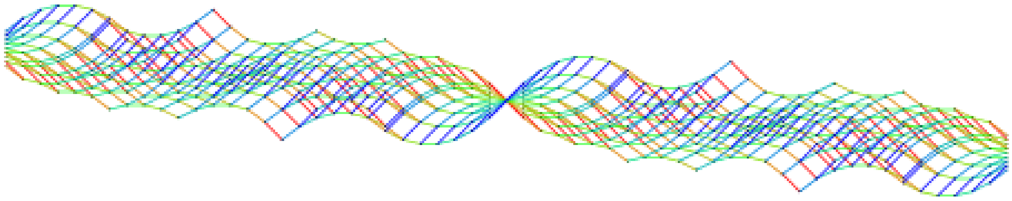


University of Groningen

Bachelor Thesis

Lie algebras and the transition to affine Lie algebras in 2 Dimensional Maximal Supergravity

Author:
Mark Boer

Supervisor:
A.V. Kiselev
D. Roest

July 14, 2013

Abstract

Finite dimensional simple and semi-simple Lie algebras will be categorized with the help of Hasse diagrams and Cartan matrices. These results will be used in the construction of a very specific Kac-Moody algebra: the affine Lie algebra. This infinite dimensional highly structured Lie algebra can be constructed using the generalized Cartan matrix. An affine Lie algebra is closely related to a semi-simple Lie algebra. The affine Lie algebra can be roughly be seen as an infinite tower of a semi-simple Lie algebra. This means that an affine Lie algebra can be constructed as the affine extension of a semi-simple Lie algebra.

Lie algebras appear in a slightly different manner in physics. They are closely related to symmetries. A close look at space-time symmetries and supersymmetry will result in a Super-Poincaré algebra. Supersymmetry can then be gauged to construct a supergravitational theory. Maximal supergravity is a supergravity theory with as many supersymmetry generators as physically possible. It can most easily be obtained by Kaluza-Klein dimensional reduction of eleven dimensional supergravity.

Keywords: affine, Lie algebra, supersymmetry, maximal, supergravity, exeptional group, , E_8, E_8^+, E_9

Contents

1 Introduction

Symmetries have always been a large part of physics. It is the similarities between observations that have allowed physicist to discover the underlying laws of nature. A famous example is Galileo's experiment where he dropped balls of different masses. Every ball would hit the ground at the same time thus confirming that the acceleration is invariant under a change of mass. If one would consider just this system, this change of mass is considered to be a symmetry.
Other theories such as relativity (both special and general) are based around a symmetry. This symmetry for special relativity being that the speed of light is independent of the inertial reference frame and that the laws of physics stay identical. The symmetry is the corner stone of this theory. As one can see from these examples physics allows for a wide range of symmetries, symmetries one would normally not think of as symmetries. In physics a symmetry is nothing more than the invariance of a system under a certain action. This action being the transition from one inertial from to another in the case of special relativity
Many conservation laws such as: momentum, angular momentum, center of mass, electric charge etc. are essentially symmetries. This follows by Noether's theorem [8], which states that every continuous symmetry has a corresponding conserved current. Integrated over all space this of course leads to a conserved charge. The electric charge is a famous example, but from this point of view all these conserved quantities are charges.
Symmetries are closely related to Lie algebras; the Lie algebra is the tangent space of a Lie group around the identity element. A Lie group is a group of continuous transformation which is used to describe a symmetry. Lie algebras can also be constructed on their own. They are basically vector spaces with an additional anticommutative binary operation $[.,.]$ [1]. Kac-Moody algebras are Lie algebras that are constructed through a generalized Cartan matrix. Strict restrictions on this Cartan matrix lead to finite dimensional (semi-) simple Lie algebras, while slightly loosening this restrictions give way to affine Lie algebras; a type of infinite dimensional Lie algebras.
Supersymmetry is an additional symmetry, which relates bosons and fermions. This symmetry has a few great properties. It solves some issues concerning the hierarchy problem and gives a candidate for dark matter. Much more interesting for our story though, is that it allows the unification of internal and spatial symmetries. The Coleman-Mandula theorem [23] states that it is impossible to combine spatial and internal symmetries in any but the trivial way (setting them to zero). Supersymmetry is not an internal symmetry, but it isn't a spatial symmetry in the conventional way either. The Lie algebra corresponding to supersymmetry is not a regular Lie algebra, but it is a graded type of algebra, which contains commutation and anti-commutation relations. A theorem is only as strong as its

assumptions and one of these assumptions is that it concerns a regular Lie algebra. Supersymmetry therefore bypasses this theorem.
The unification of spatial and internal symmetries looks promising. Unification has often proven to be a good method of constructing a theorem. The electroweak interaction (unification of weak and electromagnetic) is, among other things, used to construct the current version of the Standard Model [4]. The Standard Model (SM) is a theory of how the fundamental particles interact. It has been constructed in the early 1970's and it has successfully explained almost all experimental results. One of the problems of the SM is that only three of the four fundamental interactions have been included. Gravity is an extremely weak force in comparison to the other three (weak, strong and electromagnetic) and is not included in the SM. [24]
On the other hand there is General Relativity (GR), a theory which stands for nearly a hundred years [25]. This theory is an improvement of classical gravity and is consistent with the experimental data. This theory is however still not a quantum theory. It only works at a large enough scale. The big question that remains is how to unite the SM and GR into a quantum theory of gravity. This theory would be needed to explain the very small and very heavy.
The difficulty in this arises in the renormalization of the quantum gravity theory. These theories are known for being notoriously non-renormalizable, this means that the theory introduces infinities which cannot be worked around. [26] Supersymmetry might improve the quantum properties of such a theory and it indeed delays the divergences that arise. This is a result that some terms cancel out with respect to their superpartners. The maximally supersymmetric quantum gravity theory might be finite, thus making it a true quantum gravity theory. [26]
Another interesting property of maximal supergravity is that it can be obtained by gauging supersymmetry. This imposition of local symmetry results in additional fields; gauge fields [10]. The three forces in the SM can be obtained in exactly the same way. [4] Roughly speaking gauging the $U(1)$ symmetry results in the electromagnetic force, $SU(2)$ in the weak interaction and $SU(3)$ in the strong interaction.

Research

In this thesis Lie algebras will be studied from both a mathematical and physical point of view. We will study their basic properties and were especially interested in semisimple Lie algebras and their connection to the infinite dimensional affine Lie algebras. These are namely the type of Lie algebras that we will later encounter in the mathematical descriptions of physical systems.
The emphasis of this thesis will lay on the different symmetries that appear in maximal supergravity. First and foremost this requires a good understanding of Lie

algebras and how these are related to symmetries. Knowing that two dimensional supergravity shows an affine Lie algebra, this requires us to uncover the structure of the affine Lie algebra and how the affine Lie algebra can be constructed.
Some of the symmetries that appear in maximal supergravity are of the exceptional type (E_6, E_7, E_8). Then how is it that the Lie algebra E_9 is no longer a finite dimensional simple Lie algebra, but is in fact an affine Lie algebra also known as E_8^+.
Then we will need to focus on supergravity itself. The construction of maximal supergravity will take many steps in which symmetries play an important role. One of the big questions is of course where do the hidden symmetries come from. The symmetry differs in a different number of dimensions, how does this symmetry change and how can it be used to describe the physical system.
Finally in two dimensions an affine Lie algebra will appear. The physical construction of this symmetry differs quite bit from the mathematical construction. How does this symmetry appear and what are its physical implications?

Part I

Mathematics of Lie algebras

First the semisimple Lie algebra will be split up into root spaces. Every one of these root spaces correspond to a root. The set of these roots can be used to construct a semisimple Lie algebra, which is unique upto an isomorphism. A closer look at the root system will reveal that root system is very structured. The root system can then be identified by a Cartan matrix.
Having seen how to construct a Cartan matrix, we will study how Lie algebras can be constructed using the Cartan matrix. Finally we will get to the affine Lie algebras, these are infinite dimensional algebras with a very strict root system. They can either be obtained by loosening the bounds on the Cartan matrix or by considering a loop algebra. These loop algebras can be thought of as all the continuous maps from the S^1 sphere to the algebra.
Taking a close look at Lie algebras using the loop algebras and the Hasse diagram, a way of depicting the commutation relations. We find that the Lie algebra is an infinite tower of (semi-)simple Lie algebras.
Finally we can an element to the affine Lie algebras called the derivation. This element lifts the nondegeneracy of the Dynkin labels. It does add too much in the mathematical sens, but as it turns out these generators are related to a physical symmetry.

2 Introduction and conventions

We begin building our algebraic structures by considering a group. A group is a set of elements, with a single binary operation. Furthermore there are several group axioms that have to be satisfied. In short these are associativity, invertibility and the existence of an identity element. A common notation for a group is the tripel $(\mathcal{G}, +, 0)$ where $\mathcal{G}$ is the set of elements, $+$ the group operator and 0 the identity element. If a group furthermore is commutative it is called an abelian group $(a + b = b + a)$.
A field is an extension of an abelian group. In addition to the commutative operator there is a second binary operator and identity element (usually denoted by $*$ and 1). The notation for such a field is then a five-tuple $(F, +, *, 0, 1)$. Furthermore $1 \neq 0$ and $(F\backslash\{0\}, *, 1)$ also forms an abelian group. A Field is algebraically closed if every non constant polynomial has a root inside the field. $\mathbb{R}$ is not algebraically close while $\mathbb{C}$ is. The characteristic of a field is number (n) of times you have to add 1 to itself to get 0. In the case that there is no such n we define the characteristic to be 0. For an exact definition of groups and fields we would refer you to [13, 14]
A vector space (V) over a field (F) is a collection of vectors. It is not to be confused with a vector field. The following two binary operations are defined on a vector space: addition of vectors and multiplication of vectors by elements of the field F called scalars.
A lot of mathematically interesting structures can be build using vector spaces. Often this is done by introducing new operations. Inner product spaces and algebras are just two examples.

Example 1. *The group of elements $\{O \in \mathbb{R}^{n\times n} | O.O^T = I\}$ is called the orthogonal group. In our previous notation this would be a group of the form $(O, ., I)$, where the dot denotes regular matrix multiplication. Any two of these matrices multiplied forms a matrix that itself also is orthogonal. A group only has a single binary operation. This means it is not possible to add these orthogonal matrices.*
At the same time $\mathbb{R}^{n\times n}$ forms a vector space. A more interesting vector space that will come up a lot is the function space. This function space is a vector space (over a field) containing all functions from some set to the field. Addition and scalar multiplication goes just as one would expect. The sum of two functions is given by $(f + g)(x) = f(x) + g(x)$. The set (on which these functions work) can, but does not have to be the field. In fact, in the later stages of this thesis, we will encounter a function space called the superfield, whose elements are functions on a set called the superspace. This superspace contains ordinary and anticommuting elements.

3 Basic properties of Lie algebras

3.1 Definition of a Lie algebra

An algebra A over a field is a vector space over a field, with an additional bilinear operation $A \times A \to A$. In the case of a Lie algebra this bilinear operation is called the bracket $[,]$, which has some additional properties. A Lie algebra is of course an algebra, although the not any algebra is a Lie algebra. [1]

Definition 1. *A Lie algebra L is an algebra over a field with the binary operation $L \times L \to L, (x, y) \to [x, y]$. This operation is called the bracket or commutator. Furthermore the following axioms have to be satisfied:*

L1 The bracket is a bilinear operation

L2 $[x, x] = 0$ *for all* $x \in L$

L3 $[x, [yz]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $(x, y, z) \in L$

Axiom L1 together with axiom L2 implies that the bracket is anticommutative. $[x + y, x + y] = 0 = [x, y] + [y, x]$. If $\text{char}(F) \neq 2$ this is also true the other way around. Axiom L3 is called the jacobi identity.
A Lie subalgebra K is of course just a subspace of L such that K itself is a Lie algebra. Equivalently it is a subset of L, which is closed under scalar multiplication, addition and the bracket.
Often we will look at the set of linear transformations from a vector space V onto itself, called endomorphisms. This set (End V) quite naturally has n^2 dimensions ($n = \text{dim}V$). Now we define the bracket as $[x, y] = xy - yx | x, y \in \text{End(V)}$. It is easily checked that all three axioms hold. This algebra is called the general linear algebra and is denoted by $\mathfrak{gl}(V)$ to indicate its new algebraic structure. It is closely related to the general linear group. The general linear group only contains invertible matrices, whereas the general linear algebra contains all linear endomorphisms. Note that it is not even possible to construct a Lie algebra without the $\{0\}$ vector, the existence of the $\{0\}$ vector is one of the requirements for a vector space.
For the next sections we will only consider Lie algebras whose underlying vector space is finite dimensional and the field to be of characteristic 0 and algebraically closed, unless otherwise stated of course.

Example 2. *The Lie algebra that we will often encounter is the special linear algebra $\mathfrak{sl}(V)$. It is a subalgebra of $\mathfrak{gl}(V)$, containing all the endomorphisms with trace zero. Since $trace(xy) = trace(yx)$ it is easily seen that this forms a Lie algebra.*

The $\mathfrak{sl}(2,F)$ *has the basis* $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Their commutation relations are:* $[x,y] = h$, $[h,y] = -2y$ *and* $[h,x] = 2x$.

3.2 Homomorphism, Representations and L-modules

A homomorphism is a linear map that preserves the structure of the Lie algebra.

Definition 2. *A map* $\phi : L \to L'$ *is a homomorphism if* $\phi([x,y]) = [\phi(x), \phi(y)]$ *for all* x, y.

Clearly the map that sends every element to the zero-vector in the other Lie algebra is a homomorphism. This shows that a homomorphism not neccesarily preserve all of the information.
If a map is a homomorphism and bijective it is called an isomorphism. Two Lie algebras are isomorphic is there exists an isomorphism between the two.
A representation of a Lie algebra is a homomorphism $\phi : L \to \mathfrak{gl}(V)$. The dimensions don't have to coincide and the dimension of V can easily be infinite dimensional. An important representation is the adjoint representation, given by $ad_x : L \to \mathfrak{gl}(L)$ where $ad_x y = [x,y]$

$$[ad_x, ad_y](z) = [x,[y,z]] - [y,[x,z]] = [x,[y,z]] + [[x,z],y] = [[x,y],z] = ad_{[x,y]}(z)$$

which shows that this map is homomorphism and a representation.
A language that is equivalent to that of representations, but also very useful is the one of modules.

Definition 3. *A vector space* V *is an L-module if it is endowed with an operation* $L \times V \to V$, *which is denoted by* $(x,v) \to x.v$. *Furthermore this operation should be bilinear and* $[xy].v = x.(y.v) - y.(x.v)$.

For every representation $\phi \to \mathfrak{gl}(V)$, V may be viewed as an L-module, via the operation $x.v = \phi(x)v$. Vice versa, given an L-module, this equation defines a representation.

Example 3. *For the special linear algebra* $\mathfrak{sl}(V)$ *each element is already an endomorphism on the vector space* V. *This means that mapping each element to itself is in fact already a representation. Similar would be saying* V *is the* L *module. In case of* $\mathfrak{sl}(2,F)$ *this would be the two dimensional vector space* F^2.

Representations are often used in physics. Both representations of Lie algebras and group representation (its definition is nearly identical). Sometimes representations can be used to simplify notations, e.g. superspace.

3.3 Properties of Lie algebras

Abelian

A group $\{G, +, 0\}$ being abelian by definition means that $a+b = b+a$ for every a and b in G. However we've seen that for a Lie algebra by definition $[x, y] = -[y, x]$. A Lie algebra L is called abelian if $[x, y] = 0$ for every $x, y \in L$ or in other words $[L, L] = 0$. This definition makes a lot of sense with the usual bracket $[x, y] = xy - yx$.

Simple and Semi-simple

A subalgebra I of a Lie algebra L is called ideal if $x \in I, y \in L$ implies that $[x, y] \in I$.

Definition 4. *A Lie algebra is simple if it contains no ideals other than the zero vector and the entire algebra itself*

This automatically implies that if L is a simple Lie algebra $[L, L] = L$, because if not $[L, L]$ would be an ideal. The example in the previous chapter $\mathfrak{sl}(V)$ is a simple Lie algebra.
Let us now define a sequence of ideals of L called the derived series. $L^{(0)} = L,\ L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}] \cdots$ L is called solvable if $L^{(n)} = 0$ for some n. A Lie algebra is called semisimple if the only solvable ideal is $\{0\}$. This is not similar to stating that a Lie algebra can't have any ideals. The two statements are related, in fact, a Lie algebra is semisimple if and only if it is the direct sum of simple ideals.
A different yet useful way to define semisimplicity is through the Cartan Killing form. Let $x, y \in L$, then the Cartan Killing form is defined as $\kappa(x, y) = \text{trace}(ad_x ad_y)$. This is of course possible since the trace is independent of the basis chosen. Also the Cartan Killing form is symmetric and bilinear. A Lie algebra whose Cartan Killing form is nondegenerate is called semisimple. It is nondegenerate in the sense that the set $\{x \in L | \kappa(x, y) = 0,\ \forall y \in L\}$ has only a single component $\{0\}$. A simple check would be to consider the matrix with components $\kappa(x_i, x_j)$ where x_i forms a basis of the Lie algebra. Then κ is nondegenerate if and only if the determinant of this matrix is nonzero.
In conclusion, there are multiple equivalent ways of talking about semisimple Lie algebras. We will not proof that all these are equivalent and we will use whatever way is best suited for the current situation. [1] We will mainly study the properties of simple Lie algebra, since every semisimple Lie algebra is the direct sum of simple ones.

4 Root space decomposition

Any semisimple Lie algebra can be split up into its rootspace decomposition. This will help further categorising the Lie algebras. Starting off with the subalgebra with consisting of semisimple elements called a toral subalgebra. A semisimple element is an element whose minimal polynomial has all distinctive roots. The minimal polynomial is the monic polynomial of least degree for which $P(x) = 0, x \in L$. We should note that we use a Lie algebra of the form $End(V)$ with the bracket defined as $[x, y] = xy - yx$. This ensures the existence of $x^n, x \in L$. In case of matrices a semisimple element is just a diagonalizable element.
The maximal toral subalgebra H is the toral algebra that is not properly included in any other toral subalgebra. From this point it is possible to write L as the direct sum of the subspaces $L_\alpha = \{x \in L | [h, x] = \alpha(h)x \text{ for all } h \in H\}$. Where α ranges over the dual space of H: H^*. These are all linear functions of $H \to F$ and it forms a vector space with the same dimensions as H itself. L_0 is clearly the centralizer of H, which turns out to be H itself. Any toral subalgebra is abelian and the fact that the centralizer of H is itself implies that H is also a maximal abelian subalgebra.
The set of all the nonzero α for which rootspace L_α is nonzero will be denoted by Φ. These elements are called roots, the set is called the root system. Take all these together and it allows us to write:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

This root space decomposition has some intriguing properties. First of all the Cartan Killing form with respect to the H is nondegenerate, which implies that for an element $\phi \in H^*$ there is a unique element $t_\phi \in H$ satisfying $\phi(h) = \kappa(t_\phi, h)\ \forall h \in H$. Some of the other properties are:

- Every subalgebra L_α is one dimensional
- If $\alpha \in \Phi$ then $-\alpha \in \Phi$, but no other multiple of α is contained in Φ
- Let x_α be any nonzero element of L_α then there exists an $y_\alpha \in L_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ spans the three dimensional algebra isomorphic to $\mathfrak{sl}(2, F)$ through $x_\alpha \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y_\alpha \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h_\alpha \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Now comes the peculiar part, whereas x_α could just be chosen (upto a scalar), h_α is fixed by α and is given by $h_\alpha = 2\frac{t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. Furthermore if we have an $\alpha, \beta \in \Phi$ then $\beta(h_\alpha) = 2\frac{\beta(t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = 2\frac{\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)}$ is an integer number. These numbers are called Cartan Integers.

