpha, B_{\alpha\beta}]$. Stated differently

$$B_{\mu\nu} \to B_{\mu\nu} + [\Lambda_\mu, B_{\mu\nu}] + \mathrm{d}_\mu\Lambda_\nu + [\Lambda_\mu, \mu_\nu] \tag{10.21}$$

We can equate the variation of ω which must be

$$\begin{aligned}\delta_\omega &= \text{Terms from ordinary gauge theory} + [\mathrm{d}_\mu\Lambda_\nu, B_{\nu\lambda}] \\ &\quad + [\Lambda_\nu, \mathrm{d}_\mu B_{\nu\lambda}] + [\mu_\mu, [\Lambda_\nu, B_{\nu\lambda}]] \\ &= \text{Terms from ordinary gauge theory} + [\Lambda_\nu, \omega_{\mu\nu\lambda}] \\ &\quad + [\mathrm{d}_\mu\Lambda_\nu, B_{\nu\lambda}] + [B_{\nu\lambda}, [\mu_\mu, \Lambda_\nu]]\end{aligned}$$

The last term definitely comes from a transformation of the form $\mu_\mu \to \mu_\mu + [\Lambda_\nu, \mu_\mu] + \mathrm{d}_\nu\Lambda_\mu$ which is a strange term unless Λ_ν is treated here as n functions instead of a 1-form, where n is the dimensionality of the space.
Though this looks promising we have to note that the parameter should take values in the Lie algebra associated to μ which is the Lie algebra of $\mathrm{Aut}(G)$. This means that we have two independent $\Lambda^{(1)}$'s. One taking values in the Lie algebra of G and one taking values in the lie algebra of $\mathrm{Aut}(G)$. This means that this don't gives us anything useful to get invariant forms that can be integrated to an action. Moreover we have also shown that the algebra is closed under transformations with all possible zero and one forms and thus we expect to have found we found all gauge transformations (adding a zero form transformation taking values in the group leads to inconsistencies). This means the obvious improvement of adding extra gauge transformations, won't work. What other things can be altered in a favorable way?

Let's go back to the last simplifying assumption we made. Stated shortly we reduced the Lie algebroid structure to a product of the Lie algebra of the group with the Lie algebra of the automorphism group.
Assume that the group are represented in the adjoint representation. And assume that our algebroid is $G \times \mathfrak{Aut}(G)$, which is still a heavy restriction. A Lie algebroid is a so called Lie algebra bundle [53]. Which is a vector bundle with a lie bracket on the fibres. The assumption for the algebroid means that we demand the bundle to be globally trivial. We'll treat the transformations taking values in the Lie algebra of $\mathrm{Aut}(G)$ first. The transformations are

$$\delta\mu = -\alpha \mathrm{d}\Lambda^{(0)} + \beta[\Lambda^{(0)}, \mu] \tag{10.22}$$

$$\delta B = b[\Lambda^{(0)}, B] \tag{10.23}$$

If we integrate equation 10.22 to finite transformations we obtain

$$\mu \to U^{-1}\mu U + U^{-1}\mathrm{d}U, U \;\in\; \Omega^0 \otimes \mathrm{Aut}(G) \tag{10.24}$$

If we integrate the transformations of equation 10.23 to finite transformations we obtain

$$B \to U^{-1}BU \tag{10.25}$$

$$K \to U^{-1}KU \tag{10.26}$$

The transformation for B is the transformation for a group element since we only approximated the Lie algebra of the automorphism group with an algebra. This yields for $\nu = K^{-1}F_\mu K$

$$\nu \to U^{-1}K^{-1}UU^{-1}F_\mu UU^{-1}KU = U^{-1}\nu U \tag{10.27}$$

Since for ω our algebra structure is the same as in our earlier derivations we can conclude that these transformations are the gauge transformations for the system of equations for the curvature equations 9.33 and 9.51. This most naive way of including the non trivial group action doesn't help us out of the problems of constructing an invariant form that can be integrated to an action, since the alterations are in the transformations of ν and not in the transformations of ω that prohibit the construction of an invariant action.

10.4 Improvements

There are still some directions open for further research.

1. We have missed some vital clue telling us the commutator of ν and ω is zero. This is however not likely since this result should be included in our relations for connections on a gerbe. This is because our expansion can also be rigorously derived using simplicial methods and we have used all the freedom resulting from the group action on the automorphism group. Moreover the gauge transformations are the perturbation of abelian ones which can't describe the 2-form gauge theory we are looking for. So it's unlikely we have missed this relation.

2. We assumed the gerbe to be trivial over space-time. We were tempted to make this choice by analogy with ordinary Yang-Mills theory and the complexity of the full equations for the connection on a gerbe. This choice may simply be wrong and we have to work our way through all the additional transition function conditions from equations 9.65 to 9.76. There is one important argument against this. Gerbes are constructed from open sets, which means that locally the gerbe is trivial. Since if we look at just one U_i for which we have a labelling x_i then can take the identity arrow as a decomposition. Accordingly all the transition functions vanish and we obtain the curvature formulae derived for trivial gerbes to hold on the set U_i. So if we if we can't construct a consistent gauge theory for a trivial gerbe there is little hope that adding the transition functions will allow us

to get the theory consistent. Some work has been done in the direction of making the theory consistent by means of these functions, it can be found in [5].

