How Poincaré dualities survive dimensional reduction

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Introduction

As its title suggests, this thesis deals with Poincaré dualities, their dimensional reduction and a subsequent application to supergravity. The application is the main reason why this thesis was written, but the reduction of Poincaré dualities is interesting enough in its own right. Both will be treated in Part III, but before we are ready to discuss them we need a firm mathematical basis which will be laid down in Part II. And of course it is convenient to know *why* it is interesting to look at (Poincaré) dualities and their reductions, which is exactly the point of this first Part.

1.1 Duality

Okay, so we're about to treat dualities, but what are dualities anyway? In the absence of a decent physicist to help us out, we resort to the English dictionary and come up with the following definitions:

> du·al·i·ty (dōō-ăl'ĭ-tē, dyōō-), *n*:
>
> 1. The quality or character of being twofold.
> 2. The property of matter and electromagnetic radiation that is characterized by the fact that some properties can be explained best by wave theory and others by particle theory.

The first isn't all that helpful, but the second implies that if something exhibits duality, there are several ways of looking at it. It actually refers to the well known particle-wave duality in quantum mechanics, famously illustrated by the double-slit experiment first performed by Thomas Young. But here we stumble upon the area of physics, and this time there happens to be a physicist around who can tell us the following:

> "*Saying that a physical system exhibits duality implies that there are two complementary perspectives, formulations, or constructions of the theory.*"
>
> – J.A. Harvey [1]

Sticking to the particle-wave duality in quantum mechanics, we can crudely think of it as the relation between the position states $\langle x|\varphi\rangle$ and the momentum states $\langle p|\varphi\rangle$ given by a Fourier transform.

A simple example of this is the harmonic oscillator, which exhibits self-duality; it looks the same in coordinate space and in momentum space. This system is defined by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{1.1}$$

and we have the commutation relation $[x,p] = i$. Clearly the Hamiltonian and the commutation relation are invariant if we interchange x and p as follows:

$$\begin{aligned} x &\rightarrow \frac{p}{m\omega}, \\ p &\rightarrow -\omega m x. \end{aligned} \tag{1.2}$$

A transformation that redefines (or interchanges) one or more parameters of the theory but leaves the theory itself invariant is called a *duality transformation*. In the case of the harmonic oscillator it is nothing more than a neat gimmick, but for some other theories dualities are actually quite useful.

Consider for example the Ising model, which is defined as a set of spins σ_i taking the values ± 1 on a square two-dimensional lattice with nearest neighbor interaction J. The partition function at temperature T is given by

$$Z(K) = \sum_{\sigma} \exp(K \sum_{(ij)} \sigma_i \sigma_j), \qquad K = J/k_B T. \tag{1.3}$$

As we all know, this system becomes ferromagnetic at a critical temperature T_c. This temperature may be found by solving the system explicitly, but also by exploiting the fact the system can be expressed in terms of a *dual* lattice. This dual lattice is given by the square whose vertices are the centers of the faces of the original lattice. In order to ensure the partition function (1.3) stays the same when summing over this dual lattice, the coupling K has to be modified to

$$K \rightarrow K^*, \quad \sinh 2K^* = 1/(\sinh 2K). \tag{1.4}$$

This means that high temperature ($K \ll 1$) or weak coupling is mapped to low temperature ($K^* \gg 1$) or strong coupling on the dual lattice, and vice versa. Now if the system has only one phase transition then it has to occur at the self-dual point where we have $K = K^*$, or equivalently,

$$\sinh(2J/k_B T_c) = 1. \tag{1.5}$$

It turns out that this is exactly the same critical temperature as the one found when the system is solved explicitly!

We see that duality in this case provides non-trivial information about the critical behavior and relates a strongly coupled to a weakly coupled theory. The latter property is especially useful in theoretical physics, because one of the major tools for examining (field) theories is the perturbation expansion in the coupling parameter. For quantum electrodynamics (QED) the perturbation expansion works almost perfectly because the coupling (given by the fundamental electric charge) is very small. But for quantum chromo dynamics (QCD) this completely and utterly fails, as its coupling is quite large. Therefore it is tempting to search for dualities allowing us to describe QCD as a weakly coupled system. This is however far outside the scope of this thesis, but it may be noted that such a strong-weak duality does happen to exist in string theory [2]. It needs no explanation that this duality is readily used by theoretical physicists.

1.2 Electromagnetic duality

The electromagnetic duality is not just any duality, but it is the one which will be generalized to Poincaré duality in chapter 7. Electromagnetics is governed by Maxwell's equations, which read

$$\vec{\nabla}\cdot\vec{E} = \rho_e \qquad\qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J}_e \qquad\qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0.$$

In vacuo ($\rho_e = \vec{J}_e = 0$) these equations are invariant under the duality transformation

$$\begin{aligned}\vec{E} &\rightarrow \vec{B},\\ \vec{B} &\rightarrow -\vec{E},\end{aligned} \tag{1.6}$$

that is, the interchange of electric and magnetic fields. If we set $F^{0i} = -E^i$ and $F^{ij} = -\varepsilon^{ijk}B^k$, Maxwell's equations can be written in covariant form as

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \qquad \partial_\mu \star F^{\mu\nu} = 0. \tag{1.7}$$

where $\star F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$. The second Maxwell equation allows us to write

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{1.8}$$

If A^μ is taken as the fundamental field of electromagnetism, then the second Maxwell equation follows as an identity rather as a dynamical law. A^μ is called the vector potential, and $F^{\mu\nu}$ its field strength. The duality transformation in vacuo now is given by

$$\begin{aligned}F^{\mu\nu} &\rightarrow \star F^{\mu\nu},\\ \star F^{\mu\nu} &\rightarrow -F^{\mu\nu},\end{aligned} \tag{1.9}$$

that is, the rotation of the field strength and its dual into each other.

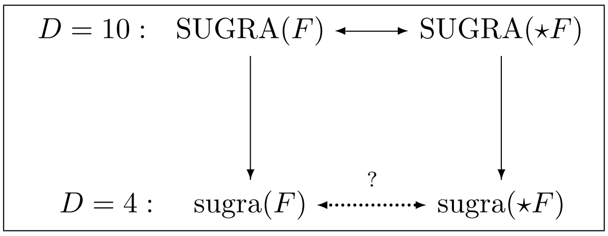

Figure 1.1: The duality in higher dimension holds, but does it also hold when both the original theory and the dualized version are reduced?

As an interesting side note, we may move away from vacuum and introduce a magnetic current term j^μ_m in (1.7). Maxwell's equations would still be invariant under the duality transformation if we also replace $j^\mu_e \longleftrightarrow j^\mu_m$. A necessary condition for the existence of magnetic monopoles is that the fundamental monopole charge be inversely proportional to the fundamental electric charge (this is known as the Dirac quantization condition). But as the latter is a small number, the former has to be large. So we see that electromagnetic duality interchanges weakly and strongly coupled regimes, similar to the duality in the Ising model.

However, we will stick to vacuum and generalize the electromagnetic duality to field strengths of arbitrary rank in arbitrary dimensions, which we will then call *Poincaré dualities*.

1.3 Motivation: duality in supergravity

In supergravity there will always be field strengths around, as we will see in the next chapter. Using Poincaré dualities these field strengths can be dualized to different field strengths, known as their duals. Of course the dynamics of the dual field strengths are the same as the original field strengths, but this might not be the case in the dimensionally reduced theory. This is schematically depicted in figure 1.1.

This is of importance when we look at some properties of the reduced theory, as the dualized version might behave different than the original. In particular, we are interested in the ability of four-dimensional supergravity to describe cosmological inflation. It is known to describe it quite well, but not good enough [3]. The hope is that the reduced dualized version does a better job, a hope which will be shattered when the question mark in figure 1.1 can be removed.

The goal of this thesis is to investigate that particular question mark.

Chapter 2

Supersymmetry and supergravity

2.1 Introduction

Supersymmetry is a symmetry that relates particles with half integer spin (fermions) to particles with integer spin (bosons). In any supersymmetric theory every fermion has a bosonic partner and vice versa. Supersymmetry relates them via

$$\delta(\text{boson}) = \text{fermion}, \qquad \delta(\text{fermion}) = d(\text{boson}), \tag{2.1}$$

where d is an ordinary translation. This symmetry extends the usual Poincaré symmetry with fermionic generators.

Despite the fact that at the moment there is no unambiguous empirical evidence for such a symmetry, theoretical physicists find the idea of supersymmetry appealing for a number of reasons [4].

One of those reasons is that supersymmetry fixes some of the renormalization problems found in quantum field theories. For instance, in the Standard Model the mass of the Higgs scalar would be of the order of the Planck mass ($\sim 10^{18}$ GeV) due to quadratic radiative corrections. However a supersymmetric version of the Standard Model the quadratic corrections of fermions cancel those of bosons, and thus this so called “Hierarchy Problem” is no longer present. Furthermore the coupling constants of the strong, electromagnetic and weak interactions are dependent on the energy scale and converge almost to a single value at approximately 10^{15} GeV. But when supersymmetry is introduced the energy dependence is modified and the coupling constants now converge much better at around 10^{16} GeV (see figure 2.1); which could be a sign of unification.

Supersymmetry also lends a helping hand in the area of cosmology, where it could account for the cold dark matter problem. Some of the supersymmetric partners of elementary particles have the properties of “Weakly Interacting Massive Particles” (or WIMPS for short). This makes them good candidates for the thus far unobserved dark matter.

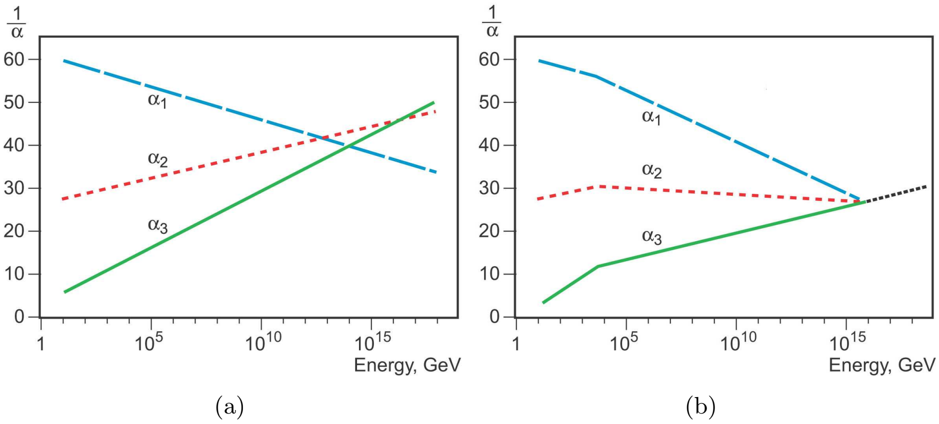

Figure 2.1: The energy dependence of the coupling constants in the Standard Model (a), and corrected for supersymmetry (b).

Apart from this 'bottom-up' approach, there is another reason why one should not ignore supersymmetry: it is an essential ingredient in string theory. In a non-supersymmetric (bosonic) string theory there are always unphysical tachyonic states present, whereas in supersymmetric string theory these states vanish. It turns out that the lower energy limit of superstring theories yield supergravities, and therefore supersymmetry also arises from a 'top-down' approach.

2.2 The Coleman-Mandula result

A Poincaré transformation is a proper Lorentz transformation Λ followed by a translation operator a. Thus if x^μ are the coordinates of a point in spacetime, then its transformed coordinates are

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{2.2}$$

Such transformations are elements of the Poincaré group, which is a semi-direct product of the Lorentz group and the translation group. Its generators obviously are the rotation generators $M_{\mu\nu}$ plus the translation generators P_μ. The Lie algebra of the Poincaré group is given by [5]

$$\begin{aligned} [M_{\mu\nu}, M^{\rho\sigma}] &= -2\delta^{[\rho}_{[\mu} M_{\nu]}{}^{\sigma]}, \\ [P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} P_{\rho]}, \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{2.3}$$

For more information on Lie groups and their associated algebras see chapter 4. Apart from Poincaré symmetry there are two other symmetries playing a role in particle physics. They are the so-called "internal" global symmetries, which are related to conserved quantum numbers, and the discrete symmetries C, P, and T.

Before 1967 physicists tried to unify these symmetries into the same group, in order to fit the observed particle spectrum into a representation of it. But in that year Coleman and Mandula showed that, given certain assumptions, the maximal Lie algebra of symmetries of the S-matrix is a direct product of the Poincaré algebra with the Lie algebra of some compact internal symmetry group [6]. As the algebra of a compact Lie group is the direct product of a semisimple and an abelian Lie algebra, the largest Lie algebra of the S-matrix symmetries is a direct product: Poincaré $\times$ semisimple $\times$ abelian. As a consequence multiplets of the internal symmetry group consist of particles with the same mass and the same spin. For this reason the Coleman-Mandula result is often referred to as their "no-go"-theorem.

The way out of this situation is not to try and weaken the initial assumptions of this theorem, but to broaden the notion of symmetry to encompass Lie superalgebras. Haag, Lopuzański and Sohnius found in their "go-go"-theorem [7] that the most general extension of the above three symmetries was the inclusion of spinor generators. These generators are fermions, and often called *supercharges*. The Coleman-Mandula result then still applies to the bosonic sector of the superalgebra, whereas the supercharges generate the fermionic sector by means of equation (2.1). It is fairly easy to see that by repeated use of (2.1) bosons get mapped to translated bosons:

$$\text{bosons} \xrightarrow{Q} \text{fermions} \xrightarrow{Q} \text{translated bosons.} \tag{2.4}$$

And thus this immediately leads to the following theorem:

Theorem 1. *There are an equal number of bosonic and fermionic degrees of freedom in any realization of the supersymmetry algebra when translations are an invertible operation.*

When the symmetry gets extended with supercharges, the super-Poincaré algebra (or superalgebra for short) contains the Lorentz generators $M_{\mu\nu}$, the translation generators P_μ, and the supersymmetry generators Q_α (which behave as spinors under the Lorentz symmetry; hence the name).

If there is only one supercharge present we are dealing with *simple* or *minimal* supersymmetry, but if there are more it is called *extended* supersymmetry. In that case the supercharges are denoted by Q^I_α, where the index $I = 1, \ldots, N$ runs over all the supercharges. In the case that the supersymmetry is global, we are dealing with *rigid* supersymmetry. Demanding that the supersymmetry be local, on the other hand, automatically leads to *supergravity*. This is due to the presence of translations the superalgebra.

We are interested in the possible realizations of supergravity in various dimensions. For that we have to know what kind of spinors are possible in those dimensions, as the field content of supergravities is dictated by representations of the superalgebra.

2.3 Spinors in different dimensions

A spinor one might be familiar with is the Dirac spinor. It is defined as a multi-component field that transforms under Lorentz symmetry according to a fermionic representation of the Lorentz group [8]. Such a representation is given by

$$S^{\mu\nu} = \frac{i}{4}[\Gamma^\mu, \Gamma^\nu], \tag{2.5}$$

where the gamma matrices Γ^μ satisfy the Clifford algebra $\mathcal{C}(D-1,1)$

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}. \tag{2.6}$$

The minimal real dimension of the representation (2.5) is given by $2^{[D/2]+1}$, where brackets denote the integer part. The number of components of Dirac spinors is equal to the dimension of the representation.

In some dimensions Dirac spinors are reducible in left and right chiral parts. These parts are called Weyl spinors and have to have an eigenvalue of respectively +1 or -1 under the chirality operator

$$\Gamma_* = i^{D/2+1}\Gamma_0 \cdots \Gamma_{D-1}. \tag{2.7}$$

Sometimes it is possible to impose a reality condition by requiring that the charge-conjugate spinor is equal to the spinor itself:

$$\Psi^c \equiv C\bar{\Psi}^T = \Psi, \tag{2.8}$$

where C is a charge conjugation matrix. This condition gives rise to so-called Majorana spinors. In some cases it is possible to impose both conditions, which leads to Majorana-Weyl spinors. For a clear treatment of Dirac, Weyl and Majorana spinors in the four dimensional case, see for example [9]. All the different types of spinors in various dimensions and their corresponding components are listed in table 2.1.

Dimension	Components (q)	Type
2 mod 8	$2^{D/2-1}$	MW
3,9 mod 8	$2^{(D-1)/2}$	M
4,8 mod 8	$2^{D/2}$	M
5,7 mod 8	$2^{(D+1)/2}$	D
6 mod 8	$2^{D/2}$	W

Table 2.1: The different minimal spinors in flat Minkowski spacetimes of dimension D. The third column specifies the types of spinors: Dirac (D), Weyl (W), Majorana (W), and Majorana-Weyl (MW).

2.4 Possible supergravity theories

When considering extended supersymmetry with N supercharges the total number of supercharge components Q is given by [10]

$$Q = Nq. \tag{2.9}$$

Supergravities with $Q > 32$ always have states with helicity higher than two. As such states cannot be consistently coupled to themselves or to other fields, such theories are usually ignored. Theories with exactly 32 supercharge components are called *maximal* supergravities.

When we combine this bound on supercharges with the results of table 2.1, we can overlook the different possibilities for N in various dimensions. A harsh inference is that the maximum dimension for supergravities is eleven, as the minimum of supercharge components in twelve dimensions is 64. Also, supergravity in eleven dimensions is necessarily maximal.

For $N = 2$ in 10 dimensions the spinors can have either the same or the opposite chirality, leading to type IIB or IIA supergravity. This is denoted by (2,0) and (1,1) respectively, where the first digit refers to the number of spinors with positive chirality and the last to negative chirality.

If we would like to know the field content realization of the superalgebras we have to know the dimension of the shortest supermultiplet and how the degrees of freedom split up on-shell, which is explicitly demonstrated in [11]. Then it is possible to assign fields with matching degrees of freedom to the supermultiplet. This has been done for $D = 11, 10, 4$ in table 2.2.

D	N	Type	Fields	n	Q
11	1	M	$g_{\mu\nu}$, ψ_μ, $B_{\mu\nu\rho}$	128	32
10	(1,1)	MW	$g_{\mu\nu}$, $\psi_{+\mu}$, $\psi_{-\mu}$, $B_{\mu\nu\rho}$, $B_{\mu\nu}$, B_μ, λ_+, λ_-, ϕ	128	32
	(2,0)	MW	$g_{\mu\nu}$, $2\,\psi_{+\mu}$, $B_{\mu\nu\rho\sigma}$, $2\,B_{\mu\nu}$, $2\,\lambda_-$, $2\,\phi$	128	32
	1	MW	$g_{\mu\nu}$, $\psi_{+\mu}$, $B_{\mu\nu}$, λ_-, ϕ	64	16
4	8	M	$g_{\mu\nu}$, $8\,\psi_\mu$, $28\,B_\mu$, $56\,\lambda$, $70\,\phi$	128	32
	6	M	$g_{\mu\nu}$, $6\,\psi_\mu$, $16\,B_\mu$, $26\,\lambda$, $30\,\phi$	64	24
	5	M	$g_{\mu\nu}$, $5\,\psi_\mu$, $10\,B_\mu$, $11\,\lambda$, $10\,\phi$	32	20
	4	M	$g_{\mu\nu}$, $4\,\psi_\mu$, $6\,B_\mu$, $4\,\lambda$, $2\,\phi$	16	16
	3	M	$g_{\mu\nu}$, $3\,\psi_\mu$, $3\,B_\mu$, λ	8	12
	2	M	$g_{\mu\nu}$, $2\,\psi_\mu$, B_μ	4	8
	1	M	$g_{\mu\nu}$, ψ_μ	2	4

Table 2.2: Supergravity multiplets for various dimensions. The subscripts ± on spinor fields denote chiralities. The column n denotes bosonic (and thus also fermionic) physical degrees of freedom. Adapted from [12]

.

The individual degrees of freedom of the all the fields in the multiplet has to sum to n, which they do, as you can check with table 2.3. As a side note, the fermions appearing in the supermultiplet are the supersymmetric partners

Name	Symbol	Spin	On-shell DOF
Graviton	$g_{\mu\nu}$	2	$\frac{1}{2}(D-1)(D-2)-1$
Gravitino	ψ_μ	3/2	$\frac{1}{2}(D-3)\cdot q$
Rank-p potential	$B_{\mu_1\cdots\mu_p}$	1	$\binom{D-2}{p}$
Dilatino	λ	1/2	$\frac{1}{2}\cdot q$
Scalar	ϕ	0	1

Table 2.3: On-shell degrees of freedom of D-dimensional supergravity fields.

of the bosons. Thus the gravitino is the fermionic partner of the graviton, as the dilatino is of the dilaton ϕ.

2.4.1 $N = 1$, $D = 10$ supergravity

There are 16 supercharge components in $N = 1, D = 10$ supergravity. Its field content consists of the graviton $g_{\mu\nu}$, the gravitino ψ_μ, an antisymmetric rank two tensor field $B_{\mu\nu}$, the dilatino λ and the dilaton ϕ. We will take the shortcut of directly truncating to the bosonic sector. The action is then given by

$$S = \int \mathrm{d}x^{10}\sqrt{-g}\ \mathrm{e}^{-\phi}\left(R + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{2.10}$$

where $H_{\mu\nu\rho}$ is the field strength associated with $B_{\mu\nu}$:

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cyclic permutations of } \mu,\nu,\rho. \tag{2.11}$$

In itself this theory is troubled by gauge and gravitational anomalies. However, if it is coupled to an appropriate supersymmetric ten-dimensional Yang-Mills theory these anomalies cancel [13]. This coupled version of the action is given by:

$$S = \int \mathrm{d}x^{10}\sqrt{-g}\ \mathrm{e}^{-\phi}\left(R + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \tfrac{1}{4}\,\mathrm{Tr}\,F^I_{\mu\nu}F^{I\mu\nu}\right), \tag{2.12}$$

where $F^I_{\mu\nu}$ denotes the non-abelian gauge field strength,

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + [A^I_\mu, A^I_\nu]. \tag{2.13}$$

The field strength $H_{\mu\nu\rho}$ now is given by

$$\begin{aligned} H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \tfrac{1}{2}\,\mathrm{Tr}\left(A^I_\mu F^I_{\nu\rho} + \tfrac{1}{3}A^I_\mu[A^I_\nu, A^I_\rho]\right) \\ + \text{cyclic permutations of } \mu,\nu,\rho. \end{aligned} \tag{2.14}$$

The Yang-Mills vectors A^I_μ are in the adjoint representation of either the $SO(32)$ or $E_8 \times E_8$ group. As it turns out, the coupled action (2.12) is also the low-energy limit of ten-dimensional heterotic string theory [14].

2.4.2 $N = 4,\ D = 4$ supergravity

The number of supercharge components in $N = 4,\ D = 4$ supergravity is 16, the same as in $N = 1, D = 10$ supergravity. This indicates that the former can be obtained from the latter by means of dimensional reduction. From table 2.2 it is apparent that the field content is given by one graviton, four gravitinos, six antisymmetric rank two tensor field, four dilatinos and two dilatons.

The Lagrangian for $N = 4,\ D = 4$ supergravity can be simply obtained from (2.10) by techniques described in chapter 6, or it can be constructed directly in four dimensions [15]. It can be coupled to $N = 4$ Yang-Mills theory; the resulting Lagrangian describes the so-called *matter coupled* supergravity. Again it can be obtained from ten-dimensional supergravity, this time from (2.12), or be constructed directly in four dimensions.

The coupling can be modified to accommodate a non-Abelian gauge group, which entails the creation of a potential for the scalar fields present in the theory. It are the extrema of this potential that determine the possible ground states of the theory, and thus the possible values of the cosmological constant Λ (which is basically the energy density of the vacuum states).

One may have noticed the lack of formulas in this section; this is because $N = 4, D = 4$ supergravity and its Lagrangian are quite complicated. Writing the latter down explicitly would not be instructive at all, instead we refer to [16] for a short result and to [15] for a more thorough treatment.

2.5 Poincaré duality in supergravity

Now that we know what the field content of $N = 1, D = 10$ supergravity is, we know what Poincaré duality to look for. It is the rank two antisymmetric potential $B_{\mu\nu}$, or rather its associated field strength $H_{\mu\nu\rho}$ given by (2.11) (or by (2.14) if it is coupled to Yang-Mills fields) we want to dualize. How this dualization exactly works will be demonstrated in chapter 7, but for now we can suffice by saying that the resulting action depends on the dualized field strength which we will denote by ${}^{d}H_{\mu\nu\rho}$. Both the original and the dualized action can be dimensionally reduced, yielding a whole set of four-dimensional field strengths. Some of these field strengths will have zero degrees of freedom and thus contribute to the scalar potential.

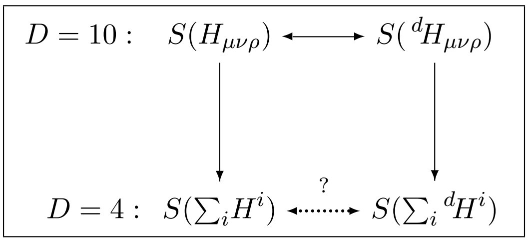

Figure 2.2: The duality scheme of section 1.3 revisited. Here $\sum_i H^i$ denotes the set of field strengths obtained from $H_{\mu\nu\rho}$.

The investigation of the question mark raised in section 1.3 now comes down to analyzing both sets of field strengths obtained from $H_{\mu\nu\rho}$ and ${}^{d}H_{\mu\nu\rho}$ to see whether they are equivalent or not (see figure 2.2).

Part II

Mathematical prerequesites

Chapter 3

Differential forms

In this chapter we will, amongst other things, describe differential forms. They are an almost mandatory tool when one considers Poincaré dualities, as these dualities involve antisymmetric tensors of arbitrary rank and thus with an arbitrary large number of indices. Using the language of differential forms one can forget about the indices, which makes life a whole lot easier. Most of the things found in this chapter come from [17] and [18].

