Abstract
When the Parrondo effect was discovered a few years ago (Harmer and Abbott 1999a, 1999b), it was hailed as a possible mechanism whereby, in a kind of collaboration of failure, losing strategies could be combined to yield profit. The precise relevance of the Parrondo effect to natural and social phenomena is however still unclear. In this paper we give specific examples, first in the artificial setting of a gambling machine, and then in more natural applications to genetics and to environmental policies. This last example touches questions of rational behaviour and expected utility in a novel setting.

1 Introduction
Ever since it was formulated, the rationality ideal of the *homo economicus* has been subjected to criticism. Psychologists have argued that individual agents do not, and in fact should not, rigorously maximize expected utility; and sociologists have claimed that the same goes for groups of agents. Many have stressed the important role of intuition and of uncertainty in decision making, and in the wake of Herbert Simon it has been argued that our actions are, and should be, aimed at ‘satisficing’ rather than maximizing (Simon 1957).

The main point of all these criticisms was clear enough: true rationality is richer than the anaemic model presented by Economic Man. But whereas the model was generally deemed too simple and too unrealistic, closer to the *modus operandi* of a robot than to that of a living being of flesh and blood, none of the critics contested its clarity and transparancy. In fact, however, the model has a strange and obscure feature. This feature was first formulated in 1999 in a different context (namely information theory) and
it is known as the Parrondo effect, after its discoverer, the Spanish physicist Juan Parrondo. Briefly, the Parrondo effect (some speak of the ‘Parrondo paradox’) consists in the fact that two losing strategies can, under certain special conditions, be combined counterintuitively to yield success. In such special cases it might be rational, according to the standards of Economic Man, to combine two clearly irrational strategies.

In most of the formulations of the Parrondo effect, the mechanism is capital dependent, the outcome being a function of the player’s current capital. While this property seems natural enough in a gambling casino, it is not very appealing in other applications. This shortcoming has however been alleviated by the introduction of games that are not capital dependent, but rather history dependent. In these games, the next move depends (deterministically or probabilistically) on the previous two (Parrondo, Harmer and Abbott 2000; Iyengar and Kohli 2004; Harmer, Abbott and Parrondo 2005). In this paper we shall give examples of this history dependent scheme, first in the artificial setting of a gambling machine (Sect. 2), and then in more interesting applications to genetics (Sect. 3) and environmental policies (Sect. 4).

The purpose of our examples is to show, in not wholly unrealistic situations, how the Parrondo effect could function. We do not arrogate to ourselves the expertise necessary for the true specification of the circumstances and parameters relevant to an empirical example, this being properly the purview of the scientific specialist. Nor do we argue that the Parrondo effect is omnipresent in nature or in society. Nevertheless it can and presumably does occur, and while the scenarios sketched below are fanciful, it is hoped that they carry the germ of a structure that is viable in the hurly-burly of the field and laboratory.

2 Crazy Horse Saloon

In the Crazy Horse Saloon there are two one-arm bandits (fruit machines) that the customers call Buffalo Bill and Calamity Jane. The rules are simple, as befits the Wild West. You put one dollar into the slot of one of the machines and pull the arm. If you win, you get two dollars back, but if you lose, you get nothing.

Gambling with Buffalo Bill generally leads to loss: on average, for every 9 pulls only 4 are wins, yielding $8, a net loss of $1. Calamity Jane is even
more of a swindler, for very few pulls give wins and, in the long run, almost all the stake is lost. Nevertheless a gambler who emulates the ambidextrous Jesse James and pulls randomly, sometimes on Buffalo Bill and sometimes on Calamity Jane, wins in the long run. On average, for every 9 pulls there are 6 wins and 3 losses, yielding $12, a net gain of $3.

How is this possible? How can random switching between two losing strategies result in a long term win? The secret of the one-arm bandits is that they give varying odds, depending on the results of the previous two pulls. After two successive losses ($LL$), Buffalo Bill always concedes a win ($LLW$), but after a loss and a win ($LW$), it always gives a loss ($LWL$). On the other hand after a win, followed either by a loss or a win, ($WL$ or $WW$), the next pull results in a win 75% of the time, ($WLW$) or ($WWW$). Summarizing,

**Buffalo Bill**

($LL$) leads to ($LLW$) and ($LW$) leads to ($LWL$) 100%
($WL$) leads to ($WLW$) and ($WW$) leads to ($WWW$) 75%

In effect, in the cases ($WL$) or ($WW$), a computer simulates the throwing of a biased coin that, when tossed, yields a head (win) three quarters of the time. In the same vein, case ($LL$) corresponds to the simulated tossing of a coin with two heads, case ($LW$) to the simulated tossing of a coin with two tails.