We could translate this into an inner product on Φ defined by $(\alpha,\beta) = \kappa(t_\alpha, t_\beta)$ so that the following statements hold:

- For any $\alpha, \beta \in \Phi$, $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in Z$.
- For any $\alpha, \beta \in \Phi$, $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$

This last item is called a reflection through α and can be abreviated by $\sigma_\alpha(\beta)$. $2\frac{(\beta,\alpha)}{\alpha,\alpha)}$ will be used so often that we will abbreviate it like $\langle\beta,\alpha\rangle$. Note that this operation is only linear in its first variable.
Given any basis of the root system β_i so any root can be written as a combination $\sum a^i\beta_i$. All these a^i turn out to be rational numbers ($a^i \in Q$). Which means that we can translate the roots to points in a Euclidian space with the same dimensions as H^*. One of the properties of the root system is that it spans H^*. With the definition of the inner product $(.,.)$ it is also possible to translate this to the inner product on an Euclidian space. This Euclidian space gives a good visualisation of the roots of low dimensional root systems.

Example 4. *Once again taking $\mathfrak{sl}(2,F)$ as an example: The names of the basis x, y, h haven't been chosen arbitrarily. The only diagonalizable elements of this Lie algebra are those generated by h (the corresponding minimal polynomial is $p(x) = x^2 - 1$). And we know that $[h,x] = 2x$ and that $[h,y] = -2y$. Let e^h be the dual basis of h such that $e^h(h) = 1$. The roots are now given by $\alpha = 2e^h$ and $-\alpha = -2e^h$. Both sides of the equation $[h,x] = 2e^h(h)x$ being linear in x and h, implies this root spans the space generated by x. In a one dimensional Euclidian space the root system would look like.*

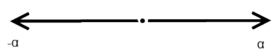


Figure 1: The root system of $\mathfrak{sl}(2)$

Base and Weights

So far we have we have seen a lot of properties of the root space decomposition and its roots. It is possible to construct a root system ϕ according to a few axioms. Using this you can construct a semisimple Lie algebra that is unique up to an isomorphism. We will however go one step further: the Cartan Matrix.
First we need a base Δ (not the same as a basis) of Φ defined by

B1 Δ is a basis for H^*

B2 Every root in Φ can be written as $\Sigma m^i \alpha_i$ where $\alpha_i \in \Delta$ and m^i either all nonnegative or nonpositive integers

All the roots in the base are called *simple roots* and for every root m^i is called the root vector. As every root can be written as distinct combination of simple roots it allows us to define the height of a root by $\mathrm{ht} = \sum_{\alpha_i \in \Delta} a^i$. Any finite dimensional simple Lie algebra has a unique highest root. [2]
This choice of basis also allows us to split up the root system into a positive and negative part. A root is called positive (resp. negative) if all $m^i \geq 0$ (resp. $m^i \leq 0$). The sets of all positive (resp. negative) are denoted by Φ^+ (resp. Φ^-). Clearly, $\Phi^+ = -\Phi^-$.
By this choice of basis for any two elements $\alpha_1, \alpha_2 \in \Delta,\ \alpha_1 \neq \alpha_2,\ (\alpha_1, \alpha_2) \leq 0$. Which means that the angle between these two elements is obtuse.

Hasse Diagram

As the root system gets more that two dimensions it becomes increasingly difficult to depict it in a Euclidian space. The Hasse diagram offers a solution. The Hasse diagram depict only the positive roots Φ^+. As we just mentioned these have the same structure as the negative ones.
In order to construct a Hasse diagram we need a couple of quick definitions. The root α is said to be bigger than β if their difference is positive

$$\alpha > \beta \quad \text{if} \quad \alpha - \beta \in E_+$$

The root α covers the root β if there does not exist another root which is both smaller than α but bigger than β.

$$\alpha \succ \beta \quad \nexists \gamma \quad \alpha > \gamma > \beta$$

These two definitions can be used to order the roots. [2] Whenever a root is bigger than another it gets a higher vertical coordinate. And if this root covers the other root it gets a line. In practice this means that the roots are sorted by their height and a line is drawn whenever the two roots differ by a simple root.
The vertical spacing is done so that every simple root gets a horizontal position x_i. Any other root then has the horizontal position $\sum m^i x_i$. This means that the slope of a line in the Hasse diagram tells us by what simple root the two roots differ

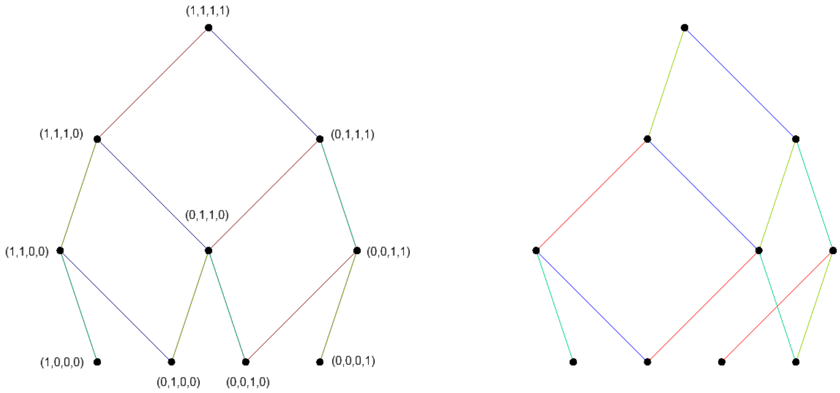


Figure 2: The Hasse diagram of $\mathfrak{sl}(5)$ (or A4 as defined in the next section). The left diagram has the root vectors added, the right one has them left out. The order of simple roots can be chosen arbitrarily, therefore these two look different, but when taking a closer look one can see that they both have exactly the same structure. The left diagram however looks much neater. In the upcoming we will choose the best looking order of simple roots.

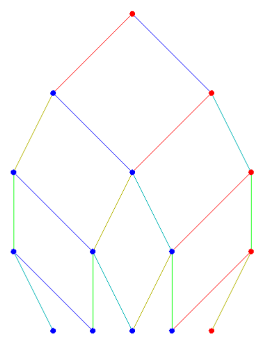


Figure 3: The algebra $\mathfrak{sl}(5)$ as a subalgebra of $\mathfrak{sl}(6)$. The blue nodes form the subalgebra.

One can also easily see some of the subalgebras within the Hasse diagram, as shown in figure 3. The Hasse diagram shows the $\mathfrak{sl}(5)$ algebra. When removing all nodes for which the last element of the rootvector $m^5 \neq 0$ one effectively ends up with the Hasse diagram for $\mathfrak{sl}(4)$, this subalgebra is shown in the picture by the blue nodes. The Hasse diagram also gives insight in the dimension of the algebra. Every root corresponds to a 1 dimensional root space. Only the positive roots are depicted and the toral subalgebra is left out. The number of positive and negative roots are of course equal and this toral subalgebra has the same number of dimensions as the amount of simple roots, as we will see in the construction of Lie algebras through the Cartan matrix. In the case of $\mathfrak{sl}(5)$ this results in a $2 \times 10 + 4 = 24$ dimensional Lie algebra.
The Hasse diagram shows the structure of a Lie algebra and it will gives a good insight in the properties of affine Lie algebras.

Cartan Matrix and Dynkin Diagrams

The cartan matrix is defined as the matrix with entries $C_{ij} = \langle \alpha_1, \alpha_2 \rangle$, $\alpha_1, \alpha_2 \in \Delta$. Right off the bat we can see a few of the properties of this algebra. Such as all its diagonal elements being 2. In the case of a semisimple finite dimensional Lie algebra the Cartan matrix satisfies the following properties and any Cartan matrix satisfying these properties will give rise to a finite dimensional semisimple Lie algebra, which is unique up to a isomorphism. Also the Cartan matrix will be one of the starting points in constructing affine Lie algebras.

- C1 $C_{ii} = 2$
- C2 $C_{ij} \leq 0$
- C3 $\det C > 0$
- C4 $M(C) > 0$
- C5 C is diagonizable in the sense that it can be written as a product BD where B is symmetric and D is a diagonal matrix.

Where $M(C)$ denotes the determinants of the principal minors of C, these are the matrices constructed by deleting one or more of the same rows and columns of C. Sometimes additional properties are given such as $C_{ij} = 0 \iff C_{ji} = 0$. This is just a result of C1 and C5. Also C4 and C5 imply that C is positive definite. Note that although not mentioned all the entries of the Cartan matrix have to be of integer value.
The order of the simple roots α_i was chosen in an arbitrary way. This means that switching any two columns and rows simultaneously has no effect on the Lie algebra.
A useful way of visualizing the different Lie algebras is the Dynkin diagram. It is basically a visualisation of the Cartan matrix. The rules of drawing such a diagram are simple:

- For every row of the Cartan matrix draw a node
- Draw $\max(C_{ij}, C_{ji})$ lines between the i$^{\text{th}}$ and j$^{\text{th}}$ node
- Whenever double or triple lines are drawn it is possible to add an arrow. If $|C_{ij}| > |C_{ji}|$ then the arrow points from node i to node j.

The nodes can be reordered in such a way that they look best. The Dynkin gives a good insight of the different subalgebras. Deleting any node gives rise to another Lie algebra which is just a subalgebra of the original one.

Example 5. *The Lie algebra* $\mathfrak{sl}(2, F)$ *which we have used consistently as an example has the one of the least interesting Cartan matrices and Dynkin diagrams possible. It has a single simple root, which means that Cartan matrix has a single element* (2) *and that the Dynkin diagram has a single node. A more interesting Dynkin diagram, which will come up later, is the diagram of* $E8$.

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

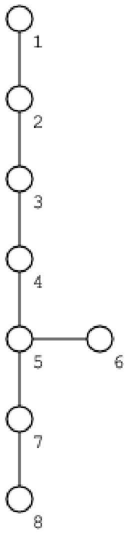
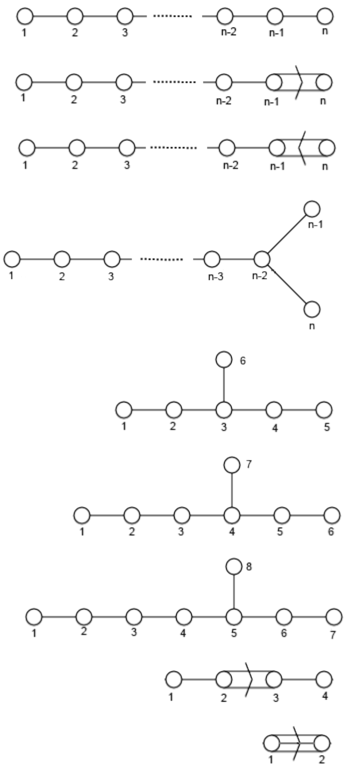


Figure 4: The Cartan matrix of the exceptional Lie algebra E8 and its Dynkin diagram.

Simple Lie algebras

Using the Cartan matrix and Dynkin diagram it is possible to classify all simple Lie algebras. We classify Lie algebras using a capital letter (A to G) indicating the type of Lie algebra and a number n indicating the dimension of the Cartan matrix. This is related but does not indicate the dimension of the Lie algebra itself.

Theorem 1. *Any finite dimensional Lie algebra is one of the following types. For a proof of this theorem see [1]*



A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2

Constructing a Lie algebra through a Cartan Matrix

Using this definition of the Cartan matrix it is possible to construct a Cartan matrix, which in turn defines a root space (up to an isomorphism), which gives rise to the Lie algebra. If the matrix is decomposable in the sense that it can be written (with possible reordering of indices) as

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

it will give rise to a reducible root system. Which then will give the direct sum of two or more simple Lie (sub)algebras. Note that this is easily seen in the Dynkin diagram, since the diagram will then have two or more distinct clusters of one or more nodes.
We start of with the 3n tuple of generators $\{h_i, e_i, f_i\}$. Where the abelian subalgebra generated by h_i is called the Cartan subalgebra, which in the finite dimensional case corresponds to the toral subalgebra.
The Chevalley-Serre relations tell us how the different generators are related through

the bracket:

$$[h_i, h_j] = 0 \tag{1}$$
$$[h_i, e_j] = C_{ji} e_j \tag{2}$$
$$[h_i, f_j] = -C_{ji} f_j \tag{3}$$
$$[e_i, f_j] = \delta_{ij} h_i \tag{4}$$
$$(\mathrm{ad}_{e_i})^{1-C_{ji}} e_j = 0 \tag{5}$$
$$(\mathrm{ad}_{f_i})^{1-C_{ji}} f_j = 0 \tag{6}$$

Note that we do not sum over the indices i and j. All other generators of the Lie algebra are constructed by considering all of the following possibilities of the form.

$$\mathrm{ad}_{e_i} \mathrm{ad}_{e_j} \cdots e_k \tag{7}$$
$$\mathrm{ad}_{f_i} \mathrm{ad}_{f_j} \cdots f_k \tag{8}$$

Example 6. *Considering the Lie algebra $\mathfrak{sl}(2, F)$, the three elements h, e, f correspond directly to the base elements h, x, y as defined in example 2.*
Also note that for every i the subalgebra spanned by h_i, e_i, f_i is the $\mathfrak{sl}(2, F)$ subalgebra constructed through $L_{-\alpha}$ and L_α as in the previous section.

The generators e_i generate the one dimensional subspace that corresponds to the simple root α_i. Similarly the root space corresponding to $-\alpha_i$ is generated by f_i. This is clearly seen by considering

$$\alpha_i(h_j) = \kappa(t_{\alpha_i}, 2\frac{t_{\alpha_j}}{\kappa(t_{\alpha_j}, t_{\alpha_j})}) = \langle \alpha_i, \alpha_j \rangle = C_{ij}$$

where we use h_i to abbreviate h_{α_i}. We already know that $[h_j, e_i] = \alpha_i(h_i) e_i$ and every (positive) root is a sum of simple roots. With a little help of the Jacobi identity one can show that

$$[h, [e_i, e_j]] = (\alpha_i(h) + \alpha_j(h))[e_i, e_j] \tag{9}$$

This process is not only true for e_i and e_j but for arbitrary elements of the Lie algebra. As any root can be constructed by continuously adding simple roots, every generator corresponding to these roots can be constructed as in equation 7. The two elements e_i and f_i both carry the same information. Like we said before the root system can be split into a positive and a negative part, where e_i correspond to the positive part of the root system and f_i correspond to the negative part. The Hasse diagram only looks at the positive roots. This together with equation 9 allows us to look at the Hasse diagrams not as roots but as spaces spanned by e_i and $\mathrm{ad}_{e_i} \mathrm{ad}_{e_j} \cdots e_k$. The slopes then correspond to the operation ad_{e_i}, where the different slopes correspond to the different ad_{e_i}. The Jacobi identity ensures that it does not matter which route is taken through the different points in the Hasse diagram.

5 Infinite dimensional Lie algebra

There are different routes one can take to construct an affine Lie algebra. We will begin by loosening some of the restrictions put on the Cartan Matrix and see how this leads to an infinite dimensional Lie algebra. As it turns out a few of the statements need to be revised.

Cartan matrices where conditions C3 and C4 are dropped are called generalized Cartan matrices. These algebras, called Kac-Moody algebras, often are infinite dimensional. A positive definite Cartan matrix gives a finite dimensional semi-simple Lie algebra. A Kac-Moody algebra where C3 is left intact and only C4 is altered in $\det C = 0$ is a positive semi definite Cartan matrix. The corresponding Lie algebra is infinite dimensional and is called an affine Lie algebra. We just consider Cartan matrices where all of its principal minors have a positive determinant. This means that is has rank $n-1$ and when deleting one or more nodes from the Dynkin diagram one obtains a semisimple Lie algebra. These affine Lie algebras have a well understood structure and are often encountered in physics. They play important roles in string theory and conformal field theory.

All elements that are not in toral subalgebra are constructed by considering all possible brackets of either e_i or f_i (equation (7)). It is the Chevalley-Serre equations (5) and (6), that will make sure that all but a few of these possibilities are in fact nonzero. In the case of infinite dimensional Lie algebras these equations can no longer ensure this and this allows for an infinite amount of generators.

The understand the cause of these infinite dimensions we will first have to take another look at the root system, this time from the viewpoint of the Cartan matrix. For the simple roots (the roots in the base) $C_{ij} = \langle \alpha_i, \alpha_j \rangle$. Every root can be expressed in terms of simple roots or fundamental weights. The fundamental weights Λ^i are defined as

$$\langle \Lambda^i, \alpha_j \rangle = \delta^i_j$$

Any root α can now be expressed as $\alpha = \sum m^i \alpha_i = \sum p_i \Lambda^i$. m^i is called the root vector and is defined to be either all positive or all negative. p^i are called the Dynkin labels. Weights and roots can be expressed as one another

$$\alpha_i = \sum C_{ij} \Lambda^j$$

From this we see that that $p_i = \sum m^j C_{ji}$.

In the case of an affine Lie algebra there no longer is a unique heighest root. Its place is taken by the null root δ. Which is defined that for every root α in Φ the element $\alpha + \delta$ is also in Φ. [1] It is easily found through

$$\sum a^i C_{ij} = 0 \quad \min(a^i) = a^0 = 1 \tag{10}$$

[1] There are so called twisted Lie algebras, where this is not entirely the case. In that $\alpha + m\delta \in \Phi$ for some integer m greater than one. We will only consider the untwisted affine Lie algebras

Where a^i is the root vector of the null root $\delta = \sum a^i \alpha_i$, the values a^i are also known as Coxeter labels. For convenience we have reordered the indices such that they now run from 0 to $n-1$. Where the zeroth Coxeter label has value 1.
This null root is of course only possible with a Cartan matrix that has determinant 0, since for every other matrix there is no other vector, other than the zero vector, which satisfies equation 10. This root truly is the nullroot in the sense that: $(\delta, \delta) = 0$.
Furthermore, any two roots differing by δ have the same Dynkin labels p_i.

$$p_i = \sum m^j C_{ji} = \sum (m^j + a_j) C_{ji}$$

Which means that when adding the null root to a root it leaves the root invariant when considering the Dynkin labels. The null root can be added an arbitrary number of times to a certain root. Given rise to infinite number of the same roots, which leads to a infinite dimensional affine Lie algebra.