3. We only tried the globally trivial Lie algebra bundle $G \times \mathrm{Aut}(G)$ in other to take into acount the full structure of the Lie algebroid. It would be worth to consider the gauge transformations for an arbitrary algebroid. Since we tried to look at new structures it's not unlikely that these new structures should eventually play a role in getting the theory consistent. This is next to skipping locallity all together the most promising amendment that can be made.

4. The language of lie algebras is not sufficient to describe higher gauge theory. One could object to the fact that we are still working with algebras, though we introduced a new differential structure called an algebroid. It is however unclear what a gauge field taking values in an algebroid should look like. Moreover Lie algebras and Lie groups appear naturally when we make take the alfa fibre tangent space, which leads to algebroids, or when we view an algebroid as a Lie algebra bundle. In literature there is however a tendency just to define a field taking values in groupoids or algebroids as giving a consitent theory without any calculation showing that it really gives an invariant action, see for example [5].

5. Finally it may be possible that Gerbes are not the object to generalize gauge theory. We have only shown that the by far simplest extension of the mathematical structure of Yang-Mills theory are gerbes. We could only change the algebraic structure, or the notion of topology. We changed the algebraic structure to obtain gerbes. That explains why an essential ingredient for connections on gerbes are the infinitesimal neighborhoods, which on a manifold asserts we are working locally. To alter the open sets will amount to changing one of the basic notions of mathematics. Which may result in a non-local theory, however there is absolutely no clue what we should do in this case.

Chapter 11

Conclusions and Prospectives

A gauge field can be formulated as a morphism of torsors. Which amounts to comparing two infinitesimal close stalks of a sheaf with a group action. This definition contains all the essential features of a gauge field but 'nothing more'. The only way to generalize gauge fields is thus either to skip the notion of open sets, which is a very unnatural thing to do in physics, or to change the structure of the sheaf. The easiest structure change that can be made and leads to new physics is to change the group of the sheaf into a groupoid. To get any sensible results we endow the morphisms and the objects with group structures. Then the connection remains a one-form field, with a (fake-)curvature defined up to a two form field. This two form field has a well defined (three form) curvature. The explicit formulas can be derived either by simplicial methods or in the case of a trivial gerbe by an infinitesimal expansion. The precise form of the infinitesimal expansion and the corresponding curvatures are derived in chapter 9. When we are dealing with a trivial gerbe the one-form field takes values in the morphism group and the two-form field takes values in the object group. The methods and formulas that work for curvature forms on gerbes are easily generalized to four-form curvatures or 2-Gerbes by using 2-categories instead of categories for the structure, the technical complication go into the definition of a two stack [18] which can largely be ignored when doing physics.
Three problems prohibit the formulation of an M-5 brane action. The first problem is partly mathematical and amount to finding a non-abelian two form gauge field. The second problem is to construct a chiral theory out of this two form field. The third problem is anomaly cancellation. In this thesis the first and second problem are treated in detail and an attempt is made to solve the first problem.
Given the curvature forms on a gerbe gauge transformations can be written down that are indicated by Yang-Mills theory. They are the ordinary Yang-Mills gauge transformations for the 1-form connection. For the two form we take all possible transformations with two space-time indices.
The curvature forms are invariant under these transformations. However it's not

possible to find an invariant integral that can be used to construct an action. It even turns out that the curvature equations could have been derived from general considerations in Yang-Mills theory by considering all two and three forms constructed out of one and two forms using exterior derivatives and the lie algebra multiplication (which is just the commutator). So all additional structure of a gerbe has been lost.

The most naive amendments, which mean either we introduce additional gauge transformations or we take the infinitesimal structure of the gerbe better into account by taking by taking a globally trivial lie algebra bundle, don't yield anything new that can cancel the problems in the construction of an invariant action. We can however still identify five ways to proceed, three of them have strong counter arguments. The two remaining ways are in short:

Firstly an improvement that could be made is to stop working with trivial products of lie algebras and Lie groups and start to consider non trivial products or non globally trivial Lie algebra bundles as the object fields take values in. This will lead to a full understanding of the conjugation in 9.33 which is unclear at present.

Secondly gerbes are the obvious generalization of torsors. It is however possible that gerbes don't describe non-abelian gauge fields. Since sheaves describe Yang-Mills theory, they depend on just the algebraic structure and open sets and changing the structure leads to gerbes, we probably need to change the foundations of mathematics by changing the notion of open sets.