The other one-armed bandit, Calamity Jane, works in a similar way, the only difference being that the two-headed and the two-tailed coins exchange their roles. That is, in the case ($LL$) the next pull always gives a loss, and in the case ($LW$) the next pull always gives a win. Summarizing,

**Calamity Jane**

($LL$) leads to ($LLL$) and ($LW$) leads to ($LWW$) 100%
($WL$) leads to ($WLW$) and ($WW$) leads to ($WWW$) 75%

It is easy to see why Calamity Jane is such an unfavourable machine, for eventually two losses will occur in a row, and then all subsequent pulls result in loss.

What are the odds when a gambler switches randomly from one machine to the other? It is understood that both one-armed bandits are run by the same computer, so that when the gambler changes from one machine
to the other, the information about the results of the previous two pulls, 
(LL), (LW), (WL), (WW), is not lost. The 75% probability of a win in 
the latter two cases is unchanged. After two losses, (LL), since on average 
Buffalo Bill is used for half of the time and Calamity Jane for the other half 
of the time, in one half of the cases there is a win, and in one half there 
is a loss. Thus the probability of a win is 50%. Similarly, after a loss and 
a win (LW), since Buffalo Bill yields a loss and Calamity Jane a gain, the 
probability of a win is also 50%. In summary, the result of an even-handed 
use of both one-armed bandits is as follows:

**Jesse James**

(LL) leads to (LLW) and (LW) leads to (LWW) 50%
(WL) leads to (WLW) and (WW) leads to (WWW) 75%

In effect, random switching between the machines amounts to the simulated 
tossing of a fair coin in the cases (LL) and (LW). It is therefore clear why 
Jesse James is a winning strategy, for the simulation is equivalent to using 
sometimes a fair coin and sometimes a coin that is biased towards a win.

![Figure 1: Win-loss curve and the 3 points BB, CJ and JJ](image)

Fig.1 is a plot between $P_{LL}$, the probability of a win in the case (LL), and 
$P_{LW}$, the probability of a win in the case (LW), given that the probability of 
a win in the cases (WL) and (WW) is $\frac{3}{4}$. The curve separating a win from 
a loss in the long term is convex downwards, so that the straight line from 
the point $P_{LL} = 0$ and $P_{LW} = 1$ (Calamity Jane) to the point $P_{LL} = 1$ and
$P_{LW} = 0$ (Buffalo Bill), both in the region of long-term loss, passes through the point $P_{LL} = \frac{1}{2}$ and $P_{LW} = \frac{1}{2}$ (Jesse James), in the region of long-term win.

3 Genetic Disorder

In an isolated tribe of Aborigines the number of individuals remains steady over the millennia. Some of the females suffer from a genetically determined mitochondrial disorder (D) that is passed from maternal grandmother to mother to daughter, along the female line. A daughter whose mother and grandmother had the disorder will certainly have it too. If only her grandmother had it, her mother being healthy (H), she will be healthy. In summary

\[(DD) \text{ leads to } (DDD) \text{ and } (DH) \text{ leads to } (DHH) \text{ 100%}\]

which may be read as follows: if the disorder is present in, say, generations $g$ and $g+1$ in a given family, then it will be present in generation $g+2$ too, whereas if it is present in generation $g$ but absent in generation $g+1$, then it will be absent in generation $g+2$.

It is discovered, on the other hand, that if the grandmother was healthy, then the probability that the daughter will be healthy is $\frac{3}{4}$, irrespective of whether the mother herself has the disorder. This may be symbolized

\[(HD) \text{ leads to } (HDH) \text{ and } (HH) \text{ leads to } (HHH) \text{ 75%}\]

The above probabilistic system is precisely that of the one-armed bandit Calamity Jane, with the following translation key: ‘D’ for ‘L’ and ‘H’ for ‘W’. As we know, the long term expectation with this machine is certain loss, which translates into the certainty of eventual disorder. Indeed, it is easy to see that, in any genealogical line, there will eventually be two succeeding generations in which the females have the disorder, after which all female descendants will also have the disorder. After some time there will never again be a healthy female in the tribe. Such is the sad conclusion, so long as the tribe remains isolated.