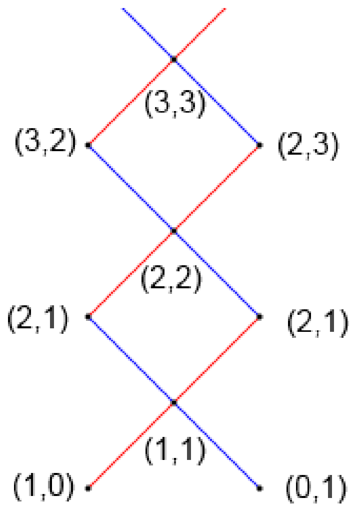


Figure 5: The Hasse diagram of A_1^+

This means that when considering Dynkin labels every rootspace now has an infinite dimensions, whereas it only had dimension 1 in the finite case. However considering the root vector every rootspace still has dimension 1. Another thing that has changed is that the Cartan subalgebra $\{h_i\}$ is no longer maximal abelian. The rootspaces of the nullroot commute with the Cartan subalgebra. $\delta(h_j) = \sum a^i \alpha_i(h_j) = \sum a^i A_{ij} = 0$

Example 7. *The simplest of all affine Lie algebras is A_1^+. The Cartan matrix looks as follows*

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

This matrix has all the properties the Cartan matrix of an affine Lie algebra should have. It has determinant 0 and its only principal minor is the matrix (2), *which is of course the subalgebra A1.*
It root system has two simple roots α_1 and α_2. Equation (10) tells us that the null root is given by $\alpha_1 + \alpha_2$. The corresponding Hasse diagram is given by figure 5. The null root in this figure is of course the center lowest "dot".
The Hasse diagram gives insight in what is happening. The Hasse diagram in figure 5 only shows the few lowest roots. There is no highest root anymore. This algebra has an infinite amount of roots all with the exact same Dynkin labels. All

the roots on the left have Dynkin labels $[2, -2]$ *while the ones on the right have* $[-2, 2]$*. The null root and all multiples of the null root have Dynkin labels* $[0, 0]$*. In other words, the Chevalley-Serre equations never cut off the production of extra generators.*

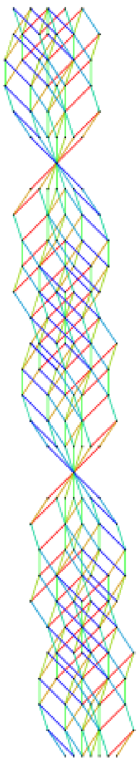


Figure 6: The Hasse diagram of $E6^+$

Structure of Affine Lie algebras

The affine Lie algebra has an elegant structure. It contains an infinite amount of copies of different semisimple subalgebras that it contains. Remember that removing any node from the Dynkin diagram of an affine Lie algebra will result in a semisimple algebra. This can easily be seen in the Hasse diagram. Any Hasse diagram of an affine Lie algebra has a similar structure to that of figure 5 as seen in figure 6. The null roots are clearly visible as the point where all the lines meet. In a section from now we will see how we can create an affine Lie algebra when starting off with a simple Lie algebra.

Derived affine Lie algebra

In the case of affine Lie algebras, we have seen that the roots are indistinguishable in terms of Dynkin labels. It is however possible to fix this by adding one additional root and one extra dimension to the Cartan subalgebra (additional roots and dimensions have to be added when considering generalized Cartan matrices with corank greater than one). These affine Lie algebras with the extended Cartan subalgebras are called derived affine Lie algebras.
To raise this nondegeneracy on the Cartan matrix we add an additional Dynkin label p_{-1}. This corresponds to an additional root γ called the root of derivation.

$$p_{-1} = \langle \alpha, \gamma \rangle \quad \text{where} \quad \langle \delta, \gamma \rangle = -1$$

This does the trick, because now $\langle \alpha + \delta, \gamma \rangle = \langle \alpha, \gamma \rangle - 1$.
The zeroth element of the root vector of the null root a^0 is always equal to 1, Therefore an easy definition of the root of derivation is

$$\langle \alpha_i, \gamma \rangle = -\delta_i^0$$

This root of derivation cannot lie in the span of simple roots, because for every of these roots the inner product with δ would be zero. We will have to manually add this root. The norm of γ is not fixed by the definition above, which means that we can choose γ in such a way that $\langle\alpha_i, \gamma\rangle = \langle\gamma, \alpha_i\rangle = -\delta_i^0$.
Using this root the Cartan matrix of the affine Lie algebra $\bar{C}$ can be extended to the extended Cartan matrix $\hat{C}$. $\hat{C}_{(-1),i} = \hat{C}_{i,(-1)} = \langle\alpha_i, \gamma\rangle = \langle\gamma, \alpha_i\rangle$. The Cartan matrix is now no longer nondegenerate. Using this matrix, however, some strict rules need to be taken into account. While adding the extra generator h_{-1}, the other generators e_{-1} and f_{-1} are left out. Also the root α_{-1} is absent, otherwise this would correspond to γ.
When allowing the elements e_{-1}, f_{-1} and h_{-1} one obtains is what a so called overextended algebra. This Cartan matrix is nondegenerate, but it has one negative and the rest positive eigenvalues. This allows for real, null and imaginary roots, in the sence that $(\alpha, \alpha) < 0$. The Hasse diagram no longer shows a steady increase of roots, but the amount of roots increases faster and faster.

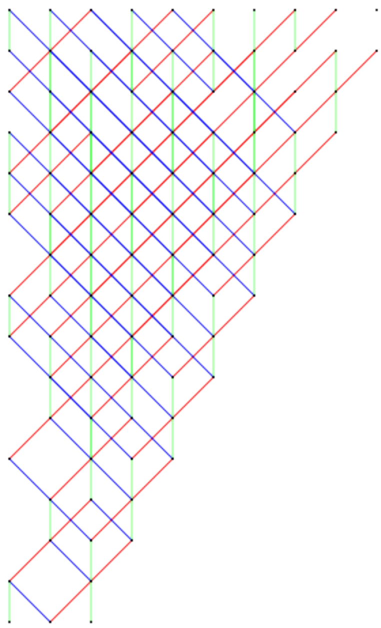


Figure 7: The Hasse diagram overextended Lie algebra of $A1$, $A1^{++}$

Loop algebra

Another way of constructing an affine Lie algebra, is by extension of a simple Lie algebra. The affine Lie algebra that follows from a semisimple Lie algebra is the direct sum of the affine Lie algebra that are constructed in this section. In this section we will have to closely pay attention of which Lie algebra we're talking about. We will use L for the original simple Lie algebra, $\mathring{L}$ is the loop algebra, $\bar{L}$ is the affine Lie algebra and finally $\hat{L}$ is the derived affine Lie algebra. Elements of the different algebras will be denoted in the same way.
First we will construct the loop algebra $\mathring{L}$ of this simple Lie algebra and then we will add one additional dimension to the Cartan subalgebra in order to make it an affine Lie algebra. This loop algebra can be thought of as all smooth maps from the 1-sphere to the simple algebra. [19] These maps can of course be parameterized in the following way:

$$\phi : S^1 \to L \quad \theta \to \sum_{n \in Z} g_n e^{in\theta}, \quad g_n \in L$$

Where clearly we have used $e^{i\theta}$ to parametrize the sphere S^1. The set of all these maps ϕ is the loop algebra $\mathring{L}$ with the bracket defined as:

$$[g_n e^{in\theta}, g_m e^{im\theta}] = e^{i\theta(m+n)}[g_n, g_m]$$

Since the bracket is bilinear it is not hard to figure out how any sum of these elements transform under the bracket. Instead of writing $e^{i\theta n}$ we can write t^n for convenience. It is a Fourier transform of sorts and it transforms the set of all functions on S^1 to the set of Laurent polynomials of the form $\mathbb{C}[x, x^{-1}]$ (here we consider the field of complex numbers)[19]. A Laurentz polynomial is a polynomial which can have both positive and negative powers, also $\mathbb{C}[x, x^{-1}]$ is a ring. This reduces the last two equations to:

$$\mathring{L} = L \otimes \mathbb{C}[x, x^{-1}] \tag{11}$$

$$[g_n \otimes t^n, g_m \otimes t^m] = [g_n, g_m] \otimes t^{n+m} \tag{12}$$

The $\otimes$ denotes the tensor product. This means that if g_n is a basis for L $g_n \otimes t^m$ forms a basis for $\mathring{L}$ and $g_m \otimes t^n + g_n \otimes t^n = (g_m + g_n) \otimes t^n$. We can now really see the grading on this Lie algebra; the infinite amount of copies of the simple Lie algebra, similar to what we saw in the Hasse diagrams of the previous section. It is however not yet an affine Lie algebra, for that we will need the central extension of this loop algebra $\mathring{L}$ by a 1-dimensional center $\mathbb{C}K$. K is called the central element. It comes as no surprise that this element is added to the Cartan subalgebra.

$$\bar{L} = \mathring{L} \oplus \mathbb{C}K \tag{13}$$

$$[g_n \otimes t^n \oplus \alpha K, g_m \otimes t^m \oplus \beta K] = [g_n, g_m] \otimes t^{n+m} \oplus \kappa(g_n, g_m) n \delta_{m+n} K \tag{14}$$

To see that this indeed corresponds to our previous definition of the affine Lie algebra, we will take a closer look at the construction of this affine algebra $\bar{L}$ (we will only consider the construction of untwisted affine Lie algebras).
We will begin by considering a simple algebra constructed through the Chevalley-Serre relations (equations 5,6) with the 3-n tuple of generators h_i, e_i, h_i $1 \leq i \leq n$. Every simple Lie algebra L has a unique highest root θ. From section 2 we know that the root θ together with the root $-\theta$ spans the subalgebra $\mathfrak{sl}(2, \mathbb{C})$ through $e_\theta \in L_\theta, f_\theta \in L_{-\theta}, h_\theta = [x_\theta, y_\theta]$. This h_θ is given by $h_\theta = 2\frac{t_\theta}{(\theta,\theta)}$.
The bracket between these elements is of course given by

$$[h_\theta, x_i] = \alpha_i(h_\theta) e_i \tag{15}$$

$$= \frac{2}{(\theta, \theta)} \alpha_i(t_\theta) e_i \tag{16}$$

$$= \langle \alpha_i, \theta \rangle e_i \tag{17}$$

and

$$[h_i, x_\theta] = \langle \theta, \alpha_i \rangle x_\theta \tag{18}$$

Now consider the centrally extended Cartan subalgebra $\bar{H} = H + \mathbb{C}K$ and we choose the generators of the affine Lie algebra to be:

$$\bar{e}_i = t^0 \otimes e_i, \quad \bar{f}_i = t^0 \otimes f_i, \quad \bar{h}_i = t^0 \otimes h_i \tag{19}$$

$$\bar{e}_0 = t \otimes f_\theta, \quad \bar{f}_0 = t^{-1} \otimes e_\theta, \quad \bar{h}_0 = -t^0 \otimes h_\theta + \frac{2}{(\theta, \theta)} K \tag{20}$$

Although this is slight alteration of what is done in lecture notes [19]. We have switched e_θ and f_θ. It is basically the same algebra. We have only ensure that the additional generators are of the form of equation 7.
The null root is the root for which $\delta(h_i) = 0$ and $\delta(K) = 0$. Now we define the additional simple root as $\alpha_0 = \delta - \theta$. With these definitions the Chevalley-Serre equations hold exactly, where of course $C_{ij} = \langle \alpha_i, \alpha_j \rangle$ $0 \leq i, j \leq n$.
All the roots of this affine Lie algebra are given by

$$\bar{\Phi} = \{i\delta + \alpha | i \in Z, \alpha \in \Phi\} \cup \{i\delta | i \in Z, i \neq 0\}$$

The Cartan matrix of the affine Lie algebra $\bar{C}$ given by this procedure is

$$\bar{C}_{ij} = C_{ij} \tag{21}$$

$$\bar{C}_{0i} = \langle \delta - \theta, \alpha_i \rangle = -\sum a^j \langle \alpha_j, \alpha_i \rangle = -\sum a^j C_{ji} \tag{22}$$

$$\bar{C}_{i0} = \langle \alpha_i, \delta - \theta \rangle = \frac{(\alpha_i, \alpha_i)}{(\theta, \theta)} C_{0i} \tag{23}$$

Where of course a^i is the root vector of the highest root $\theta = \sum a^i \alpha_i$.

Example 8. *Let's once again consider the algebra $\mathfrak{sl}(2)$. This algebra only has a single simple root, which also the highest root. The additional simple root is defined as $\bar{\alpha}_0 = \delta - \theta$, which in this case implies that $\delta = \alpha_0 + \alpha_1$, just as we saw in the previous section.*
We have the basic six elements of the 3-n tuple, where most notably $\bar{e}_1 = t^0 \otimes e_1$ and $\bar{e}_0 = t^1 \otimes f_1$. The bracket now gives $[\bar{e}_0, \bar{e}_1] = -t^1 \otimes h_1$. Which then in turn gives $[t^1 \otimes h_1, \bar{e_0}] = -2t^1 \otimes f_1$ and $[t^1 \otimes h_1, \bar{e_1}] = 2t^1 \otimes e_1$. Which clearly shows that the different levels in the Hasse diagram correspond to the different values t^n.
We could also use equations 21 to construct the affine cartan matrix of $A1^+$ and one finds $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ just as expected.

Finally if you would want to end up with the derived version of the affine Lie algebra you can immediately add an additional dimension as follows

$$L \otimes \mathbb{C}[x, x^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

$$\begin{aligned}&[g_n \otimes t^n \oplus \alpha c \oplus \beta d, g_m \otimes t^m \oplus \gamma c \oplus \sigma d] \\ &= ([g_n, g_m] \otimes t^{n+m} + \beta g_m \otimes t^m - \sigma g_n \otimes t^n) \oplus \kappa(g_n, g_m)\delta_{m+n}c\end{aligned} \tag{24}$$

Where $g_n, g_m \in L$, $t^n, t^m \in \mathbb{C}[t, t^{-1}]$, $n, m \in Z$ and $\alpha, \beta, \gamma, \sigma \in \mathbb{C}$.

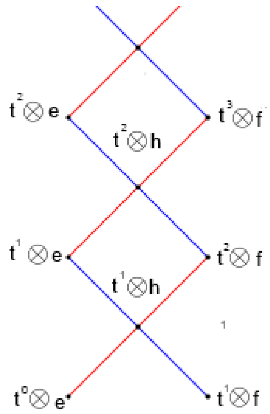


Figure 8: Example 8: the Hasse diagram $A1^+$

6 Recap and outlook

In physics symmetries are an important part of modern day theories. As we will see in the next section Lie algebras are closely related to such symmetries. We will also go into how Lie algebras are related to Lie groups, these are groups which are also differential manifolds. Most of the symmetries in the Standard Model have Lie groups and algebras.
Semi-simple Lie algebras can be most easily constructed trough the Cartan matrix. The elements of the Cartan matrix tell exactly how the different generators commute. The entire Lie algebra can easily be constructed using the Chevalley-Serre equations.
The untwisted affine Lie algebras arise when the Cartan matrix becomes positive semi-definite with a single zero eigenvalue. This is used to construct a nonzero solution for $a^i C_{ij} = 0$. This a^i is the root vector of the null root. A root which truly has length 0 and can be added an arbitrarily amount of times to a root. The resulting root will once again be a root of the root system. It is clear that this is of course not possible with an positive definite Cartan matrix.
The centrally extended loop algebra is another way to view the affine Lie algebra. This closely shows the similarities between the affine Lie algebra and the corresponding (semi-)simple Lie algebra. In fact it is this affine extension of a simple Lie algebra that occurs in two dimensional maximal supergravity. Two dimensional supergravity can be obtained by reducing three dimensional supergravity on a circle. It might be logical to view the affine Lie algebra in this way. Additionally this loop algebra allows us to view the affine Lie algebra as an infinite amount of copies of the original Lie algebra. An infinite tower of which every floor contains the (semi-)simple Lie algebra
Affine Lie algebras can further be generalized, resulting in Kac-Moody algebras.

These are all algebras that can be constructed by a generalized Cartan matrix. Finite dimensional semi-simple and affine Lie algebras are just two subclasses of Kac-Moody algebras. These algebras would for example appear in 1 dimensional maximal supergravity, but also show uses in other areas of theoretical physics. There is still much unknown about the Kac-Moody algebras, resulting from indefinite Cartan matrices.

Part II
Lie algebras in Physics

In this part of the thesis we will first quickly refresh the Lagrangian density and equations of motion, whereafter we directly dive into the different symmetries, symmetry groups and their corresponding algebras. An important algebra that will be constructed is the Poincaré algebra. When introducing supersymmetry this will be extended to the super-Poincaré algebra. This is not a Lie algebra is the conventional sense. It is a graded Lie algebra, containing odd and even elements with commutation and anticommutation relations. The Coleman-Mandula [23] theorem states that is is not possible to have a Lie algebra which combines internal and spacetime symmetries. This graded Lie algebra however is not a strict Lie algebra and allows for a combination of spatial and internal symmetries. This unification of symmetries is one of the pro's of supersymmetry.
It turns out that supersymmetry (SUSY) can contain more than one supersymmetric transformation, sometimes called extended supersymmetry. After this we will look at gauging global symmetries. Imposing local symmetries on a theory gives further restrictions. For an example we will use the $U(1)$ symmetry of electromagnetism. It is a truly elegant construction of a theory.
So far we will have assumed all these equations work in a flat Minkowski spacetime. From then on we will try to construct a local supersymmetric theory of gravity. It can be constructed in multiple ways. One way would to gauge the global (or rigid) supersymmetry. This however requires quite a bit more work that simply constructing a theory with a graviton and gravitino and requiringt this theory to be supersymmetric, so this is the route we will take.
Lastly we will take a look at the symmetries that arise in maximal supergravity. These symmetries are sometimes called hidden symmetries, because we have not imposed them in any way. The corresponding Lie algebra greatly differs in different dimensions. Maximal supergravity can live in up to eleven dimensions. A way to construct the maximal supergravity theories in alternate dimensions is by Kaluza Klein dimensional reduction of eleven dimensional supergravity. In two

dimensions maximal supergravity has a symmetry group which is infinite dimensional. The corresponding Lie algebra is of the affine type, as we have constructed in the mathematical part of this thesis.

7 Lagrangian density and eqn of motion

In classical mechanics the action, the time integral of the Lagrangian, is a fundamental quantity. In field theory this action can be written as an integral over all space-time dimensions of the Lagrangian density $\mathcal{L}$, which is a function of one or more field and their derivatives.

$$S = \int \mathrm{d}^4x \mathcal{L}(\phi, \partial_\mu \phi) \tag{25}$$

When a system evolves from one state to another, it takes a path for which the action is an extremum. This implies that

$$0 = \delta S = \int \mathrm{d}^4x (\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)) \tag{26}$$

$$= \int \mathrm{d}^4x (\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi)) \tag{27}$$

$$= \int \mathrm{d}^4x (\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)})\delta\phi) + \partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi) \tag{28}$$

The last term contains a total derivative, which can be rewritten as a surface integral over the boundary of space time. We want to know how a system evolves from a certain time to another certain time. Therefore at these two times the variation of the field ϕ can be set to zero. This leads to the equations of motion:

$$\partial_\mu(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{29}$$

Example 9. *The lagrangian density* $\mathcal{L} = \frac{1}{2}|\partial_\mu\phi|^2 - \frac{1}{2}m^2|\phi|^2$ *will give the equations of motion* $(\partial^\mu\partial_\mu + m^2)\phi = 0$ *which is the common Klein-Gordon equation.*

8 Symmetries, currents and charges

A symmetry is an invariance of the action under a certain transformation of fields and/or spacetime. A symmetry that leaves spacetime intact is called an internal symmetry. Similarly a symmetry of spacetime is called a spacetime symmetry (e.g. Lorentz symmetry). The action (and lagrangian) of example 9 is invariant

under $\phi \to e^{i\alpha}\phi$. This symmetry is an example of an internal symmetry, with the corresponding symmetry group $U(1)$.
Noethers theory states that continuous symmetries give rise to currents, which are essentially conservation laws.