3.1 Tensors, components, and bases

A tensor can be defined as an object that transforms under a coordinate transformation like the transformation itself. This is perhaps its best known definition, and it is why tensors are aptly used in the theory of relativity. But they can also be defined in another way: as a multilinear map from a collection of dual vectors and vectors to the real line:

$$T: \quad \underset{(k \text{ times})}{T_p^* \times \cdots \times T_p^*} \times \underset{(l \text{ times})}{T_p \times \cdots \times T_p} \to \mathbb{R} \tag{3.1}$$

This defines a tensor of rank (k, l). The vectors it acts on are elements of the tangent space T_p, which is the set of all possible vectors at a point p in a manifold. On the other hand, dual vectors are elements of the cotangent space T_p^*, which is the space of all linear maps from the tangent space to the real line. For instance, if $\omega \in T_p^*$ is a dual vector, then it acts on a linear combination of the vectors V and W as

$$\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbb{R}, \tag{3.2}$$

where a and b are real numbers.

Both the tangent and cotangent space are vector spaces. So there's nothing wrong with introducing a set of basis vectors $e_{(\mu)}$ for the tangent space and a set dual basis vectors $\theta^{(\nu)}$ for the cotangent space. Then every (dual) vector can be written in terms of its components:

$$\begin{aligned} V &= V^\mu e_{(\mu)}, \\ \omega &= \omega_\nu \theta^{(\nu)}. \end{aligned} \tag{3.3}$$

Furthermore we demand that the basis vectors satisfy the following orthonormality condition:

$$\theta^{(\nu)} e_{(\mu)} = \delta^\nu_\mu. \tag{3.4}$$

With this requirement the action of a dual vector on a vector now takes on a familiar form:

$$\begin{aligned} \omega(V) &= \omega_\nu V^\mu \theta^{(\nu)} e_{(\mu)} \\ &= \omega_\nu V^\mu \delta^\nu_\mu \\ &= \omega_\mu V^\mu \in \mathbb{R}. \end{aligned} \tag{3.5}$$

This is why most physicist tend to 'forget about the bases': the components alone can do all the work.

Before returning to the subject of tensors, consider as a final note on dual vectors what is the simplest example of one in spacetime: the gradient of a scalar function. It is given by the set of partial derivatives with respect to the spacetime coordinates x^μ:

$$\mathrm{d}\phi = \partial_\mu \phi \; \theta^{(\mu)} \equiv \frac{\partial \phi}{\partial x^\mu} \theta^{(\mu)}. \tag{3.6}$$

Because tensors act on elements of vector spaces, it is not surprising that the space of all (k, l) tensors also forms a vector space. For this space one can construct a basis in terms of k basis vectors and l dual basis vectors, but before doing so a new operation called the tensor product, denoted by $\otimes$, has to be defined. If T is a (k, l) tensor and S a (m, n) tensor, a new $(k+m, l+n)$ tensor $T \otimes S$ can be defined by

$$\begin{aligned} &T \otimes S(\omega^{(1)}, \ldots, \omega^{(k+m)}, V^{(1)}, \ldots V^{(l+n)}) \\ &= T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots V^{(l)}) S(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, V^{(l+1)}, \ldots V^{(l+n)}) \end{aligned} \tag{3.7}$$

The basis for the space of all (k, l) tensors can now easily be constructed by taking the tensor product of k basis vectors and l dual basis vectors:

$$e_{(\mu_1)} \otimes \cdots \otimes e_{(\mu_k)} \otimes \theta^{(\nu_1)} \otimes \cdots \otimes \theta^{(\nu_l)}. \tag{3.8}$$

The arbitrary tensor T can accordingly be written in component notation as

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \, e_{(\mu_1)} \otimes \cdots \otimes e_{(\mu_k)} \otimes \theta^{(\nu_1)} \otimes \cdots \otimes \theta^{(\nu_l)}. \tag{3.9}$$

As with vectors, one can take the shortcut of working with just the components and simply forget about the basis:

$$T(\omega^{(1)},\dots,\omega^{(k)},V^{(1)},\dots,V^{(l)}) = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\,\omega^{(1)}_{\mu_1}\cdots\omega^{(k)}_{\mu_k}V^{(1)\nu_1}\cdots V^{(l)\nu_l}. \tag{3.10}$$

So far only arbitrary bases have been considered. But there is no obstacle in defining a particular basis for the tangent space T_p, and often the so-called coordinate basis is employed. It is given by:

$$e_{(\mu)} = \partial_\mu. \tag{3.11}$$

This choice ensures that the basis vectors point along the coordinate axis. The corresponding choice of basis for the cotangent space is given the gradients of the coordinate functions x^μ:

$$\theta^{(\mu)} = \mathrm{d}x^\mu. \tag{3.12}$$

One can check that this is an appropriate basis by looking at the orthonormality requirement $\theta^{(\nu)}e_{(\mu)} = \delta^\nu_\mu$. Indeed:

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu. \tag{3.13}$$

Although this is the natural choice of basis it it sometimes more convenient to use an orthonormal basis; this is exactly what will be done later on in the Einstein-Cartan formalism (see chapter 5).

For the remainder of this section, which will deal about symmetric and antisymmetric tensors, we will simply forget about the basis. A tensor is said to be symmetric in any of its indices if it is unchanged upon interchange of those indices. For example, the $(0,3)$ tensor S is symmetric in its first two indices if

$$S_{\mu\nu\rho} = S_{\nu\mu\rho}. \tag{3.14}$$

But if it changes sign when the indices are exchanged, that is,

$$S_{\mu\nu\rho} = -S_{\nu\mu\rho}, \tag{3.15}$$

it is said to antisymmetric in those indices. When a tensor is (anti-)symmetric in all of its indices, it is simply called (anti-)symmetric. An arbitrary tensor can be symmetrized (or antisymmetrized) in any number of its indices. The symmetrization is indicated by round brackets:

$$T_{(\mu_1\cdots\mu_n)} = \frac{1}{n!}(T_{\mu_1\cdots\mu_n} + \text{sum over all permutations}), \tag{3.16}$$

where the permutations are over the indices $\mu_1\cdots\mu_n$. Conversely, antisymmetrization is given by

$$T_{[\mu_1\cdots\mu_n]} = \frac{1}{n!}(T_{\mu_1\cdots\mu_n} + \text{alternating sum over all permutations}), \tag{3.17}$$

where the permutations with an odd number of exchanges now get a minus sign. Symmetric and antisymmetric tensor satisfy respectively

$$\begin{aligned} T_{\mu_1\cdots\mu_n} &= T_{(\mu_1\cdots\mu_n)}, \\ T_{\mu_1\cdots\mu_n} &= T_{[\mu_1\cdots\mu_n]}. \end{aligned} \tag{3.18}$$

Having refreshed our knowledge of tensors, we are ready to move on to a special class of tensors known as differential forms.

3.2 Differential forms

A differential form of rank p, or simply a p-form, is nothing more than a fully antisymmetric tensor of rank $(0, p)$:

$$A^{(p)} = A_{[\mu_1\cdots\mu_p]}\mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_p}. \tag{3.19}$$

Thus a scalar is automatically a 0-form, while dual vectors are 1-forms. In fact it is the basis $\mathrm{d}x^\mu$ of dual vectors that is the 1-form, and not the components (which are merely a collection of numbers). Because the tensor basis given above is not antisymmetric in its p 1-forms $\mathrm{d}x^{\mu_i}$ the antisymmetry has to be enforced via the components. A more convenient basis for differential forms will be given in a moment.

The collection of all p-forms is a vector space, denoted by Λ^p:

$$\Lambda^p = \underset{(p \text{ times})}{T^* \wedge \ldots \wedge T^*} \subset \otimes^p\, T^*. \tag{3.20}$$

The dimension of this vector space can be determined by simple combinatorics. It is given by the number of linearly independent p-forms in a D dimensional manifold, which in turn is given by the number of independent components $A_{[\mu_1\cdots\mu_p]}$. Because the components are antisymmetric in any of the indices, no two indices can take the same value (else we would have $A = -A = 0$). Hence the number of independent components is given by the number of ways to pick p elements from a set of size D, which is just the binomial coefficient $\binom{D}{p}$. Thus:

$$\dim \Lambda^p = \binom{D}{p} = \frac{D!}{p!(D-p)!}. \tag{3.21}$$

Thus at a point in 4-dimensional spacetime there is one linearly independent 0-form, four 1-forms, six 2-forms, four 3-forms, and one 4-form. It may be obvious that there are no p-forms for $p > D$, because all of the components will be identically zero due to over-antisymmetrization.

3.2.1 The wedge product

Given a p-form A and a q-form B, we would like to define their product. Simply taking the tensor product $A \otimes B$ is not sufficient; the resulting tensor is clearly antisymmetric in its first p and last q indices, but it need not be antisymmetric in all of them. A new operator has to be defined that takes p- and q-forms into $(p+q)$-forms. This operator is the exterior or wedge product:

$$\wedge: \quad \Lambda^p \times \Lambda^q \to \Lambda^{p+q}. \tag{3.22}$$

It acts on the components as one would expect: as an antisymmetrizer:

$$A^{(p)} \wedge B^{(q)} = A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]} \mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_{p+q}}. \tag{3.23}$$

The wedge product has the following properties:

1. Bilinearity:

$$\begin{aligned} \left(A_1^{(p)} + A_2^{(p)}\right) \wedge B^{(q)} &= A_1^{(p)} \wedge B^{(q)} + A_2^{(p)} \wedge B^{(q)} \\ A^{(p)} \wedge \left(B_1^{(q)} + B_2^{(q)}\right) &= A^{(p)} \wedge B_1^{(q)} + A^{(p)} \wedge B_2^{(q)} \\ \left(f A^{(p)}\right) \wedge B^{(q)} &= A^{(p)} \left(f B^{(q)}\right) = f \left(A^{(p)} \wedge B^{(q)}\right) \end{aligned}$$

2. Anticommutativity:

$$A^{(p)} \wedge B^{(q)} = (-1)^{pq} B^{(q)} \wedge A^{(p)}$$

3. Under a pullback ϕ^*:

$$\phi^* \left(A^{(p)} \wedge B^{(q)}\right) = \phi^* A^{(p)} \wedge \phi^* B^{(q)}$$

4. Associativity:

$$\left(A^{(p)} \wedge B^{(q)}\right) \wedge C^{(r)} = A^{(p)} \wedge \left(B^{(q)} \wedge C^{(r)}\right)$$

The second property of the wedge product allows us to write a more natural basis for differential forms, in which the basis itself ensures the antisymmetry and not the components. This basis can be constructed from p 1-forms $\mathrm{d}x^{\mu_i}$ with the use of the wedge product:

$$\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{3.24}$$

A generic p-form and the wedge product between a p-form and q-form can now respectively be written as

$$\begin{aligned} A^{(p)} &= \frac{1}{p!} A_{\mu_1 \cdots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}, \\ A^{(p)} \wedge B^{(q)} &= \frac{1}{p!q!} A_{\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+q}}. \end{aligned} \tag{3.25}$$

where all the indices have been arranged in strictly increasing order (hence the extra factors in front of the components).

3.2.2 The exterior derivative

As their name suggests, differential forms can be differentiated. We have already seen the operator that differentiates a scalar (a 0-form) into a dual vector (a 1-form): it was the gradient. The gradient can be thought of as a limiting case of a more general operator known as the exterior derivative which takes p-forms into $(p+1)$-forms:

$$\mathrm{d}_p : \quad \Lambda^p \to \Lambda^{p+1}. \tag{3.26}$$

The subscript p is commonly dropped. The operator is given by

$$\mathrm{d}A^{(p)} = \frac{1}{p!} \left(\partial_{\mu_1} A_{\mu_2 \cdots \mu_{p+1}}\right) \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+1}} \tag{3.27}$$

and has the following properties:

1. Additivity: $\mathrm{d}\left(A^{(p)} + B^{(q)}\right) = \mathrm{d}A^{(p)} + \mathrm{d}B^{(q)}$
2. $\mathrm{d}\left(A^{(p)} \wedge B^{(q)}\right) = \mathrm{d}A^{(p)} \wedge B^{(q)} + (-1)^p A^{(p)} \wedge \mathrm{d}B^{(q)}$
3. Nilpotency: $\mathrm{d}^2 A^{(p)} \equiv \mathrm{d}\left(\mathrm{d}A^{(p)}\right) = 0$ for any form.

The main virtue of the exterior derivative is that it behaves as a tensor in any manifold, unlike the partial derivative. Another interesting feature is its nilpotency; this will lead to quite important notions such as closed- and exactness, and later on to De Rham cohomology. But for now we will go on and define yet another operator: the Hodge dual.

3.2.3 The Hodge dual operator

We have seen that the dimension of the space of all p-forms, Λ^p, is $\binom{D}{p}$. This is the same as the dimension of the space of all $(D-p)$ forms, Λ^{D-p}, because

$$\binom{D}{p} = \frac{D!}{p!(D-p)!} = \binom{D}{D-p}. \tag{3.28}$$

One is therefore tempted to define a map from p-forms to $(D-p)$-forms. And indeed the Hodge dual operator $\star$ is commonly defined as such a map:

$$\star : \quad \Lambda^p \to \Lambda^{D-p}. \tag{3.29}$$

It acts on a basis of 1-forms as follows:

$$\star \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} = \frac{1}{(d-n)!} \varepsilon^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_{d-p}} \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_{d-p}}. \tag{3.30}$$

Because some indices of the Levi-Civita tensor had to be raised the Hodge dual depends on the metric, unlike the wedge product and the exterior derivative.

The action of the Hodge dual on an arbitrary p-form is obtained by putting components in front of the last equation:

$$\star A^{(p)} = \frac{1}{(d-n)!} A_{\mu_1\cdots\mu_p} \varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{d-p}} \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_{d-p}}. \tag{3.31}$$

It has the following properties:

1. Distributivity: $\star\left(A^{(p)} + B^{(p)}\right) = \star A^{(p)} + \star B^{(p)}$,
2. $\star\star\, A^{(p)} = (-1)^{t+p(d-p)} A^{(p)}$,

where t is the number of timelike directions of the metric.

As a particular example, the Hodge dual of the pure number **1** (which is a 0-form) results in a D-form whose components are the Levi-Civita tensor, and be may written as:

$$\begin{aligned} \star\mathbf{1} &= \frac{1}{D!} \varepsilon_{\mu_1\cdots\mu_D} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_D} \\ &= \sqrt{|g|}\ \mathrm{d}x^0 \cdots \mathrm{d}x^{D-1} \\ &= \sqrt{|g|}\ \mathrm{d}^D x. \end{aligned} \tag{3.32}$$

Thus $\star\mathbf{1}$ is the general coordinate invariant volume element. It is the object $\mathrm{d}^D x$ that gets multiplied by a Lagrangian density (which is nothing more than a number) and integrated over spacetime in order to obtain the action. From a more geometrical point of view, it are D-forms, and not 0-forms, that can be integrated over a D-dimensional manifold. This allows for the definition of a global inner product of two p-forms A and B as

$$\left(A^{(p)}, B^{(p)}\right) = \int A^{(p)} \wedge \star B^{(p)}. \tag{3.33}$$

Elementary algebra tells us that the integrand is a D-form, which is subsequentially integrated over the manifold to give a real number. We can also define a local inner product by just contracting the components of both differential forms with each other:

$$|A^{(p)} \cdot B^{(p)}| = A_{\mu_1\cdots\mu_p} B^{\mu_1\cdots\mu_p}. \tag{3.34}$$

The local and global inner products are related by

$$\left(A^{(p)}, B^{(p)}\right) = \int \frac{1}{p!} |A^{(p)} \cdot B^{(p)}| \star \mathbf{1}. \tag{3.35}$$

Here we see a similarity between the Hodge dual and the dual vectors: both can be thought of as the space of linear maps from the original space to $\mathbb{R}$. But apart from this the notion of duality is quite different in both cases.

3.3 De Rham cohomology

Recall that the square of the exterior derivative always gives zero on any differential form (which is often abbreviated as $\mathrm{d}^2 = 0$). So if we happen to have a p-form A that is the derivative of a $(p-1)$-form B, that is $A = \mathrm{d}B$, its derivative is always zero: $\mathrm{d}A = 0$. The converse does not necessarily hold: if $\mathrm{d}A = 0$ it need not be that $A = \mathrm{d}B$. This difference the starting point of de Rham cohomology.

We begin with formalizing the difference. A p-form is said to be *closed* if its exterior derivative is zero: $\mathrm{d}A = 0$. It is said to be *exact* if it is the exterior derivative of some $(p-1)$-form: $A = \mathrm{d}B$. It is not hard to see that all exact forms are closed. This basic result of $\mathrm{d}^2 = 0$ is usually referred to as *Poincaré's lemma*.

The set of all exact p-forms makes the vector space B^p, which can be defined as the image of the exterior derivative:

$$B^p = \mathrm{Im}(\mathrm{d}_{p-1}). \tag{3.36}$$

The set all closed forms makes the vector space Z^p, which can be defined as the kernel of the exterior derivative:

$$Z^p = \mathrm{Ker}(\mathrm{d}_p). \tag{3.37}$$

Both B^p and Z^p are subspaces of Λ^p. Then the pth de Rham cohomology vector space can be defined as the closed forms modulo the exact forms:

$$H^p = \frac{Z^p}{B^p}. \tag{3.38}$$

De Rham's theorem states that this vector space is isomorphic to the pth homology group of the manifold M we are working in (what a homology group exactly is will not be explained here; see [18] for more details). In particular a corollary to his theorem says that when if M is compact, then

$$\dim H^p = b_p, \tag{3.39}$$

where b_p is the pth Betti number of M. Thus b_p is the maximal number of closed p-forms on M of which no linear combination is exact. The Betti numbers are defined as the rank of the pth homology group of M and are topological invariants; they characterize the topology of the manifold. As a few examples, the Betti numbers of $\mathbb{R}^n$ and the n-torus $\mathbb{T}^n$ are:

$$\begin{aligned} \mathbb{R}^n: &\quad b_p = 0 \;\forall\; p \\ \mathbb{T}^n: &\quad b_p = \binom{n}{p}. \end{aligned} \tag{3.40}$$

The fact that all Betti numbers of $\mathbb{R}^n$ are zero is not surprising, because that space is topologically trivial. But it does reveal that every closed form in $\mathbb{R}^n$ is necessarily exact. This leads to the converse to the Poincaré lemma:

Theorem 2 (Converse to Poincaré's lemma). *Every closed form is locally exact. More precisely, if* $\mathrm{d}A^{(p)} = 0$, $p \geq 1$, *in a neighborhood* U *of* $x \in M$, *then there is some perhaps smaller neighborhood* U' *of* x *and a* $(p-1)$*-form* $B^{(p-1)}$ *such that* $A^{(p)} = \mathrm{d}B^{(p-1)}$ *in* U'.

The proof of this theorem is trivial. From its definition every manifold M has to be locally isomorphic to $\mathbb{R}^n$. And because all closed forms are exact in $\mathbb{R}^n$, all closed forms in M are locally exact. The converse to Poincaré's lemma is an important tool for dualizing differential forms, as we will see in chapter 7.

Chapter 4

Lie groups, Lie algebras, and group manifolds

In this chapter we will discuss Lie groups, their associated algebras, and finally group manifolds. The latter will be employed in chapter 6 when we consider dimensional reduction. The discussion of these topics will be light-hearted and without any proof. A thorough mathematical treatment is given in [19], while [18, 20] take a physical approach.

4.1 Lie groups and algebras

We'll immediately start off with the very definition of a Lie group:

Definition 1 (Lie group)**.** *A Lie group is a differentiable manifold $\mathcal{G}$ endowed with a group structure such that the group operations*

1. *Product:*
$$\cdot\ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad (g, h) \mapsto g \cdot h \tag{4.1}$$

2. *Inverse:*
$$^{-1} : \mathcal{G} \to \mathcal{G}, \quad g \mapsto g^{-1} \tag{4.2}$$

are differentiable, and both $g \cdot h$ and g^{-1} are elements of G.

At each coordinate of the manifold we assign a group element $g(x) \in \mathcal{G}$, and the dimension n of the manifold equals $\dim G$. Usually the product symbol is omitted and $g \cdot h$ is simply written as gh. The identity element will be denoted by e. Because of the additional group structure defined by the product and inverse operations, a Lie group always has two families of diffeomorphisms known as left and right *translations*. They are respectively defined by

$$\begin{aligned} L_a : \mathcal{G} \to \mathcal{G}, \quad L_a g = ag, \\ R_a : \mathcal{G} \to \mathcal{G}, \quad R_a g = ga, \end{aligned} \tag{4.3}$$

where a and g are elements of $\mathcal{G}$. Because these operations are diffeomorphisms, they induce mappings from a tangent space at one point to a tangent space at another point in the manifold:

$$\begin{aligned} L_{a*} &: T_g \to T_{ag}, \\ R_{a*} &: T_g \to T_{ga}. \end{aligned} \tag{4.4}$$

Since these translations yield equivalent theories, we will only consider left translations from now on.

On a given Lie group $\mathcal{G}$ there always exists a special class of vector fields that are invariant under the group action, that is,

$$L_{a*}X|_g = X|_{ag}. \tag{4.5}$$

Such a vector field is called a *left-invariant vector field*. Once a vector $V \in T_e$ is given, it defines a unique left-invariant vector field throughout the whole of G by

$$X_V|_g = L_{g*}V. \tag{4.6}$$

Conversely, a left-invariant vector field defines a unique vector in the tangent space at the identity. The set of all the left-invariant vector fields on $\mathcal{G}$ will be denoted by the small gothic letter $\mathfrak{g}$. Note that the identification of $\mathfrak{g}$ with the tangent space at the identity would have been the same, because the map

$$T_e \to \mathfrak{g}, \quad V \mapsto X_V \tag{4.7}$$

is an isomorphism. From this notion it follows that $\dim \mathfrak{g} = \dim \mathcal{G}$. Because $\mathfrak{g}$ is a subset of all the vector fields on G, the Lie bracket is also defined on $\mathfrak{g}$. It can be shown that $\mathfrak{g}$ is closed under the Lie bracket. That is, if $X, Y \in \mathfrak{g}$, then

$$[X, Y] \in \mathfrak{g}. \tag{4.8}$$

Because of this naturally imposed extra structure on $\mathfrak{g}$, it is promoted from a vector space to an algebra. This leads us automatically to the following definition:

Definition 2 (Lie algebra)**.** *The set of all left-invariant vector fields* $\mathfrak{g}$ *with the Lie bracket*

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{4.9}$$

is called the Lie algebra of a Lie group $\mathcal{G}$*.*

4.2 Structure constants and the structure equation

Now let V_μ be a basis for T_e. This basis defines a new frame of basis X_μ for n linearly independent left-invariant vector fields at each point g in $\mathcal{G}$ by left-translation:

$$X_\mu|_g = L_{g*}V_\mu. \tag{4.10}$$

Because the Lie bracket $[X_\mu, X_\nu]$ at g returns an element of $\mathfrak{g}$, it can be expanded in terms of the induced basis as

$$[X_\mu, X_\nu] = f^\lambda{}_{\mu\nu} X_\lambda. \tag{4.11}$$

The $f^\lambda{}_{\mu\nu}$ are known as the *structure constants* of the Lie group. Lie's second theorem ensures that they indeed are constants (that is, they have no coordinate dependency whatsoever), just as their name implies. This fact will be of some importance when we consider dimensional reduction over group manifolds. Lie's third theorem states that the structure constants satisfy

1. Anti-symmetry in their lower indices:

$$f^\lambda{}_{\mu\nu} = -f^\lambda{}_{\nu\mu}. \tag{4.12}$$

2. A Jacobi identity:

$$f^\mu{}_{\nu\lambda} f^\nu{}_{\rho\sigma} + f^\mu{}_{\nu\sigma} f^\nu{}_{\lambda\rho} + f^\mu{}_{\nu\rho} f^\nu{}_{\sigma\lambda} = 0. \tag{4.13}$$

The latter may be more compactly written as $f^\mu{}_{\nu[\lambda} f^\nu{}_{\rho\sigma]} = 0$.

Just as we have defined a basis X_μ of left-invariant vector fields, we can also define a dual basis σ^μ of left-invariant 1-forms on $\mathcal{G}$ by the usual procedure: we simply demand that $\sigma^\mu X_\nu = \delta^\mu_\nu$. The most general left-invariant p-form on $\mathcal{G}$ may then be written as

$$A^{(p)} = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} \sigma^{\mu_1} \wedge \cdots \wedge \sigma^{\mu_p}. \tag{4.14}$$

Furthermore it can be shown that the left-invariant dual basis satisfies *Maurer-Cartan's structure equation*:

$$\mathrm{d}\sigma^\mu = -\tfrac{1}{2} f^\mu{}_{\nu\lambda} \sigma^\nu \wedge \sigma^\lambda. \tag{4.15}$$

Any basis for a given space has the nice property that it can be given in terms of linear combinations of another basis for that space. In particular, we can express the left-invariant dual basis σ^μ in terms of the standard dual basis $\mathrm{d}x^\mu$ by means of

$$\sigma^\mu = U^\mu{}_\nu \mathrm{d}x^\nu, \tag{4.16}$$

where $U^\mu{}_\nu = U^\mu{}_\nu(x)$ are $n \times n$ matrices. It follows that the structure constants can be in expressed in terms of $U^\mu{}_\nu$ as

$$f^\lambda{}_{\mu\nu} = -(U^{-1})^\rho{}_\mu (U^{-1})^\sigma{}_\nu \left(\partial_\rho U^\lambda{}_\sigma - \partial_\sigma U^\lambda{}_\rho \right). \tag{4.17}$$

Note that although the $U^\mu{}_\nu$'s are coordinate dependent, this particular combination of them is not, thanks to Lie's second theorem.

4.3 Group manifolds

Crudely speaking, a *group manifold* is a Lie group endowed with a specific metric that is invariant under group operations. In other words, the group multiplication gives rise to isometries of the metric. This is not the precise definition of a group manifold, but it sufficient for our purposes. For a more detailed account, see the appendix on group manifolds in [21].