In another isolated tribe, there is also incidence of the same mitochondrial disorder in the female population, but the difference is that the males carry a special gene that is missing in the first tribe. This gene affects the working of the gene responsible for the mitochondrial disorder. It interferes with the
expression of the defective gene in the maternal grandmother in such a way that a daughter whose mother and grandmother had the disorder will not have the disorder. On the other hand, if her grandmother had the defect, but her mother did not, and if her father is from her tribe, and so carries the special gene, she will indeed have the disorder. The situation in the cases $(DD)$ and $(DH)$ is therefore

$(DD)$ leads to $(DDH)$ and $(DH)$ leads to $(DHD)$ 100%

In the case that the grandmother does not have the disorder, the situation is as it was in the first tribe, namely

$(HD)$ leads to $(HDH)$ and $(HH)$ leads to $(HHH)$ 75%

This probabilistic system is the same as that of Buffalo Bill, resulting in a steady-state population, after many generations, in which, of every 9 females, 5 have the disorder and 4 do not.

On a propitious day in the outback the two tribes meet and mingle amiably, not to say amorously. The consequence is that, in a given genealogical line, sometimes the male partner carries the special gene, and sometimes he does not. If the two tribes are of the same size, and the intermingling of the populations over the generations is thorough, then

$(DD)$ leads to $(DDH)$ and $(DH)$ leads to $(DHH)$ 50%

as in the case in which there was randomization between the machines Buffalo Bill and Calamity Jane, the situation that we called Jesse James. After many generations, there will be two healthy females for every one that carries the defective gene, a significant improvement over the state of health of the second tribe, and a very great improvement over that of the first.

### 4 Pollution of the Environment

Is the Parrondo effect relevant only to the blind workings of natural processes, or could it also be germane to rational decision making? Imagine a world of the near future, slowly warming as a result of its own industrial CO$_2$ production. In this world there are two superstates, Emarica and Cathay, co-existing in a delicate balance of power. In an attempt to cope with the effects of global warming, the leaders of Emarica and Cathay, after
marathon meetings at diplomatic and ministerial level, eventually agreed to sign a treaty. The treaty includes a list of Environment-Friendly Actions (EFAs) and Environment-Hostile Actions (EHAs). EFAs are for example the reforestation of a denuded area of specified minimum size, the introduction of traffic-free Sundays in a sufficiently populous town, the recycling of effluent from particular kinds of factories, and so on. The stripping of a primeval forest is an EHA, as would be the abolition of the speed limit on the motorways of a county, and so is the building of a dirty factory or power station.

The treaty’s protocol entails that, in each superstate, central governmental permission and financing for an EFA or an EHA should depend on the nature of the two previous projects that had been approved by its own leader. The first half of the protocol specified what to do when an EFA was followed by either another EFA of an EHA. An accord was reached that the next project to be permitted would be an EFA in three out of every four cases of this type. In only one case in the four would an EHA be allowed, however urgent the need. In this way, both leaders could announce to the world that, while national economic requirements were not being stifled, the long-term health of future generations would be adequately protected.

Unfortunately the other half of the protocol, namely the specification of the action to be taken after an EHA had been followed either by another EHA or by an EFA, was never ratified by the superstates. The sous-entendu was that each superstate would have to decide for itself what policy to adopt in these cases. With an eye on his Green lobby, and in order to minimize further bookkeeping, the president of Emarica decreed that an EHA followed by another EHA should always be followed by an EFA, thus redressing the balance between environment friendly and unfriendly actions. However, an EHA followed by an EFA would, he proclaimed, always be followed by permission for an EHA. In this way the president could claim that the pre-eminent industrial strength of his great nation would be safeguarded. The electorate was however deceived by a false notion of responsibility, for the Emarican strategy is precisely mirrored by the one-armed bandit Buffalo Bill of Sect. 2, under the mapping ‘EHA’ for ‘L’ and EFA for ‘W’. With this strategy, in the long term, for every four EFAs there will be five EHAs, and so the Emarican environment will inexorably, if slowly, deteriorate. A very few Greens recognized this fact, but they were outvoted by the rest of the Emarican people.