8.1 Noether's theorem

Noether's theory states that every continuous symmetry gives rise to a current, which is essentially a conservation law. [8] We first rewrite the transformation in its infinitesimal form.

$$\phi \to \phi' = \phi + \delta\phi$$

$\delta\phi$ is a small deformation of the field. We stated before that this transformation is a symmetry, meaning that this transformation should leave the action invariant. This implies that the Lagrangian should be invariant up to a 4-divergence.

$$\mathcal{L} \to \mathcal{L} + \partial_\mu J^\mu \tag{30}$$

for some J^μ. If we now vary the Lagrangian, by varying the fields we get:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + (\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\partial_\mu\delta\phi \tag{31}$$

$$= (\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}))\delta\phi + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi) \tag{32}$$

The first term of this formula is exactly the Euler-Lagrange equation of motion for a field, which means this term vanishes. So if we now combine equation 30 and equation 32 we find that

$$\partial_\mu j^\mu = 0 \quad \text{for} \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - J^\mu \tag{33}$$

The equation $\partial_\mu j^\mu = 0$ is a simple continuity equation, where the zeroth component of j is the density and the three spatial components of j are the flux in their respective directions. The charge conserved by this continuity equation is given by

$$Q = \int_{\text{All space}} j^0 \mathrm{d}\, x^3 \tag{34}$$

To further illustrate this theorem and to see where Lie algebras come up in this story, we will work out Noether's Theorem for the following Lagrangian of n real scalar fields. [4, 8]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i\phi_i$$

Here there is summed over i. It is not hard to see that this Lagrangian density is invariant under the transformation $\phi_i \to O_{ij}\phi_j$. Where O is of the orthogonal group, meaning that $O^T = O^{-1}$. For an infinitesimal transformation we can write O as $I + \epsilon^a T_a$. ϵ is called the infinitesimal parameter and T_a are the generators of the Lie algebra as we will see in a second. Since O is orthogonal we know that

$$O^T = I + \epsilon^a (T_a)^T = O^{-1} = I - \epsilon^a T_a$$

which implies that T_a are all the skew-symmetric n by n matrices. This means that there are $n(n-1)/2$ independent infinitesimal parameters e^a and thus $n(n-1)/2$ conserved charges. It is easy to check that all skew-symmetric n by n matrices, with the bracket defined as $[x, y] = xy - yx$ indeed form a Lie algebra.
From the previous section we can see that $\delta\phi_i = \epsilon^a (T_a)_{ij}\phi_j$. Using equation 33 we get

$$Q_a = \int \mathrm{d}\, x^3 \pi_i (T_a)_{ij}\phi_j$$

If the commutation relations are worked out, one finds that Q_a has the exact same commutation relations as T_a. This means that T_a and Q_a are two different representations of the same Lie algebra.
Furthermore one can work out that

$$\epsilon^a [Q_a, \phi_i] = -i\epsilon^a (T_a)_{ij}\phi_j = -i\delta\phi_i \tag{35}$$

This is exactly the infinitesimal form of the transformation:

$$\phi' = U\phi U^{-1} \quad \text{with} \quad U = e^{i\epsilon^a Q_a} \tag{36}$$

These last two equations hold in general and not just for the example that we just treated. These equations relates elements of the algebra $\epsilon^a Q_a$ to the corresponding continuous transformation.
Using the definitions from the mathematical section of this thesis, we see that equation 35 has the exact right properties to form a L-module. This means that ϕ is the vector space or L-module and the corresponding representation is given by $Q_a \to -iT_a$

Example 10. *Lie groups and algebras are closely related. Take for example the group $SO(2)$, these are all rotations around a single axis. The corresponding Lie algebra is the algebra spanned by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In fact we could rewrite equation 35 as*

$$\phi'_i = e^{\epsilon^a (T_a)_{ij}}\phi_j$$

In fact it is easy to check

$$\exp\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} = \begin{pmatrix} cos(x) & sin(x) \\ -sin(x) & cos(x) \end{pmatrix}$$

Where we have taken the exponent over the matrix and not its individual components. We usually denote the Lie group in capital letters $SO(2)$ *and the corresponding Lie algebra* $\mathfrak{so}(2)$ *are all 2 by 2 skew-symmetric matrices.*

The result from this example is true in a more general sense. Mathematically a Lie algebra can be viewed as the tangent space around the identity element of the Lie group.

9 Poincaré algebra

In this section we will construct the Poincaré algebra. This will not only be an excellent example of an algebra, but it will also as it turns out be very relevant to the Susy algebra. The poincaré algebra is constructed through spacetime symmetries in the Lagrangian. These are four independent spacetime translations and the six independent Lorentz transformations (three rotations and three boosts). So one would expect this algebra to contain ten generators, i.e. a ten dimensional Lie algebra. As an example we will use the Lagrangian density of example 9 with a real scalar field. Although except for the explicit equations of the charges, the other equations hold in general.

Generators of translations

Let us first consider the spacetime translation, under which one would expect the theory to be invariant. The time and place has no influence on physical events.

$$x^\mu \rightarrow x^\mu - a^\mu$$

For infinitesimal translations the field will transform as follows. In these equations a^μ is the infinitesimal parameter.

$$\phi(x) \rightarrow \phi(x+a) \approx \phi(x) + a^\mu \partial_\mu \phi(x)$$

The Lagrangian density is also a spacetime dependent scalar and will transform in the same way.

$$\mathcal{L} \rightarrow \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

This implies that our Lagrangian density is not invariant under this transformation, the action however is. So there is no harm there. Our four conserved currents are now

$$T^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \delta^{\mu}_{\nu} \mathcal{L} \tag{37}$$

This is exactly the energy-momentum tensor. The charge corresponding to the time translation is the Hamiltonian

$$H = \int \mathrm{d}^3 x T_{00} = \int \mathrm{d}^3 x (\pi \dot{\phi} - \mathcal{L}) \tag{38}$$

The other three charges corresponding to the spatial translations are

$$P_i = \int \mathrm{d}^3 x T_{0i} = \int \mathrm{d}^3 x \pi \partial_i \phi \tag{39}$$

and this can be interpreted as the momentum carried by the field. These four charges are often taken together in one tensor which can be written as $P_\mu = (H, \vec{P})$. Furthermore when we work out the commutator $[P_\mu, \phi]$ with the well known relations $[\phi(x), \pi(y)] = i\delta^{(3)}(x-y)$ and $[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0$ we see that

$$[P_\mu, \phi] = -i\partial_\mu \phi \tag{40}$$

just as we expected. This means that $\phi(x+a) = U\phi(x)U^\dagger$ where $U = exp[ia^\mu P_\mu]$. Just as we have seen in the last section.
This means ∂_μ is just a representation of this theory, where ϕ is a vector space (or L-module to be precise). Oficially this is a Hilbert space, we however don't require an inner product, so we will just consider it a function space. ∂_μ is an element of the endomorphisms on this function space. Since $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ this is an abelian Lie algebra. It can be represented as the direct sum of four one dimensional algebras.

Generators of Lorentz transformations

Secondly we have our lorentz transformations $x^\mu \to \Lambda^\mu_\nu x^\nu$. For infinitesimal transformations this can be written as

$$x^\mu \to x^\mu + \omega^\mu_{\ \nu} x^\nu$$

Where $\omega^{\mu\nu}$ is antisymmetric, due to the fact that Lorentz transformation Λ^μ_ν are of the SO(1,3) group, this is similar to stating $\Lambda^{\mu\nu}$ is of the special orthogonal group. We can easily see that this generates all three rotation and boost transformations, by considering the next two examples. First we will put $\omega^{10} = -\omega^{01} = \beta$ and all other entries to null. This will result in a vector V transforming as follows:

$$V \to \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V$$

For the second case $\omega^{12} = -\omega^{21} = \theta$ and all other entries null. This will result in V transforming in the following way.

$$V \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V$$

Our first example is clearly an infinitesimal boost in the x direction, while our second example is an infinitesimal rotation around the z axis. All other boosts and rotations are generated in the same way. Now by taking a Taylor expansion around $\phi(x)$ we find that

$$\phi(x+\omega x) = \phi(x) + \omega^{\mu\nu} x_\nu \partial_\mu \phi(x) = \phi(x) + \frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x) \tag{41}$$

The resulting charge is an antisymmetric 4 by 4 tensor, given by:

$$M_{\mu\nu} = \int \mathrm{d}\, x^3 (x_\mu T_{0\nu} - x_\nu T_{0\mu}) \tag{42}$$

where $T_{\mu\nu}$ is the stress energy tensor given by equation 37. If we work this out we find that the field transforms as

$$\phi \to U\phi U^\dagger \quad \text{where} \quad U = e^{-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}$$

Note that the factor $\frac{1}{2}$ comes from the fact that $\omega^{\mu\nu}$ and $M_{\mu\nu}$ are both antisymmetric, which means that every independent charge is counted twice. Which again implies that

$$[M_{\mu\nu}, \phi] = i(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi$$

9.0.1 Commutation relations

Probably the easiest way to construct the commutation relations of this algebra, would be to work out the relations between the representation $P_\mu \to i\partial_\mu$ and $M_{\mu\nu} \to i(x_\nu\partial_\mu - x_\mu\partial_\nu)$. This results in the following relations

$$[P_\mu, P_\nu] = 0 \tag{43}$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\rho\mu}P_\nu) \tag{44}$$

$$[M_{\mu\nu}, M_{\rho\lambda}] = i(\eta_{\nu\rho}M_{\mu\lambda} - \eta_{\nu\lambda}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\lambda} + \eta_{\mu\lambda}M_{\nu\rho}) \tag{45}$$

However another interesting way of determining the commutation relations is to work out alternating transformations of the field. In some cases this can be more useful.

$$U_\alpha U_\beta \phi U_\beta^\dagger U_\alpha^\dagger - U_\beta U_\alpha \phi U_\alpha^\dagger U_\beta^\dagger \approx -[\beta Q, [\alpha Q, \phi]] + -[\alpha Q, [\beta Q, \phi]] = \delta_\alpha \delta_\beta \phi - \delta_\beta \delta_\alpha \phi$$

Where α and β stand for any infinitesimal parameter and Q for any charge. If we use the Jacobi identity we find

$$\delta_\alpha \delta_\beta \phi - \delta_\beta \delta_\alpha \phi = [[\alpha Q, \beta Q], \phi]$$

In the SUSY algebra we will use this method of determining the charges.

Example 11. *Let's consider the easiest commutation relation between P_μ and P_ν. In this case*

$$\delta_b \delta_a \phi(x^\mu) = \delta_b(\phi(x^\mu + a^\mu)) - \phi(x^\mu) \tag{46}$$
$$= \phi(x^\mu + b^\mu + a^\mu) - \phi(x^\mu + a^\mu) - \phi(x^\mu + b^\mu) + \phi(x^\mu) \tag{47}$$

This equation is symmetric under exchange of a_μ and b_μ. Which implies that

$$[[a^\mu P_\mu, b^\nu P_\nu], \phi] = 0$$

There is only one way for which this holds for arbitraty a^μ and b^μ and this is $[P_\mu, P_\nu] = 0$

10 Supersymmetry

10.1 Introduction

Supersymmetry (SUSY) is a theory that relates bosons and fermions. It transforms the bosonic field into a fermionic one and vice versa. This means that for an action to be invariant under such a transformation is needs to contain both bosonic and fermionic field. Each particle from one of the two groups is associated with a particle from the other. These particles are the same in every respect except for their spin, which differs by one half. This means supersymmetry predict a whole array of additional particles.
So far there is no direct evidence for supersymmetry. None of the predicted particles have ever been found. It does solve multiple theoretical problems. First of all it allows for an algebra that contains both spatial and internal symmetries. Unifying theories have always been a great success in physics e.g. electroweak unification.

In some more complex theories, supersymmetry allows certain terms to cancel out. One of these problems is the hierarchy problem. This problem arises that the theoretical values don't seem to correspond with the experimental values and that some of the parameters need to be tuned to an uncomfortable precise value to get the theory working.

Dirac Equation

So far we have mainly used scalar fields as examples of theories. Supersymmetry however dictates that there should be bosons and fermions involved. No single field can be invariant under a SUSY transformation. This also implies that the degrees of freedom for the boson and the fermion field should match. We will later see how this can be achieved. For now, lets look at the Dirac equation. Probably the easiest example of a fermion field.
The dirac equation is of course given by $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ where Ψ is a four component dirac spinor. The Dirac Lagrangian, which is a way of obtaining the previous formula, is given by:

$$\mathcal{L}_{Dirac} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{48}$$

This four component Dirac spinor can be split into two two component spinors.

$$\Psi = \begin{pmatrix} \xi \\ \chi^\dagger \end{pmatrix}$$

The operator $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the chirality operator and is has the two eigenstates $\begin{pmatrix} \xi \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \chi^\dagger \end{pmatrix}$, with eigenvalues 1 and -1 Chirality basically indicates how a spinor transforms under lorentz transformations. Under lorentz transformations, with an infinitesimal rotation (θ) and a boost (β) they transform as follows.

$$\xi \to (1 - \frac{i}{2}\theta_i\sigma^i - \frac{1}{2}\beta_i\sigma^i)\xi \tag{49}$$

$$\chi^\dagger \to (1 - \frac{i}{2}\theta_i\sigma^i + \frac{1}{2}\beta_i\sigma^i)\chi^\dagger \tag{50}$$

Chirality is closely linked with helicity. The helicity of a particle is the direction of it's spin with respect to the direction of the particle, which can either be positive or negative (right- or left-handed). Whenever a particle has speed c the eigenstate of the helicity is also an eigenstate of chirality. Since massless particle always travel at the speed of light this always the case for massless particles.

Notation conventions

By convention we call elements of ξ ξ_a and elements of χ χ^a. The elements of the hermitian conjugates of these are denoted by a bar and a dotted index. We use both because in some cases the explicit indices are not given. So an element of $\xi_\dagger$ is $\bar{\xi}_{\dot{a}}$. Finally indices can be lowered and raised by:

$$\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1$$

All other components are identically zero. Indices are raised in the usual manner

$$\xi^a = \epsilon^{ab}\xi_b \qquad \bar{\chi}_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\bar{\chi}^{\dot{b}}$$

The interesting thing is that if one would work out the lorentz transformations one would see that ξ^a transforms similar to χ^a, which means we have built construction that are made from left handed spinors, but transform as right handed ones. The same thing applies to dotted and undotted.
Spinor components are anticommutative which means we also have to make a convention in which order to contract them. $\xi^a\chi_a = -\xi_a\chi^a$. The convention chosen is to contract them like ${}^a{}_a$ and ${}_{\dot{a}}{}^{\dot{a}}$. Here we will use the notation of an inner product. Often in literature this dot is left out.

$$\chi \cdot \xi = \chi^a\xi_a = \xi \cdot \chi$$

$$\overline{\chi} \cdot \overline{\xi} = \overline{\chi_{\dot{a}}}\overline{\xi^{\dot{a}}} = \overline{\xi} \cdot \overline{\chi}$$

These product can be exchanged since the minus sign from the anticommutativity from the spinor components cancels against the minus sign introduced by ϵ. These two products are also related by $(\chi \cdot \xi)^\dagger = \overline{\chi} \cdot \overline{\xi}$.
Now comes the interesting part, any product between two spinors of the form above, is Lorentz invariant. The two spinors need to be of the same type, either they must both have a bar or not. These kind of product will definitely come in handy later.

10.2 Dimensional Analysis

Let us propose the following Lagrangian density for a simple supersymmetric theory. It consists of a complex scalar field and a weyl spinor field.

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^\dagger + \xi^\dagger i\bar{\sigma}^\mu\partial_\mu\xi$$

The difficulty now is to find a transformation between these two that leaves the fields intact. Here dimensional analysis is a useful tool. Since we have set c and $\hbar$ to zero $[L] = [T] = [M]^{-1} = [E]^{-1}$. It is common practice to denote the

dimension in powers of the mass. This means that length and time have dimension -1 and mass and energy have dimension 1. The action has to be dimensionless. The Lagrangian density is integrated over 4 spacetime dimension to yield the action. Every component of spacetime has dimension -1 this means that partial differentiation has dimension 1 and the Lagrangian density has dimension 4. We can now deduce that the boson field and spinor field have dimensions $[\phi] = 1$ and $[\xi] = 3/2$.
We should have a single transformation, that simultaneously transforms a boson field into a fermion field and the other way around. $\phi \to \phi + \delta\phi$ where $\delta\phi$ should be proportional to ξ and an infinitesimal parameter in some way. This can only be solved by taking the infinitesimal parameter to be a constant weyl spinor. The dot product we've seen in the last section would seem a perfect candidate. So the scalar field transforms would as

$$\delta\phi = \epsilon \cdot \xi \tag{51}$$

This means that the dimension of the infinitesimal parameter is $[\epsilon] = -1/2$. Secondly we need the transformation of the fermion field. The fermion field has dimension 3/2, this means that one needs more than just the infinitesimal parameter and the boson field. A logical extra candidate would be the partial derivative, this however has a Lorentz index so we need another dimensionless element with a Lorentz index to contract it with. A possible transformation would seem. Lastly we need to be a left handed spinor the transformation to be a left handed spinor. When all these factors are taken into account the resulting transformation is

$$\delta\xi_a = -i(\sigma^\mu)_{a\dot{b}}\bar{\epsilon}^{\dot{b}}\partial_\mu\phi \tag{52}$$

One could check that these transformations indeed leave the Lagrangian invariant. This is quite a lengthy procedure which we will not replicate here. [9] The construction of the SUSY algebra will bring enough challenges in the upcoming section.

