

# Holographic Hydrodynamics :

## A radial flow of Response function

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- *Long wavelength* effective description of strongly coupled field theory.
- It is formulated in the language of constituent equations.
- The simplest case: no global conserved currents

$$\nabla_{\mu} T^{\mu\nu} = 0$$

- Need to reduce the number of independent elements of  $T^{\mu\nu}$ .

# Hydrodynamic system

- We consider the fluid is in local thermal equilibrium and fluctuations are of small energy.
- At any given time the system is described by the following local quantities

$$\begin{aligned} \text{Temperature: } T(x) &\quad \hookrightarrow 1 \\ \text{Velocity vector: } u^\mu(x) &\quad \hookrightarrow d + 1 \end{aligned}$$

- Velocity vectors are normalized :  $u^\mu u_\nu = -1$ 
  - Total  $d + 1$  variables.
- In hydrodynamics we express  $T^{\mu\nu}$  through  $T(x)$  and  $u^\mu(x)$  through the so-called constitutive equations.

0<sup>th</sup> order

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu}$$

$\epsilon \rightarrow$  energy density and  $P \rightarrow$  pressure.

1<sup>st</sup> order

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu}.$$

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left[ \eta \left( \nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} \nabla_\lambda u^\lambda \right) + \zeta g_{\alpha\beta} \nabla_\lambda u^\lambda \right].$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \rightarrow \text{Projection operator.}$$

$\eta \rightsquigarrow$  shear viscosity coefficient

$\zeta \rightsquigarrow$  bulk viscosity coefficients

For conformal fluid  $\zeta = 0$ .

2<sup>nd</sup> order

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu} + \Theta^{\mu\nu}.$$

$$\begin{aligned}\Theta^{\mu\nu} = & \eta \pi \left[ \langle D\sigma^{\mu\nu} \rangle + \frac{1}{d-1} \sigma^{\mu\nu} (\nabla \cdot u) \right] \\ & + \kappa \left[ R^{\langle \mu\nu \rangle} - (d-2) u_\alpha R^{\alpha \langle \mu\nu \rangle \beta} u_\beta \right] \\ & + \lambda_1 \sigma^{\langle \mu}{}_\lambda \sigma^{\nu \rangle \lambda} + \lambda_2 \sigma^{\langle \mu}{}_\lambda \Omega^{\nu \rangle \lambda} + \lambda_3 \Omega^{\langle \mu}{}_\lambda \Omega^{\nu \rangle \lambda}.\end{aligned}$$

- S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, Nonlinear Fluid Dynamics from Gravity, JHEP 0802, 045 (2008) [arXiv:0712.2456 [hep-th]].
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- M. Natsuume and T. Okamura, Causal hydrodynamics of gauge theory plasmas from AdS/CFT duality, Phys. Rev. D 77, 066014 (2008) [arXiv:0712.2916 [hep-th]].

# Kubo formula: Transport coefficients from thermal correlators

- Consider the response of the fluid to small and smooth metric perturbations:  $g_{xy} = \eta_{xy} + h_{xy}(t, z)$

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu} + \Theta^{\mu\nu}$$

$$T^{xy} = -P h_{xy} - \eta \dot{h}_{xy} + \eta \tau_\Pi \ddot{h}_{xy} - \frac{\kappa}{2} [(d-3)\ddot{h}_{xy} + \partial_z^2 h_{xy}].$$

- In linear response theory

$$\langle T_{xy} \rangle \sim G_{xy,xy}^R h_{xy}$$

$$G_R^{xy,xy}(\omega, k) = P - i\eta\omega + \eta\tau_\Pi\omega^2 - \frac{\kappa}{2} [(d-3)\omega^2 + k^2].$$



# Holographic computation of Green's function

- We start with five dimensional action

$$S_{\text{EM}} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} (R + 12).$$

- The background has a black-brane solution as,

$$\begin{aligned} dS^2 &= -g_{tt} dt^2 + g_{rr} dr^2 + g_{ij} dx^i dx^j, \\ g_{tt} &= r^2 \left(1 - \frac{1}{r^4}\right), \quad g_{rr} = \frac{1}{g_{tt}} \\ g_{ij} &= r^2 \delta_{ij}. \end{aligned}$$

- Solution is asymptotically  $AdS$  and boundary topology  $\rightarrow R \times R^3$ .
- The horizon is at  $r \rightarrow 1$  and asymptotic boundary is at  $r \rightarrow \infty$ .