To make sure that left multiplication leaves the metric invariant, one can make the choice

$$\mathrm{d}s^2 = g_{\mu\nu}\sigma^\mu\sigma^\nu. \tag{4.18}$$

Because left multiplication L_a leaves σ^μ invariant it indeed is an isometry of the metric, and the metric above is appropriately called the left-invariant metric.

Right-translation, however, does not leave this metric invariant; R_a is an isometry of the metric if and only if $g_{\mu\nu}$ is given by the Cartan-Killing metric of the group $\mathcal{G}$. Such a metric is called the bi-invariant metric, and its isometry group is $\mathcal{G}_L \times \mathcal{G}_R$.

Chapter 5

The Einstein-Cartan formalism

In section 3.1 we saw that we are free to choose a basis of the tangent space T_p at a point p of the manifold, on which then objects such as tensors can be defined. At the time we chose the natural basis,

$$\begin{aligned} e_{(\mu)} &= \partial_\mu, \\ \theta^{(\mu)} &= \mathrm{d}x^\mu, \end{aligned} \tag{5.1}$$

but we will now introduce another basis in which some calculations become astonishingly simple, especially the one of the curvature scalar. For more information on the subject, see for example [17, 18, 20].

5.1 The vielbeins

The new basis will be denoted by E_a, with a Latin index replacing the Greek one, reminding us that this basis has nothing to do with any coordinate system whatsoever. It is to be orthonormal, in a way that corresponds to the signature of the manifold we are dealing with. This means that the inner product of the of basis vectors will be defined as

$$g(E_a, E_b) \equiv \eta_{ab}, \tag{5.2}$$

where η_{ab} is a diagonal metric. In a space with a positive definite metric this would be δ_{ab}, whereas in a space Lorentzian signature it will be the familiar Minkowski metric. Since we're physicists, only the latter case interests us and we will take η_{ab} to be the Minkowski metric, although this will not affect the upcoming analysis. Because η_{ab} is a diagonal metric, the new basis will be called *flat* and the old one in hindsight *curved.*

The nice thing of having a basis is that any vector can be expressed as a linear combination of basis vectors. In particular, the new basis vectors can be given in terms of the old ones:

$$E_a = E_a^\mu \partial_\mu. \tag{5.3}$$

The components E_a^μ can be seen as a $D \times D$ invertible matrix, and are called the inverse vielbeins. The 'normal' vielbeins e_μ^a are their inverse, and can be used to go from the curved dual basis to the flat dual basis:

$$e^a = e_\mu^a \mathrm{d}x^\mu. \tag{5.4}$$

From the orthonormality requirement $e^a E_b = \delta_b^a$ we indeed see that E_a^μ and e_μ^a are each other's inverse:

$$\begin{aligned} e_\mu^a E_a^\nu &= \delta_\mu^\nu, \\ e_\mu^a E_b^\mu &= \delta_b^a. \end{aligned} \tag{5.5}$$

From (5.2) it follows that

$$g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab} e^a \otimes e^b, \tag{5.6}$$

from which in turn it follows that

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b. \tag{5.7}$$

Any given vector (or tensor for that matter) can be expressed in terms of the curved basis or in terms of the flat basis:

$$V = V^\mu \partial_\mu = V^a E_a, \tag{5.8}$$

from which we see that the components in the curved and flat bases are related to each other by

$$V^a = e_\mu^a V^\mu. \tag{5.9}$$

This generalizes in the obvious way for the components of an arbitrary (k, l) tensors:

$$T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = e_{\mu_1}^{a_1} \cdots e_{\mu_k}^{a_k} E_{b_1}^{\nu_1} \cdots E_{b_l}^{\nu_l} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \, . \tag{5.10}$$

Of course we are free to keep some of the indices curved while having others flat. Furthermore we can raise and lower flat indices with the flat metric η_{ab} and its inverse η^{ab}.

5.2 The spin connection and the curvature scalar

From (5.10) it seems that translating our knowledge of tensors into the flat basis is just a matter of inserting vielbeins in the right places. But this does not hold when we begin to differentiate objects. In the ordinary curved formalism there is a metric connection $\Gamma_{\mu\nu}^\lambda$ that serves as a correction term in the covariant derivative. In the flat basis it gets replaced by the *spin connection*, denoted by $\omega_\mu{}^a{}_b$. It is defined as

$$\nabla_\mu X_b^a = \partial_\mu X_b^a + \omega_\mu{}^a{}_c X_b^c - \omega_\mu{}^c{}_b X_c^a. \tag{5.11}$$

If we compare the components of a covariant derivative of a vector X, ∇X, in a coordinate basis with the components in a flat basis, we can find a relation between the vielbeins, the spin connection, and the metric connection. It is

$$\Gamma^{\mu}_{\nu\lambda} = E^{\mu}_{a}\partial_{\nu}e^{a}_{\lambda} + E^{\mu}_{a}e^{b}_{\lambda}\omega_{\nu}{}^{a}{}_{b}, \tag{5.12}$$

or equivalently

$$\omega_{\mu}{}^{a}{}_{b} = e^{a}_{\nu}E^{\lambda}_{b}\Gamma^{\nu}_{\mu\lambda} - E^{\nu}_{b}\partial_{\mu}e^{a}_{\nu}. \tag{5.13}$$

As the spin connection replaces the metric connection, it is not surprising that the torsion and Riemann tensors (and thus also the Ricci scalar and ultimately the Einstein-Hilbert Lagrangian $\sqrt{-g}R(g)$) can be expressed solely in terms of the spin connection. We begin with rewriting the torsion tensor and the Riemann tensor as two-forms:

$$\begin{aligned} T^{a} &= \mathrm{d}e^{a} + \omega^{a}{}_{b} \wedge e^{b}, \\ R^{a}{}_{b} &= \mathrm{d}\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}. \end{aligned} \tag{5.14}$$

These re-expressions are also known as *Cartan's structure equations*. If we have a metric compatible connection, i.e. the covariant derivative of the metric vanishes, it can be shown that the spin connection is anti-symmetric in its last two indices:

$$\omega_{\mu ab} = -\omega_{\mu ba}. \tag{5.15}$$

If we furthermore have a torsion free connection, i.e. $T^{a} = 0$, it follows that the spin connection can be expressed as

$$\omega_{abc} = -\tfrac{1}{2}(\Omega_{abc} - \Omega_{bca} - \Omega_{cab}), \tag{5.16}$$

where the Ricci relational coefficients Ω_{abc} are given by

$$\Omega^{a}{}_{bc} = E^{\mu}_{b}E^{\nu}_{c}(\partial_{\mu}e^{a}_{\nu} - \partial_{\nu}e^{a}_{\mu}). \tag{5.17}$$

It is obvious from the definition that the Ricci relational coefficients are anti-symmetric in their last two indices. With a bit of manipulation, the Ricci tensor (which is a contraction of the Riemann tensor) can be given solely in terms of spin connections. A most useful expression reads

$$R = -2\partial_{a}\omega^{a} + \omega_{abc}\omega^{cab} - \omega_{a}\omega^{a}, \tag{5.18}$$

where we have defined $\omega_{a} \equiv \eta^{bc}\omega_{bca}$. The last thing to be done is rewriting the metric determinant in terms of vielbeins. As one can expect, it is not that difficult:

$$\begin{aligned} g = \det(g_{\mu\nu}) &= \det(e^{a}_{\mu}e^{b}_{\nu}\eta_{ab}) \\ &= \det(e^{a}_{\mu})^{2}\det(\eta_{ab}) \\ &= -e^{2}, \end{aligned} \tag{5.19}$$

where $e \equiv \det(e_\mu^a)$. The Einstein-Hilbert Lagrangian now becomes

$$\mathcal{L} = \sqrt{-g}R(g) = eR(\omega). \tag{5.20}$$

5.3 Local Lorentz transformations

In a D-dimensional manifold the metric $g_{\mu\nu}$ (for the moment not satisfying Einstein's equations) has $\frac{1}{2}D(D+1)$ degrees of freedom, whereas the vielbein e_a^μ has D^2 degrees of freedom. These extra degrees of freedom result in many choices for the vielbein basis which yield the same metric $g_{\mu\nu}$. Each of these choices is related to another by a local Lorentz transformation:

$$E_a \longrightarrow E_{a'} = \Lambda_{a'}{}^a(p)E_a, \tag{5.21}$$

at each point p. The vielbeins transform as

$$e_\mu^a \longrightarrow e_\mu^{a'} = \Lambda^{a'}{}_a(p)e_\mu^a. \tag{5.22}$$

The transformations leave the canonical form of the metric unaltered:

$$\Lambda_{a'}{}^a\Lambda_{b'}{}^b\eta_{ab} = \eta_{a'b'}. \tag{5.23}$$

Curved indices do not transform under these local Lorentz transformations, while flat indices on the contrary are inert under coordinate changes.

As the dimension of the Lie group of Lorentz transformations is $\frac{1}{2}D(D-1) = D^2 - \frac{1}{2}D(D+1)$, we see that these transformations are responsible for the 'missing' degrees of freedom. We can use them to write the vielbeins in upper diagonal form:

$$e_\mu^a \simeq \begin{pmatrix} \cdot & \cdots & \cdot \\ & \ddots & \vdots \\ \varnothing & & \cdot \end{pmatrix}. \tag{5.24}$$

Chapter 6

Dimensional reduction

Plainly said, dimensional reduction is nothing more than the art of making redundant dimensions disappear. This has to be done with such a skill that, beginning with a higher-dimensional theory, one chops away the pieces of the redundant dimensions such that a theory describing the real world is left in the end. In fact, Michelangelo's famous quote about sculpturing is applicable here: the right four dimensional theory is already hidden in the higher dimensional one, all we have to do is to remove (or modify) the pieces that don't fit.

But this is where the analogy stops. Where Michelangelo easily sculpted one statue after the other, string theorists are having a tremendous hard time obtaining a four dimensional theory that actually describes the world around us. The tools of the trade also differ; instead of wielding a pickax string theorists have a series of mathematical techniques at their disposition.

It are these techniques that will be discussed during this chapter. For an overview of how and why they were invented, see for example [22]. The general idea is that the underlying manifold of dimension $D+n$ is slit up into a space-time part of dimension D (usually taken to be four) and an internal part of dimension n:

$$\hat{\mathcal{M}}_{D+n} = \mathcal{M}_D \times \mathcal{M}_n \tag{6.1}$$

Here we have introduced the notation that hatted objects are higher-dimensional. The coordinates are also split up accordingly:

$$x^{\hat{\mu}} = (x^{\mu}, z^{\alpha}) \tag{6.2}$$

The unhatted higher roman indices cover the space-time directions, whereas the unhatted lower romans indices are for the internal directions. The various fields present in the theory are made independent of these so-called internal coordinates, after which the action can be integrated over the internal manifold:

$$\begin{aligned} \int \mathrm{d}x^{D+n} \hat{S} &= \int \mathrm{d}x^n \int \mathrm{d}x^D \left(S + S_z\right) \\ &= \text{constant} \times \int \mathrm{d}x^D \left(S + S_z\right) \end{aligned} \tag{6.3}$$

The appearance of the extra the extra terms S_z is due to multi-component fields that are possibly present in the action. Consider for example a vector field:

$$\hat{A}_{\hat{\mu}} = (\hat{A}_\mu, \hat{A}_\alpha) \stackrel{!}{=} (A_\mu, B_\alpha). \tag{6.4}$$

The extra internal component B_α is not only present in the higher-dimensional action, but also in the lower-dimensional one. It are exactly these kinds of internal components that make up the extra term S_z.

As you might have noticed, the fields are made independent of the internal coordinates in the reduction Ansatz (denoted by $\stackrel{!}{=}$). To see why such an Ansatz is more or less allowed we can look at a toy example.

6.1 A toy example

To get a feeling for things, we start by considering a massless scalar field $\hat{\phi}$ in flat $(D+1)$ dimensional space. It depends on the coordinates $x^{\hat{\mu}} = (x^\mu, z)$ and satisfies the Klein-Gordon equation:

$$\hat{\Box}\hat{\phi} = 0, \tag{6.5}$$

where $\hat{\Box} = \partial_{\hat{\mu}}\partial^{\hat{\mu}} = \partial_\mu\partial^\mu + \partial_z\partial^z$. The dependence on the last coordinate can be explicitly expanded in a Fourier series:

$$\hat{\phi}(x, z) = \int \mathrm{d}k \mathrm{e}^{ikz}\phi_k(x), \tag{6.6}$$

where the components ϕ_k have momentum k. Furthermore, if we set the z dimension to be compact and of period $2\pi R$ (that is, we impose the boundary condition $\hat{\phi}(x^\mu, 0) = \hat{\phi}(x^\mu, 2\pi R)$) the integral gets replaced by the sum

$$\hat{\phi}(x, z) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{inz/R}\phi_n(x). \tag{6.7}$$

The spectrum of the fields ϕ_n has become discrete instead of continuous. As said, the field is subject to the Klein-Gordon equation, from which we obtain separate equations for each component:

$$\Box\phi_n - \left(\frac{n}{R}\right)^2 \phi_n = 0. \tag{6.8}$$

These are just Klein-Gordon equations for scalar fields with a mass of $\frac{|n|}{R}$. So from one higher-dimensional scalar field we obtain an infinite tower of lower-dimensional fields (known as the *Kaluza-Klein modes*), of which only one is massless (namely ϕ_0).

Now the extra compact dimension has to be extremely small (that is, $R \ll 1$) or else we would see it in every day life. This implies that the massive fields become unphysically heavy, and are therefore usually discarded. The truncation to the massless mode results in the reduction Ansatz

$$\hat{\phi}(x, z) \stackrel{!}{=} \phi(x), \tag{6.9}$$

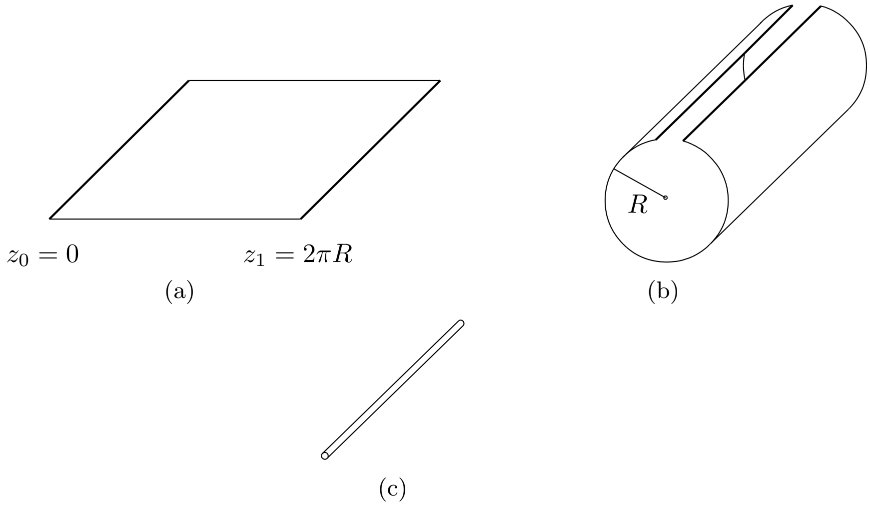

Figure 6.1: Kaluza-Klein reduction in a nutshell. We begin by making a two dimensional space compact and periodic in one dimension (a). Then the two opposite sides z_0 and z_1 can be identified with each other, allowing us to roll the sheet up to form a cylinder (b). Next the radius of the cylinder is made small, so that from a considerable distance it is indistinguishable from a line (c).

where the subscript 0 has been dropped. Although it might not be apparent at first, the consistency of this truncation is in fact guaranteed and will be discussed in section 6.4

6.2 Kaluza-Klein and Scherk-Schwarz

The method above is called *Kaluza-Klein* reduction [23, 24], and in an appropriately fashion the fields ϕ_n are named Kaluza-Klein states. Because the extra dimension is taken to be compact and periodic, the reduction corresponds to taking the underlying manifold to be

$$\hat{\mathcal{M}}_{D+1} = \mathcal{M}_D \times \mathbb{S}^1, \tag{6.10}$$

where $\mathbb{S}$ denotes the circle. This may be thought of as assigning a circle to every point in spacetime. In the case of $D = 1$ it easy to form a mental picture, as has been done in Figure 6.1. Note that the cylinder in 6.1(b) is nothing more than a collection of circles at every point on a line.

The process of Kaluza-Klein reduction over a circle can be repeated n times, which boils down to a reduction over a torus:

$$\hat{\mathcal{M}}_{D+n} = \mathcal{M}_D \times \mathbb{T}^n, \tag{6.11}$$

where $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. Instead of reducing over one circle at a time, we will reduce over the whole torus at once. Furthermore, if the torus is endowed with the additional structure of a Lie group it is possible to consistently maintain some dependence on the internal coordinates. The underlying manifold is then assumed to be of the form

$$\hat{\mathcal{M}}_{D+n} = \mathcal{M}_D \times \mathcal{G}_n, \tag{6.12}$$

where G is the group manifold in question. This procedure is called *Scherk-Schwarz* reduction [25]. The reductions of gravity over a circle, a torus, and a group manifold will now be discussed in turn. A more extensive review of them is given in [26].

6.3 Gravity reduced

When reducing anything over a circle (or a torus, or anything else for that matter) the starting point is usually to write down the reduction Ansatz for the metric. This is because the metric is present in almost all of the terms appearing in the action, most notably in the Ricci scalar. So the first and foremost thing to reduce would be gravity, and that is exactly what will be considered in this section.

6.3.1 The circle

The Ansatz for the metric can be given in terms of either the metric itself, the line element, or the vielbein basis. As this is the first reduction that will be performed, all three are given here.

In a matrix representation the $(D+1)$ dimensional split of the metric can be given by

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \left(\begin{array}{c|c} \hat{g}_{\mu\nu} & \hat{g}_{\mu z} \\ \hline \hat{g}_{z\nu} & \hat{g}_{zz} \end{array}\right). \tag{6.13}$$

The index on the internal coordinate z^α in (6.2) is suppressed, for the obvious reason that there is only one dimension to reduce over. Consequently, the internal index has been denoted by z.

From the D-dimensional point of view $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$, and $\hat{g}_{zz}$ look like a rank two symmetric tensor, a 1-form and a scalar field respectively. Following this line of thought, one can make a (very naive) reduction Ansatz:

$$\hat{g}_{\mu\nu} = g_{\mu\nu}, \qquad \hat{g}_{\mu z} = V_\mu, \qquad \hat{g}_{zz} = \varphi. \tag{6.14}$$

There is nothing wrong with doing so, although it is not a very convenient Ansatz as it does not pay respect to the underlying symmetries of gravity. A far better one is

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \mathrm{e}^{2\varphi} V_\mu V_\nu, \qquad \hat{g}_{\mu z} = -\mathrm{e}^{2\varphi} V_\mu, \qquad \hat{g}_{zz} = \mathrm{e}^{2\varphi}. \tag{6.15}$$

The reason for the extra minus sign will become clear in a moment, when we consider toroidal and group manifold reductions. With this particular parametrization the line element takes a rather elegant form:

$$\hat{\mathrm{d}s}^2 = \mathrm{d}s^2 + \mathrm{e}^{2\varphi}\left(\mathrm{d}z - V\right)^2 , \tag{6.16}$$

where $V = V_\mu \mathrm{d}x^\mu$. The simplicity of the line element allows us to come with a convenient choice of the vielbein basis:

$$\begin{aligned} \hat{e}^a &= e^a, \\ \hat{e}^m &= \mathrm{e}^{\varphi}\left(\mathrm{d}z - V\right). \end{aligned} \tag{6.17}$$

This will be the starting point of most of our calculations. The above Ansatz together with $\hat{\phi} \stackrel{!}{=} \phi + \varphi$ gives rise to the following lower-dimensional action for gravity in the string frame:

$$S = \int \mathrm{e}^{-\hat{\phi}}\left(\hat{R}\,\hat{\star}\mathbf{1} + \mathrm{d}\hat{\phi}\wedge\hat{\star}\mathrm{d}\hat{\phi}\right) \tag{6.18a}$$

$$= \int \mathrm{e}^{-\phi}\left(R\star\mathbf{1} + \mathrm{d}\phi\wedge\star\mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{2\varphi}F\wedge\star F - \mathrm{d}\varphi\wedge\star\mathrm{d}\varphi\right), \tag{6.18b}$$

where $F = \mathrm{d}V$.

The appearance of a Maxwell-kind of kinetic term for V might be surprising, and was the reason why Kaluza considered this reduction in the first place: it seemed plausible to unify general relativity with electromagnetism if one considers a $(4+1)$-dimensional space. The fact that one obtains the 'correct' terms for the metric, the 1-form and the scalar field (giving rise to respectively the Einstein equation, the Maxwell equations, and the massless Klein-Gordon equation) was referred to as the *Kaluza miracle*.

If the calculations are done in the metric formalism (as is done in appendix D) this indeed might appear to be a miracle. But when the Einstein-Cartan formalism is employed (see appendix C) resulting in Lorentz-invariance at every step, the miracle no longer seems all that mysterious. Because as the starting point was a Lorentz-invariant action, and as the Ansätze do not break it, the lower-dimensional action must also be Lorentz-invariant; which results in the Maxwell-kind of kinetic term.

The action (6.18b) has, besides the obvious Lorentz-invariance, additional symmetries which stem from its higher-dimensional origin. In particular, the higher-dimensional action (6.18a) is invariant under infinitesimal general coordinate transformations (GCT)

$$\delta x^{\hat{\mu}} = -\hat{\xi}^{\hat{\mu}} \qquad \Longrightarrow \qquad \delta\hat{e}_{\hat{\mu}}^{\hat{a}} = \hat{\xi}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{e}_{\hat{\mu}}^{\hat{a}} + \hat{e}_{\hat{\rho}}^{\hat{a}}\partial_{\hat{\mu}}\hat{\xi}^{\hat{\rho}}. \tag{6.19}$$

The form of the line element (6.16) is unchanged and remains independent of the internal coordinate z under a GCT that also has no internal dependence:

$$\begin{aligned} \hat{\xi}^{\mu}(x,z) &= \xi^{\mu}(x), \\ \hat{\xi}^{z}(x,z) &= \lambda(x). \end{aligned} \tag{6.20}$$

The effect of this Ansatz-preserving transformation on the lower-dimensional fields is as follows:

$$\delta x^\mu = -\xi^\mu \qquad \Longrightarrow \qquad \begin{aligned} \delta e_\mu^a &= \xi^\rho \partial_\rho e_\mu^a + e_\rho^a \partial_\mu \xi^\rho, \\ \delta V_\mu &= \xi^\rho \partial_\rho V_\mu + V_\rho \partial_\mu \xi^\rho, \\ \delta\varphi &= \xi^\rho \partial_\rho \varphi. \end{aligned} \tag{6.21a}$$

$$\delta z = -\lambda \qquad \Longrightarrow \qquad \delta V_\mu = -\partial_\mu \lambda. \tag{6.21b}$$

We see that e_μ^a and V_μ transform as vectors and φ transforms as a scalar under the ξ^μ transformation, just as one would expect. However, e_μ^a and φ do not transform at all under λ, while V_μ transforms as an $U(1)$ gauge field. Thus the lower-dimensional relic of the $\hat{\xi}^z(x,z)$ invariance, restricted to z-independent transformations, is an Abelian gauge invariance. For this reason V_μ is called the Kaluza-Klein gauge field.

6.3.2 The torus

As was already stated in section 6.2, the torus $\mathbb{T}^n$ comprises n circles. Thus if anything is reduced over a torus it goes from $D+n$ dimensions to D and looses n dimensions in the process, so to speak. Therefore the index on the internal coordinates in (6.2) is reinstated, and the Ansatz for the line element is modified as follows:

$$\hat{\mathrm{d}s}^2 = \mathrm{d}s^2 + G_{\alpha\beta}\big(\mathrm{d}z^\alpha - V^\alpha\big)\big(\mathrm{d}z^\beta - V^\beta\big). \tag{6.22}$$

Not only does the internal index now range over all the n internal coordinates, but the scalar φ has been promoted to the fully-fledged internal metric $G_{\alpha\beta}$. It is related to the internal Φ_α^m vielbein by

$$G_{\alpha\beta} = \delta_{mn}\Phi_\alpha^m \Phi_\beta^n, \tag{6.23}$$

just as the ordinary metric is related to the ordinary vielbein. Its inverse can easily be determined by demanding that $G_{\alpha\gamma}G^{\gamma\beta} = \delta_\alpha^\beta$. Furthermore there are n 1-forms V^α instead of the single V we had on the circle, and the Ansatz for the dilaton is appropriately changed to

$$\hat{\phi} \overset{!}{=} \phi + \frac{1}{2}\ln|\det G|. \tag{6.24}$$

Thereupon the action (6.18a) reduces to

$$S = \int \mathrm{e}^{-\phi}\left(R \star \mathbf{1} + \mathrm{d}\phi \wedge \star\mathrm{d}\phi - \frac{1}{2}G_{\alpha\beta}F^\alpha \wedge \star F^\beta + \frac{1}{4}\mathrm{d}G_{\alpha\beta} \wedge \star\mathrm{d}G^{\alpha\beta}\right), \tag{6.25}$$

with $F^\alpha = \mathrm{d}V^\alpha$.

Again it is worthwhile to consider general coordinate transformations that leave the Ansatz (6.22) unchanged. For the torus they are given by

$$\begin{aligned}\hat{\xi}^\mu(x,z) &= \xi^\mu(x),\\ \hat{\xi}^\alpha(x,z) &= \lambda^\alpha(x).\end{aligned} \tag{6.26}$$

The only difference with respect to (6.20) is that the internal part of the GCT now has n components. The effect of this GCT on the lower-dimensional fields is also mostly the same:

$$\delta x^\mu = -\xi^\mu \qquad \Longrightarrow \qquad \begin{aligned}\delta e_\mu^a &= \xi^\rho\partial_\rho e_\mu^a + e_\rho^a\partial_\mu\xi^\rho,\\ \delta V_\mu^\alpha &= \xi^\rho\partial_\rho V_\mu + V_\rho\partial_\mu\xi^\rho,\\ \delta\Phi_\alpha^m &= \xi^\rho\partial_\rho\Phi_\alpha^m.\end{aligned} \tag{6.27a}$$

$$\delta z^\alpha = -\lambda^\alpha \qquad \Longrightarrow \qquad \delta V_\mu^\alpha = -\partial_\mu\lambda^\alpha. \tag{6.27b}$$

The behavior of e_μ^a, V_μ^α, and Φ_α^m (replacing the scalar φ) is unchanged. But instead of just one $U(1)$ invariance, we now have a $U(1)^n$ invariance.