The chairman of Cathay was even less considerate of the environment and
its defenders. He chose to ignore the remonstrations of political dissidents and to use the leeway in the treaty to sanction a new EHA whenever two EHAs had occurred in succession, this being advantageous to heavy industry. He agreed to enforce an EFA after an EHA had been followed by an EFA, in order not to appear wholly antithetical to the purpose of the international treaty. In this way, as his inner council of advisors assured him, he could reach the goals of the current five-year plan, and at the same time earn the respect of the international community. Since the Cathayan strategy mirrors that of Calamity Jane, however, we know that, as time goes on, there will more and more EHAs and fewer and fewer EFAs: the air will become unbreathable and the environment irreversibly ruined.

However, in another part of that future world there is a third state, not a superstate, but a loose confederation of old colonial powers called Commarket. Commarket has also signed the treaty, but cannot agree internally on the policy to be followed in the cases not covered by the protocol. Some of the countries favoured the Emarican option and some the solution adopted by Cathay. After much currying of favours and lobbying of interests, a compromise was reached within Commarket: sometimes EHA-EHA was to be followed by EHA and sometimes by EFA, and similarly for EHA-EFA, but with this rider: central permission and financing for an EFA or an EHA was to be random. After EHA-EHA or EHA-EFA, and averaged over a long time, about the same number of EHAs as EFAs were to be permitted. This strategy is precisely that of Jesse James, and so the people of Commarket took advantage (advertently or inadvertently – that always remained unclear) of the Parrondo effect: for every EHA there would be on average two EFAs, and on the old continent the milieu would improve with each passing year.

5 Is the Parrondo Effect Paradoxical?

For reasons of expository simplicity, the imaginary scenarios sketched in Sects. 3 and 4 have both been based on the one-armed bandits of Sect. 2. Before considering in Sect. 6 how realistic systems might be susceptible to more general manifestations of the Parrondo effect, let us first try to answer the question: “How surprising is it that Buffalo Bill (BB) and Calamity Jane (CJ) are losing strategies, whilst their average, Jesse James (JJ), is a winning one?” This question is opportune, for some scholars find the fact so surprising that they call it a paradox, whilst others believe that it is merely
counterintuitive.

As we already indicated in Sect. 2, it is obvious that JJ wins on average, for it is simulated by sometimes tossing a fair coin, and sometimes one that is loaded 3 to 1 in favour of a win. Moreover, it does not require much insight to come to the conclusion that CJ is a losing strategy, for \((WL)\) gives, after a new toss, \((WLL)\) 25% of the time, that is to say \((LL)\), if we include only the last two results after the new toss. Moreover, \((LL)\) remains in this state for all subsequent tosses. Since \((WW)\) leads to \((WL)\) 25% of the time, and \((LW)\) always to \((WW)\), it is clear that, in the course of time, it is more and more likely that the cul-de-sac of the \((LL)\) state will be entered, resulting in uninterrupted loss.

It is not quite so easy to understand why BB is a losing strategy. In terms of what we shall call the diachronous probability, the chance that the state \((WW)\) leads again to \((WW)\) is \(\frac{3}{4}\), as the one-armed bandit Buffalo Bill is played, one pull after another. However, another way of describing the situation is by talking about synchronous probabilities in a large ensemble of BB machines, \(n_1\) of which are in the state \((LL)\), \(n_2\) in the state \((LW)\), \(n_3\) in the state \((WL)\) and \(n_4\) in the state \((WW)\), all at the same time, \(t = 0\). At \(t = 1\), the arm of each bandit in the ensemble is pulled once, resulting in a new configuration of states, with occupation numbers \(n'_1, n'_2, n'_3, n'_4\). This way of looking at the matter is more suited to the genetic example of Sect. 3, where a whole population of female Aborigines was involved. Here indeed we can consider the relative numbers of the four states synchronously, i.e. at one time. Now the only way that \((WW)\) can be produced at \(t = 1\) for a given machine is if its state at \(t = 0\) was also \((WW)\), since \((LW)\) always evolves into \((WL)\). Hence the number of \((WW)\) states at \(t = 1\) is on average three-quarters of the number of \((WW)\) states at \(t = 0\), the number at \(t = 2\) nine-sixteenths of that number, and so on. In short, the \((WW)\) states will die out as time progresses, and, for sufficiently large times, we can safely ignore \(n_4\) compared with \(n_1, n_2\) and \(n_3\).