Acknowledgements

I gratefully acknowledge the help of I. Moerdijk for helping me out of troubles when I was trying to cope with the various definitions of a gerbe. Private lectures by E. Bergshoeff, M. v.d. Put and J. Top have been essential for getting access to research literature on algebraic geometry and M-theory. I'm grateful to my advisor M. de Roo for this problem and being available for questions. Furthermore long discussions with M. Eenink have contributed greatly to my overview of M-theory and he introduced to me the dimensional reduction of [85] which turned out to be essential for section 10.2. Finally I'm grateful to J. Brahms for his inspiring music that accompanied me and rendered me crucial ideas during this research

Appendix A

Category Theory

A Category is a very fundamental object in mathematics and a precise definitions is complicated and has to take into account many subtle points. For anything on category theory I refer to the excellent book by MacLane [51]

A category X consists of objects $(A, B, ...)$ and morphisms $(f, g, ...)$ (or arrows).

1. For every $A, B \in \mathrm{Arr}(X)$ is a set $\mathrm{Arr}(A, B)$ given.
2. There is a multiplication $\circ : \mathrm{Arr}(A, B) \times \mathrm{Arr}(B, C) \to \mathrm{Arr}(A, C)$.
3. $\circ$ is nice: $h \circ (g \circ f) = (h \circ g) \circ f$
4. There is a unit 1_A such that $f \circ 1_A = f$ and $1_A \circ g = g$

A functor is a map between categories. It's definitions is the first time the extremely powerful notion of naturality appears. To define a functor we have to do something sensible with the object as well as with the morphisms.

A.0.1 Definition *$C_1 \xrightarrow{F} C_2$ is called a functor if for all objects A there are associated objects $F(A)$ and for all morphisms F there are associated morphisms $F(f)$ such that:*

$$F(f \circ g) = F(f) \circ F(g) \tag{A.1}$$

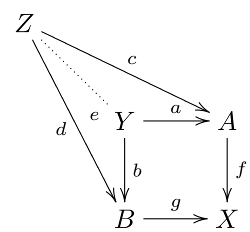
$$A \xrightarrow{f} B \Rightarrow F(A) \xrightarrow{F(f)} F(B) \tag{A.2}$$

A.0.2 Definition *A natural transormation (of functors) ϕ from F_1 to F_2 is an operation associating with each object $A \in C_1$ a morphism $\phi_C : F(A) \to G(A)$ in the category C_2 such that for any morphism $f : B \to A$ the diagram:*

$$\begin{array}{ccc} F(B) & \xrightarrow{\phi_B} & G(B) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(A) & \xrightarrow{\phi_A} & G(A) \end{array} \tag{A.3}$$

commutes.

A.0.3 Definition *given objects A, B & X in a category $\mathcal{C}$ and morphisms $f : A \to X$ and $g : B \to X$. a pull-back of (f, g) is a pair of morphisms $a : Y \to A$ and $b : Y \to B$ such that $fa = gb$ and such that the following universal property holds: given any Z and any $c : Z \to A, d : Z \to B$ such that $fc = gd$ there exists an unique $e : Z \to Y$ such that $c = ae$ and $d = be$. Stated differently, for any Z and c, d the diagram can be uniquely completed:*



(A.4)

Equivalence of Categories

Isomorphism is to restrictive in category theory. It is replaced by the notion of equivalence.

A.0.4 Definition *Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent if there exist functors $C : \mathcal{C} \to \mathcal{D}$m, $D : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\tau : C \circ D \to \mathrm{id}_{\mathcal{D}}$, $\sigma : D \circ C \to \mathrm{id}_{\mathcal{C}}$.*

A.0.5 Definition *A functor $C : \mathcal{C} \to \mathcal{D}$ is called essential surjective is for any object $y \in \mathrm{Obj}(\mathcal{D})$ there exists an object $x \in \mathrm{Obj}(\mathcal{C})$ and an isomorphism $C(x) \to y$ in $\mathcal{D}$.*

A.0.6 Definition *A functor $C : \mathcal{C} \to \mathcal{D}$ is called full and faithful if for any two objects $x, x' \in \mathrm{Obj}(\mathcal{C})$ the functor C induces a bijection $C : \mathrm{Arr}(\mathcal{C})_x^{x'}) \to \mathrm{Arr}(\mathcal{D}_{C(x)}^{C(x')})$ between the set of all arrows from x to x' in $\mathcal{C}$ and the set of all arrows from $C(x)$ to $C(x')$ in $\mathcal{D}$.*

Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists an essential surjective and fully faithful functor $C : \mathcal{C} \to \mathcal{D}$.