6.3.3 A group manifold

So far we have only considered reductions that truncate any dependence on the internal coordinates. We can however obtain a consistent z^α dependence of the higher-dimensional fields if we perform a transformation on the lower-dimensional fields, acting only on the part of the fields. The parameter of the transformation will be denoted by U and has only an internal dependence:

$$U^\alpha{}_\beta = U^\alpha{}_\beta(z). \tag{6.28}$$

Applying this transformation to all the fields in the toroidal Ansatz (6.22), we obtain

$$\begin{aligned}\hat{\mathrm{d}s}^2 &= \mathrm{d}s^2 + U^\gamma{}_\alpha U^\delta{}_\beta G_{\gamma\delta}\big(\mathrm{d}z^\alpha - (U^{-1})^\alpha{}_\epsilon V^\epsilon\big)\big(\mathrm{d}z^\beta - (U^{-1})^\beta{}_\zeta V^\zeta\big)\\ &= \mathrm{d}s^2 + G_{\alpha\beta}(U^\alpha{}_\gamma\mathrm{d}z^\gamma - V^\alpha)(U^\beta{}_\delta\mathrm{d}z^\delta - V^\beta).\end{aligned} \tag{6.29}$$

As one performs the reduction of gravity with this Ansatz, it turns out that the only combination of U's to survive the procedure is

$$(U^{-1})^\gamma{}_\alpha(U^{-1})^\delta{}_\beta\left(\partial_\gamma U^\lambda{}_\delta - \partial_\delta U^\lambda{}_\gamma\right). \tag{6.30}$$

We would like this expression to be independent of the internal coordinates z^α, so that the lower-dimensional theory is too. But as $U^\alpha{}_\beta$ depends solely on z^α, the above expression has to be a constant. This is where group manifolds come to the rescue.

We begin by noting that (6.30) looks astonishingly much like structure constants of a Lie group (4.17). In fact, it can be made sure these *are* the structure constants of some Lie group G if $U^{\alpha}{}_{\beta}$ are chosen properly. From chapter 4 we see that we have to choose $U^{\alpha}{}_{\beta}$ such that

$$\sigma^{\alpha} = U^{\alpha}{}_{\beta}\mathrm{d}z^{\beta} \tag{6.31}$$

is the left-invariant basis for G. Then, by virtue of Lie's second theorem, $f^{\lambda}{}_{\alpha\beta}$ is independent of z^{α} and so is our resulting lower-dimensional theory. The Ansatz (6.29) now becomes

$$\hat{\mathrm{d}s}^2 = \mathrm{d}s^2 + G_{\alpha\beta}(\sigma^{\alpha} - V^{\alpha})(\sigma^{\beta} - V^{\beta}). \tag{6.32}$$

We see that the internal part of the metric is given by

$$\mathrm{d}s_G^2 = G_{\alpha\beta}\sigma^{\alpha}\sigma^{\beta}, \tag{6.33}$$

that is, the left-invariant metric. Thus this reduction corresponds to the reduction over a group manifold G, using only the left-invariant metric. In principle the bi-invariant metric could be used, resulting in a reduction over $G_L \times G_R$, but it can be shown that this reduction turns out to be inconsistent.

Upon the reduction over a group manifold, the action (6.18a) reduces to

$$S = \int \mathrm{e}^{-\phi}\left(R \star \mathbf{1} + \mathrm{d}\phi \wedge \star\mathrm{d}\phi - \frac{1}{2}G_{\alpha\beta}F^{\alpha} \wedge \star F^{\beta} + \frac{1}{4}\mathcal{D}G_{\alpha\beta} \wedge \star\mathcal{D}G^{\alpha\beta} - V \star \mathbf{1}\right), \tag{6.34}$$

where we have the following generalizations of the toroidal reduction:

$$\begin{aligned} F^{\alpha} &= \mathrm{d}V^{\alpha} + \frac{1}{2}f^{\alpha}{}_{\beta\gamma}V^{\beta} \wedge V^{\gamma}, \\ \mathcal{D}G_{\alpha\beta} &= \mathrm{d}G_{\alpha\beta} + f^{\gamma}{}_{\alpha\delta}V^{\delta}G_{\gamma\beta} + f^{\gamma}{}_{\beta\delta}V^{\delta}G_{\alpha\gamma}, \end{aligned} \tag{6.35}$$

where $f^{\lambda}{}_{\alpha\beta}$ is given by (4.17). Furthermore we have a scalar potential for the internal metric:

$$V = \frac{1}{2}f^{\alpha}{}_{\beta\gamma}f^{\beta}{}_{\alpha\gamma'}G^{\gamma\gamma'} + \frac{1}{4}f^{\alpha}{}_{\beta\gamma}f^{\alpha'}{}_{\beta'\gamma'}G_{\alpha\alpha'}G^{\beta\beta'}G^{\gamma\gamma'}. \tag{6.36}$$

Thus there are two major differences when compared to toroidal reductions: the first is the non-Abelian modification of the field strength F^{α}, accompanied with the generalization of partial derivatives to covariant derivatives with respect to V^{α} on higher-dimensional indices. The second is the appearance of the scalar potential.

The former can be explained by looking at GCT's. We modify the Ansatz (6.26) so that the higher-dimensional part gets a z^{α} dependence:

$$\begin{aligned} \hat{\xi}^{\mu}(x,z) &= \xi^{\mu}(x) \\ \hat{\xi}^{\alpha}(x,z) &= (U^{-1})^{\alpha}{}_{\beta}(z)\ \xi^{\beta}(x). \end{aligned} \tag{6.37}$$

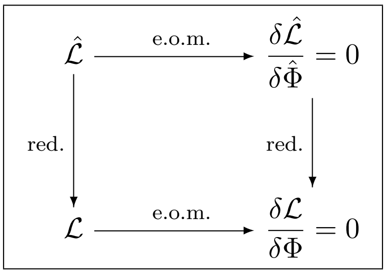

Figure 6.2: The commutation of varying the action and reducing it of a consistent reduction. It shouldn't matter if first vary the action and the reduce the equations of motion, or do it the other way around.

The transformation under $\delta x^\mu = -\xi^\mu$ is unchanged from (6.27a), but the behavior under δz^α becomes

$$\delta z^\alpha = -(U^{-1})^\alpha{}_\beta \xi^\beta \qquad \Longrightarrow \qquad \begin{aligned} \delta V^\alpha_\mu &= -\partial_\mu \xi^\alpha - f^\alpha{}_{\beta\gamma}\xi^\gamma V^\beta_\mu \\ \delta \Phi^m_\alpha &= f^\gamma{}_{\alpha\beta}\xi^\beta \Phi^m_\gamma \\ \delta \Phi^\alpha_m &= f^\alpha{}_{\beta\gamma}\xi^\beta \Phi^\gamma_m. \end{aligned} \tag{6.38a}$$

We see that V^α has exactly the same transformation properties as a gauge field of our group manifold G.

6.4 Consistency of reductions

A dimensional reduction is *consistent* if the solutions of the lower-dimensional equations of motion are also solutions of the original higher-dimensional equations of motion. It is said that the lower-dimensional solutions can be *uplifted* to the higher-dimensional theory. However, in this thesis it are not the equations of motion that are reduced, but the Lagrangian from which they stem. If one takes this approach, consistency means that the operations of varying the action and reducing it commute (see figure 6.2). It can then be shown that the lower-dimensional solutions can indeed be uplifted [27].

In the case of the group manifold reduction it turns out that a necessary and sufficient condition for consistency is that the structure constants are traceless, that is

$$f^\beta{}_{\alpha\beta} = 0. \tag{6.39}$$

An equivalent statement is that the adjoint representation of the corresponding Lie group is unimodular. Compact, semi-simple, and Abelian Lie algebras are examples that satisfy this condition. Thus we immediately see that the reduction over a torus (which is has an Abelian algebra) is consistent.

It may be noted that the number of degrees of freedom per space-time point before and after the reduction should obviously be the same if the reduction is truly consistent. This is in fact the case with every reduction we consider in this thesis.

Part III

Poincaré dualities and their reductions

Chapter 7

Poincaré dualities

7.1 Differential forms in physics

Now that we have some know-how of differential forms, we can apply it to antisymmetric tensor fields in physics. An example of such a field is the vector potential of electromagnetism, which we already encountered in chapter 1. As a recapitulation, its physics is governed by the action

$$S = -\frac{1}{4}\int \mathrm{d}x^4 F_{\mu\nu}F^{\mu\nu}, \tag{7.1}$$

and the field strength $F_{\mu\nu}$ was given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.2}$$

When we vary the action with respect to A^μ, we find $\partial_\mu F^{\mu\nu} = 0$. Furthermore we had the identity $\partial_\mu \star F^{\mu\nu} = 0$ as a result from (7.2), so the action (7.1) results in Maxwell's equations and does indeed describe electromagnetism.

The vector potential can be seen as a rank-one antisymmetric tensor field, and its field strength as a rank-two antisymmetric tensor field. We therefore have the possibility to express them in terms of differential forms:

$$\begin{aligned} A^{(1)} &= A_\mu \mathrm{d}x^\mu, \\ F^{(2)} &= \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \end{aligned} \tag{7.3}$$

The superscripts indicating the rank of the differential forms will be dropped for now, as it is clear that A is a 1-form and F is a 2-form. The field strength now has a very simple expression in terms of its potential:

$$F = \mathrm{d}A, \tag{7.4}$$

and the action (7.1) is re-expressed as follows:

$$S = \int -\frac{1}{2} F \wedge \star F. \tag{7.5}$$

A field described by such an action is called a *massless differential form*. The identity $\partial_\mu \star F^{\mu\nu} = 0$ follows simply from the nilpotency of the exterior derivative (that is, $\mathrm{d}^2 A = 0$) and the equation of motion can again be found from varying the action. They are given by

$$\mathrm{d}F = 0 \quad \text{Bianchi identity,} \tag{7.6a}$$

$$\mathrm{d}\star F = 0 \quad \text{field equation.} \tag{7.6b}$$

The identity has be rechristened to *Bianchi identity*. Also the equation of motion has been called the field equation, which will happen throughout this thesis (and in fact throughout the whole of physics literature). Finally, the duality transformation (1.9) now reads

$$\begin{aligned} F &\rightarrow \ {}^dF = \star F, \\ \star F &\rightarrow \star\, {}^dF = -F. \end{aligned} \tag{7.7}$$

But of course this specific duality only holds for 2-form field strengths in four dimensions. We would like to know how it generalizes to forms of arbitrary rank in arbitrary dimensions, which is exactly what will be discussed in the next section.

7.2 Massless Poincaré duality

Given a free massless p-form B in any given dimension D, we can construct the following action for it:

$$S = \int -\frac{1}{2} H^{(p+1)} \wedge \star H^{(p+1)}, \tag{7.8}$$

where H is the field strength for B, that is, $H = \mathrm{d}B$. The field equation for B can be obtained in the usual way, while its Bianchi identity is dictated by the nilpotency of the exterior differential operator. They respectively read

$$\mathrm{d}\star H^{(p+1)} = 0 \quad \text{field equation,} \tag{7.9a}$$

$$\mathrm{d}H^{(p+1)} = 0 \quad \text{Bianchi identity.} \tag{7.9b}$$

The action (7.8) is one of second order. The term 'second order' stems from the fact that it is not directly expressed in terms of its fundamental field B, but rather of the field strength H. There is an easy way to rewrite the action (7.8) into a first order version: just simply take H to be its fundamental field (i.e. not being $\mathrm{d}B$). The condition $\mathrm{d}H = 0$ can still be imposed by adding an appropriate Lagrange multiplier:

$$S_{\text{master}} = \int -\frac{1}{2} H^{(p+1)} \wedge \star H^{(p+1)} + (-1)^p \mathrm{d}H^{(p+1)} \wedge \star \lambda^{(p+2)}, \tag{7.10}$$

$$\boxed{S(H(B)) \underset{-\lambda}{\overset{+\lambda}{\rightleftarrows}} S_{\text{master}}(H,\lambda) \xrightarrow{-H} {}^{d}S({}^{d}H({}^{d}B \equiv \star\lambda))}$$

Figure 7.1: The dualization scheme of a massless differential form. B is the p-form we begin with, H its field strength ($H = \mathrm{d}B$), λ the Lagrange multiplier enforcing the Bianchi identity $\mathrm{d}H = 0$, ${}^{d}B$ the dualized form (${}^{d}B \equiv \star\lambda$). The arrow with $+\lambda$ indicates the addition of the Bianchi identities, whereas the arrows with minus signs indicate solving the system for the corresponding field.

Upon varying this action with respect to λ we indeed find that $\mathrm{d}H = 0$. So H must be exact, at least locally. The converse to Poincaré's lemma then tells us that $H = \mathrm{d}B$ for some p-form B; we refind our original theory.

The first order action has the subscript 'master' for a good reason. Because from it we cannot only refind (7.8), but also a different yet equivalent theory. We do this by solving the master action for H, not λ. We find

$$\star H^{(p+1)} = \mathrm{d}\star\lambda^{(p+2)}. \tag{7.11}$$

as an equation of motion for H. If we then make the redefinitions

$$\begin{aligned} {}^{d}B^{(D-p-2)} &\equiv \star\lambda^{(p+2)}, \\ {}^{d}H^{(D-p-1)} &\equiv \star H^{(p+1)}, \end{aligned} \tag{7.12}$$

the first order action (7.10) can subsequently be dualized to

$${}^{d}S = \int -\frac{1}{2}\,{}^{d}H^{(D-p-1)} \wedge \star\,{}^{d}H^{(D-p-1)}. \tag{7.13}$$

Here the field strength is given by ${}^{d}H = \mathrm{d}\,{}^{d}B$. This describes a massless $(D-p-2)$-form ${}^{d}B$. As a quick summary, we have taken the following steps:

1. Go to first order formalism with help of the Bianchi identity.
2. Solve for H.
3. Do redefinitions.

This dualization procedure is sketched roughly in figure 7.1. The action (7.13) describes the same physics as (7.8), only in a different representation. We say that a massless p-form is *dual* to a massless $(D-p-2)$-form. Because we have used the converse to Poincaré's lemma to derive this duality, it is called the *Poincaré duality*.

The Bianchi identity of ${}^{d}B$ is given by the field equation of B, and vice versa:

$$\begin{aligned} \mathrm{d}\star\,{}^{d}H^{(D-p-1)} &= 0 \quad \text{field equation}, \\ \mathrm{d}\,{}^{d}H^{(D-p-1)} &= 0 \quad \text{Bianchi identity}. \end{aligned}$$

This encourages us to look at the dualization from another angle. For example, starting from our original action (7.8) and given the fact that H is the field strength of B, we have $\mathrm{d}\star H = 0$ as a field equation for B. Using the converse to Poincaré's lemma, this is solved by

$$\star H^{(p+1)} = \mathrm{d}\, {}^{d}B^{(d-p-2)} \tag{7.14}$$

for some $(D-p-2)$-form ${}^{d}B$. When we express the Bianchi identity for B, $\mathrm{d}H = 0$, in terms of ${}^{d}B$, it reads

$$\mathrm{d}\star\mathrm{d}\, {}^{d}B^{(D-p-2)} = 0. \tag{7.15}$$

But this is just the field equation obtained from (7.13). This implies that the two methods of dualization (adding a Lagrange multiplier or looking at the roles of the field equation and the Bianchi identity) are in fact equivalent.

7.2.1 The $p = D - 1$ limiting case

In the limiting case of $p = D - 1$ there is no meaningful Bianchi identity for B, because $\mathrm{d}H$ is trivially zero. Hence the action cannot be rewritten in first-order formalism by the method previously described. We can however add a Lagrange multiplier that ensures the exact form of H:

$$S_{\text{master}} = \int -\frac{1}{2}H^{(D)} \wedge \star H^{(D)} + \left(H^{(D)} - \mathrm{d}B^{(D-1)}\right) \wedge \star\lambda^{(D)}. \tag{7.16}$$

Solving first for λ results in the second-order action we started out with. However, when we begin with varying with respect to B we find that λ is a constant. Next we solve for H, partially integrate the term with B and obtain:

$${}^{d}S = \int -\frac{1}{2}\lambda^{(D)} \wedge \star\lambda^{(D)}. \tag{7.17}$$

So the dual version of a $(D-1)$-form theory corresponds to a background field which is constant over the entire manifold.

7.3 Massive Poincaré duality

Consider the second order action

$$S = \int -\frac{1}{2}H^{(p+1)} \wedge \star H^{(p+1)} - \frac{1}{2}m^2 B^{(p)} \wedge \star B^{(p)}, \tag{7.18}$$

with $H = \mathrm{d}B$. It describes a massive p-form B. We can write it in first order formalism as

$$S_{\text{master}} = \int -\frac{1}{2}H^{(p+1)} \wedge \star H^{(p+1)} - \frac{1}{2}m^2 B^{(p)} \wedge \star B^{(p)} + H^{(p+1)} \wedge \star\mathrm{d}B^{(p)}. \tag{7.19}$$

Taking H and B to be independent fields, the field equation for H yields $H = \mathrm{d}B$. When inserted back into (7.19) this gives (7.18). We can, however, choose to eliminate B from (7.19) by solving its field equation:

$$m^2 \star B^{(p)} + (-1)^p \mathrm{d} \star H^{(p+1)} = 0. \tag{7.20}$$

It obviously solved by

$$\begin{aligned} B^{(p)} &= (-1)^{p+p(D-p)} \frac{1}{m^2} \star \mathrm{d} \star H^{(p+1)}, \\ \star B^{(p)} &= (-1)^{p+1} \frac{1}{m^2} \mathrm{d} \star H^{(p+1)}. \end{aligned} \tag{7.21}$$

Plugging this into (7.19) and setting ${}^{d}B = \frac{1}{m} \star H$ gives

$${}^{d}S = \int -\frac{1}{2} m^2 \, {}^{d}B^{(D-p-1)} \wedge \star \, {}^{d}B^{(D-p-1)} - \mathrm{d} \, {}^{d}B^{(D-p-1)} \wedge \star \mathrm{d} \, {}^{d}B^{(D-p-1)}. \tag{7.22}$$

So we see that a massive p-form is dual to a massive $(D-p-1)$-form.

7.4 Degrees of freedom

In the previous sections we saw that a massless p-form is dual to a massless $(D-p-2)$-form, and that a massive p-form is dual to a massive $(D-p-1)$-form. This can be explained by looking at their degrees of freedom.

In chapter 3 we saw that the dimension the space of all p-forms in a D dimensional manifold was $\binom{D}{p}$. One might naively think that the degrees of freedom of a massless p-forms is then also $\binom{D}{p}$, but as we already saw in chapter 2 it is less. There are two things that snatch away the missing degrees of freedom:

- A massless p-form has a gauge symmetry: $B \to B + \mathrm{d}\Lambda, \quad H \to H$.
- A massless p-form has to obey an equation of motion.

At the end of the day a massless p-form has

$$\binom{D-2}{p} = \binom{D-2}{D-p-2} \tag{7.23}$$

degrees of freedom, the same as a massless $(D-p-2)$-form. The massive p-form also has to obey an equation of motion, but the gauge symmetry is no longer there: it is broken by the mass term that appears in the action. So a massive p-form has

$$\binom{D-1}{p} = \binom{D-1}{D-p-1} \tag{7.24}$$

degrees of freedom, the same as a massive $(D-p-1)$-form.

Chapter 8

Dual reductions

8.1 Dual reduction of $N = 1, D = 10$ supergravity

We are now finally ready to discuss the dualization of the potential $B_{\mu\nu}$ appearing in $N = 1, D = 10$ supergravity. Recall that the action of this supergravity is given by

$$\hat{S} = \int \mathrm{e}^{-\hat{\phi}}\Big(\hat{R} \,\hat{\star}\, \mathbf{1} + \mathrm{d}\hat{\phi} \wedge \hat{\star}\mathrm{d}\hat{\phi} - \tfrac{1}{2}\hat{H}^{(3)} \wedge \hat{\star}\hat{H}^{(3)} - \mathrm{Tr}\ \hat{F}(\hat{A}) \wedge \hat{\star}\hat{F}(\hat{A})\Big)\,. \tag{8.1}$$

The three-form field strength reads

$$\hat{H}^{(3)} = \mathrm{d}\hat{B}^{(2)} - \mathrm{Tr}\,(\hat{A} \wedge \mathrm{d}\hat{A} + \tfrac{2}{3}\hat{A} \wedge \hat{A} \wedge \hat{A})\,. \tag{8.2}$$

This is the same action as (2.12), only now written in terms of differential forms. Although the Yang-Mills fields are required for the theory to be free of anomalies, we will not retain them in our analysis. Thus the action we want to dualize is

$$\begin{aligned} \hat{S} &= \hat{S}_{g\phi} + \hat{S}_2, \\ \hat{S}_{g\phi} &= \int \mathrm{e}^{-\hat{\phi}}\Big(\hat{R} \,\hat{\star}\, \mathbf{1} + \mathrm{d}\hat{\phi} \wedge \hat{\star}\mathrm{d}\hat{\phi}\Big), \\ \hat{S}_2 &= \int -\tfrac{1}{2}\mathrm{e}^{-\hat{\phi}}\hat{H}^{(3)} \wedge \hat{\star}\hat{H}^{(3)}, \end{aligned} \tag{8.3}$$

with $\hat{H} = \mathrm{d}\hat{B}$. The action has been split into a metric-dilaton part $\hat{S}_{g\phi}$ and a two-form part $\hat{S}_2$. We can employ the dualization technique from the previous chapter in order to obtain

$$\begin{aligned} {}^d\hat{S} &= \hat{S}_{g\phi} + \hat{S}_6, \\ \hat{S}_6 &= \int -\tfrac{1}{2}\mathrm{e}^{\hat{\phi}}\, {}^d\hat{H}^{(7)} \wedge \hat{\star}\, {}^d\hat{H}^{(3)}, \end{aligned} \tag{8.4}$$

where ${}^d\hat{H}$ is the field strength of a six-form: ${}^d\hat{H} = \mathrm{d}\, {}^d\hat{B}$, and the six-form action is the dual to the two-form action: $\hat{S}_6 = {}^d\hat{S}_2$. As the actions (8.3) and (8.4)

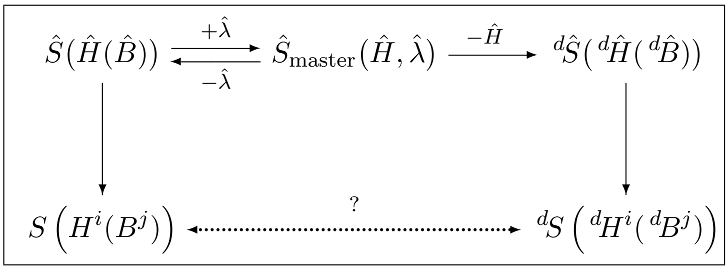

Figure 8.1: Reduction of the dualization scheme of a massless differential form. The reduced actions S and ${}^{d}S$ should a priori be equivalent, but the question remains whether or not this equivalence can be shown in the lower dimensional theory.

only differ in their second part, we will not include the metric-dilaton part $\hat{S}_{g\phi}$ (which already has been reduced in chapter 6) in our analysis. And to make things even simpler we also ignore the dilaton coupling for now and set $\hat{\phi} = 0$.

The analysis of the equivalence of the reduced versions of (8.3) and (8.4) then breaks down to the question whether or not the reduced standard action for a massless six-form can be dualized back in the lower dimension. This situation is schematically depicted for a general massless form in figure 8.1. One might hope that the equivalence still exists in the lower-dimensional theory. But because we obtain two different sets of field strengths and because some of the reduced potentials gain mass when the reduction is carried out over a group manifold (as is shown in appendix C), this is far from apparent. However, in the next section the question mark in figure 8.1 will be removed and the dotted line will be filled out.