# Holographic computation of Green's function

- We study the graviton's fluctuation in this background,

$$g_{xy} = g_{xy}^{(0)} + h_{xy}(r, x) = g_{xy}^{(0)} [1 + \epsilon \Phi(r, x)].$$

- Plugging it in the action and keeping terms to order  $\epsilon^2$ ,

$$S = \int \frac{d^4 k}{(2\pi)^4} dr (\mathcal{A}_1(r, k) \phi'(r, k) \phi'(r, -k) + \mathcal{A}_0(r, k) \phi(r, k) \phi(r, -k))$$

$$\Phi(r, x) = \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot x} \phi(r, k)$$

# Holographic computation of Green's function

- The coefficients  $\mathcal{A}_1(r, k)$  and  $\mathcal{A}_0(r, k)$  are given by

$$\mathcal{A}_1(r, k) = -\frac{\frac{1}{2}g^{rr}\sqrt{-g}}{16\pi G_5},$$
$$\mathcal{A}_0(r, k) = -\frac{\frac{1}{2}\sqrt{-g}g^{\mu\nu}k_\mu k_\nu}{16\pi G_5}.$$

# Holographic computation of Green's function

$$S = \int \frac{d^4 k}{(2\pi)^4} dr (\mathcal{A}_1(r, k) \phi'(r, k) \phi'(r, -k) + \mathcal{A}_0(r, k) \phi(r, k) \phi(r, -k))$$

- Conjugate momentum of the transverse graviton

$$\begin{aligned} \Pi(r, k_\mu) &= \frac{\partial S}{\partial \phi'(r, k)} \\ &= 2\mathcal{A}_1(r, k) \phi'(r, k) \end{aligned}$$

- The equation of motion

$$\Pi'(r, k_\mu) - 2 \mathcal{A}_0(r, k) \phi(r, k) = 0 .$$

# Holographic computation of Green's function

- Computing the gravitons action on-shell it reduces to the following surface term

$$S = \int \frac{d^4 k}{(2\pi)^4} (\mathcal{A}_1(r, k) \phi'(r, k) \phi(r, -k)) \Big|_1^\infty.$$

- Following the Minkowskian prescription (Son-Starinets), the boundary retarded Green's function is given as,

$$G_R(k_\mu) = \lim_{r \rightarrow \infty} - \frac{2\mathcal{A}_1(r, k) \phi'(r, k) \phi(r, -k)}{\phi_0(k) \phi_0(-k)}$$

- $\phi_0(k_\mu)$  is the value of the graviton fluctuation at boundary.

# A response function

- We can rewrite the boundary retarded Green's function as,

$$G_R(k_\mu) = \lim_{r \rightarrow \infty} -\frac{\Pi(r, k_\mu)}{\phi(r, k_\mu)}.$$

- Define a response function of the boundary theory

$$\bar{\chi}(k_\mu, r) = \frac{\Pi(r, k_\mu)}{i\omega\phi(r, k_\mu)} \quad \omega = k_0$$

- This function is defined for all  $r$  and  $k_\mu$ .
- Therefore the boundary Green's function is given by,

$$G_R(k_\mu) = \lim_{r \rightarrow \infty} -i\omega\bar{\chi}(k_\mu, r).$$

# Radial evolution of the response function

- Differentiate the response function w.r.t.  $r$

$$\partial_r \bar{\chi}(r, k_\mu) = \frac{1}{i\omega} \left[ \frac{\Pi'(r, k)}{\phi(r, k)} - \frac{\Pi(r, k)\phi'(r, k)}{\phi(r, k)^2} \right]$$