Special Categories

A.0.7 Definition *A category is called additive when:*

1. *There is a zero object.*
2. *Any two objects have a product.*
3. *The morphism sets $\mathrm{Arr}(C, D)$ are abelian groups such that the composition $\mathrm{Arr}(C, D) \times \mathrm{Arr}(D, E) \to \mathrm{Arr}(C, E)$ is bilinear.*

A.0.8 Definition *An Abelian category is an additive category in which:*

1. *Every morphism has a kernel and a cokernel.*
2. *Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.*
3. *Every morphism can be written as the composite of an epi- with a monomorphism.*

Appendix B

Homological Algebra

Homological algebra deals with the theory of classifying mathematical structures. Given three objects A, B & C (Rings, Groups etc) and morphisms $a : A \to B,\ b : B \to C$ such that $b \circ a = 0$. Written down like:

$$A \xrightarrow{a} B \xrightarrow{b} C \tag{B.1}$$

Central to homology theory is the question whether we can write down the structure on B using the structures on A and C. Let's call the mathematical structure we want to describe $+_{\_}$. If we take two elements b_1 and b_2 we want to know whether we can write down $b_1 +_B b_2$ using $+_A$ and $+_C$. Since a and b are morphisms we can take $b(b_1 +_B b_2) = b(b_1) +_C b(b_2)$. Stated differently we'd know that $\tilde{b} = b_1 +_B b_2 \in b^{-1}(b(b_1) +_C b(b_2))$. Unfortunately we have a whole class of possible $\tilde{b}$'s, since if $\hat{b} \in \ker(b)$ then also $\tilde{b} +_B \hat{b} \in b^{-1}(b(b_1) +_C b(b_2))$. Since $\mathrm{im}\,(a) \subset \ker(b)$ we can use $+_A$ to decern between $\tilde{b}$ and $\tilde{b} +_B \hat{b}$, this only works if $\mathrm{im}\,(a) = \ker(b)$ That's why we define the homology of B to be $\mathrm{H}(B) = \ker(b)/\mathrm{im}\,(a)$. The sequence B.1 is called exact if $\ker(b) = \mathrm{im}\,(a)$. If it is exact homologists also tend to call it a short exact sequence.

B.0.1 Example *The simplest example is an other way of characterizing injective and surjective.*
An injective map $A \overset{i}{\rightarrowtail} B$ *can be written down using the following exact sequence as the reader can easily convince himself.*

$$0 \longrightarrow A \xrightarrow{i} B \tag{B.2}$$

A surjective map $A \overset{s}{\twoheadrightarrow} B$ *can be written down using the following exact sequence*

$$A \xrightarrow{s} B \longrightarrow 0 \tag{B.3}$$

B.0.2 Example *A very nice example of an exact sequence is the so called semidirect product of two groups* A *and* C*. It is defined to be the unique group* $A \rtimes C$ *such that the following sequence is exact:*

$$A \rightarrowtail A \rtimes C \twoheadrightarrow C \tag{B.4}$$

We can write this also down as a sequence exact at all entries:

$$0 \longrightarrow A \longrightarrow A \rtimes C \longrightarrow C \longrightarrow 0 \tag{B.5}$$

The homology of B has the structure of an abelian group* as can be checked by using the properties of the kernel. The fact that it is well defined is a standard calculation familiar from algebra courses.

Some notations and notions

B.0.9 Definition *1. A monomorphism is a morphism $f : B \to C$ such that given the diagram*

$$A \overset{a}{\underset{b}{\rightrightarrows}} B \xrightarrow{f} C \tag{B.6}$$

when $af = bf$ then if follows that $a = b$. A monomorphism is denoted by $\rightarrowtail$ and in most cases can be read as injective.

2. *The definition of a epimorphism can now easily be given by the reader himself. An epimorphism is denoted by $\twoheadrightarrow$ and can in most cases be read as surjective.*

B.0.10 Definition *A cokernel for $f : A \to B$ is a set Z and a morphism $p : B \twoheadrightarrow Z$ where p is an epimorphism with $pf = 0$ such that any other morphism $q : B \to Y$ with $qf = 0$ factors through p. Stated diagrammatically:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{p} & Z \\ & & & \searrow^{q} & \uparrow \\ & & & & Y \end{array} \tag{B.7}$$

It's an easy exercise to see that a cokernel is isomorphic to $B/\mathrm{im}\,(f)$. Furthermore a similar definition can be used to write down the definition of a kernel.

B.1 Chain complexes

Homology groups are useful as soon as they are well defined, this means for any set of objects C_n together with a set of maps d_n such that for any n $d_{n+1} \circ d_n = 0$. We call this system of objects and maps a chain complex and is denoted:

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n_1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{B.8}$$

And $\mathrm{H}(C_n) = \ker d_n / \mathrm{im}\, d_{n-1}$
Often many of the C_n will be zero, we still call the system a chain complex. In this way a short exact sequence is also a chain complex.

*As soon as our objects are additive categories our homology groups are abelian. The definition of non-abelian homology requires quite a lot of work an can be found in for example [22] & [50].