8.2 Dual reduction of massless forms

Here we will show that the equivalence suggested in figure 8.1 still exists in the lower-dimensional theory by explicitly carrying out a dualization procedure. The outline is the following:

1. Find Bianchi identities for the reduced field strengths
2. Switch to first order formalism by imposing those Bianchi identities through the use of Lagrange multipliers
3. Solve the action for the reduced field strengths
4. Do appropriate redefinitions

After the last step we will end up with a dualized reduced action (that is, an action that is dualized after reduction). This actions turns out to be the same as the reduced dualized action (that is, the action that is dualized before the reduction), and hence the equivalence is proven. Let us start with the higher-dimensional action of a massless p-form $\hat{B}$. It reads

$$\hat{S} = \int -\tfrac{1}{2}\hat{H}^{(p+1)} \wedge \star\hat{H}^{(p+1)}. \tag{8.5}$$

The field strength $\hat{H}$ is given by $\hat{H} = \hat{B}$. The action reduces as

$$S = V_z \int -\tfrac{1}{2}\sum_{j=-1}^{p} \frac{1}{(p-j)!} H^{(j+1)}_{\alpha_1\cdots\alpha_{p-j}} \wedge \star H^{(j+1)\alpha_1\cdots\alpha_{p-j}}. \tag{8.6}$$

The reduced field strengths are, according to section C.3.1,

$$\begin{aligned} H^{(j+1)}_{\alpha_1\cdots\alpha_{p-j}} = {} & \underbrace{\mathcal{D}B^{(j)}_{\alpha_1\cdots\alpha_{p-j}}}_{\text{for } 0\leq j\leq p} + \underbrace{(-1)^p B^{(j-1)}_{\alpha_1\cdots\alpha_{p-j}\beta} \wedge F^\beta}_{\text{for } 1\leq j\leq p} \\ & + \underbrace{\tfrac{1}{2}(-1)^p (p-j)(p-j-1) f^\beta{}_{[\alpha_1\alpha_2} B^{(j+1)}_{\alpha_3\cdots\alpha_{p-j}]\beta}}_{\text{for } -1\leq j\leq p-2} \end{aligned} \tag{8.7}$$

The next step is to mould the action (8.6) into a first order formalism. For that the Bianchi identities for the field strengths $H^{(j+1)}$ need to be found, expressed solely in terms of those fieldstrengths. An educated guess would be to hit the fieldstrengths with a covariant exterior derivative. After some careful rearranging of the terms we find:

$$\begin{aligned} \mathcal{D}H^{(j+1)}_{\alpha_1\cdots\alpha_{p-j}} = {} & \underbrace{(-1)^p H^{(j)}_{\alpha_1\cdots\alpha_{p-j}\beta} \wedge F^\beta}_{\text{for } 0\leq j\leq p} \\ & + \underbrace{\tfrac{1}{2}(-1)^p (p-j)(p-j-1) f^\beta{}_{[\alpha_1\alpha_2} H^{(j+2)}_{\alpha_3\cdots\alpha_{p-j}]\beta}}_{\text{for } -2\leq j\leq p-2}. \end{aligned} \tag{8.8}$$

The fact that the Bianchi identities have exactly this form is not a surprise when we look at the higher-dimensional first order action $\hat{S}_{\text{master}}$. It is given by

$$\hat{S}_{\text{master}} = \int -\tfrac{1}{2}\hat{H}^{(p+1)} \wedge \star\hat{H}^{(p+1)} + (-1)^p \hat{Z}^{(p+2)} \wedge \star\hat{\lambda}^{(p+2)}, \tag{8.9}$$

where we have defined $\hat{Z} \equiv \mathrm{d}\hat{H}$. We can reduce this master action, with the result being

$$\begin{aligned} S_{\text{master}} = V_z \int & -\tfrac{1}{2}\sum_{j=-1}^{p} \frac{1}{(p-j)!} H^{(j+1)}_{\alpha_1\cdots\alpha_{p-j}} \wedge \star H^{(j+1)\alpha_1\cdots\alpha_{p-j}} \\ & + (-1)^p \sum_{j=-2}^{p} \frac{1}{(p-j)!} Z^{(j+2)}_{\alpha_1\cdots\alpha_{p-j}} \wedge \star\lambda^{(j+2)\alpha_1\cdots\alpha_{p-j}}. \end{aligned} \tag{8.10}$$

For the explicit form of the $Z^{(j+2)}$'s we again use the tools developed in section C.3.1, and find

$$\begin{aligned} Z^{(j+2)}_{\alpha_1\cdots\alpha_{p-j}} = &\underbrace{\mathcal{D}H^{(j+1)}_{\alpha_1\cdots\alpha_{p-j}}}_{\text{for } -1\leq j\leq p} + \underbrace{(-1)^{p+1}H^{(j)}_{\alpha_1\cdots\alpha_{p-j}\beta}\wedge F^\beta}_{\text{for } 0\leq j\leq p} \\ &+ \underbrace{\tfrac{1}{2}(-1)^{p+1}(p-j)(p-j-1)f^\beta{}_{[\alpha_1\alpha_2}H^{(j+2)}_{\alpha_3\cdots\alpha_{p-j}]\beta}}_{\text{for } -2\leq j\leq p-2}. \end{aligned} \tag{8.11}$$

Comparing this with the Bianchi identities found earlier (8.8), we see that they are nothing else than

$$Z^{(j+2)}_{\alpha_1\cdots\alpha_{p-j}} = 0. \tag{8.12}$$

But these are the equations of motion for the Lagrange multipliers $\star\lambda^{(j+2)}$ in (8.10). The conclusion can be drawn that going to a first order formalism and dimensional reduction commute with one another.

It may have been noted that we have boldly called (8.10) 'first order', but in advance it is not quite clear that one refinds the original action (8.6) after solving for the Lagrange multipliers $\star\lambda^{(j+2)}$. At this point there are two ways to proceed: we can either try to solve the Bianchi identities by hand, or we can attempt to plug in a solution and hope that it works. It is guaranteed that both methods yield the same result, because the Bianchi identities are differential equations of first order, and thus any given solution to them is necessarily unique.

In the latter case we do not need to guess our solution; we can simply reduce $\hat{H}$ explicitly in the second order formalism (that is, $\hat{H} = \mathrm{d}\hat{V}$), and use those results as solutions to the Bianchi identities. However, solving them by hand is in fact also a possibility. In that case we have to use a lower-dimensional analogue of the Poincaré lemma, stating that every internally closed field is locally exact:

$$f^\beta{}_{[\alpha_1\alpha_2}\Phi_{\alpha_3\cdots\alpha_{n+1}]\beta} = 0 \quad\Longrightarrow\quad \Phi_{\alpha_1\cdots\alpha_n} = f^\beta{}_{[\alpha_1\alpha_2}\Psi_{\alpha_3\cdots\alpha_n]\beta} \tag{8.13}$$

Having determined that the first order formalism actually works in the reduced theory, we are now ready continue the process of dualization. In section 7.2 the next step was to obtain the equation of motion for H, so let's do that here as well. For every $H^{(j+1)}$ we find the equation of motion to be

$$\begin{aligned} \star H^{(j+1)\alpha_1\cdots\alpha_{p-j}} = (-1)^{p-j}\Big(&\underbrace{\mathcal{D}\star\lambda^{(j+2)\alpha_1\cdots\alpha_{p-j}}}_{\text{for } -1\leq j\leq p} \\ &+ \underbrace{(p-j)F^{[\alpha_1}\wedge\star\lambda^{(j+3)\alpha_2\cdots\alpha_{p-j}]}}_{\text{for } -1\leq j\leq p-1} \\ &+ \underbrace{\tfrac{1}{2}(p-j)f^{[\alpha_1}{}_{\beta\gamma}\star\lambda^{(j+1)\alpha_2\cdots\alpha_{p-j}]\beta\gamma}}_{\text{for } -1\leq j\leq p-1}\Big). \end{aligned} \tag{8.14}$$

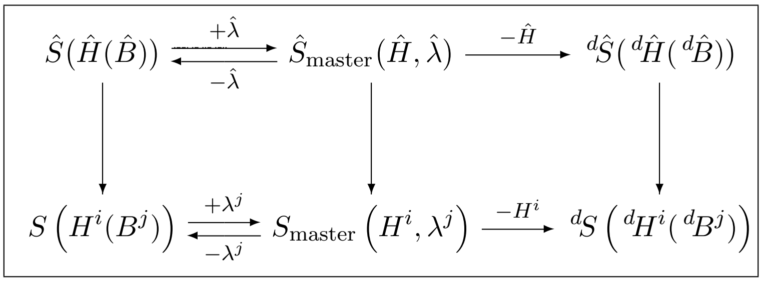

Figure 8.2: Dual reduction scheme of a massless differential form. Here we have filled in the blank of figure 8.1 where originally the question mark was.

We can solve for the reduced field strengths $H^{(j+1)}$ by plugging these equations of motion into our first order action S_{master} (8.10). The hope is that the resulting action will be the same as (or equivalent to) the reduced action of the dualized form, ${}^dS$. In order to see this equivalence, we have to make some appropriate redefinitions for $H^{(j+1)}$ and $\lambda^{(j+2)}$. The original redefinitions in the higher-dimensional theory were as follows:

$$\begin{aligned} {}^d\hat{B}^{(D+n-p-2)} &\equiv \star\hat{\lambda}^{(p+2)}, \\ {}^d\hat{H}^{(D+n-p-1)} &\equiv \star\hat{H}^{(p+1)}. \end{aligned} \tag{8.15}$$

Upon dimensional reduction, this yields

$$\begin{aligned} {}^dH^{D-j-1}_{\alpha_1\cdots\alpha_{n-p+j}} &= (-1)^{(D+n-p-1)(p-j)}\frac{1}{(p-j)!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{n-p+j}\beta_1\cdots\beta_{p-j}} \star H^{(j+1)\beta_1\cdots\beta_{p-j}}, \\ {}^dB^{D-j-2}_{\alpha_1\cdots\alpha_{n-p+j}} &= (-1)^{(D+n-p-2)(p-j)}\frac{1}{(p-j)!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{n-p+j}\beta_1\cdots\beta_{p-j}} \star \lambda^{(j+2)\beta_1\cdots\beta_{p-j}}. \end{aligned}$$

When we now plug both redefinitions into (8.14), we exactly obtain the reduced field strengths of our dual form ${}^d\hat{B}$:

$$\begin{aligned} {}^dH^{(D-1-j)}_{\alpha_1\cdots\alpha_{n-p+j}} = {} & \underbrace{\mathcal{D}\,{}^dB^{(D-2-j)}_{\alpha_1\cdots\alpha_{n-p+j}}}_{\text{for } p-n\leq j\leq D-2} + \underbrace{(-1)^{D+n-p}\,{}^dB^{(D-3-j)}_{\alpha_1\cdots\alpha_{n-p+j}\beta}\wedge F^\beta}_{\text{for } p-n\leq j\leq D-3} \\ & + \underbrace{\tfrac{1}{2}(-1)^{D+n-p}(n-p+j)(n-p+j-1)f^\beta{}_{[\alpha_1\alpha_2}\,{}^dB^{(D-1-j)}_{\alpha_3\cdots\alpha_{n-p+j}]\beta}}_{\text{for } p-n+2\leq j\leq D-1} \end{aligned} \tag{8.16}$$

Furthermore, upon inserting the equations of motion 8.14 into the first order action S_{master} (8.10) we find the reduced dual action: $S_{\text{master}} \to {}^dS$, with

$${}^dS = V_z\int -\tfrac{1}{2}\sum_{j=p-n}^{D-1}\frac{1}{(n-p+j)!}\,{}^dH^{(D-1-j)}_{\alpha_1\cdots\alpha_{n-p+j}}\wedge\star\,{}^dH^{(D-1-j)\alpha_1\cdots\alpha_{n-p+j}}. \tag{8.17}$$

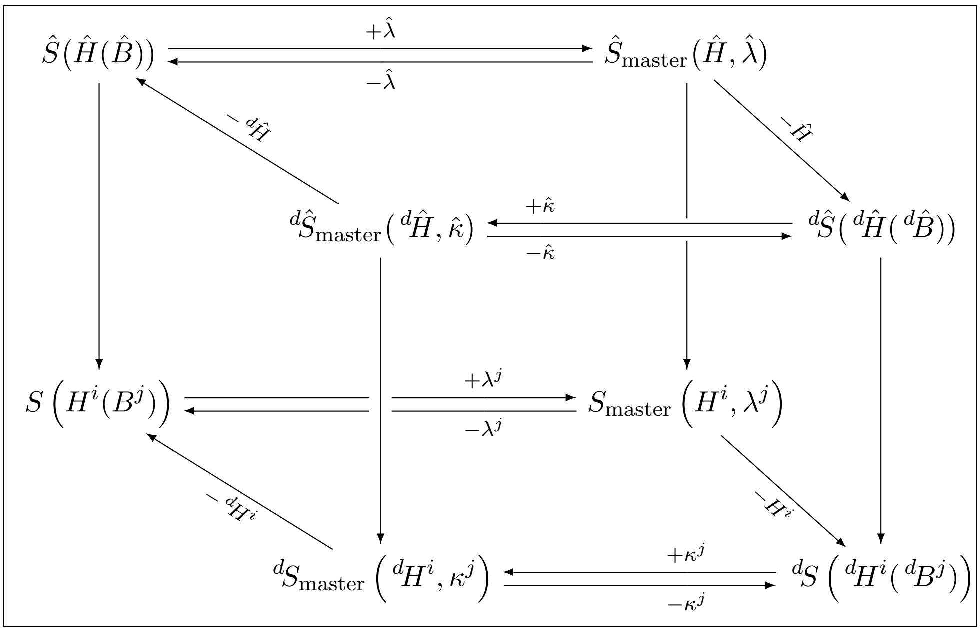

Figure 8.3: Cubed dual reduction scheme of a massless differential form.

This concludes the dualization procedure. We can now extend the dual reduction diagram of figure 8.1, the result being figure 8.2.

As a final step the dualization procedure can be applied again to the dualized action, which results in an identity operation. This can be done in both the reduced and unreduced theory, which is schematically depicted in figure 8.3.

Chapter 9

Explicit dual reductions

In this chapter we will see how the general outlines of section 8.2 apply to the reduction of a massless 2-form from ten to four dimensions. We start by reducing over a torus and neglecting the Kaluza-Klein gauge fields, then consider the torus reduction with the Kaluza-Klein gauge fields, and finally reduce over a group manifold. In the first two sections we take the route six-form → two-form, and in the last we go the other way around.

9.1 Torus reduction with no Kaluza-Klein gauge fields

To keep things simple, we'll set the Kaluza-Klein gauge fields equal to zero. This will simplify the analysis and help us fix ideas. The ten dimensional six-form ${}^d\hat{B}$ with field strength ${}^d\hat{H} = \mathrm{d}\hat{B}$ is governed by the action

$$ {}^d\hat{S} = \int -\tfrac{1}{2}\, {}^d\hat{H}^{(7)} \wedge \star\, {}^d\hat{H}^{(7)}. \tag{9.1} $$

Upon reduction this yields

$$ {}^dS = V_z \int -\tfrac{1}{2} \sum_{j=0}^{3} \frac{1}{(6-j)!}\, {}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} \wedge \star\, {}^dH^{(j+1)\alpha_1\cdots\alpha_{6-j}}. \tag{9.2} $$

The field strengths are given by

$$ \begin{aligned} {}^dH^{(1)}_{\alpha_1\cdots\alpha_6} &= \mathrm{d}\, {}^dB^{(0)}_{\alpha_1\cdots\alpha_6}, \\ {}^dH^{(2)}_{\alpha_1\cdots\alpha_5} &= \mathrm{d}\, {}^dB^{(1)}_{\alpha_1\cdots\alpha_5}, \\ {}^dH^{(3)}_{\alpha_1\cdots\alpha_4} &= \mathrm{d}\, {}^dB^{(2)}_{\alpha_1\cdots\alpha_4}, \\ {}^dH^{(4)}_{\alpha_1\cdots\alpha_3} &= \mathrm{d}\, {}^dB^{(3)}_{\alpha_1\cdots\alpha_3}. \end{aligned} \tag{9.3} $$

The 10 dimensional 6-form has $\binom{8}{6} = 28$ degrees of freedom. The ${}^dB^{(j)}_{\alpha_1\cdots\alpha_{6-j}}$'s each have $\binom{2}{j} \cdot \binom{6}{6-j}$ degrees of freedom which, as one would expect, adds up to 28. The Bianchi identities are obtained by applying the exterior derivative twice, and the field equations by varying the action with respect to the ${}^dB^{(j)}_{\alpha_1\cdots\alpha_{6-j}}$'s. They read

$$\begin{aligned} \mathrm{d}\star{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} &= 0 \quad \text{field equations}, && (9.4a)\\ \mathrm{d}\,{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} &= 0 \quad \text{Bianchi identities}, && (9.4b)\end{aligned}$$

for $j = 0, 1, 2, 3$. For $j = 3$ we have $\mathrm{d}\star{}^dH^{(4)}_{\alpha_1\alpha_2\alpha_3} = 0$, implying that

$$\begin{aligned} \star\,{}^dH^{(4)}_{\alpha_1\alpha_2\alpha_3} &= C_{\alpha_1\alpha_2\alpha_3},\\ {}^dH^{(4)}_{\alpha_1\alpha_2\alpha_3} &= \tilde{C}_{\alpha_1\alpha_2\alpha_3}, \end{aligned} \tag{9.5}$$

where the C's are antisymmetric constants. Thus the term in (9.2) corresponding to $j = 3$ gives rise to a potential V in the gravitational sector:

$$V = M^{\alpha_1\beta_1}M^{\alpha_2\beta_2}M^{\alpha_3\beta_3}C_{\alpha_1\alpha_2\alpha_3}\tilde{C}_{\beta_1\beta_2\beta_3}. \tag{9.6}$$

Subsequently, we can ignore it in the analysis of our massless 6-form. Let us now turn to the dual of (9.1). It is given by

$$\hat{S} = \int -\tfrac{1}{2}\hat{H}^{(3)} \wedge \star\hat{H}^{(3)}, \tag{9.7}$$

with $\hat{H} = \mathrm{d}\hat{B}$, and $\hat{B}$ is a two-form. This reduces as

$$S = V_z \int -\tfrac{1}{2}\sum_{j=0}^{2}\frac{1}{(2-j)!}H^{(j+1)}_{\alpha_1\cdots\alpha_{2-j}} \wedge \star H^{(j+1)\alpha_1\cdots\alpha_{2-j}}, \tag{9.8}$$

where the field strengths are given by

$$\begin{aligned} H^{(1)}_{\alpha_1\alpha_2} &= \mathrm{d}B^{(0)}_{\alpha_1\alpha_2},\\ H^{(2)}_{\alpha_1} &= \mathrm{d}B^{(1)}_{\alpha_1},\\ H^{(3)} &= \mathrm{d}B^{(2)}. \end{aligned} \tag{9.9}$$

Again, the 10 dimensional form has $\binom{8}{2} = 28$ degrees of freedom. The $B^{(j)}_{\alpha_1\cdots\alpha_{2-j}}$'s each have $\binom{2}{j} \times \binom{6}{2-j}$ degrees of freedom which, as before, adds up to 28.

In section 8.2 it was boldly stated that the reduced actions (9.2) and (9.8) are dual to each other. Let's see how it works out. We start by writing (9.2) in first order formalism:

$$\begin{aligned} S = V_z \int -\tfrac{1}{2}\sum_{j=0}^{2}\frac{1}{(6-j)!}\,{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} \wedge \star\,{}^dH^{(j+1)\alpha_1\cdots\alpha_{6-j}}\\ +\sum_{j=0}^{2}\frac{(-1)^j}{(6-j)!}\mathrm{d}\,{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} \wedge \lambda^{(2-j)\alpha_1\cdots\alpha_{6-j}}. \end{aligned} \tag{9.10}$$

Note that we indeed 'threw away' the $j = 3$ term because it contained no physical degrees of freedom. The equations of motion for the λ's yield

$$\mathrm{d}\,{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} = 0 \quad \text{for } j = 0, 1, 2. \tag{9.11}$$

Using the inverse to Poincaré's lemma, we refind (9.3). The equation of motion for the H's yield

$$\star\,{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} = \mathrm{d}\lambda^{(2-j)}_{\alpha_1\cdots\alpha_{6-j}}, \tag{9.12}$$

again for $j = 0, 1, 2$. We would like to mould this equation in the form $\star\,{}^dH = \mathrm{d}B$. There is a minor complication though: the number of internal indices of ${}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}}$ is not the same as that of $B^{(2-j)}_{\alpha_1\cdots\alpha_j}$. This is easily fixed by dualizing λ to upper indices:

$$\lambda^{(2-j)}_{\alpha_1\cdots\alpha_{6-j}} \equiv \frac{1}{j!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{6-j}\beta_1\cdots\beta_j} B^{(2-j)\beta_1\cdots\beta_j}. \tag{9.13}$$

We then have the equations of motion for the ${}^dH$'s in the desired form:

$$\star\,{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} = \frac{1}{j!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{6-j}\beta_1\cdots\beta_j} H^{(3-j)\beta_1\cdots\beta_j}, \tag{9.14}$$

where the field strength H is given by $H^{(3-j)}_{\alpha_1\cdots\alpha_j} = \mathrm{d}B^{(2-j)}_{\alpha_1\cdots\alpha_j}$. Now we are ready to plug this into (9.10), and obtain:

$$\begin{aligned} S &= V_z \int -\tfrac{1}{2}\sum_{j=0}^{2}\frac{1}{j!} H^{(3-j)}_{\alpha_1\cdots\alpha_j} \wedge \star H^{(3-j)\alpha_1\cdots\alpha_j} \\ &= {}^dS. \end{aligned} \tag{9.15}$$

Thus the higher-dimensional duality is not broken upon reduction.

9.2 Torus reduction with Kaluza-Klein gauge fields

Again we consider the reduction of a 10 dimensional massless 6-form to 4 dimensions and how it relates to the reduction of its dual, a 2-form. However, the difference with the last section is that we now keep the Kaluza-Klein gauge vectors V^α_μ instead of simply setting them equal to zero. The higher-dimensional action

$${}^d\hat{S} = \int -\frac{1}{2}\,{}^d\hat{H}^{(7)} \wedge \star\,{}^d\hat{H}^{(7)} \tag{9.16}$$

reduces as

$$S = V_z \int -\tfrac{1}{2}\sum_{j=0}^{3}\frac{1}{(6-j)!}\,{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} \wedge \star\,{}^dH^{(j+1)\alpha_1\cdots\alpha_{6-j}}, \tag{9.17}$$

where the field strength are given by

$$
\begin{aligned}
{}^{d}H^{(1)}_{\alpha_1\cdots\alpha_6} &= \mathrm{d}\,{}^{d}B^{(0)}_{\alpha_1\cdots\alpha_6},\\
{}^{d}H^{(2)}_{\alpha_1\cdots\alpha_5} &= \mathrm{d}\,{}^{d}B^{(1)}_{\alpha_1\cdots\alpha_5} + {}^{d}B^{(0)}_{\alpha_1\cdots\alpha_5\beta}\wedge F^{\beta},\\
{}^{d}H^{(3)}_{\alpha_1\cdots\alpha_4} &= \mathrm{d}\,{}^{d}B^{(2)}_{\alpha_1\cdots\alpha_4} + {}^{d}B^{(1)}_{\alpha_1\cdots\alpha_4\beta}\wedge F^{\beta},\\
{}^{d}H^{(4)}_{\alpha_1\cdots\alpha_3} &= \mathrm{d}\,{}^{d}B^{(3)}_{\alpha_1\cdots\alpha_3} + {}^{d}B^{(2)}_{\alpha_1\cdots\alpha_3\beta}\wedge F^{\beta}.
\end{aligned}
\tag{9.18}
$$

It is obvious that ${}^{d}B^{(3)}_{\alpha_1\alpha_2\alpha_3}$ contains no physical degrees of freedom. Hopefully it is possible to neglect ${}^{d}H^{(4)}_{\alpha_1\alpha_2\alpha_3}$, as this will greatly simplify the analysis. We begin with the field equation for ${}^{d}B^{(3)}_{\alpha_1\alpha_2\alpha_3}$. It yields

$$
\mathrm{d}\star\,{}^{d}H^{(4)}_{\alpha_1\alpha_2\alpha_3} = 0,
\tag{9.19}
$$

implying

$$
\begin{aligned}
\star\,{}^{d}H^{(4)}_{\alpha_1\alpha_2\alpha_3} &= C_{\alpha_1\alpha_2\alpha_3},\\
{}^{d}H^{(4)}_{\alpha_1\alpha_2\alpha_3} &= \tilde{C}_{\alpha_1\alpha_2\alpha_3}.
\end{aligned}
\tag{9.20}
$$

So again the $j = 3$ term in (9.17) gives rise to a potential, and we can ignore it here. Without the presence of ${}^{d}H^{(4)}_{\alpha_1\alpha_2\alpha_3}$ the field equations and Bianchi identities respectively read

$$
\begin{aligned}
\mathrm{d}\star\,{}^{d}H^{(1)}_{\alpha_1\cdots\alpha_6} &= -6F_{\alpha_1}\wedge\star\,{}^{d}H^{(2)}_{\alpha_2\cdots\alpha_6}\\
\mathrm{d}\star\,{}^{d}H^{(2)}_{\alpha_1\cdots\alpha_5} &= -5F_{\alpha_1}\wedge\star\,{}^{d}H^{(3)}_{\alpha_2\cdots\alpha_5}\\
\mathrm{d}\star\,{}^{d}H^{(3)}_{\alpha_1\cdots\alpha_4} &= 0
\end{aligned}
\tag{9.21}
$$

$$
\begin{aligned}
\mathrm{d}\,{}^{d}H^{(1)}_{\alpha_1\cdots\alpha_6} &= 0\\
\mathrm{d}\,{}^{d}H^{(2)}_{\alpha_1\cdots\alpha_5} &= +\,{}^{d}H^{(1)}_{\alpha_1\cdots\alpha_5\beta}\wedge F^{\beta}\\
\mathrm{d}\,{}^{d}H^{(3)}_{\alpha_1\cdots\alpha_4} &= +\,{}^{d}H^{(2)}_{\alpha_1\cdots\alpha_4\beta}\wedge F^{\beta}.
\end{aligned}
\tag{9.22}
$$

Writing (9.17) as a first order action we have to impose the Bianchi identities through the use of appropriate Lagrangian multipliers. The modified action reads

$$
\begin{aligned}
S = V_z\int & \tfrac{1}{6!}\left(-\tfrac{1}{2}\,{}^{d}H^{(1)}_{\alpha_1\cdots\alpha_6}\wedge\star\,{}^{d}H^{(1)\alpha_1\cdots\alpha_6} + \mathrm{d}\,{}^{d}H^{(1)}_{\alpha_1\cdots\alpha_6}\wedge\lambda^{(2)\alpha_1\cdots\alpha_6}\right)\\
&+\tfrac{1}{5!}\left(-\tfrac{1}{2}\,{}^{d}H^{(2)}_{\alpha_1\cdots\alpha_5}\wedge\star\,{}^{d}H^{(2)\alpha_1\cdots\alpha_5} - \left(\mathrm{d}\,{}^{d}H^{(2)}_{\alpha_1\cdots\alpha_5} - {}^{d}H^{(1)}_{\alpha_1\cdots\alpha_5\beta}\wedge F^{\beta}\right)\wedge\lambda^{(1)\alpha_1\cdots\alpha_5}\right)\\
&+\tfrac{1}{4!}\left(-\tfrac{1}{2}\,{}^{d}H^{(3)}_{\alpha_1\cdots\alpha_4}\wedge\star\,{}^{d}H^{(3)\alpha_1\cdots\alpha_4} + \left(\mathrm{d}\,{}^{d}H^{(3)}_{\alpha_1\cdots\alpha_4} - {}^{d}H^{(2)}_{\alpha_1\cdots\alpha_4\beta}\wedge F^{\beta}\right)\wedge\lambda^{(0)\alpha_1\cdots\alpha_4}\right).
\end{aligned}
\tag{9.23}
$$