10.3 The Super-Poincaré algebra

We will not work out the charges as before, instead we will go straight to an equation of the form of 35. Such an equation always holds.
In the transformations two infinitesimal parameters appear; ϵ and $\bar{\epsilon}$. Which means that there must be two spinor charges, which we choose to be left-chiral. The two possible invariant combinations are of course $\bar{\epsilon} \cdot \bar{Q}$ and $\epsilon \cdot Q$. The relation between the charges and the transformations are given by (eqn 35)

$$[\bar{\epsilon} \cdot \bar{Q} + \epsilon \cdot Q, \phi] = -i\epsilon \cdot \xi \tag{53}$$

$$[\bar{\epsilon} \cdot \bar{Q} + \epsilon \cdot Q, \xi] = -i(\sigma^\mu)_{a\dot{b}}\bar{\epsilon}^{\dot{b}}\partial_\mu\phi \tag{54}$$

The bracket is bilinear and these equations should hold for arbitrary ϵ which implies that

$$[\epsilon \cdot Q, \phi] = -i\epsilon \cdot \xi \tag{55}$$

$$[\bar{\epsilon} \cdot \bar{Q}, \xi] = -i(\sigma^\mu)_{a\dot{b}}\bar{\epsilon}^{\dot{b}}\partial_\mu \phi \tag{56}$$

$$[\bar{\epsilon} \cdot \bar{Q}, \phi] = [\epsilon \cdot Q, \xi] = 0 \tag{57}$$

Now using the same strategy as in the last part of section 9, we work out

$$\delta_\beta \delta_\epsilon \phi - \delta_\epsilon \delta_\beta \phi = [[\epsilon \cdot Q + \bar{\epsilon} \cdot \bar{Q}, \beta \cdot Q + \bar{\beta} \cdot \bar{Q}], \phi] \tag{58}$$

When working this out we have to be careful, elements of the infinitesimal parameters ϵ^a are Grassmann numbers. Whereas the charges are operators. This means that we can move the infinitesimal parameters through the other elements by introducing a minus sign. The order of Q we however cannot change. We can rewrite

$$[\epsilon \cdot Q, \beta \cdot Q] = \epsilon^a Q_a \beta^b Q_b - \beta^b Q_b \epsilon^a Q_a = -\epsilon^a \beta^b Q_a Q_b - \epsilon^a \beta^b Q_b Q_a = -\epsilon^a \beta^b \{Q_a, Q_b\}$$

If we do this for all the components of equation 58, we can write this as

$$\delta_\beta \delta_\epsilon \phi - \delta_\epsilon \delta_\beta \phi = [\mathcal{O}, \phi]$$

where

$$\mathcal{O} = -\epsilon^a \beta^b \{Q_a, Q_b\} + \epsilon^a \bar{\beta}^{\dot{b}} \{Q_a, \bar{Q}_{\dot{b}}\} + \bar{\epsilon}^{\dot{a}} \beta^b \{\bar{Q}_{\dot{a}}, Q_b\} - \bar{\epsilon}^{\dot{a}} \bar{\beta}^{\dot{b}} \{\bar{Q}_{\dot{a}}, \bar{Q}_{\dot{b}}\} \tag{59}$$

Next we can work out the left-hand side of equation 58, we find

$$\delta_\beta \delta_\epsilon \phi = \delta_\beta \epsilon^a \xi_a \tag{60}$$

$$= \epsilon^a \delta_\beta \xi_a \tag{61}$$

$$= -i\epsilon^a (\sigma^\mu)_{a\dot{b}} \bar{\beta}^{\dot{b}} \partial_\mu \phi \tag{62}$$

Here we recognize $-i\partial_\mu$ as $[P_\mu, \phi]$. This means that we can write

$$\delta_\beta \delta_\epsilon \phi - \delta_\epsilon \delta_\beta \phi = [(\epsilon^a \bar{\beta}^{\dot{b}} (\sigma^\mu)_{a\dot{b}} + \bar{\epsilon}^{\dot{b}} \beta^a (\sigma^\mu)_{a\dot{b}}) P_\mu, \phi]$$

when we compare this to 59

$$-\epsilon^a \beta^b \{Q_a, Q_b\} + \epsilon^a \bar{\beta}^{\dot{b}} \{Q_a, \bar{Q}_{\dot{b}}\} + \bar{\epsilon}^{\dot{a}} \beta^b \{\bar{Q}_{\dot{a}}, Q_b\} - \bar{\epsilon}^{\dot{a}} \bar{\beta}^{\dot{b}} \{\bar{Q}_{\dot{a}}, \bar{Q}_{\dot{b}}\} \tag{63}$$

$$= \epsilon^a \bar{\beta}^{\dot{b}} (\sigma^\mu)_{a\dot{b}} P_\mu + \bar{\epsilon}^{\dot{b}} \beta^a (\sigma^\mu)_{a\dot{b}} P_\mu \tag{64}$$

This has to hold for arbitrary ϵ and β which means that

$$\{Q_a, Q_b\} = \{\bar{Q}_{\dot{a}}, \bar{Q}_{\dot{b}}\} = 0 \tag{65}$$
$$\{Q_a, \bar{Q}_{\dot{b}}\} = (\sigma^\mu)_{a\dot{b}} P_\mu \tag{66}$$

Next up is finding the commutation relations with the components of Poincaré algebra and the supercharges. The susy transformations don't introduce any function of x on which the derivative of P_μ can act. So these commute. This is however not the case for Lorentz transformations.
In the section about chirality we have very briefly discussed how spinors translate in Lorentz transformations (equation 49). From Quantum Field theory we know that the left chiral spinor transforms as

$$\xi \to e^{\frac{i}{2}\omega^{\mu\nu}\sigma_{\mu\nu}}\xi$$

where $\sigma_{\mu\nu} = \frac{i}{4}(\sigma_\mu\bar{\sigma_\nu} - \sigma_\nu\bar{\sigma_\mu})$. For an infinitesimal transformation this results in

$$\xi \to \xi'(x) = e^{\frac{i}{2}\omega^{\mu\nu}\sigma_{\mu\nu}}\xi(x^\mu + \omega^{\mu\nu}x_\nu) \tag{67}$$
$$\approx \xi + \frac{1}{2}\omega^{\mu\nu}(i\sigma_{\mu\nu}\xi + x_\nu\partial_\mu\xi - x_\mu\partial_\nu\xi) \tag{68}$$

Now to work out the commutation relations we once again start with the equation of the form

$$-\frac{1}{2}[[\epsilon \cdot Q, \omega^{\mu\nu}M_{\mu\nu}], \phi] = (\delta_\epsilon\delta_\omega\phi - \delta_\omega\delta_\epsilon\phi) \tag{69}$$
$$= \epsilon^a \frac{1}{2}\omega^{\mu\nu} i(\sigma_{\mu\nu})_a^{\ b}\xi_b \tag{70}$$

The factor -$\frac{1}{2}$ stems from the fact that the Lorentz infinitesimal parameter and charge are both antisymmetric. Also we recognize $[Q_b, \phi] = -i\xi_b$. This together with the fact that is has to hold for arbitrary infinitesimal parameters we find.

$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu})_a^{\ b} Q_b$$

This algebra still has some problems. When looking at $\delta_\beta\delta_\epsilon\xi - \delta_\epsilon\delta_\beta\xi$ we find this contains some factors different from when it acted on the boson field ϕ. This factor however is exactly the equations of motions for the spinor. It follows that the algebra only closes for on-shell spinors, i.e. spinors satisfying the equations of motions. For off-shell spinors, this algebra does not close. There are supersymmetric theories that do not close off-shell, but these have some complications, which we will not go into.
The fact that the SUSY algebra does or does not close comes from the fact that the

fermionic and bosonic degrees of freedom have to match. In our case the bosonic complex scalar field has two degrees of freedom. The fermionic spinor field has two complex values, so four degrees of freedom. On-shell the equation of motions imposes constraints decreasing the degrees of freedom of the spinor by two. So we see that on-shell the degrees of freedom match, while off-shell they don't. [2]
The charges Q_a are the odd charges of the symmetry. The spacetime charges are the even ones. Additional charges corresponding to internal symmetries can be added to the super-Poincaré algebra. These "internal" charges are also even. And the (anti)-commutation relations are of the form:

$$[\text{even}, \text{even}] = \text{even}, \ \{\text{odd}, \text{odd}\} = \text{even}, \ [\text{even}, \text{odd}] = \text{odd}$$

10.4 Super multiplets and Extended Supersymmetry

We know that the charges are in fact operators in the same way that field in QFT are operators. These supersymmetry charges actually raise and lower the helicity of a massless state by $\frac{1}{2}$. From the anticommutation relations of Q we know that it $(Q)^2 = 0$. This means that for a minimal supersymmetric theory all particles come in pairs of a boson and a fermion, called supermultiplets. Every particle has a superpartner, which means that supersymmetry requires a whole bunch of extra particles. None of which have ever been seen. In the case of the Minimal Supersymmetric Standard Model (MSSM) all the superpartners of fermions have a prefix s: including sfermions, squarks and sleptons. The superpartners of bosons have a -ino suffix, e.g. Zino, photino.
It is possible to construct a theory with more than one supersymmetry transformation. The number of supersymmetries in a theory usually is chosen to be a power of 2 and is denoted as $\mathcal{N} = 1, 2, 4, 8$. The number of components of the irreducible spinor times the number of supersymmetry generators $\mathcal{N}$ can maximally be 32. More supersymmetry generators forces the theory to have particles of spins that are too

D	q	$\mathcal{N}_{max}$
2	1	32
3	2	16
4	4	8
5	8	4
6	8	4
7	16	2
8	16	2
9	16	2
10	16	2
11	32	1
12+	64+	0

Figure 9: The maximal number of supersymmetry generators in D dimensional spacetime and the corresponding number of components of an irreducible spinor q.

[2]Introducing an auxiliary bosonic field that has no degrees of freedom on-shell and has two degrees of freedom off-shell can solve this problem. A way of making sure this is the case is not including a kinetic term for the field (usually called F). The simplest real term depending of F and $F^\dagger$ is $FF^\dagger$.

high. It is simply not known how to incorporate such particles into a theory and therefore we disregard such theories. In four dimensions a spinor has 4 components which gives a maximum of $\mathcal{N} = 8$. It takes exactly those 8 steps of $\frac{1}{2}$ to transform a particle of helicity 2 into a particle of helicity -2. Any more generators will force the theory to have particles of spin more than two. The maximal number of dimensions that supersymmetry can live in is 11. In 11 dimensions a spinor has 32 components, meaning that a $\mathcal{N} = 1$ supersymmetry is the only possible supersymmetry.

11 Gauging global symmetries

The earliest field theory having a gauge symmetry were the Maxwell's equations. The importance of these symmetries remained unnoticed for quite some time. Nowadays, all fundamental interactions of the Standard Model (SM) are obtained by gauge symmetries ($U(1)$, $SU(2)$ and $SU(3)$). Roughly speaking $U(1)$ describes electromagnetism, $SU(2)$ the weak interaction and $SU(3)$ the strong interaction.
So far we've only encountered transformations that were performed identically at every point in space. These symmetries are called global or rigid symmetries. However, we can also demand that the action is invariant under a symmetry which is spacetime dependent, such a symmetry is called a local or gauge symmetry. This is a much stricter symmetry, in fact a global symmetry is just a local symmetry whose parameters are fixed in spacetime.
As an example we will give the construction of the Lagrangian of Quantum Electrodynamics (QED). We begin with the Dirac field (equation 48) and stipulate that it should be invariant under the transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x)$$

This transformation varies arbitrarily from point to point, making it a gauge transformation. The terms in the Lagrangian that don't have derivatives are naturally invariant under this transformation and do not give further restrictions. The difficulty arises with the terms containing derivatives. The derivative of $\psi(x)$ in the direction of the vector a^μ is given by the definition:

$$a^\mu \partial_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\psi(x + \epsilon a) - \psi(x))$$

However, in this theory, the field transforms differently at the two point x and $x + \epsilon a$. This means that this quantity has no easy transformation law. To remedy this we can define a covariant derivative of the field which transforms in a similar manner as the field itself. To start off, we define a scalar quantity $U(x, y)$ that links the local transformations from one to point to the next.

$$U(y,x) \rightarrow e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}$$

At zero separation $U(x,x) = 1$. $\phi(y)$ now transforms in the same way as $U(y,x)\psi(x)$. This let's us define a more meaningful derivative known as the covariant derivative D_μ

$$a^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\psi(x+\epsilon a) - U(x+\epsilon a, x)\psi(x))$$

If the phase is continuous, then U can be expanded in the separation of the two points

$$U(x+\epsilon a, x) = 1 - ie\epsilon a^\mu A_\mu(x)$$

This introduces a new vector field A_μ. This field contains the information of how the field transforms from point to point. Here we have extracted a constant e from A_μ. The field A_μ is called a gauge field or connection. The covariant derivative takes the form

$$D_\mu \psi(x) = \partial_\mu \psi(x) + ieA_\mu \psi(x)$$

Where A_μ transforms as $A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$. A simple check is to see how $D_\mu \psi$ transforms

$$D_\mu \psi(x) \rightarrow (\partial_\mu + ie(A_\mu - \frac{1}{e}\partial_\mu \alpha)e^{i\alpha(x)}\psi(x) \tag{71}$$

$$= e^{i\alpha(x)} D_\mu \psi(x) \tag{72}$$

This means that by just replacing the regular derivative by the covariant derivative. The Lagrangian density is invariant under a local phase transformation, it is however not the most general Lagrangian one can come up with. We can add a kinetic energy term for the field A_μ. From which we obtain the well known QED Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field A_μ is the vector potential, which is actually a spin-1 field. The very existence of this field is a consequence of local symmetry. So just by imposing local symmetry, we have gained an additional particle and all the interactions between the photon and electron are fixed by the local symmetry. These interactions are described by the term that contains both ψ and A_μ, which is $e\bar{\psi}\gamma^\mu\psi A_\mu$.

12 Supergravity

In this section we will only briefly touch the subject of general relativity, taking a look at the differences between special and general relativity. Then we will start constructing a supersymmetric gravitational theory.
Supergravity has some promising properties. The supersymmetry delays the divergences in such a theory and maximal supergravity might even be renormalizable. [26] It is also highly restricted, maximal supergravity of course being even more restricted. Just as in the case of the electromagnetic Lagrangian, all of the interactions are set by the theory.

12.1 General Relativity

Special relativity postulates that the speed of light is constant in all inertial frames of reference. As a result the length of a line element between two points that differ by $\mathrm{d}x$ is constant in every inertial reference frame

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

with $\eta = \mathrm{diag}(1, -1, -1, -1)$. As we have seen these lengths of line elements are invariant under the Lorentz transformation.
General relativity generalises special relativity. Most importantly it allows curved spacetimes [5, 10]. In this case the length of a line element is defined by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

where the (symmetric) metric tensor $g_{\mu\nu}$ is allowed to be spacetime dependent. Just like in special relativity $g_{\mu\nu}$ and $g^{\mu\nu}$ are used to lower or raise indices. A quick example is the two sphere in coordinates θ and ϕ

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi$$

It follows that $g_{\theta\theta} = 1$ and $g_{\phi\phi} = \sin^2\theta$. This is an example of a 2 dimensional space that is obtained by embedding it in a 3 dimensional space. However, $g_{\mu\nu}$ can be an arbitrary function which in general is not possible to embed in a D+1 dimensional space. This means $g_{\mu\nu}$ can be intrinsically curved.
In general relativity, gravity is not a force, it is a geometry. Objects follow geodesics, defined by the shortes path $\int_a^b \mathrm{d}s$ between two points. Since the space is not flat, these geodesics can appear to be curved.
In euclidian space a geodesic is a straight line. In general relativity these geodesics can have interesting properties. Consider for example two ants walking in a straight line on a 2 sphere. Even if these ants start of parallel, their paths will eventually

cross.
Secondly Einstein assumed matter (and energy) sources gravity, this means that matter curves the space. This is often thought of as heavy object making a ”dip” in a spanned sheet.
These assumption can be translated into two mathematically well defined principles of general relativity. These principles are

- Physics is invariant under general coordinate transformations

$$x^{\mu'} = X^{\mu'}_{\nu} x^{\nu} \quad \text{where} \quad X^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}$$

  This further generalises the Lorentz invariance of special relativity. This means that the line element in terms of $x^{\mu'}$ should be of equal length when compared to that in terms of x^{μ}.

- The Equivalence principle, which states that gravity cannot be distinguished from acceleration. Locally there is no difference between free falling and a particle not undergoing any gravity.

The general coordinate transformation is a type of local symmetry and as we have seen this leads to a gauge field. The “gauge field of GR” is the Christoffel symbol. This has to do with the fact that one cannot just freely move tensors around. They can be moved around with a procedure called parallel transport. This is fixed by introducing a covariant derivative as follows.

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}) + K^{\mu}_{\ \nu\rho}$$

so that

$$\nabla_\mu x_\nu = \partial_\mu x_\nu - \Gamma^{\lambda}_{\mu\nu} x_\lambda$$

The Christoffel symbol is expressed in terms of the metric and the so-called contorsion $K^{\mu}_{\nu\rho}$). This contorsion is a quantity that is not fixed by GR and is usually set to zero resulting in symmetric Christoffel symbols ($\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu}$). SUGRA in it simplest form, however, requires a nonzero contorsion. The contorsion can be expressed in terms of the antisymmetric (in indices ρ, ν) torsion $T^{\mu}_{\rho\nu}$ as $K^{\mu}_{\rho\nu} = \frac{1}{2}(T_{\rho\ \nu}^{\ \mu} + T_{\nu\ \rho}^{\ \mu} + T^{\mu}_{\rho\nu})$.
Finally we have the Riemann tensor, which is in essence the strength of the gauge field in every point. It shows some great similarities to the field strength of a Yang-Mills theory. Only in this case all indices are spacetime indices.

$$R_{\mu\nu\ \sigma}^{\ \ \rho} = \partial_\mu \Gamma^{\rho}_{\ \nu\sigma} - \partial_\nu \Gamma^{\rho}_{\ \mu\sigma} + \Gamma^{\rho}_{\ \mu\lambda}\Gamma^{\lambda}_{\ \nu\sigma} - \Gamma^{\rho}_{\ \nu\lambda}\Gamma^{\lambda}_{\ \mu\sigma} \tag{73}$$

By contracting two of the indices, we obtain the Ricci tensor $R_{\mu\nu} = R_{\rho\mu\ \nu}^{\ \ \rho}$. Further contractions gives the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu} = R^{\mu}_{\mu}$. The Ricci scalar is a quantity unaffected by general coordinate transformations, so this truly is a measure of the curvature at a given point in space. Taking all this information together it is possible to construct the famous Einstein equations.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$$

Where κ is a constant that has the value $-8\pi G$ (remember that c=1). $T_{\mu\nu}$ has appeared before and if of course the stress energy tensor. And it shows a relation between matter and curved spacetime.

12.2 Building a Supergravity theory

To create a theory of supergravity it is useful to start of with Einstein-Hilbert Lagrangian

$$S_{EH} = -\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-\det g}R$$

By varying this Lagrangian with respect to $g^{\mu\nu}$ this results in the equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

which are exactly the Einstein equations for an empty space. Next we can quite easily put matter in this curved space. Let's for example consider the kinetic term of an scalar field. In Minkowski space this is

$$S_{matter} = \frac{1}{2}\int \mathrm{d}^4x\partial_\mu\phi\partial_\nu\phi\eta^{\mu\nu}$$

now it becomes

$$S_{matter} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-\det g}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}$$

The $\sqrt{-\det g}$ is added in order to make it invariant under general coordinate transformations. In the case of a minkowski space this simple reduces to 1. And to be totally fair the partial derivative should be a covariant derivative. Only in the case of scalars these are identical.
This addition to the action changes the equations of motion to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{74}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\rho\phi)^2 \tag{75}$$

Although this works fine, quantum mechanically this theory has problems as we have already discussed.