B.1.1 Example *One of the most important examples is the chain complex of the differential forms on a manifold together with the exterior derivative. If we denote the set differential forms with* Ω_n *then the homology group* $\mathrm{H}(\Omega_n)$ *tells us when a closed differential form is also exact. This depends on the structure of the manifold.*

B.1.2 Example *Supersymmetry (See [25] Chapter I.'Notes on Susy' and [28] chapter 3)*
A super vector space is a vector space with a $\mathbb{Z}/2\mathbb{Z}$ *grading, i.e.* $V = V_0 \oplus V_1$ *Supersymmetric wave functions are wavefunctions taking values in this space. For example for* $\mathbb{R}^{1|1} = \mathbb{R} \oplus \mathbb{R}$ *a wavefunction on the space is a function having one even and one odd coordinate. This is naturally identified with a map consisting of a function and a differential form of degree 1 on* $\mathbb{R}$*, i.e. (if* θ *is identified with* $\mathrm{d}y$*)*

$$\psi(x,\theta) = \psi_0(x) + \psi_1(x)\theta \tag{B.9}$$

$$\psi(x,y) \in L^2(\mathbb{R}^{1|1}) \cong \Omega(\mathbb{R}) \tag{B.10}$$

More general we have the isomorphism

$$L^2(\hat{X}) \cong \Omega(X) \tag{B.11}$$

Where $\hat{X}$ *is a* $n|n$ *dimensional supermanifold, defined by the tangent bundle of* X *where the fibres are made to anti-commute.*
Define supersymmetry transformations by

$$\delta x = \theta,\ \delta\bar{\theta} = ip \tag{B.12}$$

There is an operator $Q = \theta\frac{\partial}{\partial q} = \mathrm{d}$*. This operator is a differential, i.e.* $Q^2 = 0$ *and we can take the cohomology, which is*

$$H^*_Q(\mathcal{H}) = H^*_{dR}(X) \tag{B.13}$$

B.1.1 Definition *Given two complexes* A_* *and* B_**. A chainmap* ϕ *is a sequence of homomorphisms* $\phi_n : A_n \to B_n$ *such that they commute with* d *i.e.* $\phi d = d\phi$*. State differently, the following diagram commutes:*

$$\begin{CD}
\cdots @>{d_{n+2}}>> A_{n+1} @>{d_{n+1}}>> A_n @>{d_n}>> A_{n-1} @>{d_{n-1}}>> \cdots \\
@. @V{\phi_{n+1}}VV @V{\phi_n}VV @V{\phi_{n-1}}VV @. \\
\cdots @>{d_{n+2}}>> B_{n+1} @>{d_{n+1}}>> B_n @>{d_n}>> B_{n-1} @>{d_{n-1}}>> \cdots
\end{CD} \tag{B.14}$$

Suppose now the map ϕ_n defined by $\phi_n = d_{n+1}s_n + s_{n+1}d_n$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{n+2}} & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle \phi_{n+1}} & \overset{s_n}{\swarrow} & \downarrow{\scriptstyle \phi_n} & \overset{s_{n-1}}{\swarrow} & \downarrow{\scriptstyle \phi_{n-1}} & & \\
\cdots & \xrightarrow[d_{n+2}]{} & B_{n+1} & \xrightarrow[d_{n+1}]{} & B_n & \xrightarrow[d_n]{} & B_{n-1} & \xrightarrow[d_{n-1}]{} & \cdots
\end{array} \tag{B.15}$$

If we compute

$$df - fd = d(ds + sd) - (ds + sd)d = dsd - dsd = 0 \tag{B.16}$$

So $\phi = ds + sd : A_* \to B_*$ is a chainmap. A map ψ with the property that there exists $s_n : A_n \to B_{n-1}$ such that $\psi = ds + sd$ is called null homotopic.

B.1.2 Definition *Two maps $\phi, \psi : A_* \to B_*$ are called chain homotopic if their difference is null homotopic. i.e. if there exists a set of maps (called chain homotopy) $s_n : A_n \to B_{n-1}$ such that $\phi - \psi = sd + ds$.*
A map $\epsilon : A_* \to B_*$ is called a chain homotopy equivalence if there exists a map $\zeta : B_* \to A_*$ such that $\zeta\epsilon$ and $\epsilon\zeta$ are chain homotopic to the identity maps on A and B.

The following theorem makes clear why chain homotopy equivalence is a notion more natural than isomorphism when working with chain complexes (Just like in category theory).

B.1.1 Theorem *If $\phi, \psi : A_* \to B_*$ are chain homotopic they induce the same maps in homology.*

Proof: An element in a homology group is closed. Since ϕ and ψ are chainmaps they map closed to closed ($\mathrm{d}\phi(a) = \phi(\mathrm{d}a)$). Therefore if $\phi(a)$ is closed we know $\psi(a)$ is closed. This leaves us to show that if the difference $(\phi - \psi)(a)$ is non-vanishing for a closed, it should be exact. This follows from the homotopy property: $(\phi - \psi)(a) = (\mathrm{d}s + s\mathrm{d})(a) = \mathrm{d}sa$.