When we take the ${}^dH$'s as the fundamental fields and vary with respect to the λ's, we refind the Bianchi identities (9.22). Using the converse to Poincaré's lemma the original fieldstrengths (9.18) can be recovered. Upon varying the action with respect to the H's we obtain the following equations:

$$\begin{aligned} \star\,{}^dH^{(1)}_{\alpha_1\cdots\alpha_6} &= \mathrm{d}\lambda^{(2)}_{\alpha_1\cdots\alpha_6} + 6\lambda^{(1)}_{\alpha_1\cdots\alpha_5}\wedge F_{\alpha_6}\,,\\ \star\,{}^dH^{(2)}_{\alpha_1\cdots\alpha_5} &= \mathrm{d}\lambda^{(1)}_{\alpha_1\cdots\alpha_5} + 5\lambda^{(0)}_{\alpha_1\cdots\alpha_4}\wedge F_{\alpha_5}\,,\\ \star\,{}^dH^{(3)}_{\alpha_1\cdots\alpha_4} &= \mathrm{d}\lambda^{(0)}_{\alpha_1\cdots\alpha_4}\,. \end{aligned} \tag{9.24}$$

When we now set

$$\lambda^{(2-j)}_{\alpha_1\cdots\alpha_{6-j}} \equiv \frac{1}{j!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{6-j}\beta_1\cdots\beta_j}B^{(2-j)\beta_1\cdots\beta_j} \tag{9.25}$$

and require

$$\star\,{}^dH^{(j+1)}_{\alpha_1\cdots\alpha_{6-j}} \equiv \frac{1}{j!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{6-j}\beta_1\cdots\beta_j}H^{(3-j)\beta_1\cdots\beta_j} \tag{9.26}$$

we obtain the following field strengths:

$$\begin{aligned} H^{(1)}_{\alpha_1\alpha_2} &= \mathrm{d}B^{(0)}_{\alpha_1\alpha_2}\,,\\ H^{(2)}_{\alpha} &= \mathrm{d}B^{(1)}_{\alpha} + B^{(0)}_{\alpha\beta}\wedge F^{\beta},\\ H^{(3)} &= \mathrm{d}B^{(2)} + B^{(1)}_{\beta}\wedge F^{\beta}. \end{aligned} \tag{9.27}$$

But these are just the field strengths of a 10 dimensional massless 2-form reduced to 4 dimensions. For we have:

$$\hat{S} = \int -\tfrac{1}{2}\hat{H}^{(3)}\wedge\star\hat{H}^{(3)} \tag{9.28}$$

as the higher-dimensional action, with $\hat{H} = \mathrm{d}\hat{B}$. After reduction it reads

$$S_{\mathrm{dual}} = V_z\int -\tfrac{1}{2}\sum_{j=0}^{2}\frac{1}{(2-j)!}H^{(j+1)}_{\alpha_1\cdots\alpha_{2-j}}\wedge\star H^{(j+1)\alpha_1\cdots\alpha_{2-j}}\,, \tag{9.29}$$

with the field strengths given by (9.27). In fact, plugging (9.26) into (9.23) we find that ${}^dS = S$. The field equations and the Bianchi identities of the H's respectively read

$$\begin{aligned} \mathrm{d}\star H^{(1)}_{\alpha_1\alpha_2} &= -2F_{\alpha_1}\wedge\star H^{(2)}_{\alpha_2},\\ \mathrm{d}\star H^{(2)}_{\alpha} &= -F_{\alpha}\wedge\star H^{(3)},\\ \mathrm{d}\star H^{(3)} &= 0, \end{aligned} \tag{9.30}$$

$$\begin{aligned} \mathrm{d}H^{(1)}_{\alpha_1\alpha_2} &= 0,\\ \mathrm{d}H^{(2)}_{\alpha} &= H^{(1)}_{\alpha\beta}\wedge F^{\beta},\\ \mathrm{d}H^{(3)} &= H^{(2)}_{\beta}\wedge F^{\beta}. \end{aligned} \tag{9.31}$$

Using (9.26) it follows easily that the field equations of the ${}^{d}H$'s are the Bianchi identities of the H's, and vice versa.

9.3 Dual group manifold reduction

Our starting point is the higher-dimensional master action:

$$\hat{S}_{\text{master}} = \int -\tfrac{1}{2}\hat{H}^{(3)} \wedge \hat{\star}\hat{H}^{(3)} + \mathrm{d}\hat{H}^{(3)} \wedge \hat{\star}\hat{\lambda}^{(4)}. \tag{9.32}$$

From it both the actions for the massless 2-form $\hat{B}$ and its dual ${}^{d}\hat{B}$ can be obtained. We will first do this dualization in the unreduced theory before moving on to its lower-dimensional counterpart. The second order action for the massless 2-form is easily found by solving the equation of motion for $\hat{\lambda}$. It reads

$$\hat{S} = \int -\tfrac{1}{2}\hat{H}^{(3)} \wedge \hat{\star}\hat{H}^{(3)}, \tag{9.33}$$

where $\hat{H}$ now is the field strength of $\hat{B}$, that is, $\hat{H} = \mathrm{d}\hat{B}$. If we take the other path and choose to solve the equation of motion for $\hat{H}$, which reads

$$\hat{\star}\hat{H}^{(3)} = \mathrm{d}\hat{\star}\hat{\lambda}^{(4)}, \tag{9.34}$$

we can make the redefinitions

$$\begin{aligned} {}^{d}\hat{B}^{(6)} &\equiv \hat{\star}\hat{\lambda}^{(4)}, \\ {}^{d}\hat{H}^{(7)} &\equiv \hat{\star}\hat{H}^{(3)} \end{aligned} \tag{9.35}$$

to obtain the dual action

$${}^{d}\hat{S} = \int -\tfrac{1}{2}\,{}^{d}\hat{H}^{(7)} \wedge \hat{\star}\,{}^{d}\hat{H}^{(7)}. \tag{9.36}$$

Here the field strength $\hat{H}$ for the dual form $\hat{B}$ is given by $\hat{H} = \mathrm{d}\hat{B}$. Thus the 2-form $\hat{B}$ is successfully dualized to the 6-form ${}^{d}\hat{B}$. In order to see how the lower-dimensional dualization will turn out, we begin by reducing all three actions $\hat{S}$, $\hat{S}_{\text{master}}$, and ${}^{d}\hat{S}$. The first yields upon reduction:

$$\begin{aligned} S = \int -\tfrac{1}{2}\Big(& H^{(3)} \wedge \star H^{(3)} + H^{(2)}_{\alpha} \wedge \star H^{(2)\alpha} \\ & + \tfrac{1}{2!}H^{(1)}_{\alpha_1\alpha_2} \wedge \star H^{(1)\alpha_1\alpha_2} + \tfrac{1}{3!}H^{(0)}_{\alpha_1\alpha_2\alpha_3} \wedge \star H^{(0)\alpha_1\alpha_2\alpha_3}\Big). \end{aligned} \tag{9.37}$$

The explicit form of the field strengths $H^{(j)}$ is

$$
\begin{aligned}
H^{(0)}_{\alpha_1\alpha_2\alpha_3} &= 3f^{\beta}{}_{[\alpha_1\alpha_2}B^{(0)}_{\alpha_3]\beta}, \\
H^{(1)}_{\alpha_1\alpha_2} &= \mathcal{D}B^{(0)}_{\alpha_1\alpha_2} + f^{\beta}{}_{\alpha_1\alpha_2}B^{(1)}_{\beta}, \\
H^{(2)}_{\alpha} &= \mathcal{D}B^{(1)}_{\alpha} + B^{(0)}_{\alpha\beta}\wedge F^{\beta}, \\
H^{(3)} &= \mathrm{d}B^{(2)} + B^{(1)}_{\beta}\wedge F^{\beta}.
\end{aligned}
\tag{9.38}
$$

Secondly, the reduction of $\hat{S}_{\text{master}}$ results in:

$$
\begin{aligned}
S_{\text{master}} = S + \int \Big(& Z^{(4)}\wedge\star\lambda^{(4)} + Z^{(3)}_{\alpha}\wedge\star\lambda^{(3)\alpha} + \tfrac{1}{2!}Z^{(2)}_{\alpha_1\alpha_2}\wedge\star\lambda^{(2)\alpha_1\alpha_2} \\
& + \tfrac{1}{3!}Z^{(1)}_{\alpha_1\cdots\alpha_3}\wedge\star\lambda^{(1)\alpha_1\cdots\alpha_3} + \tfrac{1}{4!}Z^{(0)}_{\alpha_1\cdots\alpha_4}\wedge\star\lambda^{(0)\alpha_1\cdots\alpha_4}\Big),
\end{aligned}
\tag{9.39}
$$

where S is given by (9.37) with the field strengths $H^{(j)}$ as fundamental fields (in other words, (9.38) is ‘forgotten’). The reduced Bianchi identities $Z^{(j)}$ are given by

$$
\begin{aligned}
Z^{(0)}_{\alpha_1\cdots\alpha_4} &= -6f^{\beta}{}_{[\alpha_1\alpha_2}H^{(0)}_{\alpha_3\alpha_4]\beta}, \\
Z^{(1)}_{\alpha_1\cdots\alpha_3} &= \mathcal{D}H^{(0)}_{\alpha_1\cdots\alpha_3} - 3f^{\beta}{}_{[\alpha_1\alpha_2}H^{(1)}_{\alpha_3]\beta}, \\
Z^{(2)}_{\alpha_1\alpha_2} &= \mathcal{D}H^{(1)}_{\alpha_1\alpha_2} - H^{(0)}_{\alpha_1\alpha_2\beta}\wedge F^{\beta} - f^{\beta}{}_{\alpha_1\alpha_2}H^{(2)}_{\beta}, \\
Z^{(3)}_{\alpha} &= \mathcal{D}H^{(2)}_{\alpha} - H^{(1)}_{\alpha\beta}\wedge F^{\beta}, \\
Z^{(4)} &= \mathrm{d}H^{(3)} - H^{(2)}_{\beta}\wedge F^{\beta}.
\end{aligned}
$$

Lastly, the reduction of ${}^{d}\hat{S}$ gives

$$
\begin{aligned}
{}^{d}S = \int -\tfrac{1}{2}\Big(& \tfrac{1}{3!}\,{}^{d}H^{(4)}_{\alpha_1\cdots\alpha_3}\wedge\star\,{}^{d}H^{(4)\alpha_1\cdots\alpha_3} + \tfrac{1}{4!}\,{}^{d}H^{(3)}_{\alpha_1\cdots\alpha_4}\wedge\star\,{}^{d}H^{(3)\alpha_1\cdots\alpha_4} \\
& + \tfrac{1}{5!}\,{}^{d}H^{(2)}_{\alpha_1\cdots\alpha_5}\wedge\star\,{}^{d}H^{(2)\alpha_1\cdots\alpha_5} + \tfrac{1}{6!}\,{}^{d}H^{(1)}_{\alpha_1\cdots\alpha_6}\wedge\star\,{}^{d}H^{(1)\alpha_1\cdots\alpha_6}\Big).
\end{aligned}
\tag{9.40}
$$

Here the field strengths ${}^{d}H^{(j)}$ are explicitly given by

$$
\begin{aligned}
{}^{d}H^{(1)}_{\alpha_1\cdots\alpha_6} &= \mathcal{D}\,{}^{d}B^{(0)}_{\alpha_1\cdots\alpha_6} + 15f^{\beta}{}_{[\alpha_1\alpha_2}\,{}^{d}B^{(1)}_{\alpha_3\cdots\alpha_6]\beta}, \\
{}^{d}H^{(2)}_{\alpha_1\cdots\alpha_5} &= \mathcal{D}\,{}^{d}B^{(1)}_{\alpha_1\cdots\alpha_5} + {}^{d}B^{(0)}_{\alpha_1\cdots\alpha_5\beta}\wedge F^{\beta} + 10f^{\beta}{}_{[\alpha_1\alpha_2}\,{}^{d}B^{(2)}_{\alpha_3\cdots\alpha_5]\beta}, \\
{}^{d}H^{(3)}_{\alpha_1\cdots\alpha_4} &= \mathcal{D}\,{}^{d}B^{(2)}_{\alpha_1\cdots\alpha_4} + {}^{d}B^{(1)}_{\alpha_1\cdots\alpha_4\beta}\wedge F^{\beta} + 6f^{\beta}{}_{[\alpha_1\alpha_2}\,{}^{d}B^{(3)}_{\alpha_3\alpha_4]\beta}, \\
{}^{d}H^{(4)}_{\alpha_1\cdots\alpha_3} &= \mathcal{D}\,{}^{d}B^{(3)}_{\alpha_1\cdots\alpha_4} + {}^{d}B^{(2)}_{\alpha_1\cdots\alpha_3\beta}\wedge F^{\beta} + 3f^{\beta}{}_{[\alpha_1\alpha_2}\,{}^{d}B^{(4)}_{\alpha_3]\beta}.
\end{aligned}
\tag{9.41}
$$

The claim made in section 8.2 was that the reduced action S (9.37) can be related to the dual action ${}^dS$ (9.40) by means of the first order action S_{master} (9.39). Indeed both can be obtained from the action S_{master}; the former by solving for the reduced Lagrange multipliers $\lambda^{(j)}$ and the latter by solving for the reduced field strengths $H^{(j)}$. If we do the first, we find as equations of motion for $\lambda^{(j)}$:

$$Z^{(0)}_{\alpha_1\cdots\alpha_4} = Z^{(1)}_{\alpha_1\cdots\alpha_3} = Z^{(2)}_{\alpha_1\alpha_2} = Z^{(3)}_{\alpha} = Z^{(4)} = 0 \tag{9.42}$$

This set of equations can be solved to exactly find the original field strengths (9.38), and thus we refind the second order action S (9.37). Conversely, if we take the second option and solve for $H^{(j)}$ instead of $\lambda^{(j)}$ the following equations of motion are found:

$$\begin{aligned}
\star H^{(0)\alpha_1\cdots\alpha_3} &= -\mathcal{D}\star\lambda^{(1)\alpha_1\cdots\alpha_3} - 3F^{[\alpha_1}\wedge\star\lambda^{(2)\alpha_2\alpha_3]} - \tfrac{3}{2}f^{[\alpha_1}{}_{\beta\gamma}\star\lambda^{(0)\alpha_2\alpha_3]\beta\gamma}\\
\star H^{(1)\alpha_1\alpha_2} &= +\mathcal{D}\star\lambda^{(2)\alpha_1\alpha_2} + 2F^{[\alpha_1}\wedge\star\lambda^{(3)\alpha_2]} + f^{[\alpha_1}{}_{\beta\gamma}\star\lambda^{(1)\alpha_2]\beta\gamma}\\
\star H^{(2)\alpha} &= -\mathcal{D}\star\lambda^{(3)\alpha} - F^{\alpha}\wedge\star\lambda^{(4)} - \tfrac{1}{2}f^{\alpha}{}_{\beta\gamma}\star\lambda^{\beta\gamma}\\
\star H^{(3)} &= \mathrm{d}\star\lambda^{(4)}
\end{aligned} \tag{9.43}$$

It is not immediately apparent that these give the field strengths (9.41) of the dual form B. However, the higher-dimensional redefinitions (9.35) can be reduced to give

$$\begin{aligned}
{}^dH^{(3-j)}_{\alpha_1\cdots\alpha_{4+j}} &= \frac{(-1)^{2-j}}{(2-j)!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{4+j}\beta_1\cdots\beta_{2-j}}\star H^{(j+1)\beta_1\cdots\beta_{2-j}},\\
{}^dB^{(2-j)}_{\alpha_1\cdots\alpha_{4+j}} &= \frac{1}{(2-j)!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{4+j}\beta_1\cdots\beta_{2-j}}\star\lambda^{(j+2)\beta_1\cdots\beta_{2-j}}.
\end{aligned}$$

And if these are plugged into the equations of motion (9.43) we indeed find (9.41). Furthermore, when these equations of motion are substituted into S_{master} (9.39) we obtain exactly the dual action ${}^dS$ (9.40).

9.4 Completing the square

Now that we know the dualization scheme of figure 8.2 actually works, we can go forth and try to apply the procedure to the dualized action. That is, we will dualize the 6-form back to the 2-form in four dimensions. According to the lower part of figure 8.3 this should yield the identity operation, which has been represented in figure 9.1. In the previous section we have taken the route $S\to S_{\text{master}}\to {}^dS$; we will now continue this path and take the route ${}^dS\to {}^dS_{\text{master}}\to S$.

As both the action for the 2-form S and the action for the 6-form ${}^dS$ have been given in the previous section (respectively by (9.37) and (9.40)),

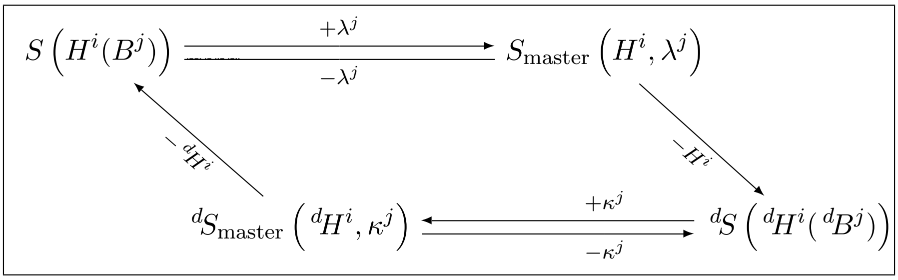

Figure 9.1: Squared dualization scheme of a reduced massless differential form.

we only need to write down the dual master action. We again start at the ten-dimensional dual master action, which is given by

$$ {}^d\hat{S}_{\text{master}} = \int -\tfrac{1}{2}\,{}^d\hat{H}^{(7)} \wedge \hat{\star}\,{}^d\hat{H}^{(7)} + \mathrm{d}\,{}^d\hat{H}^{(7)} \wedge \hat{\star}\,{}^d\hat{\kappa}^{(8)}. \tag{9.44} $$

This reduces as

$$ \begin{aligned} {}^dS_{\text{master}} = {}^dS + \int \Big(& \tfrac{1}{4!} Y^{(4)}_{\alpha_1\cdots\alpha_4} \wedge \star\kappa^{(4)\alpha_1\cdots\alpha_4} \\ & + \tfrac{1}{5!} Y^{(3)}_{\alpha_1\cdots\alpha_5} \wedge \star\kappa^{(3)\alpha_1\cdots\alpha_5} \\ & + \tfrac{1}{6!} Y^{(2)}_{\alpha_1\cdots\alpha_6} \wedge \star\kappa^{(2)\alpha_1\cdots\alpha_6} \Big), \end{aligned} \tag{9.45} $$

where ${}^dS$ has been given by (9.40) while forgetting about (9.41). Furthermore the Y's are given by

$$ \begin{aligned} Y^{(4)}_{\alpha_1\cdots\alpha_4} &= \mathcal{D}\,{}^dH^{(3)}_{\alpha_1\cdots\alpha_4} - {}^dH^{(2)}_{\alpha_1\cdots\alpha_4\beta} \wedge F^\beta - 6 f^\beta{}_{[\alpha_1\alpha_2} H^{(4)}_{\alpha_3\alpha_4]\beta} \\ Y^{(3)}_{\alpha_1\cdots\alpha_5} &= \mathcal{D}\,{}^dH^{(2)}_{\alpha_1\cdots\alpha_5} - {}^dH^{(1)}_{\alpha_1\cdots\alpha_5\beta} \wedge F^\beta - 10 f^\beta{}_{[\alpha_1\alpha_2} H^{(3)}_{\alpha_3\alpha_4\alpha_5]\beta} \\ Y^{(2)}_{\alpha_1\cdots\alpha_6} &= \mathcal{D}\,{}^dH^{(1)}_{\alpha_1\cdots\alpha_6}. \end{aligned} \tag{9.46} $$

If we vary the action (9.45) with respect to the Lagrange multipliers $\kappa^{(j)}$, we find as equations of motion the reduced Bianchi identities

$$ Y^{(4)}_{\alpha_1\cdots\alpha_4} = Y^{(3)}_{\alpha_1\cdots\alpha_5} = Y^{(2)}_{\alpha_1\cdots\alpha_6} = 0. \tag{9.47} $$

This should be enough to refind the original dual field strengths (9.41). However, if we vary the action with respect to ${}^dH^{(j)}$, we obtain the following equations of motion:

$$
\begin{aligned}
\star\, {}^{d}H^{(4)\alpha_1\cdots\alpha_3} &= -\tfrac{3}{2} f^{[\alpha_1}{}_{\beta\gamma} \star \kappa^{(4)\alpha_2\alpha_3]\beta\gamma} \\
\star\, {}^{d}H^{(3)\alpha_1\cdots\alpha_4} &= +\mathcal{D} \star \kappa^{(4)\alpha_1\cdots\alpha_4} + 2 f^{[\alpha_1}{}_{\beta\gamma} \star \kappa^{(3)\alpha_2\alpha_3\alpha_4]\beta\gamma} \\
\star\, {}^{d}H^{(2)\alpha_1\cdots\alpha_5} &= -\mathcal{D} \star \kappa^{(3)\alpha_1\cdots\alpha_5} - 5 F^{[\alpha_1} \wedge \star\kappa^{(4)\alpha_2\cdots\alpha_5]} - \tfrac{5}{2} f^{[\alpha_1}{}_{\beta\gamma} \star \kappa^{(2)\alpha_2\cdots\alpha_5]\beta\gamma} \\
\star\, {}^{d}H^{(1)\alpha_1\cdots\alpha_6} &= +\mathcal{D} \star \kappa^{(2)\alpha_1\cdots\alpha_6} + 6 F^{[\alpha_1} \wedge \star\kappa^{(3)\alpha_2\cdots\alpha_6]}
\end{aligned}
\tag{9.48}
$$

These are exactly the field strengths for the original reduced 2-form, as given by (9.38). This can be seen by reducing the ten-dimensional redefinitions

$$
\begin{aligned}
\hat{B}^{(2)} &\equiv \hat{\star}\hat{\kappa}^{(8)}, \\
\hat{H}^{(3)} &\equiv \hat{\star}\hat{H}^{(7)},
\end{aligned}
\tag{9.49}
$$

which results in

$$
\begin{aligned}
H^{(3-j)}_{\alpha_1\cdots\alpha_j} &= \frac{(-1)^{6-j}}{(6-j)!} \tilde{\varepsilon}_{\alpha_1\cdots\alpha_j\beta_1\cdots\beta_{6-j}} \star\, {}^{d}H^{(j+1)\beta_1\cdots\beta_{6-j}}, \\
B^{(2-j)}_{\alpha_1\cdots\alpha_j} &= \frac{1}{(6-j)!} \tilde{\varepsilon}_{\alpha_1\cdots\alpha_j\beta_1\cdots\beta_{6-j}} \star \kappa^{(j+2)\beta_1\cdots\beta_{6-j}}.
\end{aligned}
\tag{9.50}
$$

Furthermore, if the equations of motion (9.48) are inserted into the dual master action, we obtain the original action S (9.37). So we see that the dualization procedure indeed can be squared to the identity operator, as suggested in figure 9.1.

Discussion and conclusions

In this thesis we have looked at the dimensional reduction of Poincaré dualities in general, and the dualization of the two-form reduced from ten to four dimensions in particular. The reductions in question are all done over a unimodular group manifold. It turned out that dimensional reduction and dualization commute with one another, as can be seen in figure 8.2. This has been explicitly demonstrated for the two-form.

The reason for investigating this was to possibly obtain a new $N = 4, D = 4$ supergravity by means of dualization of $N = 1, D = 10$ two-form. Unfortunately, as dimensional reduction commutes with dualization, the resulting dualized four-dimensional supergravity is equivalent with the non-dualized four-dimensional supergravity.

The fact that dimensional reduction and dualization commute at all might be surprising. Because this has to mean that the Hodge dual does not mix the massless and massive Kaluza-Klein modes, which is far from apparent from its definition. Why this is the case requires some extra investigation.

On the other hand this might not come as a very big suprise; after all, the reduction was consistent. Thus we are able to uplift a solution to the original higher-dimensional theory, dualize it, and subsequently reduce to obtain both a reduced and dualized solution. This strongly suggests that a more direct route, that is, dualizing in the lower-dimensional theory, is also possible.

Although we have demonstrated that this direct route exists, there is a remark to be made. The method used for obtaining the Poincaré dual version involves an inner product instead of a topological term. This has been done for convienence, although it may modify the field equations for the metric. A more correct approach would be to introduce a Lagrange multiplier of the form

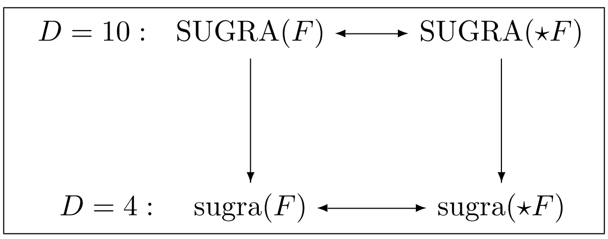

Figure 9.2: The duality in higher dimension holds, and it also holds when both the original theory and the dualized version are reduced.

$\mathrm{d}H^{(p+1)} \wedge {}^{d}B^{(D-p-2)}$ instead of the $\mathrm{d}H^{(p+1)} \wedge \star\lambda^{(p+2)}$.