12.3 The vielbein

In supergravity we do not only have bosons, but also fermions, which can cause problems in curved space. Their transformation rules are especially difficult to describe in curved space. The theory of GR can be reformulated in the so-called vielbein formalism. The idea is that at every point in space there is a tangent space around this point which is flat. The vielbein e_μ^a relates this flat spaced to the curved space through

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b n_{ab}$$

The vielbein has a "curved" index μ and "flat" index a. The curved index is transformed through general coordinate transformation, while the flat index is acted upon by Lorentz transformations. This is similar to some gauge groups that have multiple types of indices; gauge group indices and "spacetime" indices.
A description in terms of the vielbein is identical to that in terms of the metric tensor. Although the degrees of freedom don't seem to match. The metric tensor has $d(d+1)/2$ degrees of freedom; exactly that of a symmetric tensor. The vielbein seems to have d^2. However, through local Lorentz invariance $d(d-1)/2$ (number of generators of the Lorentz group) of these components can be put to zero. This results in the vielbein having exactly $d^2 - d(d-1)/2 = d(d+1)/2$ degrees of freedom.
The curved space is only locally flat, this automatically implies that the Lorentz transformations differs from point to point. This brings us to one more gauge field, called the spin connection ω_μ^{ab} and it is antisymmetric in a, b. It arises from local Lorentz invariance. The Lorentz covariant derivative is defined by

$$D_\mu = \partial_\mu + \frac{1}{2}\omega_\mu^{ab} M_{ab}$$

where M_{ab} are actually the generators of the Lorentz group as seen in previous sections. This is not a new symmetry imposed, this is just a result of the vielbein formalism. So it is expected that the spin connection are gauge fields that can be expressed in terms of the vielbein, in a similar way that the Christoffel symbol is expressed in terms of the metric. This is possible, but it can only be determined by the so-called vielbein postulate. It requires that the vielbein is covariantly constant. [11].

$$0 = D_\mu e_\nu{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a$$

This is equivalent to stating that $D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = T^a_{\mu\nu}$. We will however begin with a vanishing torsion. In this case ω is only a function of the vielbein.
Also the Riemann tensor can be expressed in terms spin connection or in essence in terms of the vielbein.

$$R^{ab}_{\mu\nu} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_{\nu c}{}^b - \omega_\nu^{ac} \omega_{\mu c}{}^b$$

This expression for the Riemann tensor is extremely similar to equation 73 and is in fact nothing else than the "regular" Riemann tensor (of equation 73) with two indices "flattened" using the vielbein.
Finally we can use all this to rewrite the Einstein-Hilbert action, we will drop the constants in front of the action for simplicity. In matrix form $g = e\eta e$ which leads to $\sqrt{-\det g} = \det e$ and

$$S_{EH} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-\det g}R = -\frac{1}{4}\int \mathrm{d}^4x \det e \; e_a^\mu e_b^\nu R_{\mu\nu}^{ab}$$

From this point out it is possible to choose different strategies to derive the supergravitational theory. A possible way would be to impose local SUSY on a system. This requires at least two fields (a boson and a fermion field) to start off with. Gauging this symmetry will result in multiple gauge fields. The gauge field(s) will be a spin 3/2 field which we call the gravitino. It is not so surprising to see that also the gauge field itself is a fermion, since the infinitesimal parameter and the SUSY charges are both spinors. There can be multiple gravitini, but there is also another gauge field. It is not possible to gauge just half a theory and all generators of translations and Lorentz transformations are also in the same algebra. The gauge field that arises here is the spin-2 graviton, which can suitably be described by the vielbein. Lastly the vielbein and the gravitini turn out to be superpartners, ensuring that theory still is supersymmetric.
The process just described requires a minimum of 4 fields, two to start off with and the gravitino and vielbein will then appear as gauge fields. A much easier way to come to the same conclusion is by continue building the Lagrangian. First adding a spin 3/2 fermion field and then requiring this Lagrangian to be locally supersymmetric. So this is exactly what we will continue to do.
In the 1940's Rarita and Schwinger [11] published a proposal for the kinetic term of a spin-3/2 field

$$\mathcal{L}_{RS} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma$$

where in curved space the partial derivative has to be replaced by a Lorentz covariant one. Here ϵ is the maximally asymmetric Levi-Civita symbol. $\epsilon^{0123} = -\epsilon_{0123} = -1$ For every odd permutation of $0, 1, 2, 3$ the sign changes. Even permutations leaves the sign the same. Odd and even simply addresses the amount of times it is needed to switch two consecutive elements. It also has to be noted the γ_μ aren't the usual gamma matrices. They have been "curved" using the vielbein: $\gamma_\mu = e_\mu^a\gamma_a$, where γ_a are the gamma matrices as we know them.
In order to have a theory that has supersymmetry we need an additional term which contains higher powers of ψ. So we expect our Lagrangian density to look like

$$\mathcal{L}(e,\psi) = \mathcal{L}_P(e) + \mathcal{L}_{RS}(e,\psi) + \mathcal{L}_{\psi^4}(e,\psi) \tag{76}$$

Where the last part contains only terms of order 4 and higher in the gravitino. For simplification we will now address the combination of the Palatini and Rarita-Schwinger action as $\mathcal{L}_P(e) + \mathcal{L}_{RS}(e, \psi) = \mathcal{L}_0(e, \psi, \omega(e))$. It will become clear in a second why we have also used ω as a variable, which is clearly not necessary.
In the case of the Palatini action the term $\frac{\delta L_p}{\delta \omega}$ vanished, because we assumed zero torsion. However now this variation revieces additional terms from the Rarita-Schwinger part of the Lagrangian density. By properly adjusting the spin connection ω we can ensure that once again $\frac{\delta \mathcal{L}_0}{\delta \omega} = 0$. From this statement it follows that

$$D_\mu e_\nu^{\ a} - D_\nu e_\mu^{\ a} = T^a_{\ \mu\nu} = -\frac{i}{2}\bar{\psi}_\mu \gamma^a \psi_\nu \tag{77}$$

This is exactly the same as the equation of the vielbein postulate. Now modifying the spin connection gives

$$\hat{\omega}_\mu^{\ ab}(e, \psi) = \omega_\mu^{\ ab}(e) + K_\mu^{a\ b}, \quad \text{where} \quad K_\mu^{a\ b} = -i(2\bar{\psi}^a \gamma^b \psi_\mu - 2\bar{\psi}^b \gamma^a \psi_\mu + \frac{1}{2}\bar{\psi}^a \gamma_\mu \psi^b)$$

So we now see that we have a nonzero contorsion. To distinguish the different spin connections, every covariant that uses $\hat{\omega}$ will be noted as $\hat{D}$.
As it turns out this Lagrangian density $\mathcal{L} = \mathcal{L}_0(e, \psi, \hat{\omega}(e, \psi))$ is already the full supersymmetric Lagrangian. There are multiple lengthy procedures to show that this statement is indeed true. However we will not go into this for much more. Finally it can be shown that the Lagrangian can be written in the form of equation 76 as follows

$$\mathcal{L} = \mathcal{L}_0(e, \psi, \omega(e)) - \frac{1}{4}\det e(K_a^{\ ac} K_{b\ c}^{\ b} + K^{abc} K_{cab})$$

where the last term clearly only contains terms of order 4 in the gravitino.

Supersymmetric transformations

Let's of course not forget the SUSY transformations. The lowest order transformations are

$$\delta_\epsilon e_\mu^a = -i\bar{\epsilon}\gamma^a \psi \tag{78}$$

$$\delta_\epsilon \psi = \hat{D}_\mu \epsilon \tag{79}$$

Higher order terms can be thought of, but these transformations work fine. Note that when you compare these to the transformation 51 and 52 and check how we defined the dot product, you will see that these transformations are nearly identical.

If we compute the commutation relations we will find

$$\delta_{\epsilon_1}\delta_{\epsilon_2}e_\mu^{\ a} - \delta_{\epsilon_2}\delta_{\epsilon_1}e_\mu^{\ a} = \hat{D}_\mu(\xi^\nu e_\nu^{\ a})$$
$$= \xi^\nu\partial_\nu e_\mu^{\ a} + e_\nu^{\ a}\partial_\mu\xi^{nu} + \xi^\nu\hat{\omega}_\nu{}^a{}_b e_\mu^{\ b} + i\xi^\nu\bar{\psi}_\nu\gamma^a\psi_\mu$$

where $\xi^\mu = i\bar{\epsilon}_2\gamma^\mu e_1$. The first two terms correspond to a general coordinate transformation of the curved index. The third term correspond to a Lorentz transformation with $\Lambda^a_b = \xi^\nu\hat{\omega}_\nu{}^a{}_b$ and the last term in an additional supersymmetry transformation with parameter $\xi^\mu\bar{\psi}_\nu$.
In a similar fashion as in section 10, the algebra only closes on shell. The variation of the gravitino

12.4 Extended Supergravity

In the section about supermultiplets and extended supersymmetries we briefly discussed the possibilities of more than a single supersymmetry generator. The same is possible with supergravity theories. We addressed that the maximal number of supersymmetry generators is 32 divided by the number of components of an irreducible spinor. A supersymmetry with the maximal amount of supersymmetry generators is of course called a maximal supersymmetry and the corresponding supergravity is called maximal supergravity.
Maximal supergravity is particularly interesting. Most of the interactions are fully defined by the requirements of local supersymmetry. It is possible to go one step beyond maximal supergravity. Some of the vectors that will arise in maximal supergravity transforms identical to the gauge field of the $U(1)$ symmetry (see section 11). A gauging (or gauge deformation) of supergravity then turns this abelian local symmetry into a non-abelian one. We will only discuss the ungauged maximal supergravities.
The maximal supergravity multiplet in four dimensions has a lot of fields, scalars, vectors, the vielbein and spin-$\frac{1}{2}$ and spin-$1\frac{1}{2}$ fermions. The amount of fields differs per dimension, but there is always just a single graviton. The number of gravitini is equal to the number of supersymmetry generators. In four dimensional maximal supergravity there are 8 gravitini, 28 vectors (spin-1 fields), 56 spin-$\frac{1}{2}$ fields and 70 scalars. We are particularly interested in the symmetries between the bosonic fields. None of the above fields have ever been discovered, but maximal supergravity also brings some positive features.

Supergravity in 11 Dimensions

In dimensions other than $D = 4$ some generalizations of the previous definitions need to be made. Most importantly the Lorentz transformations. These generators work on vectors as

$$M_{ab}V^c = \delta^c_a V_b - \delta^c_b V_a$$

and on spinors as

$$M_{ab}\psi = \frac{1}{2}\Gamma_{ab}\psi$$

where the gamma matrices can be generalized as

$$\{\Gamma_a, \Gamma_b\} = 2I\eta_{ab}, \quad \Gamma_{ab} = \frac{1}{2}\Gamma_a\Gamma_b - \Gamma_b\Gamma_a$$

where $\eta = \text{diag}(1, -1, -1, \ldots, -1)$ and I is the identity matrix in spinor indices. In the case of $D = 4$ these definitions are identical to the previous ones.
Let's now take a look at the supermultiplet in $D = 11$. In eleven dimensions the only possible supersymmetry is $\mathcal{N} = 1$. There is of course only a single graviton and gravitino.
A massless vector in 11 dimensions has 9 degrees of freedom, for exactly the same reasons as the photon having only two degrees of freedom in 4 dimensions [3]. The graviton is the symmetric traceless product of two of such vectors, this then has 44 degrees of freedom. The other bosonic field content is given by a maximally antisymmetric tensor product of three of such vectors $A_{\mu\nu\rho}$. This three-form has 84 degrees of freedom, exactly the amount of possible ways to pull three elements from 9 when order does not matter $\binom{9}{3}$. These bosonic degrees of freedom exactly match the 128 degrees of freedom of the spin-$\frac{3}{2}$ gravitino. The full maximal supergravity in 11 dimensions is given by $(e_\mu^{\ a}, \psi_\mu, A_{\mu\nu\rho})$. The full Lagrangian is given by

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}\det eR - \frac{1}{4}\det e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{96}\det eF_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} \\ - \frac{\sqrt{2}}{6912}\epsilon^{\kappa\lambda\mu\nu\rho\sigma\tau\upsilon\xi\zeta\omega}F_{\kappa\lambda\mu\nu}F_{\rho\sigma\tau\upsilon}A_{\xi\zeta\omega} \\ - \frac{\sqrt{2}}{384}\det e(\bar{\psi}_\rho\Gamma^{\kappa\lambda\mu\nu\rho\sigma}\psi_\sigma + 12\bar{\psi}^\kappa\Gamma^{\lambda\mu}\psi^\nu)F_{\kappa\lambda\mu\nu}\end{aligned} \quad (80)$$

where the fieldstrength $F_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]} = \frac{1}{6}\partial_\mu A_{\nu\rho\sigma} + 23$ terms, these 23 terms are of the same form with their indices switched around. In order the terms of this Lagrangian are: the Einstein Hilbert term, the Rarita-Schwinger term in 11 dimensions, the kinetic term of the field $A_{\mu\nu\rho}$, the topological term of $A_{\mu\nu\rho}$ and the $A_{\mu\nu\rho}$ fermionic interaction. Now do not be afraid, we will deduce most of the properties of lower dimensional supergravity using symmetry arguments. We will also mainly focus on the bosonic section of this Lagrangian density; these are the first, third and fourth term of this equation.

[3] Roughly speaking gauge invariance and the equations of motion force two degrees of freedom to be unphysical.

12.5 Kaluza-Klein reduction of $D = 11$ supergravity

Maximal supergravities can either be build from the ground up, using all the restrictions maximal supergravity lays down upon theory or one can consider dimensional reduction. Both of the resulting theories are equivalent. Having seen equation 80 we will opt for the latter.
We will use Kaluza-Klein reduction to reduce maximal supergravity in $D + n$ dimensions on a torus T^n down to D dimensions. Kaluza and Klein originally used KK reduction to unify gravity and electromagnetism in a 5 dimensional metric.
A n-dimensional torus is the product of n circles.

$$T^n = S^1 \times S^1 \times \cdots \times S^1$$

Taking x^μ as the coordinates of the D dimensional spacetimes, i.e. the coordinates we would want to keep. And y^m the coordinates of the n-torus, i.e. the coordinates we would like to "throw away". Reduction on an n-torus then corresponds to the normal Fourier node expansion

$$\Phi(x, y) = \sum_{k_1,k_1,\ldots,k_n \in Z} \phi_{k_1 k_2 \ldots k_n} e^{i2\pi k_1 y^1/R_1} e^{i2\pi k_2 y^2/R_2} \ldots e^{i2\pi k_n y^n/R_n} \tag{81}$$

and just keeping the the field $\phi_{00\ldots0}$. R_i are the radii of the torus and these are inversely proportional to masses of the fields. We live in an effectively 4 dimensional world, meaning that R_i should be small. This would however give huge masses for all the fields other than $\phi_{00\ldots0}$. We simply call these fields non-physical and throw them away.

Example 12. *Consider a massless scalar field subject to the Klein-Gordon equation* $(\partial_\mu \partial^\mu + \partial_y \partial^y)\Phi(x, y) = 0$ *in* $D+1$ *dimensions. In* D *dimensions this reduces to*

$$\partial_\mu \partial^\mu \phi_k - 4\pi^2 \frac{k^2}{R^2} \phi_k = 0$$

where clearly $4\pi^2 \frac{k^2}{R^2}$ *is identical to* m^2 *in the Klein-Gordon for a massive scalar field. Using the reasoning of the last paragraph, we would drop all the fields except for the massless* ϕ_0 *field.*

Dimensional reduction of the Vielbein

It is possible to brute force the Lagrangian (80) into a Lagrangian of a lower dimension in a similar way as the example. We will however use symmetry arguments to deduce properties of the lower dimensional supergravity theory.
First we use the local $SO(1, D+n-1)$ symmetry to write the vielbein in an upper triangular form. Capital indices are used to indicate indices that run from 0 to

$D+n-1$, regular indices run from 0 to $D-1$ and indices that have bar run from D to $D+n-1$. Finally greek letters are used for curved indices, while regular letters indicate the flat indices.

$$E_\Lambda^{\ A} = \begin{pmatrix} E_\mu^{\ a} & E_\mu^{\ \bar{a}} \\ 0 & E_{\bar{\nu}}^{\ \bar{a}} \end{pmatrix}$$

This breaks the local $SO(1, D+n-1)$ symmetry into a local $SO(1, D-1) \times SO(n)$. It can be seen that the degrees of freedom still match. The local $SO(1, D-1) \times SO(n)$ has exactly nD degrees of freedom less than the $SO(1, D+n-1)$ symmetry. This corresponds to the degrees of freedom used to set the nD components of the vielbein to zero. This can then further be parametrized into

$$E_\Lambda^{\ A} = \begin{pmatrix} \rho^\kappa e_\mu^\alpha & \rho^{1/n} V_{\bar{\nu}}^{\ \bar{a}} B_\mu^{\ \bar{\nu}} \\ 0 & \rho^{1/n} V_{\bar{\nu}}^{\ \bar{a}} \end{pmatrix} \tag{82}$$

where κ is a constant, the matrix $V_{\bar{\nu}}^{\ \bar{a}} \in SL(n)$ and $\rho = \det E_{\bar{\nu}}^{\bar{a}}$. $SL(n)$ is the special linear group, i.e. all matrices having unit determinant.
Once the supergravity in $D+n$ dimensions has been reduced to one in D dimensions, the indices of $V_{\bar{\nu}}^{\ \bar{a}}$ are no longer spacetime indices. A simple way to see this is that Lorentz transformations in D dimensions no longer act on these indices. This means that these are now in fact n^2-1 scalar fields. So in total the vielbein $E_\Lambda^{\ A}$ reduces to a single vielbein $e_\mu^{\ a}$, n vector vields $B_\mu^{\ \bar{\nu}}$ and a total of n^2 scalars $V_{\bar{\nu}}^{\ \bar{a}}$ and ρ.
In $D+n$ dimensions the Lagrangian was invariant under a general coordinate transformation. The vielbein $E_\Lambda^{\ A}$ transforms as:

$$E_\Lambda^{\ A} \rightarrow E_{\Lambda'}^{\ A} = \frac{\partial x^\Omega}{\partial x^{\Lambda'}} E_\Omega^{\ A} \approx E_\Lambda^{\ A} + \xi^\Omega \partial_\Omega E_\Lambda^{\ A} + E_\Omega^{\ A} \partial_\Lambda \xi^\Omega \tag{83}$$

Or in other words $\delta_\xi E_\Lambda^{\ A} = \xi^\Omega \partial_\Omega E_\Lambda^{\ A} + E_\Omega^{\ A} \partial_\Lambda \xi^\Omega$. The field content of the D dimensional spacetime naturally inherits these symmetries from the $D+n$ dimensional one. A general coordinate transformation is of course a local symmetry, meaning that the infinitesimal parameter ξ is a spacetime dependent.
The $D+n$ theory is invariant under any coordinate spacetime transformation. In D dimensions we can consider different "types" of such infinitesimal parameters such as those dependent on spacetime parameters of x^μ or the reduced parameters $y^{\bar{\mu}}$.
Firstly parameters of the type $\xi^\mu(x)$ just give a general coordinate transformation in D dimensions on the fields $e_\mu^a, B_\mu^{\ \bar{\nu}}, V_{\bar{\nu}}^{\ \bar{a}}, \rho$.
General coordinate transformations that are linear in coordinates y, i.e. $\xi^m(y) = g_n^m y^n$ gives a global $SL(n)$ symmetry on the fields.