B.2 Resolutions

Let M be a module a (homological) *resolution* is an exact sequence

$$\cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0 \tag{B.17}$$

A resolution is said to be *injective (projective, free)* if the modules E_n are injective (projective, free). I'll only review resolutions in the context we need them, this means that a lot essential material is skipped. I refer to [49] for a more thorough introduction.

B.3 Splittings

A short exact sequence

$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0 \tag{B.18}$$

Is said to split if there exists an idempotent morphism $i : B \to B$ (and $i^2 = i$) whose kernel or image equals $\operatorname{im} a = \ker b$. It's important to note that there is an other idempotent $1 - i$ such that $\ker(1 - i) = \operatorname{im} i$. Moreover this i turns B into a topological sum $B = \ker i \oplus \operatorname{im} i \overset{\text{If} \ker i = \operatorname{im} a}{=} \operatorname{im} a \oplus C$

B.4 Extension of Modules

Let A and B be respectively an abelian group. An extension of G by A is a short exact sequence

$$1 \longrightarrow G \longrightarrow E \longrightarrow A \longrightarrow 1 \tag{B.19}$$

Any other extension

$$1 \longrightarrow G \longrightarrow E' \longrightarrow A \longrightarrow 1 \tag{B.20}$$

is said to be equivallent if the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & G & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 1 \end{array} \tag{B.21}$$

By the five lemma ϕ is an isomorphism. The set of all extensions modulo isomorphism form a group with the semi-direct product as a unit.

Ext Functor

B.4.1 Definition *A short exact sequence of Λ-modules $0 \to R \xrightarrow{\delta} P \xrightarrow{\epsilon} A \to 0$ with P projective is called a projective presentation of A.*

Given an other Λ module, the induced sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\epsilon^*} \mathrm{Hom}_\Lambda(P, B) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(R, B) \tag{B.22}$$

is exact.

To the projective presentation we can now assign the abelian group:

$$\mathrm{Ext}^\epsilon_\Lambda(A, B) = \mathrm{coker}(\mu^* : \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(R, B)) \tag{B.23}$$

Given any other projective presentation $R' \rightarrowtail P' \twoheadrightarrow A$, the diagram

$$\begin{array}{ccccc} R' & \rightarrowtail & P' & \twoheadrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \sim} \\ R & \rightarrowtail & P & \twoheadrightarrow & A \end{array} \tag{B.24}$$

And there is a natural equivalence between the Ext functors.

B.4.1 Theorem

$$\mathrm{Ext}^1(A, B) \cong \{\textit{Extensions of module } A \textit{ by } B\} \tag{B.25}$$

The proof can be found in [81] or [41]

The Bear Sum

Let $\epsilon : 0 \to B \to E \to A \to 0$ and $\epsilon' : 0 \to B \to E' \to A \to 0$ be two extensions of A by B. Let X'' be the pullback

$$\begin{array}{ccc} X'' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X & \longrightarrow & A \end{array} \tag{B.26}$$

X'' contains three copies of $B : B \times 0, 0 \times B$ and the skew diagonal $\{(b, b^{-1}) | b \in B\}$. Take the quotient of Y of X'' with respect to the skew diagonal. The sets $B \times 0$ and $0 \times B$ are now identified. Since $X''/0 \times B \cong X$ and $X/B \cong A$ the sequence

$$0 \to B \to Y \to A \to 0 \tag{B.27}$$

is also an extension of A by B and it's equivalence class is called the *Bear sum* of ϵ and ϵ'.

B.4.2 Theorem *The set of extensions is an abelian group under the Bear sum.*

Proof. See [81] section 3.4

B.5 Group extensions

Let G be any group and A an abelian group. An extension of G by A is a short exact sequence

$$1 \longrightarrow G \longrightarrow E \longrightarrow A \longrightarrow 1 \tag{B.28}$$

Any other extension

$$1 \longrightarrow G \longrightarrow E' \longrightarrow A \longrightarrow 1 \tag{B.29}$$

is said to be equivallent if the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & G & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 1 \end{array} \tag{B.30}$$

By the five lemma ϕ is an isomorphism. The set of all extensions modulo isomorphism form a group with the semi-direct product as a unit.

B.5.1 Definition *An extension*

$$0 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1 \tag{B.31}$$

is called central if A is in the center of E.

Bibliography

[1] A. Grothendiek, Séminaire de Géométrie Algébrique de l'I. H. E. S. 1960-1961 (SGA1).

[2] M. Artin, A. Grothendiek, J.-L. Verdier, Séminaire de Géométrie Algébrique de l'I. H. E. S. 1963-1964, Cohomologie étales des schémas (SGA4).

[3] M. Aganagic, J. Park, C. Popescu, J.H. Schwartz, World-Volume Action Of the M Theory Five-Brane, hep-th/9701166

[4] P. Argyres, Introduction to supersymmetry, http://www.physics.uc.edu/ argyres/661/96/96lects.ps

[5] P. Aschieri, B. Jurčo, Gerbes, M5-Brane Anomalies and E_8 Gauge Theory, hep-th/0409200.