- Using the definition of  $\Pi(r, k)$  and field equation of motion

$$\Pi(r, k_\mu) = 2\mathcal{A}_1(r, k)\phi'(r, k) \quad , \quad \Pi'(r, k_\mu) - 2\mathcal{A}_0(r, k)\phi(r, k) = 0 \quad .$$

$$\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[ \frac{\bar{\chi}(k_\mu, r)^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right]$$

Liu+Iqbal

- Exact in  $k_\mu$ .
- First order non-linear differential equation.

# Radial evolution of the response function

$$\Sigma(r) = -2\mathcal{A}_1(r, k_\mu) \sqrt{-\frac{g_{rr}}{g_{tt}}}$$

$$\Upsilon(r) = 2\mathcal{A}_0(r, k_\mu) \sqrt{-\frac{g_{tt}}{g_{rr}}}.$$



- $$\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[ \frac{\bar{\chi}(k_\mu, r)^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right]$$
- Boundary condition,

$$\bar{\chi}(k_\mu, r)^2 \Big|_{r=1} = \frac{\Sigma(r)\Upsilon(r)}{\omega^2} \Big|_{r=1} .$$

- For two derivative gravity

$$\bar{\chi}(k_\mu, 1) = \sqrt{\frac{\Sigma(1)\Upsilon(1)}{\omega^2}} = \frac{1}{16\pi G_5}$$

# 1<sup>st</sup> order transport coefficients from flow equation

- It is trivial to see that at  $(\omega, k_i) \rightarrow 0$  limit,

$$\partial_r \bar{\chi}(k_\mu, r) = 0 \Rightarrow \bar{\chi}(k_\mu, r) = \text{constant}$$

- Using the boundary condition we get

$$\bar{\chi}(k_\mu, 1) = \bar{\chi}(k_\mu \rightarrow 0, \infty) = \eta$$

# Flow from horizon to boundary

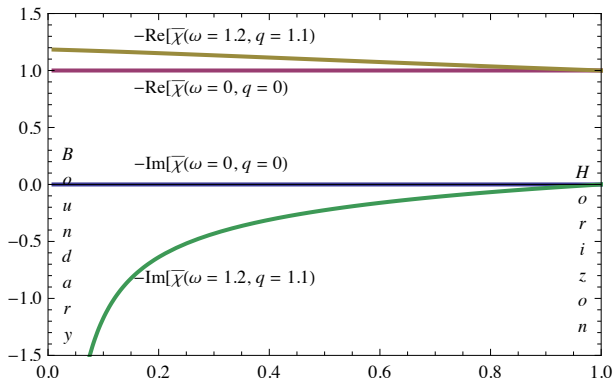


Figure: Flow of Green's function from horizon to boundary for two derivative gravity.

## 2<sup>nd</sup> order transport coefficients from flow equation

NB, S. Dutta

$$\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[ \frac{\bar{\chi}(k_\mu, r)^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right]$$

- To solve this equation up to order  $\omega \rightarrow$  replace the leading value for  $\bar{\chi}$

$$\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[ \frac{\eta^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right] + \mathcal{O}(\omega^2, k_i^2).$$

- We impose the regularity condition at the horizon.

## 2<sup>nd</sup> order transport coefficients from flow equation

✠ The solution

$$\begin{aligned} -i\omega\bar{\chi}(k_\mu, \infty) &= -i\omega \left( \frac{1}{16\pi G_5} \right) \\ &\quad + \omega^2 \left[ \frac{1}{2}(1 - \ln 2) \left( \frac{1}{16\pi G_5} \right) \right] \\ &\quad - \frac{q^2}{2} \left( \frac{1}{16\pi G_5} \right) \end{aligned}$$

- Comparing it with the expression of Green's function

$$G_R^{xy,xy}(\omega, k) = -i\eta\omega + \eta\tau_\pi\omega^2 - \frac{\kappa}{2}[(d-3)\omega^2 + q^2].$$

$$\eta = \frac{T^3\pi^3}{16\pi G_5}, \quad \kappa = \frac{\eta}{\pi T}, \quad \tau_\pi = \frac{2 - \ln 2}{2\pi T}.$$

- Thus we see that the hydrodynamic characteristic of the field theory at UV fix point is completely determined in terms of this IR boundary condition.
- The flow equation may be visualized as RG equation.