$$\delta V_{\bar{\nu}}^{\ \bar{a}} = g^{\bar{\mu}}_{\ \bar{\nu}} V_{\bar{\mu}}^{\ \bar{a}}, \quad \delta B_\mu^{\ \bar{\nu}} = -g^{\bar{\nu}}_{\ \bar{\sigma}} B_\mu^{\ \bar{\sigma}}$$

where we have chosen the matrices $g^{\bar{\mu}}_{\bar{\nu}}$ to be traceless. These matrices form the special Linear algebra $\mathfrak{sl}(n)$ corresponding to the special linear group $SL(n)$, which are all matrices of unit determinant.
This trace part is harder since this also acts on the D dimensional vielbein. Rescaling of ρ and B_μ^m is required.
Finally we should not forget the local Lorentz symmetry the theory has. We have seen that the $SO(1, D+n-1)$ symmetry splits into a local $SO(1, D-1) \times SO(n)$ symmetry. This means that the scalar fields have a local $SO(n)$ symmetry. It could for example be used to set $\frac{1}{2}n(n-1)$ of the scalar fields to zero. Thus this leaves us with a total of $\frac{1}{2}n(n+1)$ physical scalar fields (including the dilaton).
So in conclusion there are n vectors with a global $SL(n)$ symmetry, $\frac{1}{2}n(n+1)$ scalars with a global $SL(n)$ and local $SO(n)$ symmetry. As it turns out the rescaling of the dilaton which transformed under the trace of $g^{\bar{\mu}}_{\bar{\nu}}$ promotes this symmetry to a global $GL(n)$ symmetry, the is the general group consisting of all invertible matrices. This scalar fields are usually described by non-linear Coset-space σ models. [4]

Hodge duals and $p, D-p-2$ duality

To show the function of the hodge dual and the $p, D-p-2$ duality will quickly use the notations of differential form
Any p-form is a function $\omega : T_pM \times T_pM \times \cdots \times T_pM \to \mathbb{R}$ with p terms of T_pM. The wedge product $\wedge$ is the alternating bilinear form $\omega \wedge \beta(v_p, w_p) = \omega(v_p)\beta(w_p) - \omega(w_p)\beta(v_p)$. It is skew commutative, depending on the degrees of the forms it is either commutative or anticommutative. For 1-forms it is clearly anticommutative. Any p-form can be written as

$$\omega = \sum_{\mu\nu\rho\ldots} f_{\mu\nu\rho\ldots} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \ldots$$

So far we have encountered scalars and tensors. Some of these are in fact differential forms. Scalars are 0-forms and vectors 1-forms. The metric, although a tensor is not antisymmetric and thus no p-form. The tensor $A_{\mu\nu\rho}$ is a 3-form.
The exterior derivative, denoted by d, is simply the directional derivative when acting on a real function f. Acting on a 2-form $f_\mu(x)\mathrm{d}x^\mu$ it act as follows

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu, \qquad \mathrm{d}\omega = \sum_\mu \mathrm{d}f_\mu \wedge \mathrm{d}x^\mu$$

[4]There is one additional symmetry $\delta B_\mu^{\bar{\nu}} = \partial_\mu \xi^{\bar{\nu}}$ for $xi(x)$ which corresponds directly to the transformations of the spin-1 gauge field (eqn 71). This relates to the gauged supergravities as we have briefly mentioned.

The exterior derivative of a p-form is a $p+1$-form.
An important operation is the Hodge star operation. This operation relates p-forms with $D-p$ forms. In three dimensional euclidian space it is simply given by

$$\star \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z$$
$$\star \mathrm{d}y = -\mathrm{d}x \wedge \mathrm{d}z$$

In a manifold with a metric g the definition is slightly different. Let A be a p-form on a D-dimensional manifold then

$$\star A_{i_1,i_2,\ldots,i_{D-p}} = \frac{1}{!p} A^{j_1,j_2,\ldots,j_p} \sqrt{|\det g|}\, \epsilon_{j_1,j_2,\ldots,j_p,i_1,i_2,\ldots,i_{D-p}} \tag{84}$$

where ϵ is the antisymmetric Levi-Civita symbol.

Example 13. *Let's consider the Lagrangian density of QED as in section 11. The Gauge field A_μ is a one form, it can be equivalently written as $A = \sum A_\mu \mathrm{d}x^\mu$. The fieldstrength $F_{\mu\nu}$ is simply given by the exterior derivative of A; $F = \mathrm{d}A$. This term of action can then be written as*

$$\int \mathrm{d}^p x F^{\mu\nu} F_{\mu\nu} = \int \star F \wedge F$$

Note that $\star F \wedge F$ already is a 4-dimensional "volume" element.

It also allows us to write the bosonic section of the Lagrangian 80 as [2]

$$S = \int \star R - \frac{1}{2} \star \mathrm{d}A \wedge \mathrm{d}A - \frac{1}{6} \mathrm{d}A \wedge \mathrm{d}A \wedge A \tag{85}$$

As can be seen from the example this part of the Lagrangian can equivalently be described by a different field $\mathrm{d}B = \star \mathrm{d}A$, because applying the Hodge dual twice gives back the original field up to a sign. This gives a duality between p-forms and $D-p-2$-forms. A p-form is not yet physical, it still has a gauge freedom. The fieldstrength $\mathrm{d}A$, a $p+1$-form, is a physical field, which is equivalent to its dual, a $D-(p+1)$-form, which is the fieldstrength of the dual field of A $D-p-2$-form. This duality between p-forms and $D-p-2$-forms is essential for understanding the affine Lie algebras in two dimensional supergravity.

Dimensional reduction of $A_{\mu\nu\rho}$

So far we have only reduced one of the two bosonic fields in $D = 11$ supergravity. The reduction of the 3-form $A_{\mu\nu\rho}$ on a torus T^n gives 1 3-form, n 2-forms, $\binom{n}{2}$

1-forms (vectors) and $\binom{n}{3}$ 0-forms (scalars). The 1-form for example needs two "reduced" spacetime indices and only a single "real" spacetime index. There are exactly $\binom{n}{2}$ ways in which you could do this, thus the $\binom{n}{2}$ 1-forms.
We have seen the $GL(n)$ with a local $SO(n)$ of the scalar fields that arised from the vielbein, when reducing $D+n$ dimensional supergravity to D dimensional supergravity. The dimensional reduction of $A_{\mu\nu\rho}$ also gives a series of scalars. In some dimensions the p-forms might be equivalent to scalars via the p- $D-p-2$-form duality. All these scalar fields combined have a symmetry group, which is an extension of the $GL(n)/SO(n)$ group.

The hidden symmetry in various dimensions

The hidden symmetry has different forms in various dimensions. In 11 dimensions there isn't one and as the dimensions decrease the symmetry grows. In 10 dimensions it turns into an $\mathbb{R}^+$ symmetry and from 9 dimensions onwards the symmetry can most easily be depicted by a Dynkin diagram, figure **??**.

The local symmetry in maximal supergravity is always given by the maximal compact subgroup (definition follows in next section). In a $GL(n)$ symmetry, the maximal compact subgroup is $SO(n)$ just as expected. The extension of the group that arises from the dimensional reduction of $A_{\mu\nu\rho}$ increases both the global and the local symmetry as can be seen in the table 10.
The Dynkin diagram of the hidden symmetry grows steadily. It also gives a good indication that the hidden symmetry is obtained by just adding another node to the diagram thus making it an E_8^+.

D	G	K
11	1	1
10	$\mathbb{R}^+$	1
9	$GL(2)$	$SO(2)$
8	$SL(3)\times SL(2)$	$SO(3)\times SO(2)$
7	$SL(5)$	$SO(5)$
6	$SO(5,5)$	$SO(5)\times SO(5)$
5	E_6	$USp(8)$
4	E_7	$SU(8)$
3	E_8	$SO(16)$
2	E_8^+	$K(E_8^+)$

Figure 10: The global symmetry group G and the local symmetry K.

Example 14. *As an example we will have a look at the supermultiplet of $D=4$ $\mathcal{N}=8$ maximal supergravity. As we have seen we can derive the supermultiplet simply from dimensional reduction of the 11 dimensional supergravity. This reduction on T^7 gives a single graviton, 7 vector fields and $\frac{1}{2}n(n+1)$ with a global $GL(7)$ and a local $SO(7)$ symmetry. Furthermore $A_{\mu\nu\rho}$ splits into a single 3-form, 7 2-forms, 21 vectors and 35 scalars. The 2-forms are equivalent to scalar fields and the 3-form is non-propagating; it is constant and it can be set to zero. The total bosonic multiplet has 70 scalars, 27 vectors and a single graviton. The 70 scalars are described by a global E_7 (the group corresponding to the E_7 algebra) symmetry and a local $SU(8)$ symmetry,* $\dim E_7 - \dim SU(8) = 133-63$ *exactly as expected.*

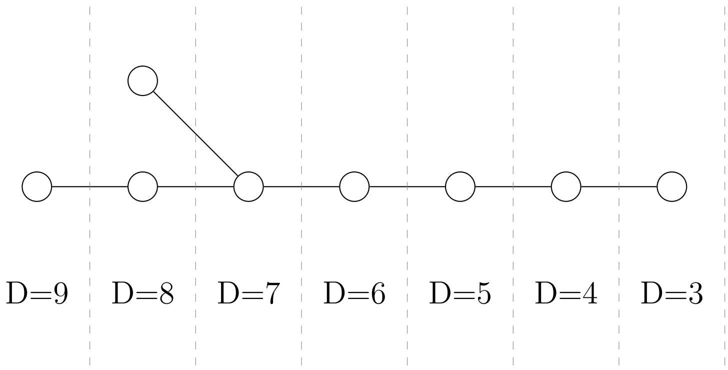


Figure 11: The Dynkin diagram of the hidden symmetry in various dimensions. The Dynkin diagram corresponding to a certain number of dimensions is diagram left of the corresponding striped line. In $D = 9$ it is A_1, in $D = 8$ A_2 etc.

13 Two dimensional maximal supergravity

Two dimensional theories of gravity have been studied for quite a while. At first to find solutions of the Einstein equation that were independent of one or more coordinates. An additional hidden symmetry appeared in the lower dimensional theory. In 1970, Geroch showed that the hidden symmetry was in fact infinite dimensional. This group is now known as the Geroch group. B. Julia was the first to realize the connection between these symmetries and the symmetries that appeared in maximal supergravity. He made the conjecture that the infinite dimensional symmetries in two dimensions is something generic for gravity. He was also the first to show that the Geroch group is actually the affine Lie algebra $A1^+$ and the first to realise that the central element of the Lie algebra acts as a shift operator on the conformal factor. [20]

Additional symmetries in low dimensional supergravity could also provide hints for the formulation of a proper theory of quantum gravity. Both the underlying symmetries and the correct conceptual framework. [20] The trivial low dimensional solutions could be lifted to a tiny subsection of the higher dimensional solutions.

In order to see the symmetries on the scalar section of the supergravity multiplet in two dimensions we will first change notation to that of a non-linear coset space sigma model. This will greatly reduce the number of terms in the Lagrangian density and will make it easy to write down the action of the different elements in the Lie algebra.

Non-linear Coset space sigma models

We will use non-linear coset space sigma models to describe the scalar section of the supergravitational theories. It was originally used to describe a spin-0 meson

called σ.
We will view the scalars as function from the spacetime manifold M to a coset space G/K. In the case of the previous example this coset space would be $E_7/SU(8)$. By definition, Lie groups are differential manifolds. We will view the scalars ϕ^a as functions from this spacetime M to the coset space G/K. Let x^μ be coordinates of this spacetime and ϕ^a coordinates of the coset space. The action corresponding to these fields takes the form

$$S = \int_M \mathrm{d}x \sqrt{\det g} g^{\mu\nu} \mathcal{G}_{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b = \int \star \mathrm{d}\phi^a \mathrm{d}\phi^b \tag{86}$$

In case of the manifold M being 1 dimensional parameterized by t one finds nothing more than the equations of motion of a geodesic

$$\frac{\mathrm{d}\phi^a}{\mathrm{d}t} \frac{\mathrm{d}\phi^b}{\mathrm{d}t} \mathcal{G}_{ab} = 0$$

The Lie algebra corresponding can be split up into an even and odd eigenspace $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$. $\mathfrak{l}$ is the part of the $\mathfrak{g}$ that is left invariant by the involution operator. Recall that this this sends the elements e_i, f_i, h_i to $-f_i, -e_i, -h_i$ respectively. The elements of $\mathfrak{p}$ receive a minus sign under the involution. This decomposition is orthogonal with respect to the Cartan-Killing form. Let θ be the involution operator be θ then

$$\mathfrak{l} = \{x \in \mathfrak{g} | \theta(x) = x\}$$
$$\mathfrak{p} = \{x \in \mathfrak{g} | \theta(x) = -x\}$$

The commutation relations that follow from this split are:

$$[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}, \quad [\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$$

It immediately follows that $\mathfrak{p}$ is a subset of $\mathfrak{g}$, but it is not a subalgebra. The algebra $\mathfrak{l}$ corresponds to the local Lie group K. In supergravity K is always the maximally compact subgroup as constructed here.
Let $V(x)$ be an element of G. It transforms under a global transformation as $V(x) \to gV(x), \quad g \in G$. In a coset space two elements are identical is $g' = gk$ for some $k \in K$. As example one could take the modulo 2 group, in that case 3 is identical to 1, altough this group is definitely not a Lie group. The action should of course also be invariant under a local transformation $V(x) \to V(x)k(x)$.
Now one could fix the gauge, often $V(x)$ is chosen to be upper triangular also called the Borel gauge (if K is maximal compact this is always possible). This Borel gauge only considers the Cartan subalgebra and the elements corresponding to the positive roots. The transformation under G generally does not conserve this

gauge choice. We could then fix $k(x)$ in such a way that it conserves this gauge choice.

$$V(x) \to gV(x)k(V(x), g)$$

In order to preserve the gauge choice of $V(x)$, $k(V(x), g)$ depends nonlinearly on $V(x)$, hence the name; nonlinear coset space sigma models.
The Maurer-Carter form of the group element $V(x)$ is given by

$$V(x)^{-1}\partial_\mu V(x)\mathrm{d}x^\mu$$

This element is a 1-form and an element of the algebra $\mathfrak{g}$ at the same time. Just consider the group $SO(2)$ parameterized by $O = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ gives

$$O^{-1}\partial_\mu O = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_\mu \alpha \mathrm{d}x^\mu$$

This is simultaneously a 1-form and an element of $\mathfrak{so}(2)$. Such an element can then be split into an odd and even part

$$V(x)^{-1}\partial_\mu V(x) = Q_\mu(x) + P_\mu(x) \qquad Q_\mu(x) \in \mathfrak{l},\ P_\mu(x) \in \mathfrak{p}$$

This is easily done by defining

$$P_\mu = \frac{1}{2}(V(x)^{-1}\partial_\mu V(x) - \theta(V(x)^{-1}\partial_\mu V(x))) \tag{87}$$

$$Q_\mu = \frac{1}{2}(V(x)^{-1}\partial_\mu V(x) + \theta(V(x)^{-1}\partial_\mu V(x))) \tag{88}$$

The Maurer-Carter form is of course left intact under global transformations. The local transformations of K however do not.

$$V(x)^{-1}\partial_\mu V(x) \to V(x)^{-1}k(x)^{-1}\partial_\mu k(x)V(x)$$

It follows directly that P_μ and Q_μ transform as

$$Q_\mu \to k(x)^{-1}Q_\mu k(x) + k(x)^{-1}k(x)$$
$$P_\mu \to k(x)^{-1}P_\mu k(x)$$

To construct a Lagrangian that is invariant under global, local and general coordinate transformations is now quite simple. "Squaring" P_μ with respect to the Cartan Killing form gives

$$S = \int_M \mathrm{d}x\sqrt{\det g}g^{\mu\nu}\kappa(P_\mu, P_\nu) \tag{89}$$

where κ is the Cartan-Killing form. It is possible to check that this is indeed similar to equation (86). Note that the degrees of freedom of P_μ exactly match those of the cosetspace G/K.
A last thing which we will need in the upcoming section is the K covariant derivative D_μ. It is easy to check that if $D_\mu P_\nu$ is defined as follows it indeed transforms $k(x)^{-1}D_\mu P_\nu k(x)$

$$D_\mu P_\nu = \partial_\mu P_\nu + [Q_\mu, P_\nu]$$

Finally if we work out the equations of motion we will find that the conserved current of this Lagrangian is

$$\partial^\mu j_\mu = 0 \quad \text{for} \quad j_\mu = V(x)^{-1}P_\mu V(x)$$

Example 15. *This is shows the reduction of the 4 dimensional Einstein equations to 3 dimensions and how the fields can be described by the nonlinear sigma coset space model. [18]*
First we start with the vielbein that is parameterized as follows

$$E_\Lambda^{\ A} = \begin{pmatrix} \Delta^{-1/2}e_\mu^{\ a} & B_\mu \\ 0 & \Delta^{1/2} \end{pmatrix}$$

The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}ER = -\frac{1}{4}eR(e) + \frac{1}{8}g^{\mu\nu}\Delta^{-1}\partial_\mu\Delta\Delta^{-1}\partial_\nu\Delta - \frac{1}{16}e\Delta^2 g^{\mu\nu}g^{\kappa\rho}F_{\mu\kappa}F_{\nu\kappa}$$

where $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The second term of this equation already shows close resemblance to the sigma models. If we use duality to define the dual scalar $\tilde{B}$ of vector B_μ, $\epsilon_{\mu\nu\rho}\partial^\rho\tilde{B} = \Delta^2 F_{\mu\nu}$ we have also slightly rearranged this field with the extra Δ^2 term.

$$\mathcal{L} = -\frac{1}{4}eR(e) + \frac{1}{8}g^{\mu\nu}\Delta^{-2}(\partial_\mu\tilde{B}\partial_\nu B + \partial\mu\Delta\partial_\nu\Delta) \tag{90}$$

This Lagrangian density got a global $SL(2)$ and local $SO(2)$ symmetry.
Take the group $SL(2)$ with a local $SO(2)$. We choose the upper triangular gauge. As you can see $SL(2)$ does not generally preserve this upper triangular gauge. However with local $SO(2)$ any element of $SL(2)$ can easily be put back into this form. Choose V of the form:

$$V = \begin{pmatrix} \Delta^{1/2} & \tilde{B}\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix}$$

The Maurer-Cartan form now reads:

$$V^{-1}\partial_\mu V = \begin{pmatrix} \frac{1}{2}\Delta^{-1}\partial_\mu\Delta & \Delta^{-1}\partial_\mu B \\ 0 & -\frac{1}{2}\Delta^{-1}\partial_\mu\Delta \end{pmatrix}$$