[6] M.F. Atiyah, R. Bott, Yang-Mills Equations over Riemann Surfaces, Phi. Trans. R. Soc. Lond. A 308, 523-615 (1982)

[7] H. Awata, M. Li, D. Minic, T. Yoneya, On the quantization of Nambu Brackets, hep-th/9906248

[8] X. Bekaert, M. Henneaux, A. Sevrin, Chiral Forms and Their Deformations, hep-th/0004049

[9] I. Bengtsson, A. Kleppe, On Chiral P-Froms, hep-th/9609102

[10] E. Bergshoeff, D.S. Berman J.P. van der Schaar P. Sundell, A Noncommutative M-Theory Five-Brane, hep-th/0005026

[11] E. Bergshoeff, E. Sezgin, P.K. Townsend, Supermembranes in 11 dimensinal supergravity, Phys. Lett. 189B (1987 75

[12] E. Bergshoeff, E. Sezgin, P.K.Townsend, Properties of the eleven-dimensional supermembrane theory, Ann. Phys. 185 (1988) 330

[13] M. Berkooz, M. R. Douglass, Five-branes in M(atrix) Theory, hep-th/9610236

[14] R. Bott & Tu, Differential Forms in Algebraic Topology, Graduate texts in Mathematics 82, Sringer-Verlag, Berlin 1982.

[15] G. E. Bredon, Topology and Geometry, Graduate text in Mathematics 139, Springer-Verlag, New York 1993

[16] L. Breen, On the Classification of 2-Gerbes and 2-Stacks, Astérisque 225, Société Mathématique de France 1994.

[17] L. Breen & W. Messing, Differential Geometry of Gerbes, math.ag/0106083

[18] L. Breen & W. Messing, Combinatorial Differential Forms, Advances in Mathematics, 614 (2001) 203-282.

[19] A. Cannas da Silva, Lectures on Symplectic Geomtry, Lecture Notes in Mathematics 1764, Springer-Verlag, Berlin 2001.

[20] M. Cederwall, B.E.W. Nilsson, P. Sundell, An Action for the Super-5-Brane in D=11 Supergravity, hep-th/9712059

[21] S.R. Coleman, J. Mandula, All Possible Symmetries of the S-Matrix, Phys. Rev. D159 (1967) 1251.

[22] A. Connes, Non Commutative Geometry, Academic Press, London, 1994.

[23] A. Connes, M.R. Douglas, A. Schwarz, Noncommutative Geometry and Matrix Theory: Compactification on Tori, hep-th/9711162

[24] P. Cotta-Ramusino, Geometria Differenziale e Teorie di Gauge, available at: http://www.teor.mi.infn.it/~cotta/methodi-geom.ps

[25] P. Deligne et al., Quantum Fields and Strings: A Course for Mathematicians, AMS, 1999.

[26] F.P. Devecchi, M. Henneaux, Phys. Rev. D45 (1996) 1606.

[27] E. Diaconescu G. Moore D. S. Freed, The M-Theory 3-Form and E_8 Gauge Theory, hep-th/0312069

[28] R. Dijkgraaf, Les Houches Lectures on Fields, Strings and Duality, hep-th/9703136

[29] R. Dijkgraaf, The Mathematics of Fivebranes, hep-th/9810157

[30] R. Dijkgraaf E. Verlinde H. Verlinde, BPS Quantization of the Five-Brane, hep-th/9604055

[31] M. R. Douglas, C. Hull, D-branes and the Noncommutative Torus, hep-th/9711165

[32] M.R. Douglas, N.A. Nekrasov, Noncommutative Field Theory, hep-th/0106048

[33] Y. A. Gelfand, E.P. Likhtman, Extension of the Algebra of Poincaré Group Generators and Violation of P Invariance, JETP Lett. 13 (1971) 323

[34] P. Ginsparg, Applied Conformal Field Theory, hep-th/9108028

[35] Gireaud, Cohomologie non Abelienne, Grundlehren der Mathematische Wissenschaften 179, Springer Verlag, Berlin 1971.

[36] P. G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, Progress in Mathematics 174, Birkh auser, Basel 1999

[37] A. Grothendiek & Dieudonne, Elements de Geometrie Algebraic, Grundlehren der Mathematische Wissenschaften 166, Springer Verlag, Berlin 1971.

[38] B.R. Green, String Theory on Calabi-Yau Manifolds, hep-th/9702155

[39] M. Henneaux & C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, Princeton 1992.

[40] M. Henneaux & C. Teitelboim, Phys Lett. B206 (1988) 650

[41] P.J. Hilton, U. Stammbach, A Course in Homological Algebra, Graduate Text in Mathematics 4, Srpinger-Verlag, Berlin 1971.

[42] P.S. Howe, E. Sezgin, D=11, p=5, hep-th/9611008

[43] P.S. Howe, E. Sezgin, P.C. West, Covariant Field Equations of the M Theory Five-Brane, hep-th9702008

[44] B. Iversen, Cohomology of Sheaves, Universitext, Springer-Verlag, Berlin 1986.