The only way that \((LL)\) can be produced at \(t = 1\) is from a state \((WL)\) at \(t = 0\), and this occurs with probability \(\frac{1}{4}\), so \(n'_1 = \frac{1}{4}n_3\). At \(t = 1\), \((WL)\) can be produced from either \((LW)\) or \((WW)\) at \(t = 0\), but we agreed to neglect the latter, on the grounds that the number of \((WW)\) states will rapidly become negligible. Since \((LW)\) becomes \((WL)\) with certainty, \(n'_3 = \frac{1}{4}n_2\). In the course of time, the relative populations of the four states stabilizes, and at equilibrium we may omit the primes, so \(n_1 = \frac{1}{4}n_2 = \frac{1}{4}n_3\) and \(n_4 = 0\). The total number of machines is \(n = n_1 + n_2 + n_3\), and thus \(n_1 = \frac{1}{9}n\),
\[ n_2 = n_3 = \frac{4}{9}n \text{ and } n_4 = 0. \] Since the \( n_1 (LL) \) states and the \( n_3 (WL) \) states have just resulted from losses, while the \( n_2 (LW) \) states have resulted from a win, we see that the average number of losses is \( n_1 + n_3 = \frac{5}{9}n \), whereas the average number of wins is only \( n_2 = \frac{4}{9}n \). This concludes the demonstration that BB is indeed a losing strategy. Inasmuch as the losing nature of the BB strategy is not obvious, we conclude that the occurrence of the Parrondo effect is surprising.

One puzzling aspect of the Parrondo effect deserves closer attention, since it is easily neglected. It may seem odd, especially in the diachronous interpretation, why it should make a difference whether one adopts the JJ strategy or simply takes BB and CJ to constitute one entity. For example, consider all the actions of Emarica and Cathay together over a certain period of time. Does this not amount to averaging, and should not the Parrondo effect imply that there will be twice as many EFAs (i.e. wins) as EHAs (i.e. losses)? The answer is no, for what is missing in such a spurious averaging on paper is the bookkeeping involved in keeping track of the two previous actions. For example, if the last two actions were both EFAs, then the next will be an EHA, only if all three actions appertain to Emarica. If, in the attempted averaging on paper, the list includes an EFA by Emarica, followed by an EFA by Cathay, then the probability that the next action will be an EHA, whether by Emarica or Cathay, is not determined by the given data. If all the actions were those of Commarket, on the other hand, the probability in question would be \( \frac{1}{2} \).

6 Generalization

The probabilities that were chosen for BB and CJ were very special, indeed extreme examples. A generalization consists in taking different points in the ‘loss’ region of Fig. 1 to define the probabilities for BB and CJ. For example, BB could be moved to \( P_{LL} = 0.8 \) and \( P_{LW} = 0.05 \), and CJ might be placed at \( P_{LL} = 0.03 \) and \( P_{LW} = 0.6 \). Both of these new points are in the ‘loss’ region, but the mid-point of the straight line drawn between them is still in the ‘win’ region, so there is still a Parrondo effect. The precise position of the rectangular hyperbolic curve that separates the ‘win’ and the ‘loss’ regions is dictated by the common probability that was chosen for the transitions \((WL)\) to \((LW)\) and \((WW)\) to \((WW)\). In Sects. 2-4 this was assumed to be \( \frac{3}{4} \); but it can be changed to any value strictly between 0 and 1. For example,
if it is \( \frac{1}{2} \), corresponding to simulation by a fair coin, the hyperbola passes through the point \( P_{LL} = \frac{1}{2} = P_{LW} \), but a Parrondo effect is still possible by astute choice of the BB and CJ probabilities. Further, the probabilities for the two transitions \((WL)\) to \((LW)\) and \((WW)\) to \((WW)\) need not be equal, in which case the model is more complicated. Finally, the restriction that the probabilities associated with the next step depend only on the last two steps, while that may be natural enough in the genetic context, is an unnecessary simplification in certain other applications. Relaxation of this restriction is possible, leading to a richer Parrondo structure. In short, much generalization is possible: the occurrence of the counterintuitive Parrondo effect does not depend on the fine tuning of parameters, and it may well be a more common phenomenon than we presently realise.

References