As you can see this is indeed an element of $\mathfrak{sl}(2)$*.*
When these are split into an even and odd part the result is.

$$P_\mu = \begin{pmatrix} \frac{1}{2}\Delta^{-1}\partial_\mu\Delta & \frac{1}{2}\Delta^{-1}\partial_\mu B \\ \frac{1}{2}\Delta^{-1}\partial_\mu B & -\frac{1}{2}\Delta^{-1}\partial_\mu\Delta \end{pmatrix}, \qquad Q_\mu = \begin{pmatrix} 0 & \frac{1}{2}\Delta^{-1}\partial_\mu B \\ -\frac{1}{2}\Delta^{-1}\partial_\mu B & 0 \end{pmatrix}$$

The Cartan killing form in the case of $\mathfrak{sl}(2)$ *is just* $\kappa(P_\mu, P_\nu) = 4Tr(P_\mu P_\nu)$*, which results in*

$$\kappa(P_\mu, P_\nu) = 4\frac{1}{2}\Delta^{-2}(\partial_\mu B\partial_\nu B + \partial\mu\Delta\partial_\nu\Delta)$$

This indeed shows that the second part of the Lagrangian in equation (90) can be written as

$$\mathcal{L}' = \frac{1}{4}eg^{\mu\nu}Tr(P_\mu P_\nu) = \frac{1}{16}eg^{\mu\nu}\kappa(P_\mu P_\nu)$$

Two Dimensional Gravity

In two dimensions the Lagrangian density of the bosonic section can be written as a coset sigma space model.

$$\mathcal{L} = \frac{1}{2}\rho eR(e) - \frac{1}{2}\rho e\mathrm{Tr}(P^\mu P_\mu) \tag{91}$$

It is important to note that we no longer have vector bosons. By similar reasoning which was used to get rid of the 3-forms in the 4 dimensional theory. Now the vectors (1-forms) also dissappear. The $p, D-p-2$ form duality causes the vectors to be non-propagating.
Using local $SO(1,1)$ and general coordinate transformation we can the zweibein in the conformal gauge.

$$e_\mu^{\ a} = \lambda\delta_\mu^{\ a} = e^\sigma\delta_\mu^{\ a} \tag{92}$$

It directly follows that $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$. It will also be extremely useful to switch to light-cone coordinates

$$x^\pm = \frac{1}{\sqrt{2}}x^0 \pm x^1, \quad \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$$

And the Minkowski metric changes to $\eta_{+-} = \eta_{-+} = 1$. The Louisville degree of freedom does not transform as genuine scalar. The expression $\hat{\sigma} = \ln\lambda - \frac{1}{2}\ln(\partial_+\rho\partial_-\rho)$ does. [21]
Using this sigma we can further reduce the Lagrangian to

$$\mathcal{L} = \partial_\mu\hat{\sigma}\partial^\mu\rho - \frac{1}{2}\rho\mathrm{Tr}P^\mu P_\mu \tag{93}$$

where $\partial_\mu \hat{\sigma} \partial^\mu \rho$ (from now on we will drop the bar over σ) is all that remains of the factor ρR. This Lagrangian is invariant under any global G and local K transformation. Also, indices can now be raised and lowered by the Minkowski metric.
The equations of motions of the different field are

$$\partial_+\partial_-\rho = 0 \tag{94}$$

$$\partial_+\partial_-\sigma + \frac{1}{2}\mathrm{Tr}(P_+P_-) = 0 \tag{95}$$

$$D^+(\rho P_+) + D^-(\rho P_-) = 0 \tag{96}$$

The conformal factor σ also satisfies the two first order equations

$$\partial_\pm\rho\partial_\pm\sigma - \frac{1}{2}\rho\mathrm{Tr}(P_\pm P_\pm) \tag{97}$$

These are not exactly equations of motions. They are remnants of the of the two unimodular degrees of freedom. [21]
The dual $\tilde{\rho}$ given by equation (84), which in two dimensions reduces to

$$\partial_\mu\tilde{\rho} = -\epsilon_{\mu\nu}\partial^\nu\rho, \quad \partial_\pm\tilde{\rho} = \pm\partial_\pm\rho$$

The equations of motions (96) can equivalently be written as the coservation law $I_\mu = \rho V P_\mu V^{-1}$ where $\partial^\mu I_\mu$. Note that this conservation law holds for both the original and the light cone metric. The first dual potential Y_1 is then given by

$$\partial_\pm = \mp I_\pm = \mp\rho V P_\mu V^{-1} \tag{98}$$

This is just the $p, D-p-2$ duality, which in two dimensions turns a scalar into a scalar. In two dimensions however this is the start of an infinite hierarchy of dual potentials. The next one being given by

$$\partial_\pm Y_1 = (\pm\rho\tilde{\rho} + \frac{1}{2}\rho^2)VP_\pm V^{-1} \tag{99}$$

The matter field equations (96) can be obtained from a linear system. These techniques are often used in flat space integrable systems. It introduces a spectral parameter γ and it gives rise to infinite set of conserved charges. This ensures the integrability of the model.
The linear system introduces a family of group valued elements $\hat{V}(\gamma)$ given by

$$\hat{V}^{-1}\partial_\mu\hat{V} = Q_\mu + \frac{1+\gamma^2}{1-\gamma^2}P_\mu + \frac{2\gamma}{1-\gamma^2}\epsilon_{\mu\nu}P^\nu \tag{100}$$

Or equivalently

$$\hat{V}^{-1}\partial_{\pm}\hat{V} = Q_\mu + \frac{1 \mp \gamma}{1 \pm \gamma}P_\mu \tag{101}$$

The essential difference between flat and curved space is that we are trying to get the equations of motion $D^\mu(\rho P_\mu)$ instead of $D^\mu(P_\mu)$. This implies that the spectral parameter is not a constant, but rather is given function of spacetime and

$$\gamma = \frac{1}{\rho}(\omega + \tilde{\rho} - \sqrt{(\omega + \tilde{\rho})^2 - \rho^2} \tag{102}$$

ω is an integration constant and is also known as the constant spectral parameter.

We only consider the family of which $|\gamma| < 1$, demanding regularity around $\gamma = 0$ allows us to expand the original group V as

$$\tilde{V} = \dots e^{\omega^{-3}Y_3}e^{\omega^{-2}Y_2}e^{\omega^{-1}Y_1} \tag{103}$$

$$= \dots e^{\omega^{-3}Y_3}e^{\omega^{-2}Y_2}e^{\omega^{-1}Y_1}e^{Y_0} \tag{104}$$

An easy check is to take the expansion of γ around $\omega = \infty$

$$\gamma = \frac{1}{2}\rho\omega^{-1} - \frac{1}{2}\rho\tilde{\rho}\omega^{-2}....$$

and substituting this into equation (101). The zeroth order just yields the Maurer Cartan form. The first order gives back equation (99) and all other dual potentials can easily be constructed in a similar way.
An alternate way to define the dual potentials is as $\tilde{V} = \dots e^{\gamma^3 Y_3}e^{\gamma^2 Y_2}e^{\gamma^1 Y_1}e^{Y_0}$. This instead results in $\partial_\mu Y_1 = \epsilon_{\mu\nu}\partial^\nu Y_0 +$ nonlinear terms, which shows great resemblance to equation (84). Since the group element $V = e^{Y_0}$, these results are nearly identical.
Elements of the affine lie algebra can now simply be identified with $\Lambda = \sum_{n\in N} w^{-n} \otimes g_n$. First of all this is only half of the affine Lie algebra. Only non negative integers are included. The reason for this will become clear in a second. Since ω is a scalar these are in fact represented in the finite dimensional Lie algebra $\mathfrak{g}$

The symmetries of the affine Lie algebra

Remember that the commutation relations of an affine Lie algebra are given by equation (14). We have only substituted the Cartan-Killing form by the $\text{Tr}(g_n, g_m)$. In this case these are identical.

$$[g_n \otimes t^n \oplus \alpha K, g_m \otimes t^m \oplus \beta K] = [g_n, g_m] \otimes t^{n+m} \oplus \text{Tr}(g_n, g_m) n\delta_{m+n} K$$

To be consistent with existing literature we change notation to t_α for the generators the finite dimensional algebra $\mathfrak{g}$ and $T_{\alpha,m}$ the generators of the affine Lie algebra $\hat{\mathfrak{g}}$.
We will need the Witt-Virasoro generator L_1, which can easily be build using the derivation D and the generator corresponding to the null root.

$$[L_1, T_{\alpha,m}] = -mT_{\alpha,m+1}$$

The dual potentials $\tilde{\rho}$ and Y_m are only defined up to a constant. The corresponding symmetry is given by

$$\delta_1 \tilde{\rho} = 1 \tag{105}$$

$$\delta_{\alpha,m} Y_n^\beta = \begin{cases} \delta_\alpha^\beta \text{ if } m = n \\ 0 \text{ if } m > n \end{cases} \tag{106}$$

Where Y_n^β is the component of Y_n in the direction of t_β such that $Y_n = Y_n^\alpha t_\alpha$. The definition of the dual potential also includes $\tilde{\rho}$, therefore the shift symmetries L_1 and $T_{\alpha,m}, m > 0$ also work on the higher dual potentials $Y_n, m < n$. The lowest examples being given by

$$\delta_1 Y_2 = -Y_1, \quad \delta_1 Y_3 = -2Y_2 \tag{107}$$

$$\delta_{\alpha,1} Y_2 = \frac{1}{2}[t_\alpha, Y_1], \quad \text{etc.} \tag{108}$$

None of these shift symmetries work on any of the physical fields though, so we have not yet introduced any new physics. But so far we have left out the other family of generators $T_{\alpha,m}, m < 0$. Their action on the physical field contains the dual potentials. The action of the lowest generator is given by

$$\delta_{\alpha,-1} V = [t_\alpha, Y_1]V - \tilde{\rho} V [V^{-1} t_\alpha V]_{\mathfrak{p}} \tag{109}$$

where $[V^{-1} t_\alpha V]_{\mathfrak{p}} \in \mathfrak{p}$ is the odd part of this element. It follows from the first order constraints (97) and the definition of the dual potential that

$$\delta_{\alpha,-1} \sigma = \mathrm{Tr}[t_\alpha, Y_1] \tag{110}$$

$$\delta_{\alpha,-1} Y_1 = [t_\alpha, Y_2] + \frac{1}{2}[[t_\alpha, Y_1], Y_1] + \frac{1}{2}\rho^2 V [V^{-1} t_\alpha V]_{\mathfrak{p}} V^{-1} \tag{111}$$

Using the this results for the generators of $\delta_{\alpha,1}$ and $\delta_{\alpha,-1}$ we find that the central extension K works exclusively on the conformal factor as a shift symmetry.

$$\delta_0 \sigma = -1 \tag{112}$$

An elegant way to combine all the symmetries on the physical field is to use the group valued family of matrices $\hat{V}(\gamma)$. We represent elements of the affine Lie algebra $\Lambda^{\alpha,m}T_{\alpha,m}$ as $\mathfrak{g}$ valued functions $\Lambda(\omega) = \Lambda^{\alpha,m}\omega^{-m}t_\alpha$. The actions on the physical field can now be given by [22]

$$V^{-1}\delta_\Lambda V = \frac{2\gamma(\omega)}{\rho(1-\gamma^2(\omega))\hat{\Lambda}(\omega)} \tag{113}$$

$$\delta_\Lambda\sigma = -\mathrm{Tr}(\Lambda(\omega)\partial_\omega\hat{V}(\gamma(\omega))\hat{V}^{-1}(\gamma(\omega)) \tag{114}$$

where have defined

$$\hat{\Lambda} = \hat{V}^{-1}(\omega)\Lambda(\omega)\hat{V}(\gamma(\omega)) = \hat{\Lambda}_\mathfrak{k} + \hat{\Lambda}_\mathfrak{p}$$

The Lagrangian (93) is indeed invariant under these transformations.
So we have seen that although the symmetry group is now infinite dimensional there are no infinite number of scalars. The number of real scalars no longer matches the dimensions of the coset space. Apart from the conformal factor there are in fact no additional scalars in two dimensions than there were in three.
But do not let this fool you. There are an infinite number of conserved currents and charges associated with this symmetries. And discovering this infinite dimensional symmetry we have most certainly uncovered additional physics.

14 Conclusion & discussion

In this thesis we have constructed a Maximal supergravity theory starting of with a basic bachelor graduate level of physics. We have covered a lot of different subjects ranging from basic symmetries to general relativity, from gauging symmetries to the mathematical properties of (affine) Lie algebras.
In order to obtain maximal supergravity in a variety of dimensions we have reduced the 11 dimensional theory on a torus. This lead to additional fields resulting from the bosonic section ($A_{\mu\nu\rho}$ and $e_\mu^{\ a}$ of the 11 dimensional Lagrangian. These field contained a symmetry, which is called the hidden symmetry. It arises from spacetime symmetries in the 11 dimensional theory; the general coordinate transformations and the local Lorentz invariance. The resulting symmetries were part global and part local symmetries and can best be described by the coset space G/K. The number of scalar fields is identical to $\dim G/K$
Maximal supergravity shows some beautiful properties. This highly symmetric theory sets a lot of the interactions, which comes with high predictive value. So not only is the theory more elegant, it is also has increased falsifiability. In fact maximal supergravity in its current form is falsified. Maximal supergravity does solve some issues that the quantification of gravity offers. It might therefore be a

valid theory in a slightly altered version.
We have also seen how the affine Lie algebras arise in 2 dimensional supergravity. In the mathematical section of this thesis we have seen how the affine Lie algebra is an infinite stack of semi-simple Lie algebras. It could also be seen as the maps from a circle to the semi-simple Lie algebra. Although this looked promising, considering the Kaluza-Klein reduction from 3 to 2 dimensions reduces the symmetry on a circle, I have found it easier to look at the affine Lie algebra as an infinite number of copies of the original Lie algebra. 3 dimensional maximal supergravity was described by the $E_8/SO(16)$ coset space. In two dimensions E_8 turns into $E_8^+ = E_9$. The scalar fields in two dimensions are however not described by the corresponding coset space. The number of scalar fields is with the exception of the conformal factor identical to that of the three dimensional theory. The infinite dimensional symmetry group E_9 is just a symmetry on the physical fields. In some sense it is a hidden symmetry within the hidden symmetry and although it feels obsolete, it does uncover additional physics, e.g. in the sense of an infinite amount of conserved charges.
It is however worthy to note that the affine Lie algebras are not unique to maximal supergravity. Plain general relativity also has an infinite dimensional Lie algebra A_1^+ in two dimensions. This is however not the affine extension of an exceptional group, as is the case in maximal supergravity. It seems to be a generic feature of gravity in two dimensions.
Obviously two dimensional maximal supergravity is not a realistic theory for our 4 dimensional spacetime. However, for any theory it is good to look how a theory behaves in extreme cases. Back in 1987 two-dimensional field theories were quite popular, because of their direct relevance to (super)string theories [18]. Today, they can still provide us with hints for a quantum gravity theory. Maximal supergravity might even be a quantum theory itself. [22]
The tactics used in this thesis can of course be taken one step further. This would result in the over-extended affine Lie algebra E_8^{++} in 1 dimension. [2]

A Notations and Conventions

Throughout this section we will use the following conventions. For the metric tensor the mostly minus convention is used:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The Pauli matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And we define

$$\sigma^\mu = (1, \vec{\sigma}) \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

Using the Weyl basis for the gamme matrices this gives the easy formulation.

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$

This means that the fifth gamme matrix is given by $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$

References

[1] James E. Humphreys, *Introduction to Lie algebras and Representation theory*, Springer-Verlag, New York, 3rd Printing, 1980.

[2] Teake Nutma, *Kac-Moody symmetries and gauged supergravity*, Proefschrift, Groningen, 2010.

[3] Stephen P. Martin, *A Supersymmetry Primer*, arXiv:hep-ph/9709356v6, 2011.

[4] M. de Roo, *Elementaire Deeltjesfysica*, Geheel herziene versie, Groningen, 1993.

[5] James Foster & J. David Nighting, *A Short Course in General Relativity*, Springer, New York, Third edition, 2006.

[6] Viktor G. Kac, *Infinite Dimensional Lie Algebras*, Birkhäuser, Boston, 1983.

[7] Diederik Roest, *M-Theory and Gauged Supergravity*, Proefschrift, Goningen, 2004.

[8] Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press, 1995.

[9] Patrick Labelle, *Supersymmetry Demystified*, McGraw-Hil, 2010.

[10] Horatiu Nastase, *Introduction to supergravity*, arXiv:1112.3502v3, 2011.

[11] Henning Samtleben, *Introduction to Supergravity*, http://www.itp.uni-hannover.de/ lechtenf/Events/Lectures/samtleben.pdf, 2007.

[12] Kevin Wray, *Affine Lie Algebras*, http://math.berkeley.edu/˜kwray/papers/affine_ lie_ algebras.pdf, 2008.

[13] Jaap Top, *lecturenotes, Groepentheorie*, http://www.math.rug.nl/˜top/alg1.pdf.

[14] Jaap Top, *lecturenotes, Algebra*, http://www.math.rug.nl/˜top/dic.pdf.

[15] Marc Henneaux, Daniel Persson & Philippe Spindel, *Space-like Singularities and Hidden Symmetries of Gravity*, http://relativity.livingreviews.org/Articles/lrr-2008-1/download/lrr-2008-1Color.pdf.

[16] Henning Samtleben & Martin Weidner, *Gauging Hidden Symmetries in Two Dimension*, http://www.math.rug.nl/˜top/dic.pdf.

[17] Peter Breitenlohner & Dieter Maison, *On the Geroch group*, http://archive.numdam.org/ARCHIVE/AIHPA/AIHPA_1987__46_2/AIHPA_1987__46_2_215_0/

[18] H. Nicolai, *The integrability of N=16 supergravity*, http://inspirehep.net/record/21766.

[19] Kevin Wray, *Affine Lie algebras*, http://math.berkeley.edu/ kwray/papers/affine_lie_algebras.pd

[20] H. Nicolai, *Two-Dimensional Gravities and Supergravities as Integrable Systems*, http://pubman.mpdl.mpg.de/pubman/item/escidoc:153337:1/component/escidoc:15333

[21] H. Nicolai, D.Korotkin & H. Samtleben, *Integrable Classical and Quantum Gravity*, http://arxiv.org/abs/hep-th/9612065

[22] Henning Samtleben & Martin Weidner, *Gauging hidden symmetries in two dimensions*, http://arxiv.org/abs/0705.2606

[23] Oskar Pelc & L. P. Horwitz, *Generalization of the Coleman-Mandula Theorem to Higher Dimensions*, http://arxiv.org/abs/hep-th/9605147

[24] CERN, http://home.web.cern.ch/about/physics/standard-model

[25] The Official Web Site of the Nobel Prize, http://www.nobelprize.org/nobel_prizes/physics/laureates/1921/einstein-bio.html

[26] Henning Samtleben, *Introduction to Supergravity* (lecturenotes), http://www.itp.uni-hannover.de/˜lechtenf/Events/Lectures/samtleben.pdf