# Generalized flow equation in HD gravity

NB, S. Dutta

- We consider a gravity set-up with  $n$  derivative action.

$$\mathcal{I} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[ R + 12 + \alpha' R^{(n)} \right]$$

- Not clear how to define the conjugate momentum and response function.
- Way out:

$$S_{eff} = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left[ \mathcal{A}_1^{HD}(r, k) \phi'(r, k) \phi'(r, -k) \right. \\ \left. + \mathcal{A}_0^{HD}(r, k) \phi(r, k) \phi(r, -k) \right]$$

NB, S. Dutta, JHEP 0903:116,2009

# Generalized flow equation in HD gravity

## § Steps to write the flow equation in HD gravity

- Generalized momentum

$$\Pi^{HD}(r, k) = 2\mathcal{A}_1^{HD}(r, k)\phi'(r, k)$$

- Boundary Green's function

$$\begin{aligned} G_R^{HD}(k_\mu) &= \lim_{r \rightarrow \infty} - \frac{2\mathcal{A}_1^{HD}(r, k)\phi'(r, k)\phi(r, -k)}{\phi_0(k)\phi_0(-k)} \\ &= \lim_{r \rightarrow \infty} - \frac{\Pi^{HD}(r, k_\mu)}{\phi(r, k_\mu)} \end{aligned}$$

- Define a response function in higher derivative theory

$$\bar{\chi}^{HD}(k_\mu, r) = \frac{\Pi^{HD}(r, k_\mu)}{i\omega\phi(r, k_\mu)}.$$



# Generalized flow equation in HD gravity

★ Therefore the flow equation is given by,

$$\partial_r \bar{\chi}^{HD}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[ \frac{\bar{\chi}^{HD}(k_\mu, r)^2}{\Sigma^{HD}(r, k)} - \frac{\Upsilon^{HD}(r, k)}{\omega^2} \right]$$

Here we define

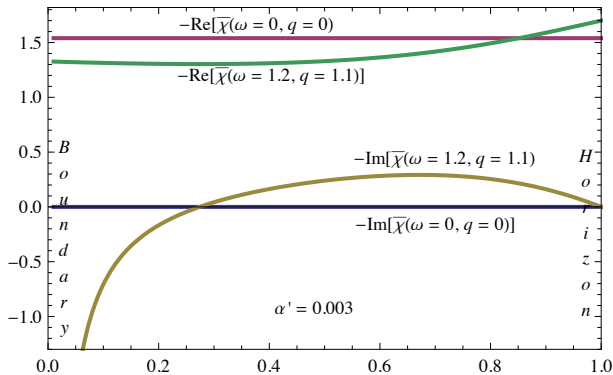
$$\begin{aligned} \Sigma^{HD}(r, k) &= -2\mathcal{A}_1^{HD}(r, k_\mu) \sqrt{-\frac{g_{rr}}{g_{tt}}} \\ \Upsilon^{HD}(r, k) &= 2\mathcal{A}_0^{HD}(r, k_\mu) \sqrt{-\frac{g_{tt}}{g_{rr}}}. \end{aligned}$$

## Boundary Condition

- The response function  $\bar{\chi}^{HD}$  should be well-defined at horizon. This implies,

$$\bar{\chi}^{HD}(k_\mu, r) \Big|_{r=r_h} = \sqrt{\frac{\Sigma^{HD}(r)\Upsilon^{HD}(r)}{\omega^2}} \Big|_{r=r_h}$$