[45] C.V. Johnson, D-Brane Primes, hep-th/0007170

[46] M. Kontsevich, Y. Manin, Gromov-Witten Classes, Quantum Cohomology and Enumerative Geometry, Commun. Math. Phys. **164** 525-562 (1994).

[47] S. Kobayashi & K. Nomizu, Foundations of Differential Geometry, Interscience publishers, 1963.

[48] A. Kock, Synthetic Differential Geomtry, London Mathematical Society Lecture Note Series 51, Cambridge University Press 1981.

[49] S. Lang, Algebra (3th edition), Graduate texts in Mathematics 211, Springer Verlag, New York 2002.

[50] J-L Loday, Cyclic Homology, Grundlehren der Mathematische Wissenschaften 301, Springer Verlag, Berlin 1992.

[51] S. MacLane, Categories for the Working Mathematician, Graduate Texts in Mathematics 5, Springer-Verlag, Berlin 1971

[52] S. MacLane, I. Moerdijk, Sheaves in Geometry and Logic, Universitext, Springer Verlag, Berlin 1992.

[53] K. Mackenzie, Lie Groupoids and Algebroids in Differential Geometry, London Mathematical Society Lecture Note Series 124, Cambridge University Press 1987.

[54] B. McClain, Y.S. Wu, F. Yu, Nucl. Phys. B343 (1990) 689.

[55] I. Moerdijk, Introduction to the Language of Stacks and Gerbes, math.AT/0212266

[56] I. Moerdijk, Lie Groupoids, Gerbes and Non-Abelian Cohomology, K Theory 28 207-258, Math.dg/0203099.

[57] I. Moerdijk, J. Mrčun, Introduction to Foliations and Lie Groupoids, Cambridge Studies in Advanced Mathematics 91, Cambridge University Press 2003.

[58] M. K. Murray, Bundle Gerbes, J. London Math. Soc. 54(2) (1996) 403-416

[59] M. Nakahara, Geometry, Topology and Physics, Institute of Physics, London 2003 (second edition).

[60] P. Pasti, D. Sorokin, M. Tonin, On Lorentz Invariant Actions for Chiral P-Forms, hep-th/9611100

[61] P. Pasti, D. Sorokin, M. Tonin, Covariant Action for a D=11 Five-Brane with the Chiral Field.

[62] M. E. Peskin, D. V. Schroeder, An Introduction to Quantum Field Theory, Westview Press, 1995.

[63] J. Polchinski, String Theory, Cambridga momographs on mathematical physics, Cambridge University Press 1998.

[64] I.R. Shafarevich, Basic Algebraic Geometry, Springer-Verlag, Berlin, 1997 (third edition).

[65] A.N. Schellekens, Introduction to Conformal Field Theory, NIKHEF-H/96-011

[66] J.H. Schwarz, Coupling a Self-Dual Tensor to Gravity in Six Dimensions, hep-th/9701008

[67] E. Sezgin, Topics in M-Theory, hep-th/9809204

[68] N. Seiberg, E. Witten, String Theory and Noncommutative Geometry, hep-th/9908142

[69] E. Sezgin P. Sundell, Aspects of the M5-Brane, hep-th/9902171

[70] W. Siegel, Nucl. Phys. B238 (1984) 307

[71] A. Strominger, Open P-branes, hep-th/9512059

[72] W. Taylor, M(atrix) Theory: Matrix Quantum Mechanics as a Fundamental Theory, hep-th/0101126

[73] B. Toën G. Vezzosi, Homotopical Algebraic Geometry I: Topos theory, math.AG/0207028.

[74] P.K. Townsend, Four Lectures on M-Theory, hep-th/9612121

[75] P.K. Townsend, D-branes from M-branes, hep-th/9512062

[76] P.K. Townsend, M-Theory from it's Superalgebra, hep-th/9712004

[77] A.A. Tseytlin, R^4 Terms in 11 dimensions and Conformal Anomaly of (2,0) Theory, hep-th/0005072

[78] C. Vafa, Lectures on Strings and Dualities, hep-th/9702201

[79] P. di Vecchia A. Liccardo, String Theories and D-Branes, Nordita-2001/15 HE

[80] W. Walter, Ordinary Differential Equations, Graduate texts in Mathematics 182, Springer-Verlag, Berlin 1998.

[81] Weibel, An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press 1994.

[82] A. Weinstein, Groupoids: Unifying Internal and External Symmetry, Math.RT/9602220

[83] P.C. West, Supergravity, Brane Dynamics and String Duality, hept-th/9811101

[84] E. Witten, Five-Brane Effective Action in M-Theory, hep-th/9610234

[85] E. Witten, Bound States of Strings and p-Branes, hep-th/9510135

[86] E. Witten, String Theory Dynamics in Various Dimensions, hep-th/9503124