As last, but certainly not least, we must not forget that the analysis presented here only concerns uncoupled field strengths. However, in the $N = 1, D = 10$ supergravity in question the field strength is coupled to Yang-Mills fields. Although these have been set to zero in chapter 8, a full analysis would require them to be included. If in that case reduction and dualization still commute might be the subject of future research.

Acknowledgments

I would like to thank Mees de Roo for assigning me on this project. Although I was not very pleased with my first assignment (which can be found in appendix D), I realise in hindsight that this was an excellent introduction into the fascinating world of high energy physics and dimensional reductions. On the whole I have enjoyed working on this project very much, not in the least because of his supervision.

Furthermore I would like to thank Martijn Eenink and Dennis Westra, who I have frequently harrassed with my questions about this or that. Without their kind answers this thesis undoubtedly would have been much less accurate.

Part IV

Appendices

Appendix A

Conventions

A.1 Notation

Hats denoted higher-dimensional objects. Spacetime coordinates run from 1 to D, internal coordinates run from $D+1$ to $D+n$. They are denoted by $x^{\hat{\mu}} = (x^\mu, z^\alpha)$.

Curved spacetime indices are denoted by later Greek letters (e.g. μ, ν, ρ), and the internal indices by early Greek letters (e.g. α, β, γ). Flat spacetime indices are denoted by early Roman letters (e.g. a, b, c), and the internal indices by later Roman letters (e.g. m, n, p).

Dualized objects will be denoted by the superscript d; for example, ${}^dH$ denotes the Poincaré dual of H.

Every manifold we consider will be of Lorentzian signature and we use 'mostly-plus' convention, in which the flat metric is $\eta_{ab} = (-1, +1, +1, \ldots, +1)$.

A.2 Levi-Civita

We define the Levi-Civita symbol such that

$$\begin{aligned} \tilde{\varepsilon}_{12\ldots d} &= +1 \\ \tilde{\varepsilon}^{12\ldots d} &= (-)^t, \end{aligned} \tag{A.1}$$

holds in any coordinate system, for both curved and flat indices. The number of timelike directions is denoted by t. In flat indices we have the following relation:

$$\begin{aligned} \tilde{\varepsilon}_{a_1\cdots a_p c_1 \cdots c_{d-p}} \tilde{\varepsilon}^{b_1 \cdots b_p c_1 \cdots c_{d-p}} &= (-)^t (d-p)! \delta^{b_1 \cdots b_p}_{a_1 \cdots a_p} \\ &= (-)^t p!(d-p)! \delta^{[b_1}_{[a_1} \cdots \delta^{b_p]}_{a_p]} \end{aligned} \tag{A.2}$$

where $\delta^{b_1\cdots b_p}_{a_1\cdots a_p} = \pm 1$. We get a plus sign if $\{b_1, \ldots, b_p\}$ is an even permutation of $\{a_1, \ldots, a_p\}$, and a minus sign if it is an odd permutation. In curved indices:

$$\begin{aligned} \tilde{\varepsilon}_{\mu_1\cdots\mu_d} &= e^{-1} e^{a_1}_{\mu_1} \cdots e^{a_d}_{\mu_d} \tilde{\varepsilon}_{a_1\cdots a_d} \\ \tilde{\varepsilon}^{\mu_1\cdots\mu_d} &= e^{+1} e^{\mu_1}_{a_1} \cdots e^{\mu_d}_{a_d} \tilde{\varepsilon}^{a_1\cdots a_d} \end{aligned} \tag{A.3}$$

And the Levi-Civita tensor is defined as:

$$\begin{aligned} \varepsilon_{\mu_1\cdots\mu_d} &= \sqrt{|g|}\tilde{\varepsilon}_{\mu_1\cdots\mu_d} \\ \varepsilon^{\mu_1\cdots\mu_d} &= \frac{1}{\sqrt{|g|}}\tilde{\varepsilon}^{\mu_1\cdots\mu_d} \end{aligned} \tag{A.4}$$

A.3 Differential forms

This is a short and technical recapitulation of section 3.2. A differential form of rank p, often simply called a p-form, will be defined as

$$\begin{aligned} A^{(p)} &= \frac{1}{p!} A_{\mu_1\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}, \\ &= \frac{1}{p!} A_{a_1\cdots a_p} e^{a_1} \wedge \cdots \wedge e^{a_p}, \end{aligned} \tag{A.5}$$

in curved and flat coordinates respectively. The exterior product of a p- and a q-form is given by:

$$A^{(p)} \wedge B^{(q)} = \frac{1}{p!q!} A_{\mu_1\cdots\mu_p} B_{\mu_{p+1}\cdots\mu_{p+q}} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+q}} \tag{A.6}$$

The exterior derivation of a p-form is given by:

$$\mathrm{d}A^{(p)} = \frac{1}{p!} \left(\partial_{\mu_1} A_{\mu_2\cdots\mu_{p+1}}\right) \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+1}}, \tag{A.7}$$

or, equivalently, in terms of components:

$$(\mathrm{d}A)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}. \tag{A.8}$$

We define the Hodge dual operator on a basis of one-forms:

$$\begin{aligned} \star e^{a_1} \wedge \cdots \wedge e^{a_p} &= \frac{1}{(d-p)!} \eta^{a_1 b_1} \cdots \eta^{a_p b_p} \tilde{\varepsilon}_{b_1\cdots b_p b_{p+1}\cdots b_d} e^{b_{p+1}} \wedge \cdots \wedge e^{b_d} \\ &= \frac{1}{(d-p)!} \tilde{\varepsilon}^{a_1\cdots a_p}{}_{b_1\cdots b_{d-p}} e^{b_1} \wedge \cdots \wedge e^{b_{d-p}}. \end{aligned} \tag{A.9}$$

It acts on a p-form as

$$\begin{aligned}\star A^{(p)} &= \star\left(\frac{1}{p!}A_{a_1\cdots a_p}e^{a_1}\wedge\cdots\wedge e^{a_p}\right)\\ &= \frac{1}{p!}A_{a_1\cdots a_p}\star e^{a_1}\wedge\cdots\wedge e^{a_p}\\ &= \frac{1}{p!(d-p)!}A_{a_1\cdots a_p}\hat{\varepsilon}^{a_1\cdots a_p}{}_{b_1\cdots b_{d-p}}e^{b_1}\wedge\cdots\wedge e^{b_{d-p}}\\ &\equiv (\star A)^{(d-p)}.\end{aligned} \tag{A.10}$$

It follows that it acts on the components of a p-form as

$$(\star A)_{a_1\cdots a_{d-p}} = \frac{1}{p!}A_{b_1\cdots b_p}\hat{\varepsilon}^{b_1\cdots b_p}{}_{a_1\cdots a_{d-p}}. \tag{A.11}$$

A.4 Gauge fields

Gauge-fields:

$$F^\alpha = \mathrm{d}A^\alpha + \frac{1}{2}f^\alpha{}_{\beta\gamma}A^\beta\wedge A^\gamma.$$

Covariant exterior derivative on a p-form with anti-symmetric indices:

$$\begin{aligned}\mathcal{D}\Phi^{(p)}{}_{\alpha_1\cdots\alpha_n} &= \mathrm{d}\Phi^{(p)}{}_{\alpha_1\cdots\alpha_n} - nf^\gamma{}_{\beta[\alpha_1}A^\beta\wedge\Phi^{(p)}{}_{\gamma|\alpha_2\cdots\alpha_n]}\\ \mathcal{D}\Phi^{(p)\alpha_1\cdots\alpha_n} &= \mathrm{d}\Phi^{(p)\alpha_1\cdots\alpha_n} - nf^{[\alpha_1}{}_{\gamma\beta}A^\beta\wedge\Phi^{(p)\gamma|\alpha_2\cdots\alpha_n]}\end{aligned}$$

Furthermore:

$$\mathcal{D}^2\Phi^{(p)}_{\alpha_1\cdots\alpha_n} = -nf^\gamma{}_{\beta[\alpha_1}\Phi^{(p)}_{\gamma|\alpha_2\cdots\alpha_n]}\wedge F^\beta$$

The structure constants f and the field strength F are constants with respect to covariant exterior differentiation:

$$\begin{aligned}\mathcal{D}f^\alpha{}_{\beta\gamma} &= 0\\ \mathcal{D}F^\alpha &= 0.\end{aligned}$$

Appendix B

Weyl rescaling

In this section we will show how the Einstein-Hilbert Lagrangian $\sqrt{-g}R[g]$ differs in one metric frame from another, using a so-called Weyl rescaling. Given a metric $\bar{g}_{\mu\nu}$ we can define a new metric $g_{\mu\nu}$ by

$$\bar{g}_{\mu\nu} = \mathrm{e}^{2\alpha\phi} g_{\mu\nu}, \tag{B.1}$$

or, equivalently,

$$\begin{aligned} \bar{e}_\mu^a &= \mathrm{e}^{\alpha\phi} e_\mu^a, \\ \bar{E}_a^\mu &= \mathrm{e}^{-\alpha\phi} E_a^\mu, \end{aligned} \tag{B.2}$$

As barred derivatives contain barred vielbeins, we have to be a bit careful when taking barred derivatives on unbarred objects. In particular it means that $\bar{\partial}_a = \mathrm{e}^{-\alpha\phi}\partial_a$. We begin with calculating the vielbein determinant, which is very straightforward:

$$\bar{e} = \mathrm{e}^{D\alpha\phi} e, \tag{B.3}$$

where D is the number of dimensions we are working in. The relation coefficient Ω is a bit harder, but straightforward nonetheless. It and the spin connection ω derived from it turn out to be

$$\begin{aligned} \bar{\Omega}^a{}_{bc} &= \mathrm{e}^{-\alpha\phi}\big(\Omega^a{}_{bc} + \alpha(\delta_c^a \partial_b\phi - \delta_b^a \partial_c\phi)\big), \\ \bar{\omega}_{abc} &= \mathrm{e}^{-\alpha\phi}\big(\omega_{abc} + \alpha(\eta_{ab}\partial_c\phi - \eta_{ac}\partial_b\phi)\big). \end{aligned} \tag{B.4}$$

Using that $\eta_{ab}\eta^{ab} = D$ and being careful with all the coefficients of α, the Einstein-Hilbert Lagrangian becomes

$$\bar{e}\bar{R} = \mathrm{e}^{\alpha(D-2)\phi} e\Big(R - \alpha^2(D-1)(D-2)(\partial\phi)^2 - 2\alpha(D-1)\nabla_a\partial^a\phi\Big). \tag{B.5}$$

Appendix C

Reductions

C.1 Pure gravity

With the results of the section 5, reducing gravity comes down to calculating the Ricci relational coefficients, obtaining the spin connections from them, and finally inserting the spin connections in (5.18). Only the most crucial results of these steps are given here, although the full calculation is in fact rather simple. We begin with reducing gravity over the circle, proceed to the torus, and conclude with the group manifold.

C.1.1 The circle

The Ansatz for the line element is given by

$$\hat{\mathrm{d}s}^2 = \mathrm{d}s^2 + \mathrm{e}^{2\varphi} (\mathrm{d}z - V)^2 \,. \tag{C.1}$$

The vielbeins are

$$\hat{e}_{\hat{\mu}}^{\hat{a}} = \begin{pmatrix} e_\mu^a & -\mathrm{e}^{\varphi} V_\mu \\ 0 & \mathrm{e}^{\varphi} \end{pmatrix}, \qquad \hat{E}_{\hat{a}}^{\hat{\mu}} = \begin{pmatrix} E_a^\mu & V_a \\ 0 & \mathrm{e}^{-\varphi} \end{pmatrix}. \tag{C.2}$$

The only non-zero Ricci relational coefficients are

$$\begin{aligned} \hat{\Omega}^z{}_{az} &= \partial_a \varphi, \\ \hat{\Omega}^z{}_{ab} &= -\mathrm{e}^{\varphi} F_{ab}, \\ \hat{\Omega}^a{}_{bc} &= \Omega^a{}_{bc}. \end{aligned} \tag{C.3}$$

The only non-zero spin connections are

$$\begin{aligned} \hat{\omega}_{zza} &= \partial_a \varphi, \\ \hat{\omega}_{abz} &= -\hat{\omega}_{zab} \\ &= -\tfrac{1}{2} e^{\varphi} F_{ab}, \\ \hat{\omega}_{abc} &= \omega_{abc}. \end{aligned} \tag{C.4}$$

Making the $D+1$ split for the Ricci scalar, it is given by

$$\begin{aligned}\hat{R} &= -2\partial_{\hat{a}}\hat{\omega}^{\hat{a}} + \hat{\omega}_{\hat{a}\hat{b}\hat{c}}\hat{\omega}^{\hat{c}\hat{a}\hat{b}} - \hat{\omega}_{\hat{a}}\hat{\omega}^{\hat{a}} \\ &= -2\partial_a\hat{\omega}^a + \hat{\omega}_{abc}\hat{\omega}^{cab} - \hat{\omega}_{zab}\hat{\omega}^{zab} - \hat{\omega}_{zza}\hat{\omega}^{zza} - \hat{\omega}_a\hat{\omega}^a.\end{aligned} \tag{C.5}$$

Inserting the spin connections given above, it becomes

$$\hat{R} = R - \tfrac{1}{4}\mathrm{e}^{2\varphi}F^2 - 2\partial_a\varphi\partial^a\varphi - 2\nabla_a\partial^a\varphi. \tag{C.6}$$

C.1.2 The torus

Having seen that the reduction of gravity in the Einstein-Cartan formalism is relatively easy, we can try and attempt to reduce it at once over an n-torus. To do so we modify the original line element Ansatz(D.3) to

$$\hat{ds}^2 = ds^2 + G_{\alpha\beta}(\mathrm{d}z^\alpha - V^\alpha)(\mathrm{d}z^\beta - V^\beta), \tag{C.7}$$

where $V^\alpha = V^\alpha_\mu dx^\mu$, and $G_{\alpha\beta}$ is an $n\times n$ scalar matrix playing the role of an internal metric (and thus replacing the scalar φ). If we choose the flat basis as

$$\begin{aligned}\hat{e}^a &= e^a, \\ \hat{e}^m &= \Phi^m_\alpha(\mathrm{d}z^\alpha - V^\alpha),\end{aligned}$$

we see that G and its inverse are given by

$$G_{\alpha\beta} = \delta_{mn}\Phi^m_\alpha\Phi^n_\beta, \qquad\qquad G^{\alpha\beta} = \delta^{mn}\Phi^\alpha_m\Phi^\beta_n, \tag{C.8}$$

where the scalars Φ^m_α and their inverse Φ^α_m are the internal vielbeins. Without further postponement, we shall proceed and see how the results of the $\mathbb{S}^1$ reduction generalize to those of one over $\mathbb{T}^n$. The vielbeins and their inverse are given by

$$\hat{e}^{\hat{a}}_{\hat{\mu}} = \begin{pmatrix} e^a_\mu & -\Phi^m_\alpha V^\alpha_\mu \\ 0 & \Phi^m_\alpha \end{pmatrix}, \qquad\qquad \hat{E}^{\hat{\mu}}_{\hat{a}} = \begin{pmatrix} E^\mu_a & V^\alpha_a \\ 0 & \Phi^\alpha_m \end{pmatrix}. \tag{C.9}$$

The only non-zero Ricci relational coefficients are

$$\begin{aligned}\hat{\Omega}^m{}_{an} &= \Phi^\alpha_n\partial_a\Phi^m_\alpha, \\ \hat{\Omega}^m{}_{ab} &= -\Phi^m_\alpha F^\alpha_{ab}, \\ \hat{\Omega}^a{}_{bc} &= \Omega^a{}_{bc}.\end{aligned} \tag{C.10}$$

The spin connections are

$$
\begin{aligned}
\hat{\omega}_{mnp} &= 0, \\
\hat{\omega}_{amn} &= \tfrac{1}{2}\left(\Phi^{\alpha}_{m}\partial_a\Phi_{\alpha n} - \Phi^{\alpha}_{n}\partial_a\Phi_{\alpha m}\right), \\
\hat{\omega}_{man} &= -\tfrac{1}{2}\Phi^{\alpha}_{m}\Phi^{\beta}_{n}\partial_a G_{\alpha\beta}, \\
\hat{\omega}_{abm} &= -\hat{\omega}_{mab} \\
&= -\tfrac{1}{2}\Phi_{m\alpha}F^{\alpha}_{ab}, \\
\hat{\omega}_{abc} &= \omega_{abc}.
\end{aligned}
\tag{C.11}
$$

where $\Phi_{m\alpha} = \delta_{mn}\Phi^{n}_{\alpha} = G_{\alpha\beta}\Phi^{\beta}_{m}$.
Again, the $D+n$ split of the Ricci-scalar reduces nicely because of the symmetries of the spin connections. It is given by

$$
\begin{aligned}
\hat{R} = &- 2\partial_a\hat{\omega}^a - \hat{\omega}_a\hat{\omega}^a - \hat{\omega}_m\hat{\omega}^m \\
&+ \hat{\omega}_{abc}\hat{\omega}^{cab} - \hat{\omega}_{mab}\hat{\omega}^{mab} - \hat{\omega}_{man}\hat{\omega}^{man} + \hat{\omega}_{mnp}\hat{\omega}^{pmn}.
\end{aligned}
\tag{C.12}
$$

Filling in the spin connections above, the reduced Ricci scalar reads

$$
\begin{aligned}
\hat{R} = R &- \tfrac{1}{4}G_{\alpha\beta}F^{\alpha}F^{\beta} + \tfrac{1}{4}\operatorname{Tr}\partial_a G\partial^a G^{-1} \\
&- \nabla_a \operatorname{Tr} G^{-1}\partial^a G - \tfrac{1}{4}\left(\operatorname{Tr} G^{-1}\partial G\right)^2.
\end{aligned}
\tag{C.13}
$$

C.1.3 A group manifold

The starting point is the line element of the group manifold reduction (6.32):

$$
\hat{\mathrm{d}s}^2 = \mathrm{d}s^2 + G_{\alpha\beta}(\sigma^\alpha - V^\alpha)(\sigma^\beta - V^\beta). \tag{C.14}
$$

The vielbeins now read

$$
\hat{e}^{\hat{a}}_{\hat{\mu}} = \begin{pmatrix} e^a_\mu & -\Phi^m_\alpha V^\alpha_\mu \\ 0 & U^\beta{}_\alpha \Phi^m_\beta \end{pmatrix}, \qquad \hat{E}^{\hat{\mu}}_{\hat{a}} = \begin{pmatrix} E^\mu_a & (U^{-1})^\alpha{}_\beta V^\beta_a \\ 0 & (U^{-1})^\alpha{}_\beta \Phi^\beta_m \end{pmatrix}. \tag{C.15}
$$

The non-zero Ricci relational coefficients are

$$
\begin{aligned}
\hat{\Omega}^m{}_{np} &= -f^\alpha{}_{\beta\gamma}\,\Phi^m_\alpha\Phi^\beta_n\Phi^\gamma_p, \\
\hat{\Omega}^m{}_{an} &= \Phi^\alpha_n\mathcal{D}_a\Phi^m_\alpha, \\
\hat{\Omega}^m{}_{ab} &= -\Phi^m_\alpha F^\alpha_{ab}, \\
\hat{\Omega}^a{}_{bc} &= \Omega^a{}_{bc},
\end{aligned}
\tag{C.16}
$$

where we have the following covariant generalizations of the toroidal case:

$$\begin{aligned} f^{\gamma}{}_{\alpha\beta} &= -(U^{-1})^{\delta}{}_{\alpha}(U^{-1})^{\epsilon}{}_{\beta}(\partial_\delta U^{\gamma}_{\epsilon} - \partial_\epsilon U^{\gamma}_{\delta}) \\ \mathcal{D}_a \Phi^m_\alpha &= \partial_a \Phi^m_\alpha + f^{\gamma}{}_{\alpha\beta} V^{\beta}_a \Phi^m_\gamma \\ F^{\alpha}_{ab} &= \partial_a V^{\alpha}_b - \partial_b V^{\alpha}_a + f^{\alpha}{}_{\beta\gamma} V^{\beta}_a V^{\gamma}_b . \end{aligned}$$

The spin connections are

$$\begin{aligned} \hat{\omega}_{mnp} &= \tfrac{1}{2} f^{\alpha}{}_{\beta\gamma} \left(\Phi_{\alpha m} \Phi^{\beta}_n \Phi^{\gamma}_p - \Phi_{\alpha n} \Phi^{\beta}_p \Phi^{\gamma}_m - \Phi_{\alpha p} \Phi^{\beta}_m \Phi^{\gamma}_n \right), \\ \hat{\omega}_{amn} &= \tfrac{1}{2} \left(\Phi^{\alpha}_m \mathcal{D}_a \Phi_{\alpha n} - \Phi^{\alpha}_n \mathcal{D}_a \Phi_{\alpha m} \right), \\ \hat{\omega}_{man} &= -\tfrac{1}{2} \Phi^{\alpha}_m \Phi^{\beta}_n \mathcal{D}_a G_{\alpha\beta}, \\ \hat{\omega}_{abm} &= -\hat{\omega}_{mab} \\ &= -\tfrac{1}{2} \Phi_{m\alpha} F^{\alpha}_{ab}, \\ \hat{\omega}_{abc} &= \omega_{abc}. \end{aligned} \tag{C.17}$$

The $D+n$ dimensional split of the Ricci scalar looks of course the same as in the toroidal case, and is given by (C.12). The only differences are the modifications of the field strengths and the appareance of a scalar potential. This potential arises from the extra term $\hat{\omega}_{mnp}\hat{\omega}^{pmn}$ in (C.12). The reduced Ricci scalar reads

$$\begin{aligned} \hat{R} = R &- \tfrac{1}{4} G_{\alpha\beta} F^{\alpha} F^{\beta} + \tfrac{1}{4} \operatorname{Tr} \mathcal{D}_a G \mathcal{D}^a G^{-1} \\ &- \nabla_a \operatorname{Tr} G^{-1} \mathcal{D}^a G - \tfrac{1}{4} \left(\operatorname{Tr} G^{-1} \mathcal{D} G \right)^2 \\ &- \tfrac{1}{4} \left(2 f^{\alpha}{}_{\beta\gamma} f^{\beta}{}_{\alpha\gamma'} G^{\gamma\gamma'} + f^{\alpha}{}_{\beta\gamma} f^{\alpha'}{}_{\beta'\gamma'} G_{\alpha\alpha'} G^{\beta\beta'} G^{\gamma\gamma'} \right). \end{aligned} \tag{C.18}$$

Note that the previous results for the torus and the circle can be obtained from this one in the respective limits

$$\begin{aligned} f^{\alpha}{}_{\beta\gamma} &\longrightarrow 0 \qquad \text{(torus and circle)}, \\ G_{\alpha\beta} &\longrightarrow \mathrm{e}^{2\varphi} \qquad \text{(circle)}. \end{aligned} \tag{C.19}$$

From now on we will only consider reductions over group manifolds; use the limits above if you're interested in the corresponding results for the torus or the circle.

C.2 Metric-dilaton system

So far we have only considered pure gravity, but in string theory the metric is always coupled to a dilaton. The $(D+n)$ dimensional action in question is given by

$$\hat{S} = \int \mathrm{d}x^{D+n} \; \hat{e} \; \mathrm{e}^{-\hat{\phi}} \left(\hat{R} + (\hat{\partial}\hat{\phi})^2 \right). \tag{C.20}$$

The vielbein determinant reduces as

$$\begin{aligned}\hat{e} &= \det \hat{e}_{\hat{\mu}}^{\hat{a}} \\ &= \det e_{\mu}^{a} \det \Phi_{\alpha}^{m} \\ &= e\sqrt{\det G}.\end{aligned} \tag{C.21}$$

The out of place factor of $\sqrt{\det G}$ as well as the non-standard terms for G coming out of the Ricci scalar can be removed by a convenient Ansatz for the string dilaton. It is given by

$$\hat{\phi} \overset{!}{=} \phi + \frac{1}{2} \ln \det G, \tag{C.22}$$

making it independent of the internal dimensions. Indeed

$$\hat{e}\ \mathrm{e}^{-\hat{\phi}} = e\ \mathrm{e}^{-\phi}. \tag{C.23}$$

Furthermore we see that

$$\begin{aligned}(\hat{\partial}\hat{\phi})^2 &= \left(\partial\phi + \tfrac{1}{2}\frac{\mathcal{D}(\det G)}{\det G}\right)^2 \\ &= \left(\partial\phi + \tfrac{1}{2}\operatorname{Tr} G^{-1}\mathcal{D}G\right)^2 \\ &= (\partial\phi)^2 + \operatorname{Tr} G^{-1}\mathcal{D}G\partial\phi + \tfrac{1}{4}(\operatorname{Tr} G^{-1}\mathcal{D}G)^2\end{aligned} \tag{C.24}$$

The second term combines with another term from the Ricci scalar into a total derivative (which will be dropped), while the last term cancels against another term. Combining this with (C.18), we find for (C.20)

$$\hat{S} = \int \mathrm{d}x^{D+n} e\ \mathrm{e}^{-\phi}\Big(R + \partial_a\phi\partial^a\phi - \tfrac{1}{4}G_{\alpha\beta}F^{\alpha}_{ab}F^{\beta ab} + \tfrac{1}{4}\mathcal{D}_a G_{\alpha\beta}\mathcal{D}^a G^{\alpha\beta} - V\Big), \tag{C.25}$$

with the potential V given by

$$V = \tfrac{1}{2} f^{\alpha}{}_{\beta\gamma} f^{\beta}{}_{\alpha\gamma'} G^{\gamma\gamma'} + \tfrac{1}{4} f^{\alpha}{}_{\beta\gamma} f^{\alpha'}{}_{\beta'\gamma'} G_{\alpha\alpha'} G^{\beta\beta'} G^{\gamma\gamma'}. \tag{C.26}$$

C.3 Differential forms

Here we will see how we can reduce differential forms and the operators acting on them over a group manifold. The reductions will be carried out in a flat basis, as this ensures that the resulting lower-dimensional forms do not transform under higher-dimensional GCTs.