# Flow from horizon to boundary in HD gravity



**Figure:** Flow of Green's function from horizon to boundary in higher derivative gravity.

- Exact Gauss-Bonnet theory

$$\eta = \frac{1}{16\pi G_5} (1 - 4\lambda_{gb})$$

$$\kappa = \frac{2\lambda_{gb} (8\lambda_{gb} - 1)}{(1 - \sqrt{1 - 4\lambda_{gb}}) (4\lambda_{gb} - 1)}$$

$$\begin{aligned} \tau_\pi T = & \frac{1}{4\pi(-1 + 4\lambda_{gb})} \left[ -8\lambda_{gb}^2 + 12\sqrt{1 - 4\lambda_{gb}}\lambda_{gb} \right. \\ & + 10\lambda_{gb} - 2\sqrt{1 - 4\lambda_{gb}} - 4\log(2)\lambda_{gb} \\ & + (1 - 4\lambda_{gb}) \log \left( -4\lambda_{gb} + \sqrt{1 - 4\lambda_{gb}} + 1 \right) \\ & \left. + (4\lambda_{gb} - 1) \log(1 - 4\lambda_{gb}) - 2 + \log(2) \right]. \end{aligned}$$

- To preserve causality of a conformal fluid there exists a bound [Buchel+Myers]

$$\tau_{\pi} T - 2 \frac{\eta}{s} \geq 0 .$$

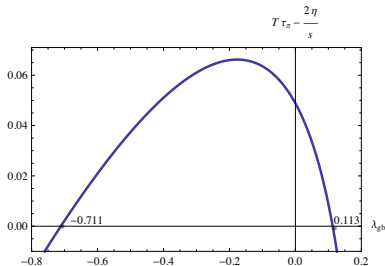


Figure: Bound on  $\lambda_{gb}$

- The action

$$S_{\text{EM}} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} (R + 12 - F^2) + S_{\text{CS}}$$

Kraus + Dhoker

- Three possible extremal near-horizon geometry.

- 

$$qB(\zeta^2 - 1) = 0$$

- Shear viscosity to entropy density

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

- The higher derivative term

$$S_{\text{HD}} = \frac{\alpha'}{16\pi G_5} \int d^5x \sqrt{-g} \left[ c_1 \text{Rim}^2 + c_2 \text{Rim} \cdot F^2 + c_3 (F^2)^2 + c_4 F^4 + c_5 \text{GCS} \right]$$

Myers + paulos + Sinha

- Shear viscosity to entropy density

$$\frac{\eta}{s} = \frac{1}{4\pi} + \frac{\alpha'}{\pi} \left[ \frac{c_1}{3} \left( \left( q^2 - \frac{B^2}{r_h^4} \right) - 6 \right) + \frac{c_2}{2} \left( q^2 - \frac{B^2}{r_h^4} \right) \right] + \mathcal{O}(\alpha'^2)$$

- Second order

$$\begin{aligned}\kappa &= \frac{\eta}{\pi T} \left[ \left( 1 - \frac{q^2}{2r_0^6} \right) - \alpha' \left( 10c_1 - \frac{q^2}{3r_0^6} (37c_1 - 48c_2) \right) \right] \\ \tau_\pi T &= \frac{2 - \ln 2}{2\pi} - \frac{q^2(5 - 3 \ln 2)}{4\pi r_0^6} \\ &\quad + \alpha' \left[ -\frac{11c_1}{2\pi} + \frac{q^2}{4\pi r_0^6} (-16c_2 + 5c_1(11 - 4 \ln 2)) \right]\end{aligned}$$



# Summary and Outlook

- We study the flow of retarded Green's function of energy momentum tensor in presence of generic matter coupling and higher derivative gravity.
- The flow equation is valid for any momentum:  $k_\mu \rightarrow$  Any higher order transport coefficient can be computed from this flow equation.
- It would be interesting to understand the connection between low frequency behavior of membrane fluid and boundary fluid using this flow equation. *Any UV/IR relation??*

**Thank you**