The reduction Ansatz is given by

$$\hat{A}_{a_1\cdots a_j m_1\cdots m_{p-j}}(x,z) \overset{!}{=} A_{a_1\cdots a_j m_1\cdots m_{p-j}}(x) \tag{C.27}$$

The $D+n$ dimensional decomposition of differential forms is straightforward when working with indices. Let's first write a p-form in terms of its components and basis, and then make the dimensional split:

$$\begin{aligned}\hat{A}^{(p)} &= \frac{1}{p!}\hat{A}_{\hat{a}_1\cdots\hat{a}_p}\hat{e}^{\hat{a}_1}\wedge\cdots\wedge\hat{e}^{\hat{a}_p}\\ &= \frac{1}{p!}\sum_{j=0}^{p}\binom{p}{j}\hat{A}_{a_1\cdots a_j m_1\cdots m_{p-j}}\hat{e}^{a_1}\wedge\cdots\wedge\hat{e}^{a_j}\wedge\hat{e}^{m_1}\wedge\cdots\wedge\hat{e}^{m_{p-j}}\\ &= \sum_{j=0}^{p}\frac{1}{(p-j)!}A^{(j)}_{\alpha_1\cdots\alpha_{p-j}}\wedge\hat{e}^{\alpha_1}\wedge\cdots\wedge\hat{e}^{\alpha_{p-j}}.\end{aligned} \tag{C.28}$$

Here we use the shorthand notation

$$A^{(j)}_{\alpha_1\cdots\alpha_{p-j}} = \frac{1}{j!}A_{a_1\cdots a_j\alpha_1\cdots\alpha_{p-j}}e^{a_1}\wedge\cdots\wedge e^{a_j}, \tag{C.29}$$

which is a lower-dimensional j-form with $p-j$ antisymmetric internal indices. Take also notice of the somewhat sloppy notation

$$\hat{e}^{\alpha} = \sigma^{\alpha} - V^{\alpha} = \Phi^{\alpha}_{m}\hat{e}^{m}. \tag{C.30}$$

C.3.1 The exterior derivative

First we will derive a most useful identity. Recall that the reduction Ansatz in terms of the flat basis is

$$\begin{aligned}\hat{e}^{a} &= e^{a},\\ \hat{e}^{m} &= \Phi^{m}_{\alpha}(\sigma^{\alpha} - V^{\alpha}).\end{aligned}$$

Furthermore we have the Maurer-Cartan equation:

$$\mathrm{d}\sigma^{\alpha} = -\tfrac{1}{2}f^{\alpha}{}_{\beta\gamma}\sigma^{\beta}\wedge\sigma^{\gamma}. \tag{C.31}$$

And so we have

$$\begin{aligned}\mathrm{d}\hat{e}^{\alpha} &= \mathrm{d}\sigma^{\alpha} - \mathrm{d}V^{\alpha}\\ &= -\tfrac{1}{2}f^{\alpha}{}_{\beta\gamma}\sigma^{\beta}\wedge\sigma^{\gamma} + \tfrac{1}{2}f^{\alpha}{}_{\beta\gamma}V^{\beta}\wedge V^{\gamma} - F^{\alpha},\end{aligned} \tag{C.32}$$

which leads to the identity we are looking for:

$$\mathrm{d}\hat{e}^{\alpha} = -F^{\alpha} - f^{\alpha}{}_{\beta\gamma}V^{\beta}\wedge\hat{e}^{\gamma} - \frac{1}{2}f^{\alpha}{}_{\beta\gamma}\hat{e}^{\beta}\wedge\hat{e}^{\gamma} \tag{C.33}$$

Now we can proceed to reduce the exterior derivate. The decomposition of a field strength $\hat{H} = \mathrm{d}\hat{B}$ is

$$\hat{H}^{(p+1)} = \sum_{j=-1}^{p}\frac{1}{(p-j)!}H^{(j+1)}_{\alpha_1\cdots\alpha_{p-j}}\wedge\hat{e}^{\alpha_1}\wedge\cdots\wedge\hat{e}^{\alpha_{p-j}}. \tag{C.34}$$

But this is, by definition, the same as the decomposition of $\mathrm{d}\hat{B}$, which yields

$$\begin{aligned}\mathrm{d}\hat{B}^{(p)} =& \sum_{j=0}^{p} \frac{1}{(p-j)!} \mathrm{d}B^{(j)}_{\alpha_1\cdots\alpha_{p-j}} \wedge \hat{e}^{\alpha_1} \wedge \cdots \wedge \hat{e}^{\alpha_{p-j}} \\ &+ \sum_{j=0}^{p-1} \frac{(-1)^j}{(p-j-1)!} B^{(j)}_{\alpha_1\cdots\alpha_{p-j}} \wedge \mathrm{d}\hat{e}^{\alpha_1} \wedge \hat{e}^{\alpha_2} \wedge \cdots \wedge \hat{e}^{\alpha_{p-j}}\end{aligned} \tag{C.35}$$

Upon plugging (C.33) into the equation above, we see that we get four contributions to the second term. The one with $f^{\alpha}{}_{\beta\gamma} V^{\beta} \wedge \hat{e}^{\gamma}$ combines with the first term into a covariant exterior derivative, for we have:

$$\mathcal{D}B^{(j)}_{\alpha_1\cdots\alpha_{p-j}} = \mathrm{d}B^{(j)}_{\alpha_1\cdots\alpha_{p-j}} - (p-j) f^{\gamma}{}_{\beta[\alpha_1|} V^{\beta} \wedge B^{(j)}_{\gamma|\alpha_2\cdots\alpha_{p-j}]}. \tag{C.36}$$

To cast the remaining three terms into the same form as $\mathcal{D}V$, we reshuffle the summation- and dummy indices. After doing so we can instantly read of the components of the reduced field strength:

$$\begin{aligned}H^{(j+1)}_{\alpha_1\cdots\alpha_{p-j}} =& \underbrace{\mathcal{D}B^{(j)}_{\alpha_1\cdots\alpha_{p-j}}}_{\text{for } 0\leq j\leq p} + \underbrace{(-1)^p B^{(j-1)}_{\alpha_1\cdots\alpha_{p-j}\beta} \wedge F^{\beta}}_{\text{for } 1\leq j\leq p} \\ &+ \underbrace{\frac{1}{2}(-1)^p (p-j)(p-j-1) f^{\beta}{}_{[\alpha_1\alpha_2} B^{(j+1)}_{\alpha_3\cdots\alpha_{p-j}]\beta}}_{\text{for } -1\leq j\leq p-2}\end{aligned} \tag{C.37}$$

C.3.2 The inner product

Given the inner product of any two higher-dimensional p-forms,

$$\left(\hat{A}^{(p)}, \hat{B}^{(p)}\right) = \int \hat{A}^{(p)} \wedge \star\hat{B}^{(p)}, \tag{C.38}$$

we would like to know how it reduces in terms of its lower-dimensional components. This is not quite trivial in form notation, so let's switch to and from flat indices:

$$\begin{aligned}\hat{A}^{(p)} \wedge \star\hat{B}^{(p)} =& \frac{1}{p!}\ \hat{A}_{\hat{a}_1\cdots\hat{a}_p} \hat{B}^{\hat{a}_1\cdots\hat{a}_p} \hat{\star} 1 \\ =& \frac{1}{p!}\ \star 1\ \Omega_z \sum_{j=0}^{p} \left(\binom{p}{j} \prod_{k=1}^{j} \eta^{a_k a'_k} \prod_{l=j+1}^{p} \delta^{m_l m'_l} \times \right. \\ & \left. \times\ \hat{A}_{a_1\cdots a_j m_{j+1}\cdots m_p} \hat{B}_{a'_1\cdots a'_j m'_{j+1}\cdots m'_p} \right) \\ =& \Omega_z \sum_{j=0}^{p} \frac{1}{(p-j)!} A^{(j)}_{\alpha_1\cdots\alpha_{p-j}} \wedge \star B^{(j)\alpha_1\cdots\alpha_{p-j}},\end{aligned} \tag{C.39}$$

where Ω_z is the volume invariant form of the internal space. Furthermore we have used the shorthand notation

$$B^{(j)\alpha_1\cdots\alpha_{p-j}} = \left(\prod_{k=1}^{p-j} G^{\alpha_k\alpha'_k}\right) B^{(j)}_{\alpha'_1\cdots\alpha'_{p-j}}. \tag{C.40}$$

Integrating over the internal volume form, the inner product becomes

$$\left(\hat{A}^{(p)}, \hat{B}^{(p)}\right) = V_z \int \sum_{j=0}^{p} \frac{1}{(p-j)!} A^{(j)}_{\alpha_1\cdots\alpha_{p-j}} \wedge \star B^{(j)\alpha_1\cdots\alpha_{p-j}}, \tag{C.41}$$

where V_z is the volume of the internal space.

We have not yet considered the fact that one cannot antisymmetrize over more than D spacetime and n internal indices in the reduced theory. This will set some of the field strengths equal to zero. This can be expressed as an extra condition on the sum, for we have

$$\left.\begin{array}{rl} p-j & \leq n \\ j & \leq D \end{array}\right\} \rightarrow p-n \leq j \leq D. \tag{C.42}$$

The sum in the inner product can be rewritten as

$$\left(\hat{A}^{(p)}, \hat{B}^{(p)}\right) = V_z \int \sum_{j=\lceil 0,p-n\rceil}^{\lfloor p,D\rfloor} \frac{1}{(p-j)!} A^{(j)}_{\alpha_1\cdots\alpha_{p-j}} \wedge \star B^{(j)\alpha_1\cdots\alpha_{p-j}}, \tag{C.43}$$

where $\lceil x, y\rceil = \max(x, y)$ and $\lfloor x, y\rfloor = \min(x, y)$.

C.3.3 The Hodge dual operator

The higher-dimensional Hodge dual operator is defined as

$$\begin{aligned} \hat{\star}\hat{A}^{(p)} &= \frac{1}{p!}\hat{A}_{\hat{a}_1\cdots\hat{a}_p}\hat{\star}\left(\hat{e}^{\hat{a}_1} \wedge \cdots \wedge \hat{e}^{\hat{a}_p}\right) \\ &= \frac{1}{p!}\sum_{j=0}^{p}\binom{p}{j}\hat{A}_{a_1\cdots a_j m_1\cdots m_{p_j}}\hat{\star}\left(\hat{e}^{a_1} \wedge \cdots \wedge \hat{e}^{a_j} \wedge \hat{e}^{m_1} \cdots \wedge \hat{e}^{m_{p-j}}\right). \end{aligned} \tag{C.44}$$

It acts on the mixed vielbein basis as

$$
\begin{aligned}
&\hat{\star}\big(\hat{e}^{a_1}\wedge\cdots\wedge\hat{e}^{a_j}\wedge\hat{e}^{m_1}\cdots\wedge\hat{e}^{m_{p-j}}\big) = \\
&\quad= \frac{1}{(D+n-p)!}\hat{\varepsilon}^{a_1\cdots a_j m_1\cdots m_{p-j}}{}_{\hat{b}_1\cdots\hat{b}_{D+n-p}}\,\hat{e}^{\hat{b}_1}\wedge\cdots\wedge\hat{e}^{\hat{b}_{D+n-p}} \\
&\quad= \frac{1}{(D+n-p)!}\sum_{i=0}^{D+n-p}\binom{D+n-p}{i}\hat{\varepsilon}^{a_1\cdots a_j m_1\cdots m_{p-j}}{}_{b_1\cdots b_i l_1\cdots l_{D+n-p-i}}\times \\
&\qquad\qquad\times\,\hat{e}^{b_1}\wedge\cdots\wedge\hat{e}^{b_i}\wedge\hat{e}^{l_1}\wedge\cdots\wedge\hat{e}^{l_{D+n-p-i}} \\
&\quad= \frac{1}{(n-p+j)!(D-j)!}\hat{\varepsilon}^{a_1\cdots a_j m_1\cdots m_{p-j}}{}_{b_1\cdots b_{D-j} l_1\cdots l_{n-p+j}}\times \\
&\qquad\qquad\times\,\hat{e}^{b_1}\wedge\cdots\wedge\hat{e}^{b_{D-j}}\wedge\hat{e}^{l_1}\wedge\cdots\wedge\hat{e}^{l_{n-p+j}}.
\end{aligned}
\tag{C.45}
$$

In the last line we used the fact that you cannot antisymmetrize over more than D spacetime and n internal indices. This condition picks out the $i = D - j$ contribution from the sum. In the expression for $\hat{\star}\hat{A}^{(p)}$ we can split the Levi-Civita symbol in two, and use the spacetime part for the lower-dimensional Hodge dual. The rather intimidating looking result is

$$
\hat{\star}\hat{A}^{(p)} = \sum_{j=0}^{p}\frac{(-1)^{(D-j)(p-j)}}{(n-p+j)!(p-j)!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{p-j}\beta_1\cdots\beta_{n-p+j}}\star A^{(j)\alpha_1\cdots\alpha_{p-j}}\wedge\hat{e}^{\beta_1}\wedge\cdots\wedge\hat{e}^{\beta_{n-p+j}}.
\tag{C.46}
$$

If we define $\hat{B}^{(D+n-p)} \equiv \hat{\star}\hat{A}^{(p)}$ and reduce B, we obtain

$$
\hat{B}^{(D+n-p)} = \sum_{j=p-n}^{D}\frac{1}{(n-p+j)!}B^{(D-j)}_{\alpha_1\cdots\alpha_{n-p+j}}\wedge e^{\alpha_1}\wedge\cdots\wedge e^{\alpha_{n-p+j}}.
\tag{C.47}
$$

Comparing terms, we find

$$
B^{(D-j)}_{\alpha_1\cdots\alpha_{n-p+j}} = \frac{(-1)^{(D+n-p)(p-j)}}{(p-j)!}\tilde{\varepsilon}_{\alpha_1\cdots\alpha_{n-p+j}\beta_1\cdots\beta_{p-j}}\star A^{(j)\beta_1\cdots\beta_{p-j}}
\tag{C.48}
$$

Appendix D

Gravity on a circle, old-school style

This chapter should be a good pointer as to why one wants to use the Einstein-Cartan formalism instead of the metric formalism when doing dimensional reductions. A quick look at the wretched-looking components of the Christoffel symbol and the Ricci scalar below will probably be enough; even over the mere circle this method yields far more work than the reduction over an arbitrary group manifold.

It may be noticed that the reduction Ansatz is a bit different that the one used elsewhere in this thesis. This is because the system considered here is pure gravity, without a string dilaton. Furthermore the sign of the Kaluza-Klein vector V doesn't really matter when reducing over the circle, and will positive here. The coefficient α (to be introduced in a moment) will be used for an immediate Weyl rescaling in $(D+1)$ dimensions, and the coefficient β for getting a canonical kinetic term for the Kaluza-Klein scalar.

We start from Einstein gravity in $(D+1)$ dimensions, described by the Einstein-Hilbert action

$$\hat{S} = \frac{1}{2k_{D+1}^2} \int \mathrm{d}\hat{x} \hat{\mathcal{L}}, \tag{D.1}$$

with

$$\hat{\mathcal{L}} = \sqrt{-\hat{g}} \hat{R}. \tag{D.2}$$

We will take the the $(D+1)$ dimensional line element as

$$\hat{\mathrm{d}s}^2 = \mathrm{e}^{2\alpha\phi} \mathrm{d}s^2 + \mathrm{e}^{2\beta\phi} (\mathrm{d}z + V)^2, \tag{D.3}$$

where $V = V_\mu \mathrm{d}x^\mu$ and $\mathrm{d}z = \mathrm{d}x^z$. V_μ and ϕ are respectively the Kaluza-Klein vector and scalar, and α and β constants still to be chosen. We instantly read off the values of the metric, and using $\hat{g}_{\hat{\mu}\hat{\rho}} \hat{g}^{\hat{\rho}\hat{\nu}} = \delta_{\hat{\mu}}^{\hat{\nu}}$ we can also determine its inverse. They are respectively given by

$$\hat{g}_{\mu\nu} = \mathrm{e}^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} V_\mu V_\nu, \quad \hat{g}_{\mu z} = \mathrm{e}^{2\beta\phi} V_\mu, \quad \hat{g}_{zz} = \mathrm{e}^{2\beta\phi}, \tag{D.4}$$

$$\hat{g}^{\mu\nu} = \mathrm{e}^{-2\alpha\phi} g^{\mu\nu}, \quad \hat{g}^{\mu z} = -\mathrm{e}^{-2\alpha\phi} V^\mu, \quad \hat{g}^{zz} = \mathrm{e}^{-2\beta\phi} + e^{-2\alpha\phi} V^2. \tag{D.5}$$

We can fairly easy see that the $D+1$ dimensional metric determinant nicely relates to the D dimensional metric determinant by

$$\hat{g} \equiv \det(\hat{g}_{\hat{\mu}\hat{\nu}}) = \mathrm{e}^{2(\beta + D\alpha)\phi} \det(g_{\mu\nu}) = \mathrm{e}^{2(\beta+D\alpha)\phi} g. \tag{D.6}$$

Now that easier part of the Einstein-Hilbert Langrangian is accounted for, we will turn to the Ricci-scalar. The first step is to decompose it in $(D+1)$ dimensions by

$$\hat{R} = \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} + 2\hat{g}^{\mu z} \hat{R}_{\mu z} + \hat{g}^{zz} \hat{R}_{zz}. \tag{D.7}$$

Recalling that the Ricci-scalar is constructed in a non-trivial manner from first derivatives and squares of Christoffel symbols, and that the Christoffel symbols on their turn are constructed from first derivatives and squares of the metric, you can probably guess that reducing $\hat{R}$ is a bit more tricky than reducing $\hat{g}$. Still, before doing so, we need to decompose the Christoffel symbol the same way we did the Ricci-tensor. Given

$$\hat{\Gamma}^{\hat{\mu}}_{\hat{\nu}\hat{\rho}} = \tfrac{1}{2} \hat{g}^{\hat{\mu}\hat{\sigma}} (\partial_{\hat{\nu}} \hat{g}_{\hat{\sigma}\hat{\rho}} + \partial_{\hat{\rho}} \hat{g}_{\hat{\sigma}\hat{\nu}} - \partial_{\hat{\sigma}} \hat{g}_{\hat{\nu}\hat{\rho}}) \tag{D.8}$$

we can calculate the $D+1$ split for the Christoffel symbol. It is given by

$$\begin{aligned}
\hat{\Gamma}^z_{zz} &= \beta \mathrm{e}^{2(\beta-\alpha)\phi} V \cdot \partial\phi \\
\hat{\Gamma}^\mu_{zz} &= -\beta \mathrm{e}^{2(\beta-\alpha)\phi} \partial^\mu \phi \\
\hat{\Gamma}^z_{\mu z} &= \tfrac{1}{2} \mathrm{e}^{2(\beta-\alpha)\phi} V^\sigma F_{\sigma\mu} + \beta \mathrm{e}^{2(\beta-\alpha)\phi} V_\mu V \cdot \partial\phi + \beta \partial_\mu \phi \\
\hat{\Gamma}^\mu_{\nu z} &= \tfrac{1}{2} \mathrm{e}^{2(\beta-\alpha)\phi} F_\nu{}^\mu - \beta \mathrm{e}^{2(\beta-\alpha)\phi} V_\nu \partial^\mu \phi \\
\hat{\Gamma}^z_{\mu\nu} &= -V_\sigma \Gamma^\sigma_{\mu\nu} + \tfrac{1}{2} \mathrm{e}^{2(\beta-\alpha)\phi} V^\sigma (F_{\sigma\mu} V_\nu + F_{\sigma\nu} V_\mu) \\
&\quad + \beta \mathrm{e}^{2(\beta-\alpha)\phi} V_\mu V_\nu V \cdot \partial\phi + (\beta - \alpha)(V_\mu \partial_\nu \phi + V_\nu \partial_\mu \phi) \\
&\quad + \tfrac{1}{2}(\partial_\mu V_\nu + \partial_\nu V_\mu) + \alpha g_{\mu\nu} V \cdot \partial\phi \\
\hat{\Gamma}^\mu_{\nu\rho} &= \Gamma^\mu_{\nu\rho} + \alpha(\partial_\nu \phi \delta^\mu_\rho + \partial_\rho \phi \delta^\mu_\nu - \partial^\mu \phi g_{\nu\rho}) \\
&\quad + \tfrac{1}{2} \mathrm{e}^{2(\beta-\alpha)\phi} (F_\nu{}^\mu V_\rho + F_\rho{}^\mu V_\nu) - \beta V_\nu V_\rho \partial^\mu \phi,
\end{aligned} \tag{D.9}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. Notice that

$$\begin{aligned}
\hat{\Gamma}^{\hat{\mu}}_{z\hat{\mu}} &= \hat{\Gamma}^\mu_{z\mu} + \hat{\Gamma}^z_{zz} = 0, \\
\hat{\Gamma}^{\hat{\mu}}_{\nu\hat{\mu}} &= \hat{\Gamma}^\mu_{\nu\mu} + \hat{\Gamma}^z_{\nu z} = \Gamma^\mu_{\nu\mu} + (\beta + D\alpha)\partial_\nu \phi.
\end{aligned} \tag{D.10}$$

This can be used to simplify the expression for tensor, vector and scalar components of the Ricci-scalar a bit. They can be calculated through the use of

$$\hat{R} = \hat{g}^{\hat{\mu}\hat{\nu}} (\partial_{\hat{\rho}} \hat{\Gamma}^{\hat{\rho}}_{\hat{\mu}\hat{\nu}} - \partial_{\hat{\nu}} \hat{\Gamma}^{\hat{\rho}}_{\hat{\mu}\hat{\rho}} + \hat{\Gamma}^{\hat{\sigma}}_{\hat{\mu}\hat{\nu}} \hat{\Gamma}^{\hat{\rho}}_{\hat{\sigma}\hat{\rho}} - \hat{\Gamma}^{\hat{\sigma}}_{\hat{\mu}\hat{\rho}} \hat{\Gamma}^{\hat{\rho}}_{\hat{\sigma}\hat{\nu}}) \tag{D.11}$$

The actual calculation remains rather lengthy, but straightforward nonetheless. In the end one finds that

$$
\begin{aligned}
\hat{R}_{zz} &= \mathrm{e}^{2(\beta-\alpha)\phi}\Big(\tfrac{1}{4}\mathrm{e}^{2(\beta-\alpha)\phi}F^2 - \beta\Box\phi - \beta\big(\beta+\alpha(D-2)\big)(\partial\phi)^2\Big)\\
\hat{R}_{\mu z} &= V_\mu\hat{R}_{zz} + \tfrac{1}{2}\mathrm{e}^{2(\beta-\alpha)\phi}\Big(\nabla_\sigma F_\mu{}^\sigma + \big(3\beta+\alpha(D-4)\big)\partial_\sigma\phi F_\mu{}^\sigma\Big)\\
\hat{R}_{\mu\nu} &= R_{\mu\nu} + V_\mu\hat{R}_{\nu z} + V_\nu\hat{R}_{\mu z} - V_\mu V_\nu\hat{R}_{zz} - \tfrac{1}{2}\mathrm{e}^{2(\beta-\alpha)\phi}F_{\mu\sigma}F_\nu{}^\sigma\\
&\quad - \big(\beta+\alpha(D-2)\big)\nabla_\mu\partial_\nu\phi - \alpha\Box\phi g_{\mu\nu}\\
&\quad - \big(\beta^2-\alpha(2\beta+\alpha D-2\alpha)\big)\partial_\mu\phi\partial_\nu\phi\\
&\quad - \alpha(\beta+\alpha D-2\alpha)(\partial\phi)^2 g_{\mu\nu},
\end{aligned}
\tag{D.12}
$$

where $\Box\phi = \nabla_\mu\partial^\mu\phi$, $F^2 = g^{\mu\gamma}g^{\nu\sigma}F_{\mu\nu}F_{\gamma\sigma}$ and $(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.
Now we are ready to plug this all into the $D+1$ dimensional Lagrangian. It becomes

$$
\begin{aligned}
\hat{\mathcal{L}} = \mathrm{e}^{\big(\beta-\alpha(2-D)\big)\phi}\sqrt{-g}\Big(&R - 2\big(\beta+\alpha(1-D)\big)\Box\phi\\
&- \Big(2\beta\big(\beta+\alpha(D-2)\big) + \alpha^2(D-1)(D-2)\Big)(\partial\phi)^2\\
&- \tfrac{1}{4}\mathrm{e}^{2(\beta-\alpha)\phi}F^2\Big).
\end{aligned}
\tag{D.13}
$$

We would like to have ended up with a Lagrangian of the Einstein-Hilbert form, that is, $\hat{\mathcal{L}} = \sqrt{-g}R + \cdots$. To achieve this we have to set $\beta = (2-D)\alpha$. Furthermore, we would like the Kaluza-Klein scalar ϕ to have a kinetic term with canonical normalization. This term would look like $-\frac{1}{2}\sqrt{-g}(\partial\phi)^2$. So we see that we have to choose the constants α and β as follows:

$$
\alpha^2 = \frac{1}{2(D-1)(D-2)}, \quad \beta = (2-D)\alpha. \tag{D.14}
$$

With this choice the reduced Lagrangian becomes

$$
\hat{\mathcal{L}} = \sqrt{-g}\left(R - 2\alpha\Box\phi - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}\mathrm{e}^{2(D-1)\alpha\phi}F^2\right). \tag{D.15}
$$

Since the $\Box\phi$ term only gives a total derivate in $\hat{\mathcal{L}}$, we can drop it. If we also perform the integration over the internal coordinate, the action becomes

$$
\hat{S} = \frac{1}{2k_D^2}\int dx\sqrt{-g}\left(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}\mathrm{e}^{2(D-1)\alpha\phi}F^2\right), \tag{D.16}
$$

where $k_D^2 = \frac{k_{D+1}^2}{\int dz}$.

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